Arbitrarily Small Amount of Measurement Independence Is Sufficient to Manifest Quantum Nonlocality

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(Received 25 July 2014; published 6 November 2014)

The use of Bell’s theorem in any application or experiment relies on the assumption of free choice or, more precisely, measurement independence, meaning that the measurements can be chosen freely. Here, we prove that even in the simplest Bell test—one involving 2 parties each performing 2 binary-outcome measurements—an arbitrarily small amount of measurement independence is sufficient to manifest quantum nonlocality. To this end, we introduce the notion of measurement dependent locality and show that the corresponding correlations form a convex polytope. These correlations can thus be characterized efficiently, e.g., using a finite set of Bell-like inequalities—an observation that enables the systematic study of quantum nonlocality and related applications under limited measurement independence.

DOI: 10.1103/PhysRevLett.113.190402

PACS numbers: 03.65.Ud, 03.67.—a

Since Bell’s seminal work [1], quantum nonlocality has gathered more and more interest, not only from a foundational point of view but also as a resource in several tasks like quantum key distribution [2,3], randomness expansion [4,5], randomness extraction [6], or robust certification [7] and quantification [8] of quantum entanglement. It has led to the notion of device independence (see, e.g., [9]), where the violation of a Bell inequality alone certifies properties that are useful to the task at hand, e.g., nondeterminism of the outputs. In such a scenario, it is enough to consider black boxes that the parties give an input to and get an outcome from instead of having to consider the complex physical description of the implementation.

However, an important assumption has to be made in order for violations of Bell inequalities to exclude any local, and, in particular, deterministic explanation. Let us consider an adversarial scenario in which the boxes were in the hands of an adversary Eve before being given to the parties performing the experiment or protocol. The inputs for the boxes are chosen by local random number generators. If the adversary could influence these random number generators, then she can prepare the boxes with local strategies that appear to be nonlocal to the parties. The violation of a Bell inequality therefore does not imply that the outcomes of the boxes are unknown to the adversary unless we assume that the inputs are independent of the adversary and that she cannot gain any information about them. This assumption is commonly referred to as measurement independence [10,11]. Similar, but slightly stronger assumptions [12] are the assumptions of free choice [13] and free will [14]. Ensuring measurement independence in a Bell test seems impossible. However, if we abandon measurement independence completely and place no restriction at all on the adversary’s influence, then it is impossible to show and exploit quantum nonlocality [15].

In light of this, relaxations of this assumption have gathered attention and been studied in recent works. Hall [10], Barrett and Gisin [11], and, recently, Thinh, Sheridan, and Scarani [16] studied how different possible relaxations influence well-known Bell inequalities. Colbeck and Renner [13] introduced the idea of randomness amplification, in which a quantum protocol produces random outcomes even though complete free choice is not given. This was further developed by Gallego et al. [17] and others [18–22].

A common denominator of these works is that they study and use well-known Bell inequalities. On the contrary, in this Letter, we derive Bell-like inequalities that are specifically suited for a measurement dependent scenario. Using these we show that quantum nonlocality allows for correlations that cannot be explained by any local models exploiting measurement dependence, even when the dependence is arbitrarily strong, as long as some measurement independence is retained (in the sense that we explain more precisely below).

Bell locality.—In a Bell test, two spacelike separated parties, usually referred to as Alice and Bob, have access to two boxes. They can give these boxes an input, denoted by the random variables $X$ and $Y$, respectively, and each of the boxes gives back an outcome, $A$ and $B$ as depicted in Fig. 1. In a quantum mechanical scenario, each box is given by a quantum system and the inputs determine measurements that are performed on this system. By performing many runs, the parties collect data that allow them to estimate with what probability a given input pair $xy$ leads to an outcome pair $ab$; i.e., they estimate the conditional probability distribution $P_{AB|XY}$. Note that we use capital letters for random variables and lower case letters for the values that the corresponding random variable can take. The question a Bell test is trying to answer is whether these
correlations could be explained by a (Bell-) local [9] model, allowing for the existence of some underlying hidden strategy, denoted by \( \Lambda \). We say that a correlation is local if [1]

\[
P(ab|xy) = \int d\lambda \rho(\lambda) P(a|x\lambda)P(b|y\lambda).
\] (1)

A correlation cannot be written as such an integral if and only if it violates a Bell inequality.

