On Algebraic Thinking

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Robert Davis (1975)

$$\frac{3}{x} = \frac{6}{3x+1}$$

“Henry cannot divide 3 by x, because he doesn’t know what x is.”

The problem-solving process “is NOT linearly **sequential**.”

Al-Khwarizmi

- “By the division of thing by thing and two dirhams, half a dirham appears as quotient.”

Modern notations:

$$\frac{x}{x+2} = \frac{1}{2}$$

$$\frac{a}{d} = q$$

$$\rightarrow q \times d = a$$

- Multiply, therefore, **thing and two dirhams** by half a dirham [and the **thing** is restaured]."

$$\frac{\boxed{x}}{\boxed{x+2}} = \frac{1}{2}$$

Antonio de Mazzinghi (14th century)

$$\frac{4000}{x+6000} - \frac{3000}{x+5000} = \frac{1}{15}$$

By analogy with fractional numbers:

$$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$$

What does it mean?

- Does it mean that algebraic thinking is an ***extension*** of arithmetic thinking and that algebra is a generalized arithmetic?
- Or is algebraic thinking something different from arithmetic thinking while keeping a similar underpinning structure?

Agenda of my presentation

- What are the differences between arithmetic and algebraic thinking?
- Remarks on the historical development of algebraic thinking
- Three distinctive interrelated features of Algebraic Thinking
- Application to Early Algebra

Arithmetic and Algebraic Thinking

- 588 passengers must travel from one city to another. Two trains are available. One train consists only of 12-seat cars, and the other only of 16-seat cars. Supposing that the train with 16-seat cars will have eight cars more than the other train, how many cars must be attached to the locomotives of each train?

Bednarz, Radford, Janvier, & Lepage (PME 1992)

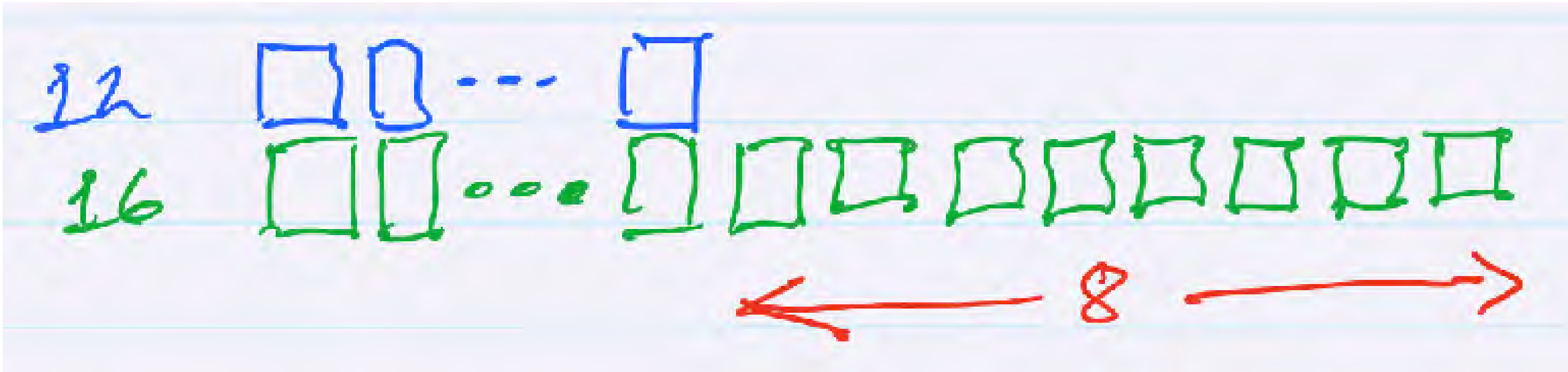
Arithmetic Thinking

- 588 passengers
- 12-seat cars
- 16-seat cars
- The train with 16-seat cars will have eight cars more than the other train

Procedure:
8 x 16 = 128 passangers
588 - 128 = 460 passengers
460 ÷ 28 = 16.4
Answer: 17 12-seat cars and 25 16-seat cars.

(14- and15-year olds)

Bednarz, Radford, Janvier, & Lepage (PME 1992)



$8 \times 16 = 128$

$588 - 128 = 460$

$460 \div 28 = 16.4$ cars

Answer = 17 12-Seat cars

Algebraic Thinking

- 588 passengers
- 12-seat cars
- 16-seat cars
- the train with 16-seat cars will have eight cars more than the other train

1st x 12; 2nd (x+8)12

$588 = x.12 + (x+8)16$

$588 = 12x + 16\,x + 128$

$-12x - 16x = 128 - 588$

$-28x = -460$

$28x=460$

$x= 16.42$

1st $\Rightarrow 16.42 \times 12 = 197.14$

2nd $\Rightarrow (16.42+8)16= 390.72$

What is the difference?

- Arithmetic:
- successive calculations with the *given known numbers*
- *semantic control* throughout the problem-solving procedure

- Algebra:
- *Introduction of the unknown quantity* at the very beginning
- Global representation of the problem
- *detachment* from the mean quan

$x = 16.42$

1st $\Rightarrow 16.42 \times 12 = 197.14$

Bednarz, Radford, Janvier, & Lepage (PME 1992)

Arithmetic Thinking - Algebraic Thinking

- Is it a question of rupture or filiation?
- Is algebra a generalized arithmetic or is it something else?

Two routes to algebra

- Word-problems (equations) and
- Patterns (sequence generalization)

Rupture

Continuity

Algebraic Thinking = Generalizing?

"For some authors (e.g., Open university, 1985), the main idea of algebra is that it is a means of representing and manipulating generality and, thus they see algebraic thinking everywhere — even in the recording of geometric transformations."

