Continuous quantum jumps and infinite-dimensional stochastic equations

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From a mathematical point of view, a class of infinite-dimensional stochastic differential equations describing continuous spontaneous localization in quantum dynamics will be studied. Existence and uniqueness of weak and strong solutions of respective equations are proven via Cameron–Martin–Girsanov transformation. The case of Gaussian initial states is explicitly solved.

I. INTRODUCTION

The interest for stochastic modifications of the Schrödinger equation has greatly increased the last years among physicists. One reason for this interest comes from the Ghirardi–Rimini–Weber–Bell1,2 proposal to unify micro- and macrodynamics. This proposal naturally meets Pearle’s work3–5 on the quantum “measurement problem,” and the Ito stochastic differential equation proposed by one of the authors.5 These ingredients have led to a rapidly expanding literature, see Refs. 6–18 and references therein. Diosi6 first proposed a stochastic equation containing the main physically appealing features of the proposals:

$$d\psi_t = -i(p^2/2m)\psi_t dt + (q - \bar{q}_t)\psi_t dW_t,$$

where $p$ and $q$ are the momentum and position operator in $L^2(\mathbb{R})$, respectively, $\bar{q}_t = \langle \psi_t | q | \psi_t \rangle / \langle \psi_t | \psi_t \rangle$ and $W_t$ is the standard Wiener process. The mathematical meaning of this equation, however, was suspicious, because of the unbounded operators $q$ in the stochastic equation.

The problem of existence and uniqueness of solution of Eq. (1.1) is also interesting from a mathematical point of view, as a case study of Hilbert space valued nonlinear stochastic differential equations. Nonlinear stochastic partial differential equations (NSPDE) were studied since the early 1970s in a number of papers and in many contexts, for example, Refs. 19–29 and references therein. Almost all the papers were restricted to the notion of the strong solution of NSPDE. Here, we apply the Cameron–Martin–Girsanov measure transformation,20,21,23,30 further called Girsanov transformation, and therefore prove the existence of a weak solution of Eq. (1.1). We apply this result to prove existence and uniqueness of a strong solution of (1.1). To the authors’ knowledge this is the first mathematically precise study of these types of equations.

Two further introductory remarks are needed. The first one is about Girsanov transformations20,21,23,30 and Pearle’s “raw” and “physical”7 ensembles. We show that the two ensembles distinguished by Pearle are related by a Girsanov transformation; the distinction is thus made precise. We also show that this transformation, well known by mathematicians, appears naturally in our physically motivated derivation of the equation. The second remark is about the connection with Hudson and Parthasarathy “quantum Itô calculus”.31 The basic equations look quite different, due to the nonlinearity in Eq. (1.1). But the conceptual frameworks are related, and Belavkin32,33 has indeed derived Eq. (1.1) by continuously applying the “projection postulate.” More work in that direction is ongoing.

II. DERIVATION OF THE EQUATION

Let us start by recalling that not all stochastic modifications of the Schrödinger equation are physically acceptable if one aims at preserving the “peaceful coexistence”5–8 between quantum mechanics and relativity.5,35,8,36 Indeed, it is well known that a (nonpure) density operator has nonunique decompositions into mixtures of pure states. Accordingly, there are different mixtures of pure states that are undistinguishable, as for instance half the spin 1/2 up and half down in different directions. These different mixtures can be prepared in a finite time at an arbitrarily large distance, thanks to EPR like quantum correlations, as for instance in the Bohm77 version of the EPR paradox. But since, according to quantum mechanics, the mixtures are undistinguishable, these “preparations at a distance” do not violate relativity (but “only” Bell’s inequality). Now, if we modify the dynamics, then, in general, different mixtures will be distinguishable, even if they correspond to a same density operator. This is what Mielnik called “mobility.”38,39 The new dynamics would thus contradict relativity, since it allows for arbitrary fast communications by preparations at a distance. There are, however, some nonlinear stochastic differential equations, like Eq. (1.1), for which the density operator $\rho_t$, obtained by averaging over the Wiener process $W_t$, follows a closed (deterministic) evolution equation.5,6,8,11
For $\rho_t = \langle \langle \psi, \psi^+ \rangle \| \psi \| ^2 \rangle$, where the double brackets mean "average over the Wiener process $W_i$," one has
\[
\frac{d\rho_t}{dt} = -i \left( \frac{p^2}{2m} \rho_t - \frac{1}{2} [q, [q, \rho_t]] \right)
\]

In this section we shall apply this constraint to derive the class of equations we are interested in.