However, when performing an actual Bell test, an additional assumption has to be made: the inputs \( X \) and \( Y \) have to be chosen freely, i.e., uncorrelated to the hidden strategy \( \Lambda \) [14].

\[
P(xy|\lambda) = P(xy) \quad \forall \ x, y, \lambda.
\] (2)

Following Hall [10] and Barrett and Gisin [11], we call this assumption measurement independence.

Measurement dependence.—We now analyze the case where complete measurement independence is not given. It is useful to consider, for this case, the full distribution \( P_{ABXY} \), which—contrary to the conditional distribution \( P_{AB|XY} \)—takes into account the distribution of the inputs \( X \) and \( Y \). We say that a correlation \( P_{ABXY} \) is measurement dependent local (MDL) if

\[
P(abxy) = \int d\lambda \rho(\lambda)P(xy|\lambda)P(a|x\lambda)P(b|y\lambda).
\] (3)

As stated previously, if we allow measurement dependence and make no further assumptions, it is impossible to show that quantum mechanics is nonlocal. However, if one bounds the correlations between the inputs and the hidden strategy by imposing upper and lower bounds on the conditional distribution

\[
\ell' \leq P(xy|\lambda) \leq h,
\] (4)

then interesting conclusions can be derived. It is common to refer to such an assumption as a condition on the input source [13]. Examples of such sources are the min-entropy sources [16], which have been studied in recent works [6].

We say that a correlation is measurement dependent nonlocal for a given \( \ell' \) and \( h \) if it cannot be expressed in the integral form given by (3) when assuming the lower and upper bounds coming from (4).

The set of MDL correlations.—The set of measurement dependent local distributions for a given \( \ell' \) and \( h \), i.e., the set of \( P_{ABXY} \) satisfying (3) and (4), turns out to be a convex set with a finite number of extremal points: a convex polytope (see the Supplemental Material [23]). The set can thus be fully characterized using a finite set of Bell-like inequalities: a distribution is measurement dependent nonlocal if and only if it violates at least one of these MDL inequalities.

Quantum violation of a specific MDL inequality.—In the following, we focus on analyzing the simplest possible nonlocality scenario: the inputs \( X \) and \( Y \) and outputs \( A \) and \( B \) of both parties are taken to be binary random variables, taking values 0 or 1. In terms of \( P(abxy) \), one useful parametrized MDL inequality derived using the polytope structure of the MDL set is given by (see the Supplemental Material [23])

\[
\ell'P(0000) - h(P(0101) + P(1010) + P(0011)) \leq 0.
\] (5)

This inequality allows us to prove our main result.

Main result.—Quantum mechanics is measurement dependent nonlocal for any \( \ell' > 0 \) and for any \( h \). A state that exhibits this property is the 2-qubit state

\[
|Au\rangle = \frac{1}{\sqrt{3}} \left( \sqrt{5} \frac{1}{2} |00\rangle + \frac{\sqrt{5} + 1}{2} |11\rangle \right),
\] (6)

on which Alice and Bob perform the rank 1 projective measurements defined by

- \( |A_0(\theta)\rangle = \cos \theta |0\rangle + \sin \theta |1\rangle \),
- \( |A_1(\theta)\rangle = |A_0(\theta - (\pi/4))\rangle \),
- \( |B_0(\theta)\rangle = |A_0(-\theta)\rangle \),
- \( |B_1(\theta)\rangle = |A_1(-\theta)\rangle \) with \( \theta = \arccos \sqrt{\frac{1}{5} + (1/\sqrt{5})} \).

Evaluating the left-hand side of inequality (5) for this state and these measurements, we find \( \ell' \frac{1}{12} P_{XY}(00) \) where \( P_{XY}(00) \) is the probability of choosing the inputs \( x = 0 \) and \( y = 0 \). This proves that measurement dependent local distributions cannot explain quantum correlations, as long as it is impossible for any hidden strategy to exclude the possibility that a certain input pair occurs, i.e., \( \ell' > 0 \) [in fact, any quantum correlation violating Hardy’s paradox [25] violates inequality (5)]. Note that this condition also excludes the possibility of having fully dependent inputs for one of the two parties since \( P(x|\lambda) = 0 \) implies \( P(xy|\lambda) = 0 \). A visual representation of inequality (5) can be found in Fig. 2.