(C. Kieran, PME 1989, p.170)

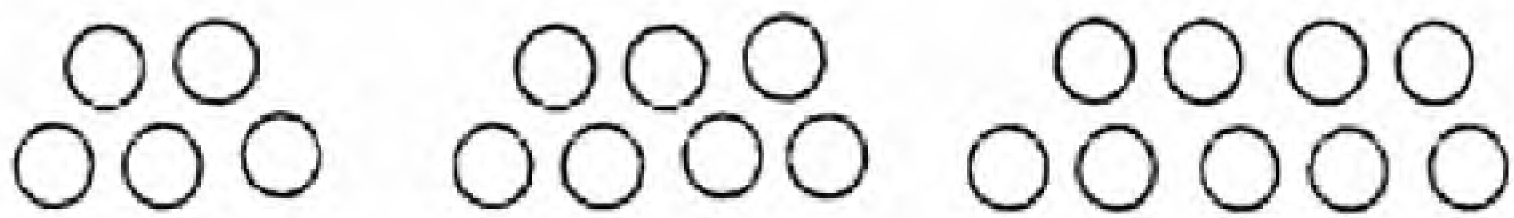


Figure 1 Figure 2 Figure 3

Trial and error: “times 2 plus 1”, “times 2 plus 2” or “times 2 plus 3” and check their validity on a few cases.

One group of students suggested: “nx2(+3)”. How come? “We found it by accident.”

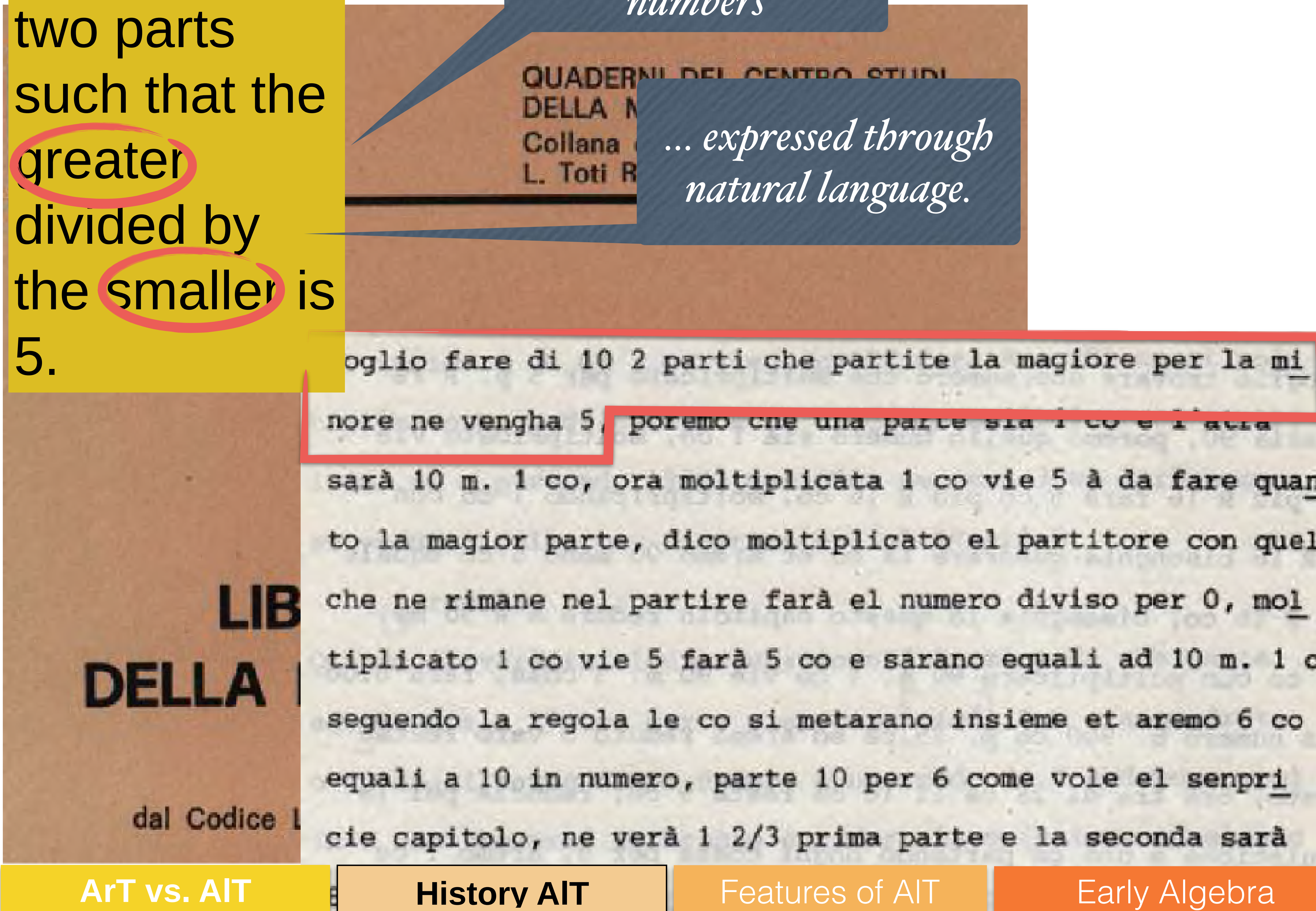
Is this algebraic thinking?
I do not think so…

(Radford, PME 2006)

I want to make 10 into two parts such that the greater divided by the smaller is 5.

Unknown or indeterminate numbers

… expressed through natural language.



oglio fare di 10 2 parti che partite la magiore per la mi_
nore ne vengha 5, poremo che una parte sia 1 co e l'atra
sarà 10 m. 1 co, ora moltiplicata 1 co vie 5 à da fare quan_
to la magior parte, dico moltiplicato el partitore con quel
che ne rimane nel partire farà el numero diviso per 0, mol_
tiplicato 1 co vie 5 farà 5 co e sarano equali ad 10 m. 1 c
seguendo la regola le co si metarano insieme et aremo 6 co
equali a 10 in numero, parte 10 per 6 come vole el senpri_
cie capitolo, ne verà 1 2/3 prima parte e la seconda sarà

I want to make 10 into two parts such that the greater times the smaller is 5

Voglio fare di 10 2 parti che partite la magiore per la mi
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8 1/3.

