What VIEW of Mathematics should we learn at school?

by

Derek Holton

Abstract

I want to suggest that mathematics is more than skills such as addition, solving equations and differentiating basic functions. I believe this because to a research mathematician, mathematics is more than skills. The processes that are required to link skills together in a useful way and to generate new skills are equally as important as the skills themselves. If students are to get a rounded view of what mathematics is they need to see more than skills alone. I illustrate this by means of a problem.

The Sum Problem and Its Conjectures

The Sum Problem: What numbers cannot be expressed as the sum of a string of two or more consecutive positive integers?

The only way to get started on this, that I can think of, is to try a few small numbers to see what can and what can't be done. To help in the discussion, I'll call a number 'nice' if it can be expressed as the sum of a string of two or more consecutive positive intergers. You can see pretty quickly that

$3 = 1 + 2$; $5 = 2 + 3$; $6 = 1 + 2 + 3$; and $9 = 4 + 5$

are nice numbers.

It's clear too that 1 and 2 aren't nice – they're just too small to find positive integers that will add up to them. And 8 isn't nice either. Try a few connective starts. $1 + 2 + 3 = 6$ but $6 + 4 = 10 > 8$; $2 + 3 = 5$ but $5 + 4 = 9 > 8$; $3 + 4 = 7$ but $7 + 5 = 12 > 8$; and $4 + 5 > 8$.

It might be useful drawing up the following table, where we show the numbers less than or equal to 20 and their niceness.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
		nice		nice	nice	nice		nice	nice	nice	nice	nice	nice	nice		nice	nice	nice	nice

At this point we see that only 1, 2, 4, 8 and 16 aren't nice. That should make us sufficiently suspicious to make a conjecture.

Conjecture 1: If $n = 2^\alpha$ for $\alpha = 0, 1, 2, \ldots$, then n is not nice.

But while we're thinking about, that it's worth noting that, in our table, all the other numbers are nice. So we can make a stronger conjecture, which may not of course be true.

Conjecture 2: n is not nice if and only if $n = 2^\alpha$.

Naturally you might prefer the next conjecture, even though it is identical to Conjecture 2.

Conjecture 3: n is nice if and only if $n \neq 2^{\alpha}$.

The thing about Conjecture 1 is that it only requires us to prove one thing. Conjectures 2 and 3 require us to prove two things, one with the "if" and one with the "only if". However these last two conjectures give a much more powerful result, if they're true. They actually tell us **all** numbers that are nice. Conjecture 1 doesn't assure us that there are no other "not nice' numbers.

The Conjectures and Their Proofs

It's by no means clear either that any of our conjectures is true nor how we might go about proving any of them. So maybe we can find a counter-example.

The first power of 2 that we don't have in our table is 32. So let's assume that 32 **is** nice. That means that

$$\begin{aligned} 32 &= a + (a + 1) + (a + 2) + \ldots + (a + r - 1) \\ &= \frac{r(2a + r - 1)}{2} \end{aligned}$$

where $a \geq 1$ and $r \geq 2$.

We know then that

$$64 = r\,(2a + r - 1).$$

What does this tell us? Since r and $2a + \mathrm{r} - 1$ are factors of and $r \geq 2$ and $2a + r - 1 \geq 2a \geq 2$, then r and $2a + r - 1$ must both be powers of two. So 32 looks like being nice after all! Until you realize that if r is a power of 2 it's even. So $r - 1$ is odd and $2a + r - 1$ is also odd. No way that r and $2a + r - 1$ can both be powers of two.

But look at what we've done. Where did it matter that we assumed that it was 32 that was nice? Wouldn't the same proof apply equally well to 64, 128, 256, … and any old power of 2 we liked? Well, maybe we'd need to do something slightly different for $2^0 = 1$. Putting that aside though we see that any power of two is not nice. We've actually **proved** Conjecture 1!