Let $A$ and $B$ be two self-adjoint operators and consider the following linear Stratonovich stochastic differential equation (in this section all computations are merely formal, the next sections will be rigorous):
\[
d\psi_t = A\psi_t dW_t + B\psi_t dt,
\]
where $W_t$ is a standard Wiener process: $(dW_t)^2 = dt$. We are interested in the one-dimensional projector
\[
P_t = \frac{\psi_t, \psi_t^*}{\| \psi_t \|^2}
\]
and in the corresponding density operator
\[
\rho_t = \langle \langle \psi, \psi^+ \rangle \| \psi \| ^2 \rangle
\]
where the double brackets mean "average over some process $W^*_t". We require the process $W^*_t$ to be a standard Wiener process. (2.1)

Moreover, the probability for any eigenstate $|\alpha\rangle$ is equal to the quantum probability $\langle \alpha | P_0 | \alpha \rangle$ (see, e.g., Ref. 8).

III. EXISTENCE AND UNIQUENESS THEOREMS FOR THE EQUATION

Let $H$ be a separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, $A$ a self-adjoint operator from $D(A)$ to $H$, and $W$ a standard one-dimensional Wiener process on the probability space $(\Omega, P)$. By the Girsanov formula
\[
d\psi_t = A\psi_t dW_t + B\psi_t dt,\]
for certain stochastic process $\xi_t$. The equation for $P_t$ can be computed with the help of the Ito formula
\[
dP_t = \{A - a_t, P_t\}dW_t + \{B - b_t, P_t\}dt
\]
where the double brackets mean "average over some process $W^*_t." We require the process $W^*_t$ to be a standard Wiener process. (2.1)

The density operator $\rho_t$ undergoes a closed evolution if and only if the drift of Eq. (2.2) is linear in $\rho_t$ and the quadratic term $\{A - a_t, P_t\}dW_t$ is a closed evolution if and only if (i) $B = -A^2$ and (ii) $dW^*_t = dW_t - 2a_t dt$.

In this case, the drift operator $\rho_t$ undergoes a closed evolution if and only if $\{A - a_t, P_t\}$ is a bounded Brownian motion under the constraint $\rho_t$.

Note that $\rho_t = \langle \langle \psi, \psi^+ \rangle \| \psi \| ^2 \rangle$, where the double brackets mean "average over the Wiener process $W_i". This is the subject of the next section.
for \( k = 0,1 \). Hence, \( Z \) satisfies Eq. (3.3), \( Z \in L^1(\Omega;C(0,T;D(A))) \) and \( Z(0) = \psi(0) \). Let

\[
\begin{align*}
\alpha(t) &= \langle AZ(t),Z(t)\rangle \|Z(t)\|^{-2}, \\
\zeta(t) &= \exp\left\{ \int_0^t \alpha(s) dW(s) - \int_0^t \alpha^2(s) ds \right\},
\end{align*}
\]

(3.8)

and \( \psi(t) = \zeta^{-1}(t)Z(t) \). By the Ito differentiation rule \( \psi \) satisfies:

\[
\begin{align*}
d\psi(t) &= \langle \psi(t), -a(t) \rangle \psi(t) dt + \langle \psi(t), A \psi(t) \rangle dW(t) + \frac{1}{2} \|\psi(t)\|^2 \psi(t) dt \\
&= \|A - a(t)\|^2 \psi(t) dt + \langle \psi(t), (A - a(t)) \psi(t) \rangle dW(t) \\
&\quad + \langle a(t) \psi(t), \psi(t) \rangle (dW(t) - 2a(t) dt).
\end{align*}
\]

(3.9)

By the definition of \( \psi \)

\[
\langle (A\psi(t),\psi(t))\|\psi(t)\|^{-2} = \langle (AZ(t),Z(t))\|Z(t)\|^{-2}
\]

and therefore \( \psi \in C(0,T;D(A)) \) is a unique solution of Eq. (3.9) with \( a(t) \) given by (3.2). Moreover, \( \|\psi(t)\| = 0 \) for any \( t \geq 0 \). Thus \( \zeta(t) = \|Z(t)\| \) and \( \psi \) is a normalization of \( Z \). We will apply Girsanov measure transformation to prove that \( \psi \) is a weak solution of Eq. (3.1). Recall the formulation of the Girsanov theorem.\(^{30}\)

**Proposition 3.1:** For any \( t \in \mathbb{R}_+ \), \( E_2 < 1 \). If \( \xi(T) = 1 \) then

\[
W(t) = W(t) - 2 \int_0^t a(s) ds
\]

is a standard Brownian motion under the probability measure \( P_*(dw) = P(dw)\xi(T,\omega) \). \( \square \).

