Participant paper for WG1 – Disciplinary Mathematics and School Mathematics

What do we wish students to know with respect to mathematics when they come to us from school to the university?

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A student is not a vessel to be filled,
but rather a lamp to be lit.
-- attributed to A.D. Aleksandrov (Russian mathematician, 1912-1999)

I want to focus on one aspect of the topic of our WG1 – the aspect contained in the title of this paper. Certainly, mastery of concepts and techniques in mathematics is important; but I suspect the we may have trouble agreeing on such a list. However, I would argue that there is something else that maybe we can agree on and that should be a part of our discussions -- I will illustrate with the story of a visit to a classroom:

Story of a Classroom Visit:

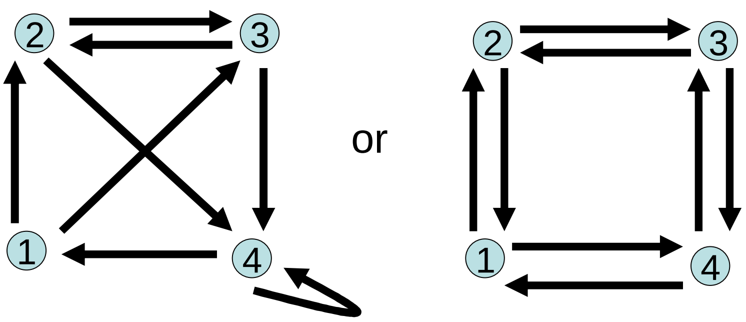
In 2006, I visited a 9th grade classroom near Chicago. The school was a so-called "low-performing" school and these students were considered to be in the bottom half of this school. The class met the first period of the day and I went to the assigned classroom. The teacher was delayed for 40 minutes and not in the classroom, so I introduced myself as a mathematician to the students. Then I asked them to tell me what they had been studying – and they did!

- The students were **confident in their thinking** about important mathematics,
- Their thinking was **based on common physical experiences,** and
- **They were articulate in talking about their thinking** with a visiting mathematician whom they had never seen before.

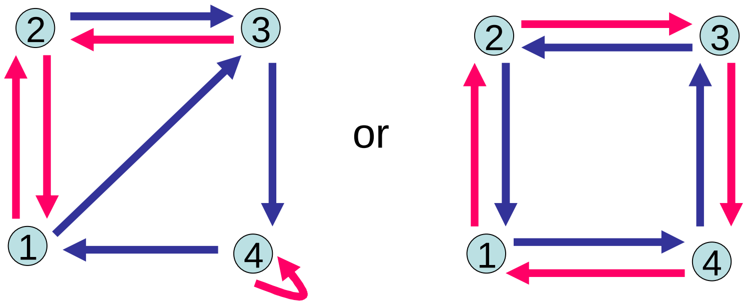
What the students showed me:

A. A **city** is a collection of **buildings** connected by **one-way roads** and that has two features:
1. there is a **walk** along the roads from any building to any other building, and
2. exactly **two roads go out** from each building.

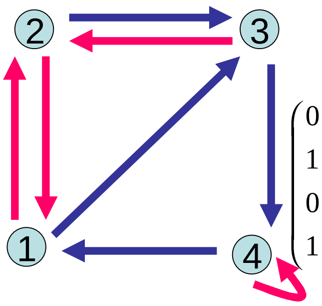
{A *2-out strongly connected digraph* in graph theory terminology.}



B. A city is **correctly colored** if each road is colored either **Red** or **Blue** and each city has exactly 1 **Red** road and 1 **Blue** road leading away from it.
Examples:



C. Multiple representations for correctly colored cities using diagrams (on paper and/or classroom floor), matrix representations, and arrow diagrams:

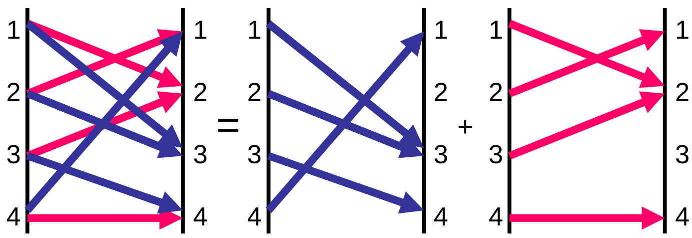


Matrix representation:

$$M \quad = \quad B \quad + \quad R$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

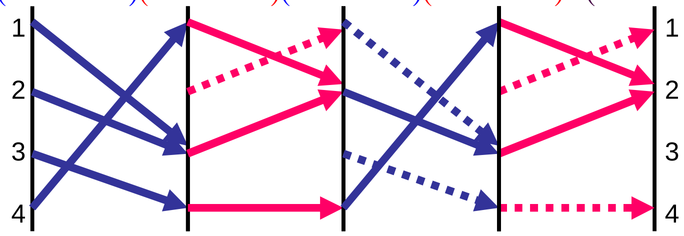
Arrow diagrams:



D. The Problem: Find a set of instructions ("*synchronizing instructions*") that get all the people to same building at the same time. *Solution*: **BRBR.**