Can we go the other way? Is it possible to show that every number that's **not** a power of two **is** nice? Perhaps we should go the same way as we did with 32. How would we show that 367 is nice? We'll assume it is and see what happens.

$$368 = \frac{r(2a + r - 1)}{2}$$

$$\therefore 736 = r(2a + r - 1)$$

What factors does 736 have? Can they give us an r and a $2a + r - 1$? Perhaps we should look for one odd and one even factor? Hello, $8 \times 91 = 736$. So let $r = 8$ and then $2a + 7 = 91$; $a = 42$. That was easy enough!

Sure but how can you convert that into a proof for **all** natural numbers not equal to a power of 2? That needs a little work but the stage has been set.

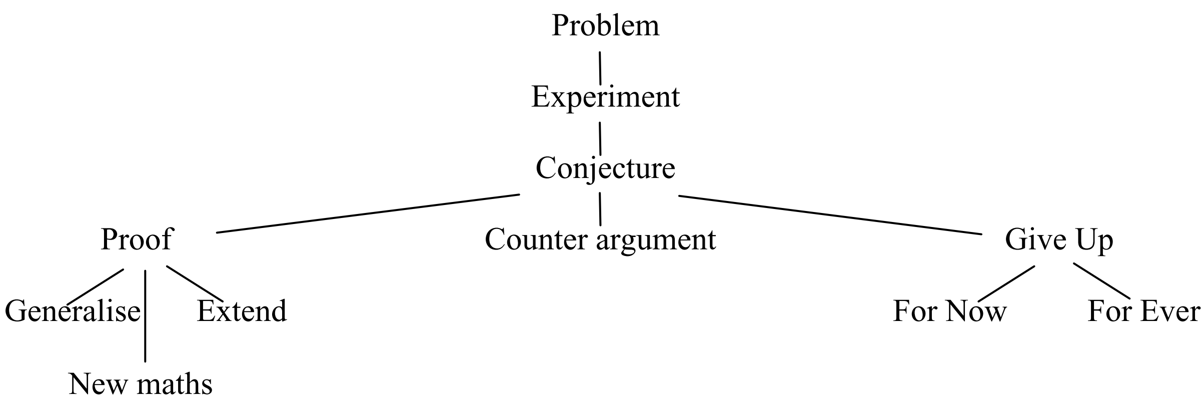
What Next?

This is one of the fun parts of mathematics – generalizing and extending. What questions could you ask now? Are there any numbers that are 1-nice? That is, are there any numbers that can only be made up of one sequence of consecutive numbers? It's almost certain that 3 is 1-nice. And $9 = 3 + 4 + 5 = 4 + 5$ isn't! What can be said about 1-nice numbers or 2-nice or 3-nice or …?

Maybe there are other things that can be done with this problem. It's time to use the imagination.

What's It All About Then?

Let's go back over the steps we've just been through. If we do, and there is license to imagine other steps, we find the following diagram.



All maths starts with a problem, whether it's a long known and revered one or something someone just made up. But a problem is something that you can't solve immediately. You have do scratch around for a while, experiment in some way. Here we experimented by seeing what nice numbers we could found.

This experimentation led us to a guess, a conjecture. Now these are not always good guesses, research mathematicians especially get them wrong all the time. But guessing itself is good if the problem is hard, as it enables us to whittle the problem away and get successively better conjectures.

When we've got a pretty decent conjecture we might think of trying to prove it. How would we do that? It's not easy, especially at first. It definitely needs practice. At this point, then, we either find a proof or we construct a counter-argument. In the latter case it's back to experimenting and onto yet another conjectures. In the former case there's hopefully a proof and a theorem. Out has possibly popped some new maths. And then it's time to think about generalizing or extending the original problem and so you are back at a new problem and the whole process starts again.

Why Do this?

First of all it's important to say that something like the process described in the last section is gone though by pure mathematicians when they are doing research. There's a fundamental series of steps that are gone through, though generally in a very haphazard manner – this is no linear process.