Recall also the Novikov condition.

**Proposition 3.2:** If

\[
E \exp\left\{ 2 \int_0^t a(s) ds \right\} < \infty \quad \text{then} \quad E_2^2(T) = 1.
\]

\( \square \)

Let \( \tau_n = \inf\{t \geq 0; \|A\psi(t)\| \geq n\} = \inf\{t \geq 0; \langle A\psi(t),\psi(t) \rangle \geq n\} \). Notice that

\[
E \exp\left\{ 2 \int_{T \wedge \tau_n} a(s) ds \right\} < \infty
\]

by boundness of \( a(s) \). Hence, \( E_2^2(T \wedge \tau_n) = 1 \) and by Girsanov theorem

\[
W_n(t) = W(t) - 2 \int_0^{t \wedge \tau_n} a(s) ds
\]

is a standard Brownian motion under the probability measure \( P^*(dw) = P(dw)\xi^2(T \wedge \tau_n,\omega) \). Since

\[
\|A\psi(t)\| < \|A\psi(t)\| \|\psi(t)\| = \|A\psi(t)\|
\]

then \( \tau_n > T_n \), where \( T_n = \inf\{t \geq 0; \|A\psi(t)\| = n\} \). By Itō differentiation rule

\[
d\|A\psi(t)\|^2 = \langle (A\psi(t),\psi(t)) - a(t) \|A\psi(t)\|^2 \rangle dW_n(t)
\]

for any \( t \leq \tau_n \). Hence, \( \|A\psi(t \wedge \tau_n)\|^2 \) is a positive supermartingale under the measure \( P^* \) and

\[
\text{const} = C = \|A\psi(0)\|^2 E_2^2 \|A\psi(t \wedge \tau_n)\|^2 > n^2 P^n(T_n < t).
\]

Therefore, \( P^n(T_n < t) < C/n^2 \) for any \( t \in \mathbb{R}_+ \). Compute that

\[
E_2^2(T) = E \left[ \zeta^2(T \wedge \tau_n) I_{\{\tau_n > T\}} \right]
\]

\[
= P^n(T_n > T) > 1 - C/n^2
\]

for any \( n > 0 \). Thus \( E_2^2(T) = 1 \) and in consequence we have the following proposition.

**Proposition 3.3:** The process \( \psi(t) \) is a solution of the equation

\[
\begin{align*}
d\psi(t) &= \langle \psi(t), (A - a(t)) \psi(t) \rangle dW^*(t) \\
&\quad - \frac{1}{2} \|A - a(t)\|^2 \psi(t) dt
\end{align*}
\]

where

\[
W^*(t) = W(t) - 2 \int_0^t a(s) ds
\]

is a standard Brownian motion under the probability measure \( P^*(dw) = P(dw)\zeta(T,\omega) \).

**Corollary:** There exists a weak solution of Eq. (3.1).

We close the section with the following existence and uniqueness result for strong solutions of (3.1).

**Proposition 3.4:** There exists a unique strong solution \( \psi \in C(0,T;D(A)) \cap L^2(0,T;D(A^2)) \) of Eq. (3.1).

**Proof:** We fix the Brownian motion \( W \). We will prove pathwise uniqueness of solutions of Eq. (3.1). Let \( \psi \) and \( Y \) be two solutions of Eq. (3.1). Let \( \sigma^* = \inf\{t \geq 0; \|A\psi(t)\| \|\psi(t)\| > n\} \). Let \( \gamma^*(t) = E \|\psi(t \wedge \sigma^*) - Y(t \wedge \sigma^*)\|^2 \). Compute that

\[
\begin{align*}
\gamma^*(t) &= \frac{1}{2} E \int_0^{\wedge \sigma^*} \langle \psi(s), Y(s) \rangle \langle (A\psi(s),\psi(s)) \rangle ds \\
&\quad - \langle A Y(s), Y(s) \rangle ds \\
&\quad - \frac{1}{2} E \int_0^{\wedge \sigma^*} \langle \psi(s), Y(s) \rangle \langle (A\psi(s) + Y(s),\psi(s) - Y(s)) \rangle ds
\end{align*}
\]

\[
< 2n^2 \int_0^{\wedge \sigma^*} \gamma^*(s) ds.
\]