MDL correlations satisfying additional physical constraints.—Motivated by the idea that information needs a physical carrier, most nonlocal experiments are conducted under the assumption that no information can be
transmitted between the parties by the use of such boxes, for example, by performing the experiment in spacelike separation. In other words, the input to Alice’s box cannot influence the outcome on Bob’s side and vice versa, i.e.,

\[
P(a|xy) = P(a|x'y) \quad \forall \ a, x, y, x', y'.
\]

\[
P(b|xy) = P(b|x'y) \quad \forall \ b, x, x', y.
\]  \hspace{1cm} (7)

These are the nonsignaling assumptions [26,27]. Nonsignaling, as opposed to measurement independence, can in principle be verified in a protocol by checking the equalities (7).

The measurement dependent local correlations given by (3) are not inherently nonsignaling due to the hidden strategy \( \Lambda \) establishing correlations between Alice’s input \( X \) and Bob’s output \( B \) and vice versa. Since, as stated above, these equalities can, in principle, be verified, we impose that they are satisfied in the following.

Additionally, the experimenters can observe the input distribution \( P_{XY} \) given by \( P(xy) = \int d\rho(\lambda)P(xy|\lambda) \). Therefore any experiment or protocol involving Bell tests can make use of this knowledge. We consider, from here onwards, the case in which the input distribution is observed to be uniform, meaning that

\[
P(xy) = P(x'y') \quad \forall \ x, x', y, y'.
\]  \hspace{1cm} (8)

Comparison with CHSH.—Instead of using inequality (5), one can try to show the measurement dependent nonlocality of quantum theory by using the well-known Clauser-Horne-Shimony-Holt (CHSH) expression [28]

\[
\text{CHSH} = \sum_{abxy} (-1)^{a+b+xy} P(ab|xy).
\]  \hspace{1cm} (9)

It is well known that quantum mechanics respects Cirel’son’s bound [29]

\[
\text{CHSH} \leq 2\sqrt{2}.
\]  \hspace{1cm} (10)

Therefore, if for a given \( \ell \) and \( h \), the MDL set given by (3) and (4) allows for correlations with \( \text{CHSH} \geq 2\sqrt{2} \), then the inequality cannot be used to reveal measurement-dependent nonlocality.

Using the polytope structure of the MDL set, we find that MDL correlations, even with the additional constraints of nonsignaling (7) and uniform inputs (8), can reach (see the Supplemental Material [23])

\[
\text{CHSH} = 4(1 - 2\ell').
\]  \hspace{1cm} (11)

where \( \ell' = \max(\ell, 1 - 3h) \). Comparing this value to the quantum bound of \( 2\sqrt{2} \), we find that for \( \ell' \leq ((2 - \sqrt{2})/4 \) it is impossible for CHSH to reveal the measurement dependent nonlocal behavior of quantum mechanics. Inequality (5), on the other hand, is able to reveal this \( \forall \ell' > 0 \).

Input sources with \( \ell = 0 \).—One specific input source is the min-entropy source [16]. The conditional min-entropy is defined as

\[
H_{\min}(XY|\lambda) = -\log_{2}\max_{xy}P(xy|\lambda).
\]  \hspace{1cm} (12)

Using a min-entropy source means that \( H_{\min}(XY|\lambda) \) is lower bounded by some value \( H \forall \lambda \). In our language it corresponds to setting the lower bound \( \ell = 0 \) and the upper bound \( h = 2^{-H} \) in condition (4).

For \( X, Y, A, B \in \{0, 1\} \) and specific values of \( h \), we obtain a complete set of MDL inequalities. We say that a set of inequalities is complete if every measurement dependent nonlocal distribution violates at least one inequality in this set while measurement dependent local distributions, cf. (3), respect all inequalities. For example, in the corresponding Bell-locality scenario it is known that the CHSH inequalities form a complete set.