Second we should say that students of all ages can go through this process and experience the joy of discovery. At the same time they are experiencing an important part of mathematics that is often hidden away both by text book writers and by research mathematicians when they are writing up their papers.

Further, I believe that it's possible to go through many of the individual steps with school students – of all ages. There are problems of varying difficulty for varying ability. One of the individual steps is proof. It's a fundamental part of mathematics. I think it should be introduced to all children in all mathematics right from the start. It's very important for children to be able to justify things if for no other reason than the power it gives them over the content of the subject.

Now I have taken an interesting and none too easy a problem here. But it is artificial in some sense – there is no obvious link with the normal school curriculum. However, there are certainly things in the secondary curriculum that can be posed as problems that students can work through in the way I've suggested here. For instance, after students have explored Pythagoras' Theorem they can be asked to consider what happens if the triangle doesn't have a right angle. This can lead them to the Cosine Rule.

Finally I want to say that it may not be possible to do this kind of thing with every piece of mathematics at every level of the curriculum. First there may not be time. You may need to cover more material than you can reach by allowing the students to explore. Second, some result/skills at a given level are too hard to be discovered at that level. Much of what is in the secondary syllabus took years to develop after a great deal of effort by the finest minds. And then, of course, students should begin to learn how to learn to use books and people as sources.

But at least **some** of every course should be approached this way, otherwise the students are being short-changed. They are not seeing for themselves part of the mathematical process and worst of all, they are being denied the joy of discovery.

So I strongly advocate a wider view of mathematics in school (and possibly even at university too) that encompasses more than simply skills. As a step in this direction I would propose that examinations be set that are not mere repetitions of learned algorithms. That way students might be tempted to learn about a wider and more interesting view of mathematics.

The Process of an ICMI Study.

Derek Holton

Abstract

We will discuss the origins of ICMI Studies and the various stages that those Studies go through. This will cover the reasons for the Study, the Study conference, the publications relating to the Study, and the outcomes of the Study. We will also reflect on a number of issues including the efficiency of Studies and ways that the Study process might be kept alive even after the Study volume is published.

1. The Start of All Studies.

The idea of ICMI Studies developed from discussions held in the early 1980s at the time when Jean-Pierre Kahane became President of ICMI. Influential in these discussions were Geoffrey Howson the Secretary of ICMI and Bent Christiansen, ICMI Vice-President.

From its inception, ICMI had been involved in determining themes for discussion at the congresses (ICMs) organised by the International Mathematical Union (see [1]). ICMI representatives were asked to prepare reports on these themes. Unfortunately many of the representatives could not be relied upon to publicise these themes and to have national reports on them prepared. Accordingly, one of the motivations for the ICMI studies was to have interested people directly involved.

ICMI, too, had always given its support to other meetings, not always under its direct control, that were devoted to specific issues. For example in the 1970s, one such meeting was held in Luxembourg on the applications of mathematics and another in Nairobi on language and the teaching of mathematics.

While these conferences were acknowledged to be good things for ICMI to be involved in, there was a feeling that improvements could be made in the form of those conferences. For instance, it was believed that

- the meetings should have greater publicity;
- there should be better preparation for the meetings;
- participation should be open to anyone who was interested in the topic under consideration, following an invitation from the planning committee based on a submitted contribution; and
- reports of the meetings should be properly prepared and edited and be available commercially as widely and as cheaply as possible.

The ICMI Executive Committee, EC, of the early 1980s, then, decided to become directly involved in the major contemporary practical and theoretical issues in mathematics education. In practice this resulted in the ICMI study series. As a result of their deliberations, the first Study, on computers, was initiated. An International Programme Committee, IPC, was set up and with the help of its members a discussion document was written that was sent to all national delegates and published in a selection of journals.