(3.10)

Hence, \( \gamma^* \equiv 0 \). Since \( \sigma^* \to + \infty \), letting \( n \) tend to infinity we obtain that \( \psi = Y \). By the Hilbert space version of the Yamaida–Watanabe theorem, weak existence and pathwise uniqueness imply existence of a unique strong solution of Eq. (3.1).

Propositions 3.3 and 3.4 can be easily generalized for a multidimensional Brownian motion; we have restricted here to the one-dimensional Brownian motion for the simplicity of presentation.

**IV. THE EQUATION WITH A FREE HAMILTONIAN**

In this section we specialize to localization in space and add a free Hamiltonian term. Let \( H = L^2(\mathbb{R};\mathbb{C}), p = -i \) be the momentum operator and let \( q \) be the position operator, i.e., \( qf(x) = xf(x) \) for any \( f \in L^2(\mathbb{R};\mathbb{C}) \).

We consider the following stochastic equation:
\[ d\psi(t) = -i\rho^2\psi(t)dt - \frac{1}{2}(q - a(t))^2\psi(t)dt + (q - a(t))\psi(t)dW(t) \]  
(4.1)

with initial condition \( \psi(0) \in D(\rho^2) \cap D(q^2) \), such that \( \|\psi(t)\| = 1 \),

\[ a(t) = \langle q\psi(t),\psi(t) \rangle \|\psi(t)\|^{-2}. \]  
(4.2)

Consider the auxiliary equation

\[ dZ(t,x) = -i\rho^2Z(t,x)dt - 2i\rho Z(t,x)dt + qZ(t,x)dW(t) \]  
(4.3)

with initial Gaussian condition (compare with Sec. V).

Compute that \( Z(t,x) = e^{\frac{1}{2}a(t)x^2 + \frac{1}{2}P(t)x + r(t)}W(x) \), where \( a, P, \) and \( r \) satisfy:

\[ a'(t) + 1 = 4ia^2(t), \]
\[ P'(t) = 4ia(t)P(t) + W(t), \]

and

\[ r'(t) = i(2a(t) + (\beta(t) + W(t))^2). \]

By the Heisenberg formalism:

\[ d\langle B \rangle = -i(\{B,p^2\})dt + \{q,B\}dW(t) - \frac{1}{2}\langle q,B^2 \rangle dt, \]  
(4.4)

where \( [\cdot, \cdot] \) is the commutator and \( \{\cdot, \cdot\} \) the anticommutator. Hence,

\[ d\langle I \rangle = 2\langle q \rangle dW(t), \]  
(4.5)

\[ d\langle p^2 \rangle = \{q,p^2\}dW(t) + \{q,\langle p, q \rangle \}dt, \]  
(4.6)

\[ d\langle q^2 \rangle = 2\langle q \rangle dW(t) + 2\{q,\langle p, q \rangle \}dt, \]  
(4.7)

and

\[ d\langle q^4 \rangle = 2\langle q^2 \rangle dW(t) + 2\langle q^2,\langle p, q \rangle \rangle dt. \]  
(4.8)

Applying the Cauchy inequality and the Gronwall lemma we derive that:

\[ E\langle q^4 \rangle_{Z(t)} + E\langle q^4 \rangle_{Z(t)} \leq K(\langle q^4 \rangle_{Z(t)} + \langle p^4 \rangle_{Z(t)} + \langle q^2 \rangle_{Z(t)} + \langle q \rangle_{Z(t)}). \]