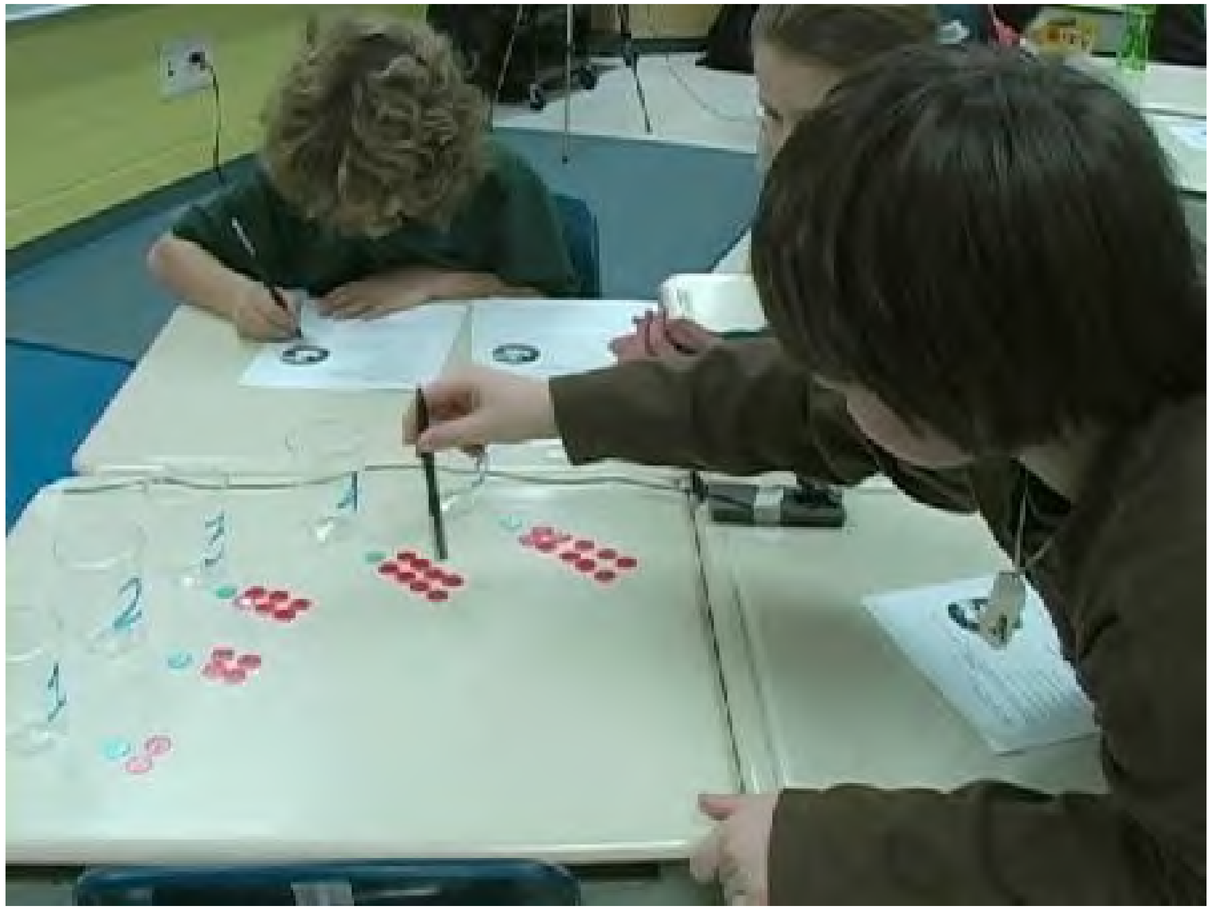
Let one part be 1 co [thing] and the other part be 10 m 1 co.

Expression of the indeterminate quantities through a different, technical language

Expression

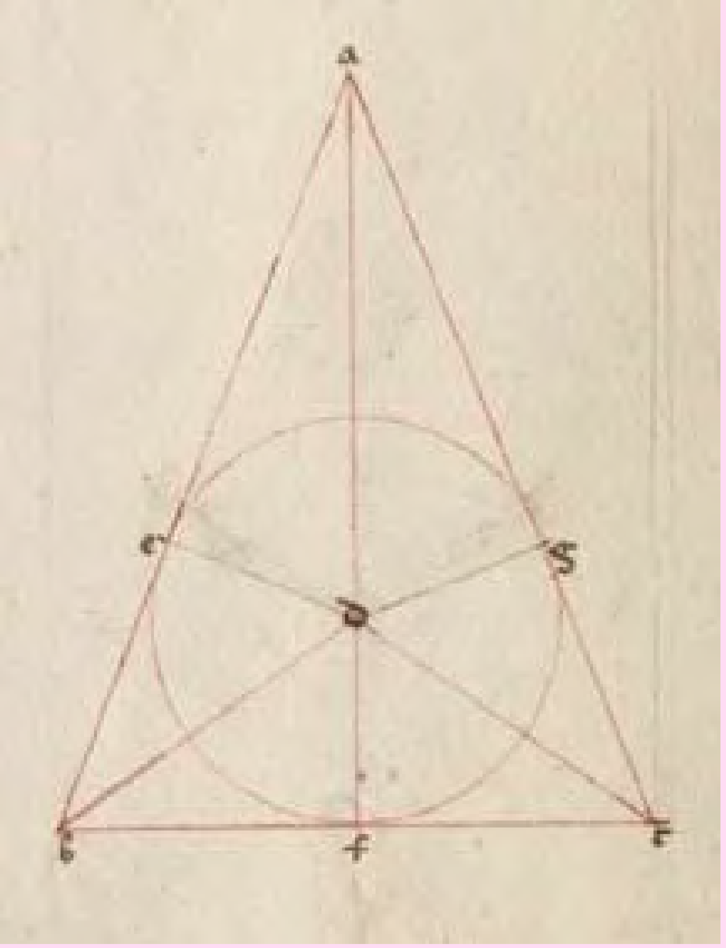
- The indeterminate numbers involved in the situation must be expressed in some way.
- You can use alphanumeric characters, but not necessarily.