In order to ensure that conference participants would be genuinely involved in the meeting, potential participants were asked to submit a paper on some aspect of the Study. The members of the IPC read the papers and invited to the Study Conference those authors whom they felt had most to offer. Their papers were not read at the meeting but formed the basis of general or working group sessions. In addition various experts were invited as plenary speakers.

On the matter of publication, the model used was that of the Proceedings of ICME 2. These were published by Cambridge University Press, CUP, on the condition that there had been a strict vetting of all the papers. This selection was achieved by an international editorial board. It was noted that "in some cases a paper has been selected because of its originality, in others because it represents a theme treated in several contributions – in such cases our choice reflects our wish to give prominence to the theme, and not necessarily our support for the arguments advanced" [2]

The first study volume and its four successors were produced very quickly and in both an extremely cheap paperback form and in hardback. As there were strict constraints on the number of pages these volumes could contain, only a small selection of the submitted papers was included in them. Thus the Discussion Document plus 42 other papers, which were not published in the CUP volume, appeared in a collection that was published by Strasbourg University [3]. This first study volume was reprinted and also appeared in translation, as did Studies 2, 3, and 4: indeed Study 2 was translated into five languages.

More on the early history of ICMI studies and an outline of the first five studies can be found in [4].

2. The Start of a Study.

The pattern set down by the first ICMI study is essentially the same as that followed by subsequent studies with the exceptions of Study 2 (which was based on an international seminar with invited participants) and Study 4 (which was directed entirely by the PME group). First a given study topic is decided upon by the EC of the day. At any one time, there are several ideas being considered for study topics. These ideas come both from previous ECs as well as from the current one. Possible topics are discussed by the EC and those that seem more important at the time, of general interest, and, in some way, ready for a study, are chosen to be taken further. This choice is made with an eye for balance on mathematical area, on level of education, and what may be relevant to a worldwide audience. There seem to be no 'types of topics' followed. There is an aim to be as all encompassing as possible. Naturally there is a certain amount of personal taste in the choice of study themes. At the end of the day the studies are expected, through their study volumes, to produce state-of-the-art discussions of the study topic. More specifically, as Hodgson, not then Secretary-General, said, [4],

> "Le but de ces Etudes n'est pas de fournir des solutions portant le 'sceau de garantie-CIEM', pour reprendre les mots mêmes du président de la CIEM …, mais bien de faire le point sur un thème

> donné de façon à animer la réflexion et à susciter de nouvelles initiatives tant au plan national que régional ou institutionnel."[1]

As Galbraith reminds us, [5], the aims of ICMI, in particular through the study programme, are to offer

> "… researchers, teachers, curriculum designers, policy makers and others interested in mathematical education a forum promoting reflection, collaboration, exchange and dissemination of ideas and information on all aspects of the theory and practice of contemporary mathematical education as seen from an international perspective."

Not that every individual study will be relevant to all of these groups but that these groups will find something of interest in most studies.

Having decided on a topic, the EC next chooses a Study Chair, or recently two co-Chairs, for their expertise in the theme of the study. It is the Chairs' job to make sure that the study proceeds to a satisfactory conclusion. Often, though not always, the Chairs edit the final study volume.

The EC also draws up a list of possible members for the IPC, whose job it is to determine the overall direction of the study. This list is finalised with the involvement of the study chairs. The IPC typically consists of people with knowledge in one or more areas of the study and is chosen with an eye for gender and regional balance. Usually the IPC is around a dozen strong and includes the ICME Secretary-General ex-officio. A wide range of countries and cultures have been represented on IPCs to date. However, the African continent seems to have been under-represented and Australasia has been over-represented. Further, a closer examination of the authors of articles in study volumes reveals that 65% were from six Western countries [5].

The IPC rarely meets more than once before the study conference so most of its work is conducted by email. Its major roles are outlined in section 2.

Nineteen studies have now been completed, are under way, or are planned. We have listed these below. It can be seen that they cover a wide range of mathematical topics. Further details are available in [6] where it can also be seen that Conferences have been held in a range of countries.