The students showed me how to prove that a set of instructions was synchronizing in 3 different way for the city in **C**: Following the city diagram (on paper and/or on the floor), multiplying matrices, or composing arrow diagrams:

$$\begin{pmatrix}0&0&1&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}0&1&0&0\\0&1&0&0\\0&1&0&0\\0&1&0&0\end{pmatrix}$$ matrices



arrow diagrams:

{There is a conjecture that mathematicians have been working on for about 20 years: **Road Coloring Conjecture:** *Does every city have a coloring for which there exists synchronizing instructions.*}

Mathematics topics the students used (in our 40 minute discussion)

- **Graphs** (directed, nodes, …) on paper and/or on the floor
- **Matrices** (addition, multiplication,…)
- **Functions** (composition, range, domain,…)
- **Arrow diagrams**
- Equivalent **alternative verifications** using matrices, arrow diagrams, graphs.

What the students talked about (in our 40 minute discussion)

- **Definitions**
- **Axioms** (“Features”)
- **Conjectures**
- **Examples/Counterexamples**
- **Mathematical models**
- Discussing **alternative means of verification** of a conjecture and confirming that they gave the same conclusion.

Summary of Classroom Visit

9th grade students in a **below average** algebra class in a below average urban school:

- Were **confident in their thinking** about important mathematics,
- Used **alternate modes of representations** starting with a physical activity, and
- Were **articulate in talking about their thinking** with each other and a visiting mathematician whom they had never seen before.

What do we want?

We want that our students come to us from school with

- **positive experiences** with important mathematics

- **confidence** in their own mathematical thinking
- **ability in talking** about their thinking with others

My classroom visit shows that it is possible, but I see little evidence of what we want, even among the students who come to Cornell University and these are the best high school students in the USA and most have passed the equivalent of Calculus I in high school, but most still repeat the first calculus course at Cornell.

What do we get?

According to the 2005 CBMS survey of mathematics courses in USA 4-year colleges and universities:

Enrollments in mathematics courses:

- Courses lower than calculus: 906,000 **(57%)**
- Calculus I (first semester): 309,000 **(19%)**
- Courses above Calculus I: 389,000 **(34%)**

Calculus I is a course taught in most high schools in USA. Note that these are all students who have graduated from high school and who have passed state mathematics exams and who are taking mathematics course in college.

It appears that most USA students in college take mathematics courses whose content they have on-paper already learned in high school. There are different reasons why students repeat the courses:

- They are required to repeat the course because they scored too low on a placement exam administered by the college.
- They repeat the course because they do not feel confident of their understanding from high school.
- Even though they understand the mathematics, they repeat the course with the hope of easily an "easy A" (the highest grade without much work).

How might we get what we want?

{Here I speak only about the situation in the USA, since this is what I am most familiar with.}

- Many topics entered the school mathematics curricula when schools were training human calculators during the late 19th and early 20th Centuries. Many of these topics are no longer necessary because we have electronic calculators and computers. I propose weeding these topics out and instead concentrating on the human understanding of the meanings of the computations. Examples of topics that can be weeded: complicated manipulations of trigonometric and logarithmic identities, emphasis on algebraic "simplest terms" (Why is $\frac{\sqrt{2}}{2}$ better than $\frac{1}{\sqrt{2}}$ or $2^{-1/2}$?).
- Believe that school students can think confidently about mathematics
    - if encouraged to do so, and
    - in the appropriate context starting from common experiences.
- Much interesting and important mathematics is either not taught in schools at all or is presented in ways that are inaccessible to most students. Examples:

- Spherical geometry (navigation, surveying, non-Euclidean geometry)
- Cryptology (number theory, finite geometries)

- Important notions in mathematics are formally defined in ways that separate them from the students' experiences.
  - Many USA high school texts *define* a rotation as the product of two reflections – students know that this is not their meaning of "rotation".
  - Real numbers are separated from their emergence in measurement
- Many important and basic mathematical questions are not asked.
  - What does it mean to be "straight"? Most texts say it is an "undefined term", but students know that is has meaning.
  - Connections between linear algebra, geometric transformations, symmetries, and Euclidean geometry – this can be accessible if not restricted inside fixed formalizations.
  - Is .999… = 1? Why? What are we assuming? What does it mean?
  - Why do we get the same symbolic answer in these two questions: I have a piece of yarn that has length L. $Q1$. If I divide the yarn equally among 3 people how much does each person receive. $Q2$. If I divide the yarn into 3cm pieces, how many pieces will I obtain? The physical actions and mathematical meanings are very different, but the symbolic answer in each case is "$L/3$". Why is this the case?
- Promote in schools "alive mathematical reasoning", which is:
  - paying attention to meanings behind the formulas/words
  - convincing communications that answer the question —*Why?*
  - using a variety of mathematical contexts
  - combining together all parts of mathematics
  - using the world of experiences (including images/imagination, physical models, drawings) to develop notions and meanings in mathematics
  - applying mathematics to the world of experiences
  - making conjectures, counterexamples, connections
  - always asking — *WHY?*