- The expression of indeterminate numbers can also be made through gestures, unconventional or conventional signs (graphics, for example), or even a combination of all these.



An essential idea …

- **It is not because we use letters that we are thinking algebraically.**



One can think algebraically without necessarily using letters.

Now multiply 1 co by 5 which results in 5 co that will be equal to 10 m 1 co

r Voglio fare di 10 2 parti che partite la magiore per la mi
nore ne vengha 5, ponamo che una parte sia 1 co e l'atra
sarà 10 m. 1 co, ora moltiplica 1 co via 5 è da fare quan
to la magior parte, dico moltiplicato el partitore con quel
che ne rimane nel partire farà el numero diviso per 0, mol
tiplicato 1 co vie 5 farà 5 co e sarano equali ad 10 m. 1 co
seguendo la regola le co si metarano insieme et aremo 6 co
equali a 10 in numero, parte 10 per 6 come vole el senpri
cie capitolo, ne verà 1 2/3 prima parte e la seconda sarà
8 1/3.

$$\frac{10 - x}{x} = 5$$

AlT is analytic

- Although they are unknown, indeterminate numbers are treated in the same way as known numbers: they are added, subtracted, multiplied, divided, and so on.

"... Without distinction between known and unknown numbers " (Descartes, La Géometrie)

Three distinctive interrelated features of Algebraic Thinking

Al.T.

- resorts to:
  - **indeterminate quantities** and
  - specific culturally and historically evolved **modes of representing/symbolizing** these indeterminate quantities and their operations,
- and deals with:
  - indeterminate quantities in an **analytical** manner.

Algebraic Thinking

indeterminate quantities

- 588 passengers
- 12-seat cars
- 16-seat cars.
- the train with 16-seat cars will have eight cars more than the other train.

symbolized

1st **x** 12; 2nd (**x+8**)12

$588 = x.12 + (x+8)16$

$588 = 12x + 16x + 128$

$-12x - 16x = 128 - 588$

$-28x = -460$

$28x = 460$

$x = 16.42$

dealt with in an analytical manner

1st ⇒ $16.42 \times 12 = 197.04$

2nd ⇒ $(16.42+8)16 = 390.72$

Algebraic Thinking

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1st ⇒ 16.42 x 12 = 197.04

2nd ⇒ (16.42+8)16= 390.72

Symbolizing the sought-after numbers

Nominalization

a “theoretical tool to examine how symbolic expressions become endowed with meaning”
(Radford, PME 2002)

Algebraic Thinking

576

- ~~588~~ passengers
- 12-seat cars
- 16-seat cars.
- the train with 16-seat cars will have eight cars more than the other train.

1st **x** 12; 2nd (**x+8**)12

576 = x.12 + (x+8)16

Nominalization

576 12x+16x+8

I. Demonty (2017)

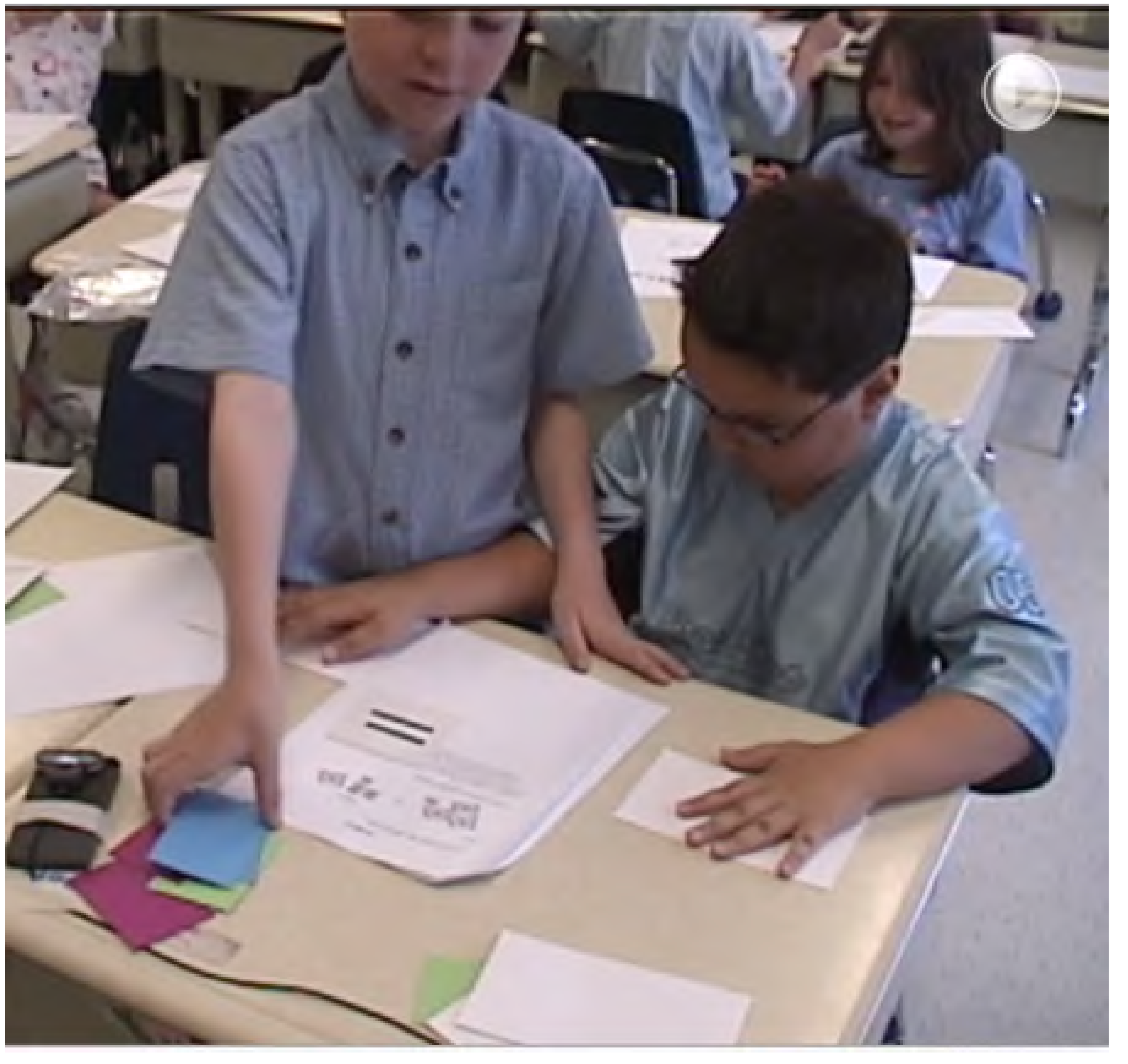
$$\frac{x}{x+2} = \frac{1}{2} \qquad \frac{10-x}{x} = 5$$

The unknown will appear in both sides of the equation.