1. *The Influence of Computers and Informatics on Mathematics and its Teaching.* Study Conference 1985; Study Volume 1986.
2. *School Mathematics in the 1990s.* International seminar 1986; Study Volume 1986.
3. *Mathematics as a Service Subject.* Study Conference 1987; Study Volume 1988; Selected Papers 1988.
4. *Mathematics and Cognition.* No Study Conference; Study Volume 1990.
5. *The Popularization of Mathematics.* Study Conference 1989; Study Volume 1990.
6. *Assessment in Mathematics Education.* Study Conference 1991; Two Study Volumes 1993.
7. *Gender and Mathematics Education.* Study Conference 1993; Proceedings published 1995; Study Volume 1996.
8. *What is Research in Mathematics Education and What are its Results?* Study Conference 1994; Study Volume 1998.

[1] The aim of these Studies is not to give any "ICMI-labeled" solution, to use the ICMI President's own words, ..., but to provide an up-to-date presentation and analysis of a given theme so as to promote reflection and foster new initiatives at the national, regional or institutional level.

9. *Perspectives on the Teaching of Geometry for the 21st Century.* Study Conference 1995; Study Volume 1998.
10. *The Role of the History of Mathematics in the Teaching and Learning of Mathematics.* Study Conference 1998; Study Volume 2000.
11. *The Teaching and Learning of Mathematics at University Level.* Study Conference 1998; Selected papers published 2000; Study Volume 2002.
12. *The Future of the Teaching and Learning of Algebra.* Study Conference 2001; Study Volume 2004.
13. *Mathematics Education in Different Cultural Traditions: A Comparative Study of East Asia and the West.* Study Conference 2002; Study Volume 2006.
14. *Applications and Modelling in Mathematics Education.* Study Conference 2004; Study Volume 2007.
15. *The Professional Education and Development of Teachers of Mathematics.* Study Conference 2005; Study Volume planned for 2008.
16. *Challenging Mathematics in and Beyond the Classroom.* Study Conference 2006; Study Volume planned for 2008.
17. *Digital Technologies and Mathematics Teaching and Learning: Rethinking the Terrain.* Study Conference 2006; Study Volume planned for 2009.
18. *Statistics Education in School Mathematics: Challenges for Teaching and Teacher Education.* Study Conference planned for 2008.
19. *The Role of Mathematical Reasoning and Proving in Mathematics Education.* Study Conference planned for 2009.

3. The Main Steps of a Study.

We note here that there is no formula for the process of a study but they all studies have much in common. For instance, there are three main steps in any study and these are the main responsibilities of the IPC. These are steps are, in order, the Discussion Document, the Study Conference, and the Study Volume (see also [4]). Other optional extras will also be considered.

3.1 Discussion Document.[2] The first job of the IPC is to prepare and disseminate the study discussion document , DD. The DD broadly sets out the aims of the study and calls for submissions from those interested in the area.

There are no rules about the structure of a DD but they usually start with a section setting the scene of the study within the mathematical or statistical community and within the ICMI study framework.

This is followed by a justification for the study. What has taken place in the area in recent times that suggests that now is a good time to review the area? For instance, Study 11 followed on the heels of some significant changes. In the mid 90s there had been a considerable increase in the number of students attending tertiary institutions in many parts of the world; major pedagogical and curriculum changes had taken place at secondary level; there was an increasing difference between secondary and tertiary mathematics education; there was an increase in the use of technology; and there was a general demand for universities to be publicly accountable. Consequently it was felt that there was a need to examine the current and future state of the teaching and learning of mathematics at university level.

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[2] The Discussion Documents for Studies 11, 12, 13 and 14, for example, can be found on-line at http://www.mathunion/ICMI/bulletin/43/, http://www.mathunion/ICMI/bulletin/48/, http://www.mathunion/ICMI/bulletin/49/, http://www.mathunion/ICMI/bulletin/51.pdf/.