Thus by linearity of Eq. (4.3), analogously as in Sec. III, we extend initial conditions to any \( Z(0) \in D(q^2) \cap D(p^2) \) and we derive that for any \( Z(0) \in D(q^2) \cap D(p^2) \) there exists a unique solution

\[ Z \in L^2(0,T;D(q^2) \cap D(p^2)) \cap C(0,T;D(q) \cap D(p)) \]

of Eq. (4.3). Let

\[ a(t) = \langle qZ(t),Z(t) \rangle \|Z(t)\|^{-2}, \]
\[ \zeta(t) = \exp\left[ \int_0^t a(s)dW(s) - \int_0^t a^2(s)ds \right], \]  
(4.10)

and \( \psi(t) = \zeta^{-1}(t)Z(t) \). By Ito differentiation rule \( \psi \) satisfies:

\[ d\psi(t) = -i\rho^2\psi(t)dt - \frac{1}{2}(q - a(t))^2\psi(t)dt + (q - a(t))\psi(t)dW(t) \]  
(4.11)

and therefore \( \psi \in C(0,T;D(q)) \) is a unique solution of Eq. (4.11) with \( a(t) \) given by (4.2). Moreover \( d\|\psi(t)\|^2 = 0 \) and hence \( \|\psi(t)\|^2 = 1 \) for any \( t \geq 0 \). Thus \( \zeta(t) = \|Z(t)\| \) and \( \psi \) is a normalization of \( Z \). We will apply Girsanov measure transformation to prove that \( \psi \) is a weak solution of Eq. (4.1). Let

\[ \tau_n = \inf\{t > 0: |a(t)| > n\} = \inf\{t > 0: |\langle q\psi(t),\psi(t) \rangle| > n\}. \]

Notice that

\[ E\left[ \int_0^{\tau_n} a^2(s)ds \right] < +\infty \]

by boundness of \( a(s) \).

Hence \( E\zeta^2(T \wedge \tau_n) = 1 \) and by Girsanov theorem

\[ W_n(t) = W(t) - 2 \int_0^{\tau_n} a(s)ds \]

is a standard Brownian motion under the probability measure \( P_n(\omega) = P(\omega)\zeta^2(T \wedge \tau_n|\omega) \).

Since

\[ \{q(t)|,\psi(t)\} \leq \|q(t)|\|\|\psi(t)\| = \|q(t)|\|, \]

then \( \tau_n > T_n \), where \( T_n = \inf\{t > 0: |\langle q\psi(t),\psi(t) \rangle| = n\} \). To simplify notation we write \( \langle B \rangle \) instead of \( \langle B \rangle_{X(t)} \). By the Heisenberg formalism:

\[ d\langle B \rangle = -i(\{B,p^2\})dt + \{q - \langle q, B \rangle\}dW_n(t) \]

\[ - \frac{1}{2}\|q,\langle q, B \rangle\| dt. \]  
(4.12)

Hence,

\[ d\langle p^2 \rangle = \{q - \langle q, p^2 \rangle\}dW_n(t) + dt \]  
(4.13)

and

\[ d\langle q^2 \rangle = \{q^2, q - \langle q \rangle\}dW_n(t) + 2\{p, q\}dt \]  
(4.14)

for any \( t < T_n \). By (4.13),

\[ E_n\langle p^2 \rangle_{X(t)} < t + \langle p^2 \rangle_{X(t)} \]  
(4.15)

Denote

\[ \gamma_n(t) = E_n\langle q^2 \rangle_{X(t)} \]

By (4.14) and the Cauchy inequality

\[ \gamma_n(t) < t^2 + 2t\langle p^2 \rangle_{X(t)} + \langle q^2 \rangle_{X(t)} + 2\int_0^t \gamma_n(s)ds \]

for any \( t > 0 \). Hence,

\[ E_n\langle q^2 \rangle_{X(t)} < E_n\|q\psi(t \wedge T_n)\|^2 < K + \infty \]

independently of \( n > 0 \) and by the Chebyshev inequality

\[ \text{const} = K > E_n\|q\psi(t \wedge T_n)\|^2 > n^2P_n(T_n < t). \]
Therefore, \( P^n(\tau_n \leq t) \leq P^n(T_n \leq t) \leq Cn^{-2} \) for any \( n \in \mathbb{R}_+ \).

Compute that
\[
E \xi^2(T) > E \left[ \xi^2(T \wedge \tau_n) I_{(T_n \leq T)} \right] = P^n(\tau_n > T) > 1 - Cn^{-2}
\]
for any \( n > 0 \). Thus \( E \xi^2(T) = 1 \) and in consequence we have the following.