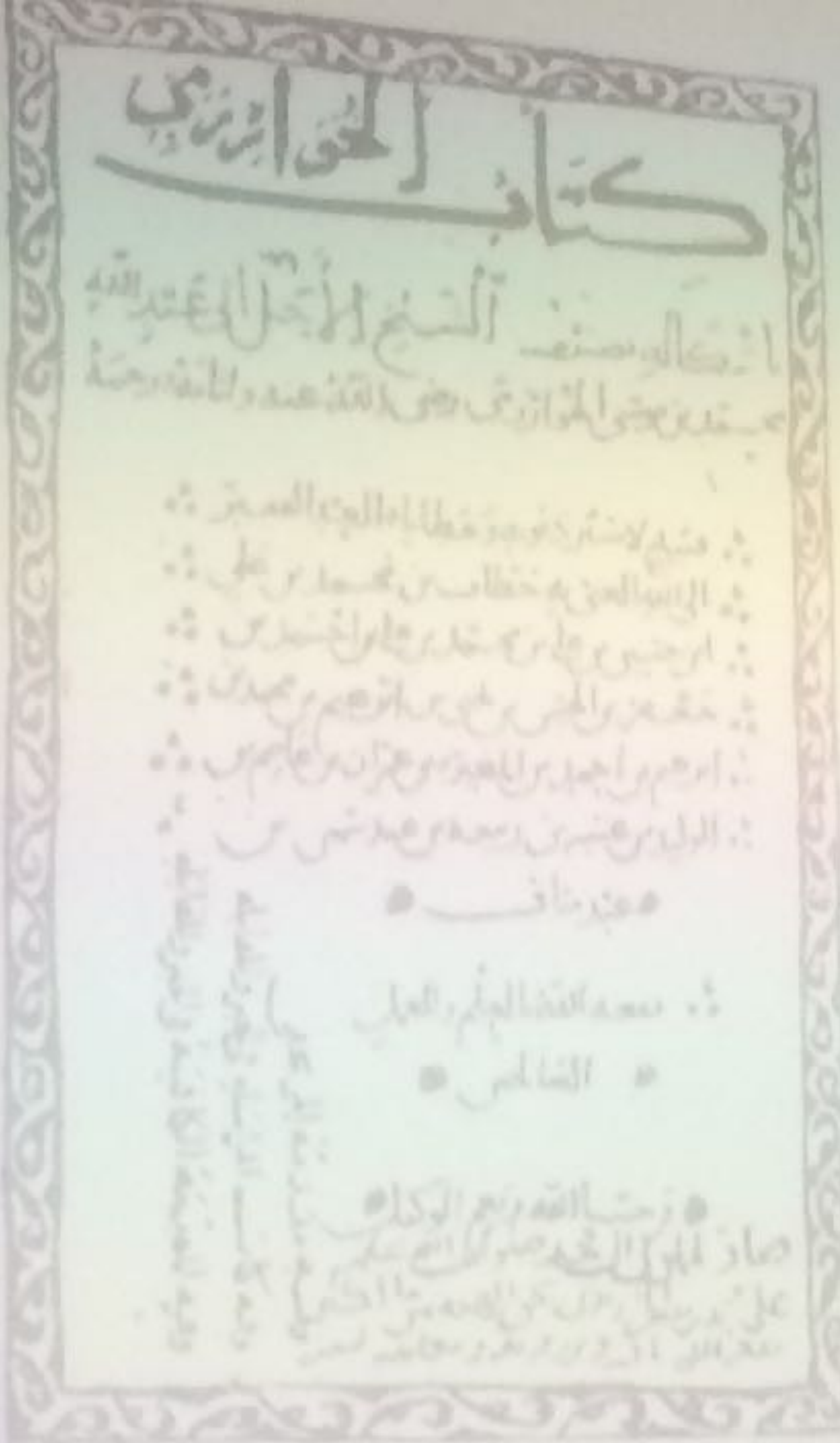
$$ax + b = c$$
$$ax + b = cx + d$$

(Filloy & Rojano, FLM 1989)

EARLY ALGEBRA



Sylvain and Chantal have some hockey cards. Chantal has three cards and Sylvain has two cards. Their mother puts some cards in three envelopes making sure to put the same number of cards in each envelope. She gives Chantal one envelope and two to Sylvain. Now, the two kids have the same amount of hockey cards. How many hockey cards are in an envelope?



Abrégé de calcul selon les méthodes de l'al-jabr et l'al muqabala rédigé entre 813 et 833.

