Beyond verification: Proof can teach new methods

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Abstract

The fact that proofs can convey new mathematical techniques to students effectively, as shown in recent literature, is an important advantage of the classroom use of proof, but it is one that mathematics educators seem to have overlooked to a large extent. The paper argues that teachers should make use of the potential of proof for presenting new techniques and demonstrating their value, and that mathematics educators in general should accord this potential its due importance among the many reasons for teaching proof.

The teaching of proof in schools has been the topic of extensive investigations over the last two decades in the scholarly literature on mathematics education and in particular in the proceedings of the International Group for the Psychology of Mathematics. In her survey of research on proof in mathematics education, Mariotti (2006) found that most of the investigations on this topic have dealt primarily with the logical aspects of proof and with the cognitive problems encountered in having students follow deductive arguments. Other aspects of proof that have also been investigated include the role of intuition and schemata in proving, the usefulness of heuristics for the teaching of proof, the explanatory power of proof, the various functions of proof, and justification and proof as seen in the context of dynamic software (Hanna, 2000).

There has been little scrutiny however, of an idea recently discussed in Rav's inspiring paper "Why do we prove theorems?" (1999). He states that proofs do much more than verify mathematical claims, that they are actually bearers of mathematical knowledge and also indispensable to the broadening of that knowledge. Rav argues that the very act of devising a proof contributes to the development of mathematics, and sees proofs as the primary focus of mathematical interest. He goes on to say that proofs can not only yield new mathematical insights, giving them a value far beyond establishing the truth of new propositions, but can also convey new mathematical strategies and new methods for solving problems.

Rav is not the only one who assigns to proofs a role that goes well beyond demonstrating *that* a theorem is true and *why* a theorem is true. Avigad (2006) lends support to Rav's central thesis when he says that mathematicians value a proof when it "exhibits methods that are powerful and informative; that is, we [mathematicians] value methods that are generally and uniformly applicable, make it easy to follow a complex chain of inference, or provide useful information beyond the truth of the theorem that is being proved" (Avigad, 2006, p. 2).

Dawson (2006), having analysed the reasons why mathematicians re-prove theorems, lends additional support to Rav's claim that the innovative strategies and methods often embodied in proofs, rather than the theorems proved, are the primary value that proofs bring to mathematics. Dawson shows persuasively that there are eight reasons that propel mathematicians to seek new proofs to theorems that have already been accepted, and most of these reasons have to do with methods, such as "To demonstrate the power of different methodologies", "To discover a new route", and "Concern for methodological purity" (Dawson, 2006, pp. 275- 281).

Corfield (2003) would also appear to support this assessment of proof when he says that ''What mathematicians are largely looking for from each other's proofs are new concepts, techniques, and interpretations'' (p. 56). He clearly shares with Rav the view that there is more to proof than establishing the truth or falsity of a proposition. It is also enlightening to note the following comment by Zeilberger: "The value of a proof of an outstanding conjecture should be judged, not by its cleverness and elegance, and not even by its 'explanatory power,' but by the extent in which it enlarges our toolbox." (as cited in Bressoud, 1999, p. 190)

The idea that proof might be most valuable in the school curriculum because it conveys methods worth teaching, thus enlarging the students' toolbox, is unfortunately largely absent from curriculum materials that discuss the reasons for teaching proof. Indeed, in most documents addressed to teachers, such as the ones written by the National Council of Teachers of Mathematics (NCTM, 1998) and the Education Development Center (EDC), the reasons for teaching proof are the following: 1) to establish a fact with certainty; 2) to gain understanding; 3) to communicate ideas to others; 4) for the challenge; 5) to create something beautiful, meaning "the development of a proof that possesses elegance, surprises us, or provides new insight is a creative act."; 6) to construct a larger mathematical theory (EDC, pp. 3-7). Clearly these items in this list encompass the valid considerations of justification, understanding, new insights and aesthetics, but they make no mention of the contribution of proof in presenting new methods and demonstrating their value.