**Proposition 4.1:** The process \( \psi(t) \) is a solution of the equation
\[
d\psi(t) = -ip^2\psi(t)dt + (q - a(t))\psi(t)dW^* (t) - \frac{1}{2}(q - a(t))^2\psi(t)dt,
\]
where
\[
W^* (t) = W(t) - \int_{0}^{t} a(s)ds
\]
is a standard Brownian motion under the probability measure \( P^*(d\omega) = P(d\omega)\xi^2(T, \omega) \).

**Corollary:** There exists a weak solution of Eq. (4.1).

We close the section with the following existence and uniqueness result for strong solutions of (4.1).

**Proposition 4.2:** There exists a unique strong solution
\[
\psi = C(0,T; D(q) \cap D(p)) \cap L^2(0,T; D(q^2) \cap D(p^2))
\]
of Eq. (4.1).

**Proof:** Identical as in Sec. III.

Analogously as in Sec. III, Propositions 4.1 and 4.2 can be easily generalized to the multidimensional case \( H = L^2(\mathbb{R}^n) \) and multidimensional position and momentum operators; we have restricted here to the one-dimensional case for the simplicity of presentation.

**V. INITIAL GAUSSIAN WAVE**

In this section we give the explicit solution of Eq. (1.1) for an initial Gaussian wave function \( \psi_0 \):
\[
\psi_0(x) = N_1 e^{-\beta_1 x - \beta_1^* x^2 + \beta_1 x},
\]
where the complex function \( \beta_1 \) and the real function \( \beta_1^* \) and \( \beta_1^* \) satisfy:
\[
d\beta_1 = \beta_1^*/m dt + (2\beta_1^*)^{-1} dW_1,
\]
\[
d\beta_1^* = -\left( \beta_1^* \right)^{-1} \beta_1^* dW_1,
\]
and
\[
-2i \beta_1^* = \left[ 1 - i(2/m)\beta_1^2 \right]dt,
\]
where \( \beta_1 \) and \( \beta_1^* \) denote the real and imaginary part of \( \beta_1 \) and \( N_1 \) contains the normalization factor and an irrelevant global phase. The check that (5.1) satisfies (4.1) is straightforward, especially if one rewrites (4.1) as a Stratonovich stochastic equation:
\[
d\psi_1 = -i\left( \frac{\partial^2}{2m} \right) \psi_1 dt + (q - a_1) \psi_1 dW_1 + 2\beta_1^* dW_1
\]
\[
- (q^2 - \langle q^2 \rangle) \psi_1 dt,
\]
where \( \langle q^2 \rangle_1 = \langle \psi_1^* \xi_1 \psi_1 \rangle / \langle \psi_1^* \psi_1 \rangle \) and if one notices that
\[
\langle q^2 \rangle_1 = (4\beta_1^*)^{-1} + q_1^{-2}.
\]
Equation (5.4) is an ordinary differential equation with solution (see also Ref. 33):
\[
\beta_i = c \tanh(at + \kappa) = \frac{e^{a t} - e^{-a t}}{e^{a t} + e^{-a t}},
\]
where \( c = \sqrt{m} (1 - i)/2 \), \( \alpha = (1 + i)/\sqrt{m} \), \( \kappa = \text{arctanh}(\beta/c) \), \( \gamma = e^{-2\kappa} \) This solution can now be inserted into Eqs. (5.2) and (5.3).

Notice that in the long time limit we recover Diosi's result:
\[
\beta_i = -\beta_i = \frac{v}{2m},
\]
\[
dq_i = \frac{p_i}{m} dt + \frac{1}{\sqrt{m}} dW_i, \quad dp_i = -dW_i.
\]

The spreading of the Schrödinger wave packet is thus balanced by the additional stochastic terms of Eq. (4.1). The mean-square deviation \( \Delta q \) stabilizes for large times:
\[
\Delta q = m^{-1/4}.
\]

**VI. CONCLUSIONS**

The remarkable properties of Eq. (2.3) are of interest to anyone looking at the "quantum measurement problem," not only as a problem of interpretation, but also as a guide to new physics to be discovered. It is thus encouraging that the equations are mathematically sound, as we proved in this contribution.

The mathematical importance of the paper lies in showing the power of the Girsanov measure transformation in proving existence of solutions of stochastic differential equation. It could be thought surprising that the Girsanov transformation, intuitively more associated to the notion of subjective probability, has also a good physical interpretation as in Sec. II.

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