Al-Jabr : réduction, restauration, consiste à supprimer les expressions retranchées :

$x^2 + 3 = 5 - 10x \rightarrow x^2 + 10x + 3 = 5$

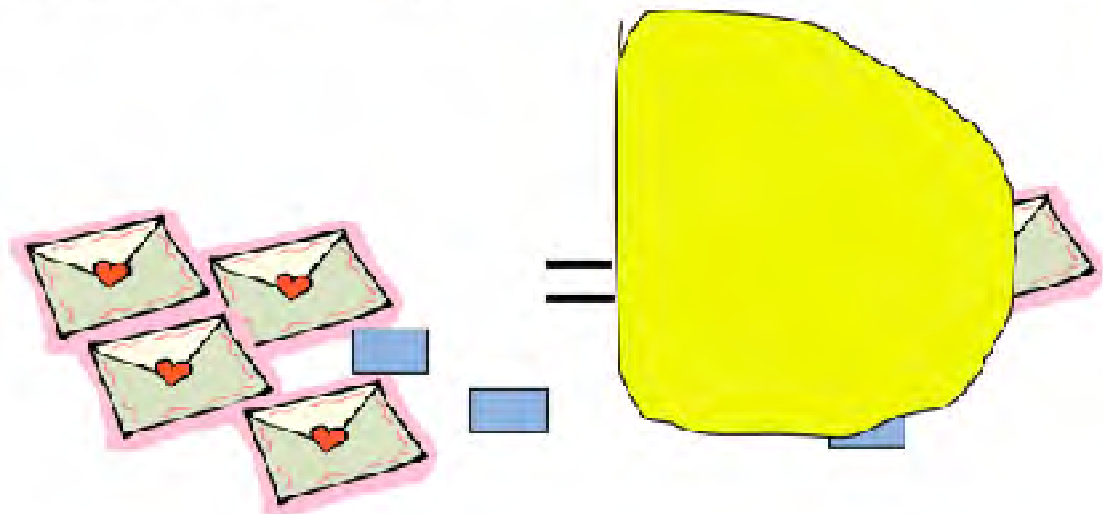
Al muqabal veut dire simplification

$x^2 + 10x + 3 = 5 \rightarrow x^2 + 10x = 2$

Problème 3

Marianne se préparait à résoudre l'équation ci-dessous quand elle a renversé du jus d'orange dessus.

On sait que la solution était égale à 2. Quel était le côté droit de l'équation? Trouve au moins deux possibilités.



Grade 3

b) $2x + 3 = x + 5$

Problème 6

Marianne se préparait à résoudre l'équation ci-dessous quand elle a renversé deux gouttes de jus d'orange dessus.

On sait que la solution était égale à **2**. Quel était le côté droit de l'équation? Trouve au moins deux possibilités.

Vérifie qu'elles fonctionnent.

$3x + 2 = \quad x + $

Grade 5

a) $3n + 2 = 1n + 8$

Grade 4

b)