A first observation to make is that maximal min-entropy, meaning that \( h = \frac{1}{4} \) implies that \( P_{XY|\lambda=\lambda'} \) is uniform \( \forall \lambda \). This conclusion follows from the fact that every probability distribution has to be normalized, i.e., \( \sum_{x,y}P(xy|\lambda) = 1 \forall \lambda \), and that probabilities are non-negative. Since the inputs are not biased by \( \lambda \), this corresponds to imposing measurement independence and is therefore equivalent to the standard Bell locality. As already stated, the CHSH inequalities form a complete set in this case.

Another special value is \( h = \frac{1}{4} \). It turns out that if \( h \geq \frac{1}{4} \), measurement dependent local correlations can reproduce any nonsignaling distributions. Since the set of quantum
correlations is a strict subset of the set of non-signaling correlations, it is therefore impossible to see measurement dependent nonlocality in this case. The reason this does not occur for \( h < \frac{1}{2} \) is due to the fact that for these values of \( h \) the normalization and non-negativity of probabilities imply that no input pair can be excluded, i.e., \( P(xy|\lambda) > 0 \forall x, y, \lambda \).

The interesting case is therefore \( h \in \left[ \frac{1}{3}, \frac{1}{2} \right] \). For each fixed value of \( h \) in this interval, one can make use of a standard software package \([30]\) to obtain the complete set of inequalities characterizing the set of MDL correlations. We performed this computation for several values of \( h \in \left[ \frac{1}{3}, \frac{1}{2} \right] \). For each of these chosen values (see the Supplemental Material \([23]\)), we always found 7 families of inequalities, where we say that two inequalities belong to the same family if one can be obtained from the other by simply relabeling the inputs and outputs or by exchanging the roles of the two parties. As a function of \( h \), the inequalities we found can be expressed as in Table I. Based on the above observation, we conjecture that the inequalities of Table I form a complete set for all \( h \in \left[ \frac{1}{3}, \frac{1}{2} \right] \).

A visual representation of the evolution of the MDL-polytope as \( h \) goes from \( \frac{1}{4} \) to \( \frac{1}{2} \) can be seen in Fig. 3.

It is interesting to note that the well-known CHSH inequality is not among these 7 families. From (11), we see that for \( h \geq (2 + \sqrt{2})/12 \approx 0.2845 \), quantum mechanics can no longer outperform the measurement dependent local correlations when looking at CHSH as given by (9). This was already shown by Thinh, Sheridan, and Scarani \([16]\). CHSH is therefore only useful up to this critical value of \( h \). On the other hand, all 7 families introduced in Table I can be violated for values larger than \((2 + \sqrt{2})/12\). In fact, inequalities (6) and (7) can reveal quantum nonlocality for all \( h \) below the critical value of \( \frac{1}{3} \). This shows that the complete set presented here is better suited for the task of witnessing measurement dependent quantum nonlocality than CHSH.

**Conclusions.**—Bell locality, the essential concept when working in any kind of device independent scenario, includes the untestable assumption of measurement independence. We have analyzed what happens when this assumption is relaxed and found that, as with Bell locality, it is sufficient to work with a finite number of Bell-like inequalities. Using one such inequality, we showed that the nonlocality of quantum mechanics can be manifested as long as an arbitrarily small amount of free choice is guaranteed. Surprisingly, the simplest nontrivial scenario involving only two parties (and binary-outcome measurements) is already sufficient to arrive at this conclusion.

