Bell’s inequality holds for all non-product states

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We prove that any non-product state of two-particle systems violates a Bell inequality.

In 1964 Bell [1] surprised many physicists by proving that there are states of two-quantum-particle systems that do not satisfy a certain inequality which he derived from very plausible assumptions about locality and realism in the spirit of Einstein. A huge literature has covered lots of aspects, ranging from philosophy to experimental physics, of the new field opened by Bell’s 1964 paper. See, for instance, the valuable mark review of Clauser and Shimony [2], and the more recent reviews by Greenberger and co-workers [3], and by Mermin [4]. The two latter reviews also contain the more recent results on a version of Bell’s result without inequalities, but valid only for systems with more than two particles.

It is well known that not all states of two-particle systems violate the Bell inequality \(^{\text{81}}\), the product states, for instance, do satisfy the inequality. In this brief note I prove that the product states are the only states that do not violate any Bell inequality. When I had the chance to discuss this equivalence between “states that violate the inequality” and “entangled states” (i.e. “non-product states”) with John Bell last September, just before his sudden tragic death, I was surprised that he did not know this result. This motivates me to present today this little note which I have had on my shelves for many years and which may be part of the “folklore”, known to many people but (apparently) never published. I would like to dedicate this Letter to John Bell, not only as the person who discovered the inequality and thus opened the field of “experimental metaphysics”, but also as the man who taught me so much during our discussions and who amazed me many times by his capability to immediately focus on the central point under investigation.

**Theorem.** Let \(\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2\). If \(\psi\) is entangled (i.e. \(\psi\) is not a product), then \(\psi\) violates the Bell inequality, that is there are projectors \(a, a', b, b'\), such that

\[
|P(a, b) - P(a, b')| + P(a', b) + P(a', b') > 2,
\]

where

\[
P(a, b) = \langle (2a - 1) \otimes (2b - 1) \rangle_\psi.
\]

**Proof.** Let \(\{\varphi_i\}\) and \(\{\theta_i\}\) be orthonormal bases of \(\mathcal{H}_1\) and \(\mathcal{H}_2\), respectively, such that

\[
\psi = \sum_i c_i \varphi_i \otimes \theta_i,
\]

for some real \(c_i\), with \(c_1 \neq 0 \neq c_2\). Notice that the above sum runs over only one index (polar or Schmidt decomposition); the existence of two non-zero \(c_i\)'s comes from the entanglement of \(\psi\). One has

\[
\psi = \chi + \chi_\perp,
\]

where

\[
\chi = c_1 \varphi_1 \otimes \theta_1 + c_2 \varphi_2 \otimes \theta_2 \in \mathbb{C}^2 \otimes \mathbb{C}^2
\]

and \(\chi_\perp \perp \chi\).

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\(^{\text{81}}\) There are many Bell inequalities, we shall use one due to Clauser, Horne, Shimony and Holt [5].
For notational convenience, we apply a product unitary operation on $\psi$ such that
$$\chi = c_1 |+\rangle + c_2 |\rangle.$$
(In the standard case of two spin-1/2, this amounts to the choice of an appropriate reference frame in the two regions where the spins are located.) A straightforward computation leads to
$$\langle a\sigma \otimes b\sigma \rangle \chi = -2c_1 c_2 (a_x b_x + a_y b_y) - a_z b_z,$$
for all vectors $a, b \in \mathbb{R}^3$, where $\sigma$ are the usual Pauli matrices.

Let $a_y = b_y = 0$, $a_z = \sin(\alpha)$, $b_z = \sin(\beta)$, $b_z = \cos(\beta)$ and similarly for $a'$ and $b'$. Furthermore let $\alpha = 0$ and $\alpha' = \pm \pi/2$ where the sign is opposite to that of the product $c_1 c_2$:
$$|P(a, b) - P(a', b')| + P(a', b) + P(a', b') = |\cos(\beta) - \cos(\beta')|$$
$$+ 2 |c_1 c_2| (1 + 4 |c_1 c_2|)^{-1/2}.$$
This is maximum for $\cos(\beta) = -\cos(\beta') = (1 + 4 |c_1 c_2|)^{-1/2}$, $\sin(\beta) > 0$, $\sin(\beta') > 0$, and for this value one has
$$|P(a, b) - P(a', b')| + P(a', b) + P(a', b')$$
$$= 2 (1 + 4 |c_1 c_2|)^{-1/2}.$$
This is strictly greater than 2 for all non-zero $c_1$ and $c_2$, that is for all entangled $\psi$. Q.E.D.

A natural related question is whether a similar result holds for the Bell theorem without inequality [3]. I shall argue that no such result holds. Let $\psi \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. In order to derive the Bell theorem without equality, i.e. a contradiction between locality and perfect correlation, we need to find enough triple-product operators $A_i \otimes B_j \otimes C_k$ admitting $\psi$ as a common eigenvector. In $\mathbb{C}^2$ the most general self-adjoint operators are of the form $b + a\sigma$. Moreover the values of the eigenvalues are irrelevant, we can thus choose $b = 0$ and $|a| = 1$. Consequently we have
$$a\sigma \otimes b\sigma \otimes c\sigma \psi = \psi.$$
Hence
$$a = \frac{\langle \psi | \sigma \otimes 1 \otimes 1 | \psi \rangle}{\langle \psi | \sigma \otimes 1 \otimes 1 | \psi \rangle}.$$
Now, consider the following non-product state:
$$\psi = \alpha |111\rangle + \beta |000\rangle - |000\rangle.$$ If $|\alpha| \neq |\beta|$, then the only possibilities for $a$ are $\pm \epsilon_1$; and similarly for $b$ and $c$. Therefore, for such states $\psi$ with $|\alpha| \neq |\beta|$, there is only one experimental arrangement leading to perfect correlations. It is not difficult to produce local hidden variables that reproduce the correlations of a unique experimental arrangement. We conclude that no Bell theorem without inequality holds for such $\psi$'s. However, clearly, such $\psi$'s would violate a Bell inequality.

References