$2+2n=2+3n$

$2+2n=2+3n$

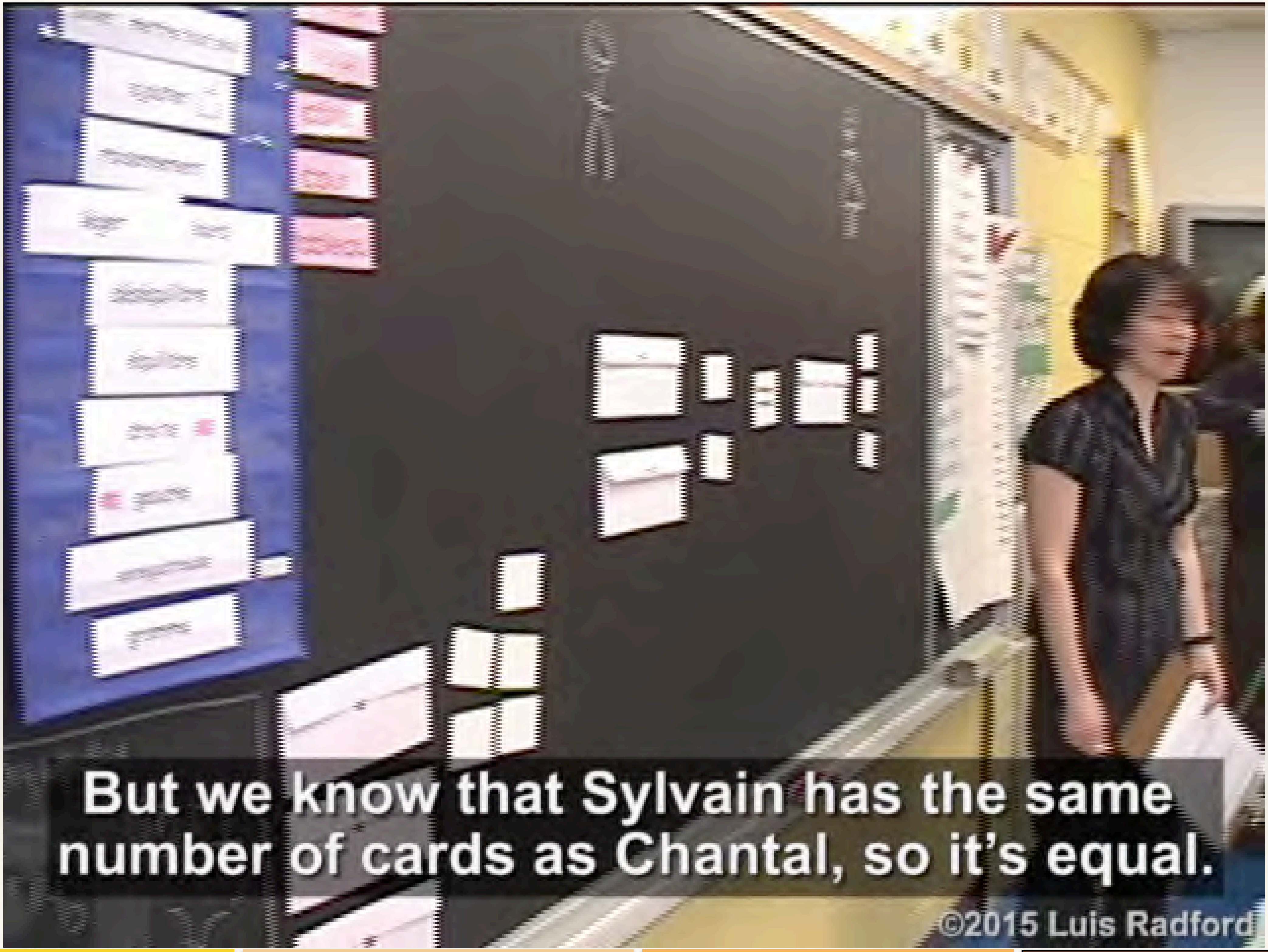
$0+2n=0+3n$

$0+1n=0+2n$

$0=0+1n$

$1n=0$

Grade 6


But we know that Sylvain has the same number of cards as Chantal, so it's equal.
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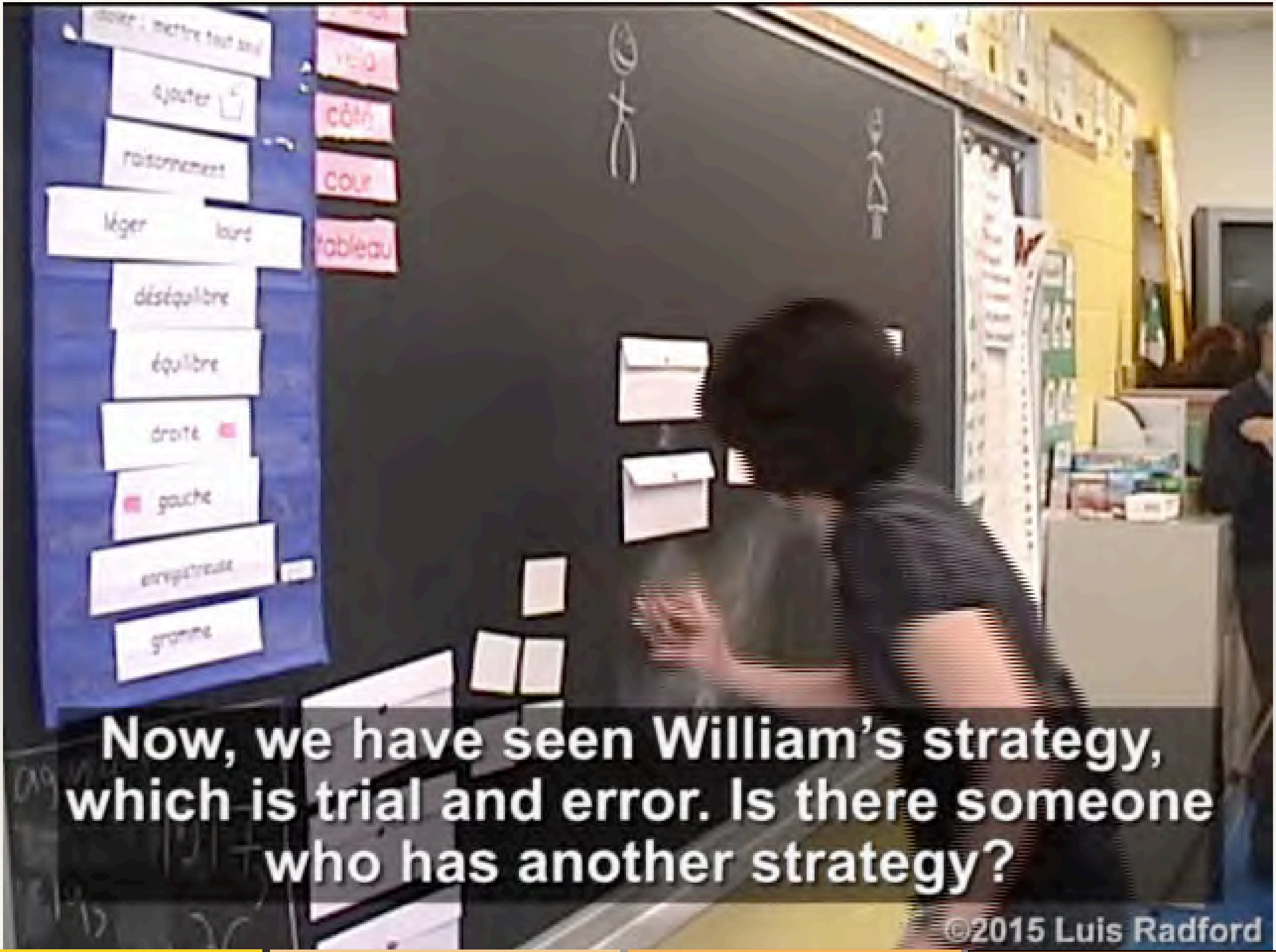
The teacher reformulates W's linguistic expression and in doing so she makes explicit the ideas.
But she reformulates W's *thinking* as well.
The reformulation carries an ideological valence.
There is an opening towards a new *theoretical awareness.*

- T : So if I t
trial and e

- W: uhhuh…

- T: That's it, you said: ah! I am going to *pretend* that there is a card here, a card here, a card here, that is what you did?

