Bell inequality for arbitrary many settings of the analyzers

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Abstract

A generalization of the CHSH-Bell inequality to arbitrary many settings is presented. The singlet state of two spin $\frac{1}{2}$ systems violates this inequality for all numbers of setting. In the limit of arbitrarily large number of settings, the violation tends to the finite ratio $\frac{5}{4} = 1.27$. © 1999 Elsevier Science B.V. All rights reserved.

1. Introduction

One of the most remarkable features of our basic physical science, quantum mechanics, is certainly its holistic description of Nature. In particular, when we divide her into parts, then, according to quantum theory, these parts are generally entangled. As a consequence, what happens here cannot be considered as independent of what happens there, as demonstrated by Bell’s theorem (i.e. that entangled quantum states violate Bell inequality) [1]. Considering the importance of quantum non-locality, both for the worldview quantum physics presents and for the potential applications of entanglement in quantum computing and communication [2], it is important to examine the basis of entanglement as closely as possible. Let us, for example, notice that despite that Bell’s result is now 35 years old, no loophole free experiment has yet been realized [3,4]!

The most well-known Bell inequality is the CHSH inequality [5]:

$$S_2 = E(a_1b_1) + E(a_1b_2) + E(a_2b_1) - E(a_2b_2) \leq 2 \tag{1}$$

This inequality refers to 2 2-dimensional systems (spin $\frac{1}{2}$) with 2 settings on each of the 2 sides. Clearly, there are many possible generalizations. Generalizations to n 2-dimensional systems with 2 settings are known [6]. In this short Letter, we present a generalization to 2 2-dimensional systems with n settings.

2. CHSH-Bell inequality for $n$ settings

Let $a_j = \pm 1$ and $b_j = \pm 1$ for all indices $j = 1 \ldots n$. The following inequalities can be easily checked by inspection:

$$a_1(b_1 + b_2) + a_2(b_1 - b_2) \leq 2 \tag{2}$$
a_{1}(b_{1} + b_{2} + b_{3}) + a_{2}(b_{1} + b_{2} - b_{3})
+ a_{3}(b_{1} - b_{2} - b_{3}) \leq 5
(3)

a_{4}(b_{1} + b_{2} + b_{3} + b_{4}) + a_{2}(b_{1} + b_{2} + b_{3} - b_{4})
+ a_{3}(b_{1} + b_{2} - b_{3} - b_{4})
+ a_{4}(b_{1} - b_{2} - b_{3} - b_{4}) \leq 8
(4)

Note that minus signs appear only below the diagonal.

From the first of these inequalities, i.e. from inequality (2), the usual CHSH-Bell inequality (1) can be deduced with $E(a,b) = \langle \rho(\lambda) d\lambda a(a,\lambda) \rangle \times b(b,\lambda)$, where $a(a,\lambda) = \pm 1$ is the outcome given the setting $a$ and the local hidden variable $\lambda$ and similarly for $b(b,\lambda) = \pm 1$ (note the important assumption that $a(a,\lambda)$ and $b(b,\lambda)$ are independent of the other setting $b$ and $a$, respectively).

From the inequalities (3) and (4), the following generalization of the Bell-CHSH inequality is obtained

$S_{3} = E(a_{1},b_{1}) + E(a_{1},b_{2}) + E(a_{1},b_{3})
+ E(a_{2},b_{1}) + E(a_{2},b_{2}) - E(a_{2},b_{3})
+ E(a_{3},b_{1}) - E(a_{3},b_{2}) - E(a_{3},b_{3}) \leq 5$
(5)

$S_{4} = E(a_{1},b_{1}) + E(a_{1},b_{2}) + E(a_{1},b_{3})
+ E(a_{2},b_{1}) + E(a_{2},b_{2}) + E(a_{2},b_{3})
+ E(a_{3},b_{1}) - E(a_{3},b_{2}) + E(a_{3},b_{3})
+ E(a_{4},b_{1}) - E(a_{4},b_{2}) - E(a_{4},b_{3})
- E(a_{4},b_{3}) \leq 8$
(6)

This set of inequalities generalizes to arbitrary many settings of the analyzers:

$S_{n} = \sum_{j=1}^{n} \left( \sum_{k=1}^{n+1-j} E(a_{j},b_{k}) - \sum_{k=n+2-j}^{n} E(a_{j},b_{k}) \right)$

$\leq \left[ \frac{n^{2} + 1}{2} \right] \equiv S_{n_{lhv}}$
(7)

where $[x]$ denotes the largest integer smaller or equal to $x$. The above inequality (7) is the main result of this letter.

For any product state and any number of settings $n$, there are settings that saturate inequality (7) 2 On the opposite, for maximal entanglement, like in the singlet state $\psi$, the inequality (7) can be violated by quantum correlations. Numerical evidence shows that it suffice to consider co-planar settings $a_{j}$ and $b_{j}$: $E(\alpha,\beta) = \langle a_{j} \otimes \psi | b_{j} \rangle = -\cos(\alpha - \beta)$, where $a_{j} = (\sin(\alpha_{j}),\cos(\alpha_{j}))$ and $b_{j} = (\sin(\beta_{j}),\cos(\beta_{j}))$. Indeed, the following settings, $j = 1,\ldots,n$:

$\alpha_{j} = \frac{\pi}{n}, \quad \beta_{j} = \frac{3 - n - 2j}{2n}$
(8)

lead to the maximal violation of (7):

$S_{n_{QM}} = 2n \cos\left( \frac{\pi}{2n} \right) \sin\left( \frac{\pi}{n} \right) > \left[ \frac{n^{2} + 1}{2} \right] \equiv S_{n_{lhv}}$
(9)

The ratio of violation is shown in Fig. 1 as a function of the number of settings. Asymptotically, i.e. for

2 It is not clear whether arbitrary entangled states violate all the inequalities (7) for some settings (as is the case for the CHSH inequality, see [7]). The case of odd numbers of setting appears special.
large $n$, the sums in (7) converge into integrals and the ratio of violation tends to:

$$\frac{S_{QM}}{S_{inv}} = \lim_{n \to \infty} 2n \cos \left( \frac{\pi}{2n} \right) \sin \left( \frac{\pi}{n} \right) \sqrt{n^2 + 1} \over 2$$

$$= \frac{4}{\pi} = 1.273 > 1$$

(10)

3. Conclusion

A generalization of the CHSH-Bell inequality to arbitrary many settings has been presented. For large numbers of settings, the ratio by which this generalized inequality is violated by quantum mechanics is smaller than the usual $\sqrt{2}$ factor (i.e. 41%) valid for 2 settings. However, this ratio is still significant: 27.3%. Consequently, experimental data reproducing the entire quantum correlation function could violate the generalized inequality by a larger number of standard deviations, thanks to improved statistics [8].

Another possible advantage of this generalized inequality could have been that it is less sensitive to the detection loophole [3]. Admittedly, this was our original motivation for this work. However, it turns out that the corresponding generalization for the CH inequality [9] does not allow to lower the detector’s efficiency threshold necessary to close the detection loophole. Actually, recently we found a local hidden variable model based on the detection loophole which reproduces exactly the quantum correlation function [4]! Consequently, this model does also violate the generalized inequality (7) for any number of settings.

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References