Following are two examples from mathematics at the school level. Their aim is to show that proofs have the capacity to expand the students' toolbox of techniques and strategies for problem solving and to provide new mathematical insights. Note that the emphasis here is on properties intrinsic to the proof, not on the ways in which the proof might be taught or understood by the students. Nor are the examples about the logical features of the proof or about the degree to which a proof might be convincing (though of course it is taken for granted that the proof must justify the correctness of its conclusion). The first example is one that is discussed in Rav (1999).

Example I:

Euclid's Proposition 20 says that the number of primes is infinite ("Prime numbers are more than any assigned multitude of prime numbers", Book IX). In other words, there is no largest prime number, just as there is no largest number. There are several proofs of this proposition, each with its own concepts and method.

Proof

The idea is to show that given any finite list of primes, it is possible to find a prime number q distinct from the primes given in the list.

Let $p_1, p_2, p_3, ..., p_n$ be prime numbers. Multiply them together and add 1, calling this number a new integer N.

$$N = p_1 p_2 p_3 ... p_n + 1$$

If N is a prime number, then we have a new prime.

If N is not a prime, it must be divisible by a prime number q. But q cannot be p_1 or any other from our original list of prime numbers, because if we were to divide N by any of p_1 , p_2 , p_3 , ... p_n we would get a remainder 1, which means that N is not divisible by any of these prime numbers. So q is a new prime. So either there is a new prime N, or if N is not a prime, then it has a new prime for a prime factor. Hence there is always a prime distinct from any number of primes.

The point that Rav (1999) wishes to emphasize in this proof is that there is a key idea, that of forming the new number N, which is a creative idea that is specific to the topic of this proof, not stemming from any other axiom or proposition. Thus the proof contains a method, novel to the students, which could be used in problem solving or in proving other propositions when appropriate.

Example II

Some series are referred to as “telescoping” series. Their sums or the proof of their convergence can be found by noticing that every term cancels with a succeeding or preceding term and using a technique known as the method of differences. To be able to do this, one makes use of the method of partial fractions to decompose the fraction that is common in some telescoping series. This “telescoping” process of collapsing terms in a series so that they are removed from the calculation allows us to manipulate series into telescoping forms and greatly simplifies the proof or the determination of the sum.
An example of the last is the finite sum of the series $\sum_{n=1}^{N} \frac{1}{n(n+1)}$ which can be treated as a telescoping sum, as follows:

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right)$$

$$= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{N} + \frac{1}{N}\right) - \frac{1}{N+1} = 1 - \frac{1}{N+1}$$

The same telescoping technique can be applied in determining the convergence of an infinite series:

Proof:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Look at the partial sum: $\sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$$= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{n-1}\right) = 1 - \frac{1}{n+1} = 1 \quad \text{as} \quad n \to \infty$$

So the sum of the series which is the limit of the partial sums is 1.

Again the point here is not the actual proof of the mathematical fact that this series is convergent, but the way in which the proof introduces a new technique (new to the students) and demonstrates its power. Inherent in the use of this proof is the opportunity for the students to gain a piece of knowledge important for mathematical practice.

Conclusion

The recognition that proofs can convey new mathematical techniques effectively, and thus should be treated as important bearers of mathematical knowledge, is a fertile point of view that mathematics educators seem to have overlooked to a large extent. Adopting this approach to proof in the classroom does not challenge in any way the accepted "Euclidean" definition of a mathematical proof (as a finite sequence of formulae in a given system, where each formula of the sequence is either an axiom of the system or is derived from preceding formulae by rules of inference of the system), nor does it challenge the teaching of proof as a Euclidean derivation. It is rather an acknowledgment that the teaching of proof has the potential to further students' mathematical knowledge in other ways. It offers an opportunity to make new connections between the process of proving and mathematical techniques, and also gives us an additional reason for keeping proof in the mathematics curriculum.

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