- Mhu mhu


Now, we have seen William's strategy, which is trial and error. Is there someone who has another strategy?
ajouter
raisonnement
léger
lourd
déséquilibre
équilibre
droite
gauche
gramme
tableau
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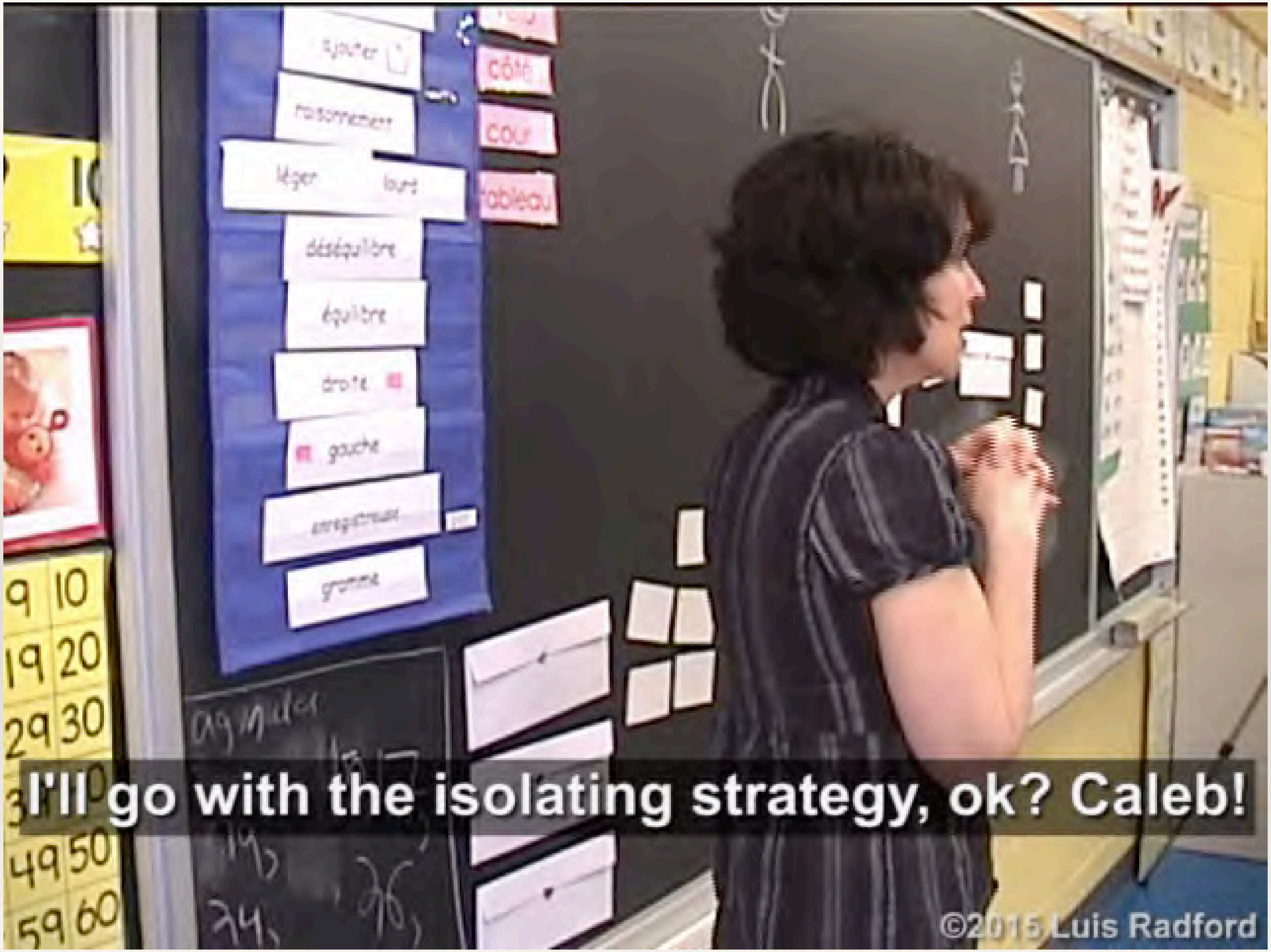
The teacher reformulates again the student's strategy. *She brings the "isolation-of-the-unknown" idea to the fore (the al-muquabala of Al-Khwarizmi).*

- P: Ok, so you found the solution like that? You, you isolated a little bit, but you didn't isolate completely, eh? That was your solution, you removed envelopes eh?
- J: Yes


I'll go with the isolating strategy, ok? Caleb!
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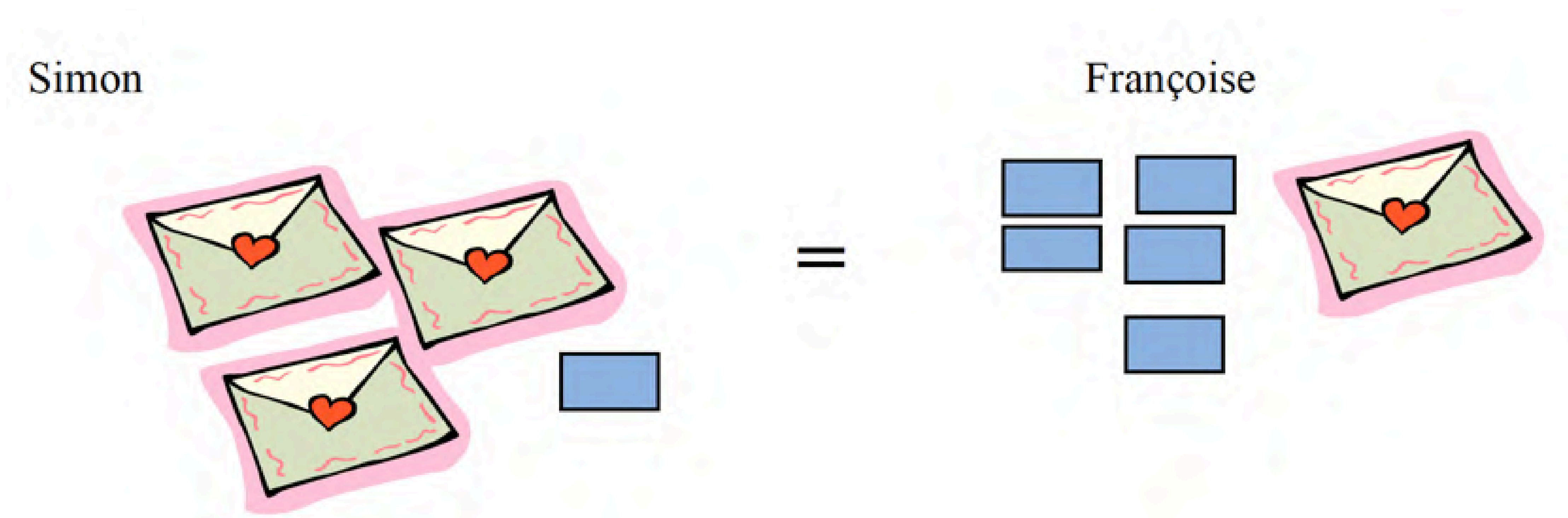
Simon
Françoise
=


Look! Pretend that, look. Just give me the paper.
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Equations in Kindergarten