We have also presented inequalities that are better suited to measurement dependent scenarios than the CHSH inequality. In fact, with the additional assumption of nonsignaling and uniform observed inputs, we obtained a set of Bell-like inequalities—which we conjecture to be complete—for the measurement dependent local set of two parties, two inputs (with a min-entropy input source), and

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**TABLE I.** Conjectured families of MDL inequalities for \( h \in \left[ \frac{1}{3}, \frac{1}{2} \right] \). The Table contains the coefficients belonging to each term (given in the first row). We denote by \( P_{\text{AX}}(a|x) \) the marginal distribution over Alice’s output \( A \) conditioned on her input \( X \) and similarly for Bob. The expression being \( \leq 0 \) is a representative MDL inequality from each family.

| \( h \) | \( P_{\text{AX}}(0|0) \) | \( P_{\text{AX}}(0|1) \) | \( P_{\text{BY}}(0|0) \) | \( P(0000) \) | \( P(0010) \) | \( P_{\text{BY}}(0|1) \) | \( P(00|0) \) | \( P(00|1) \) | \( P(00|11) \) |
|---|---|---|---|---|---|---|---|---|---|
| \( 12h^2 - 11h + 2 \) | \( 2h - 1 \) | \( 4h - 1 \) | \( 2h - 1 \) | \( 2h \) | \( 2 - 6h \) | \( 4h - 1 \) | \( 2 - 6h \) | \( -2h \) |
| \( 12h^2 - 11h + 2 \) | \( 4h - 1 \) | \( 3h - 1 \) | \( 4h - 1 \) | \( -h \) | \( 1 - 3h \) | \( 3h - 1 \) | \( 1 - 3h \) | \( 1 - 3h \) |
| \( 11h^2 - 8h + 1 \) | \( -4h^2 + 5h - 1 \) | \( 5h^2 - 4h + 1 \) | \( -4h^2 + 5h - 1 \) | \( -3h^2 - 2h + 1 \) | \( 3h^2 - 2h \) | \( 5h^2 - 4h + 1 \) | \( 3h^2 - 2h \) | \( -9h^2 + 9h - 2 \) |
| \( 8h^2 - 7h + 1 \) | \( -4h^2 \) | \( 0 \) | \( -4h^2 + 5h - 1 \) | \( -h \) | \( 1 - 3h \) | \( -4h^2 + 2h \) | \( -h \) | \( 3h - 1 \) |
| \( 13h^2 - 8h + 1 \) | \( -8h^2 + 6h - 1 \) | \( -5h^2 + 2h \) | \( -h^2 + h \) | \( 5h^2 - 2h \) | \( h^2 - h \) | \( 0 \) | \( 3h^2 - 4h + 1 \) | \( -3h^2 + 4h - 1 \) |
| \( 20h^2 - 13h + 2 \) | \( -8h^2 + 6h - 1 \) | \( -7h^2 + 5h - 1 \) | \( -8h^2 + 6h - 1 \) | \( -82 - 2h \) | \( 3h^2 - 4h + 1 \) | \( -7h^2 + 5h - 1 \) | \( 3h^2 - 4h + 1 \) | \( -h^2 + h \) |
| \( 1 - 4h \) | \( 3h - 1 \) | \( 0 \) | \( 3h - 1 \) | \( 1 - 3h \) | \( 0 \) | \( h \) | \( 0 \) | \( -h \) |
two outputs. In general, our observations that MDL correlations can be fully characterized using Bell-like inequalities provides a powerful framework for the study of measurement dependent quantum nonlocality and related applications. For instance, the MDL polytope presented in this Letter may become a useful tool for the analysis of tasks like randomness extraction [6] and amplification [13]. So far, inequalities suitable for such tasks were guessed or inspired by the local polytope. Inequalities derived or inspired from the MDL polytope should be better suited for the task. Specifically, it would be interesting to see whether the bipartite scenario with binary inputs and outputs could indeed be sufficient to perform a randomness amplification protocol using one of the inequalities presented here.

The framework introduced in this Letter allows one to study further other possible assumptions on the input source. A natural possibility would be that any correlations between the random number generators that the two parties use to determine their inputs must come from a local hidden variable, i.e., $P(xy|\lambda) = P(x|\lambda)P(y|\lambda)$, a problem that we shall leave for future research.

We acknowledge helpful discussions with Valerio Scarani and Jean-Daniel Bancal as well as financial support from the European project CHIST-ERA DIQIP, the European Cooperation in Science and Technology (COST) Action MP1006, the European Research Council (ERC) Grant No. 258932, the European Union Seventh Framework Programme via the RAQUEL project (Grant Agreement No. 323970), and the Swiss National Centre of Competence in Research Project NCCR-QSIT.

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