Actually the DD for Study 11 then listed specific points that that study would cover. These included:

- to identify, review, encourage and disseminate, research in educational matters at the tertiary level;
- to identify and describe major approaches to tertiary mathematics teaching within different cultures and traditions;
- to discuss equity and other issues relating to mathematics education at university level;
- to discuss the goals of teaching mathematics to a range of students with different backgrounds and needs, and who should be responsible for that teaching; and
- to identify, publicise, and expose to scrutiny, new teaching methods and the positive use of technology.

As far as Study 18 is concerned, statistics is taking a more central place in schools in many countries because of the need for a statistically literate society. However, the curricula recommendations for the data oriented teaching of statistics are often not followed and there are challenges in the initial training and ongoing development of teachers. So the current study, in association with the International Association for Statistical Education, aims to consider the teaching of statistics in some depth. This is to be achieved around six topics:

Topic 1: The current situation of teaching statistics in schools.
Topic 2: Teachers' attitudes, knowledge, conceptions and beliefs in relation to statistics education.
Topic 3: Analysing current practices in teacher education regarding the teaching of statistics.
Topic 4: Empowering teachers to teach statistics.
Topic 5: Training teachers in developing countries.
Topic 6: Building collaboration between mathematics and statistics educators in teacher education.

From here DDs usually go into depth looking at the themes, issues and focus of the study. In the case of Study 11, this section considered research, practice and policy in some detail, ending with a number of questions such as

- What are the major research findings of mathematics education?
- In what ways can teaching change to take into account the different background, abilities and interests of the learner?
- What are the professional aspirations of our student population?
- Should the profession through its national bodies, show that it recognises the importance of teaching at the university level?
- What does the mathematical community need to do to make mathematics more visible in a competitive environment?

The DD of Study 18 developed each of the six topics above in some depth using a set of questions. Some of the questions under Topic 1 are:

- What status does data handling and statistics have in the curricula of different countries?
- What are the main current problems in the way statistics are taught?
- What is the difference between teaching statistics and teaching statistics literacy?
- What are good examples of teaching statistics in schools?

At some point too the DD indicates the kind of groups that might contribute either with a paper or by attendance at the study conference. For instance in the Study 11 DD, the groups suggested were "those concerned with the teaching of mathematics at the university level, to mathematics educators undertaking research in related areas, and to many other people with an interest in university level mathematics." On the other hand, Study 18 proposes “mathematics and statistics educators, students preparing to be teachers, teacher educators, people involved in curricular development in statistics as well as researchers in statistics and mathematics education.”

Then study conference submissions from interested people are called for. It is expected that theoretical and practical experts in the field will offer papers on the topic of the study.

Finally a list of the IPC members is given to complete the DD.

3.2 The Study Conference. One more decision that is made by the EC, in cooperation with the co-chairs, is the location of the study conference. Often a proposal to host a conference is made by members of the IPC. Selection of a site for the conference is made on the basis of costs and local support as well as a desire by the EC to increase the involvement of developing countries. Once the venue of the conference is decided the Local Organising Committee, LOC, then takes full responsibility for the running of the study conference though the IPC decides on the format and the speakers.

As the study DD and related interactions proceed, the IPC develops a better idea of what they expect from the conference. They will begin to make decisions on their plenary speakers and their working group leaders. It should be noted at this point that study conferences are very definitely working conferences whose immediate goal is to produce the skeleton from which the study volume will ultimately be produced. So the potential working groups will each take responsibility for a different aspect of the study.

The papers that are prompted by the DD will be considered by the IPC and a limited number of authors will be invited to the study conference. It should be underlined that study conference attendance is by invitation only, typically to some 70 or so people. The restriction is because of the need for an optimum number to be engaged in a working conference.

The papers are generally organised into proceedings that may or may not be published in any sense. Sometimes they are put into book form just for the study conference participants. Sometimes they are put on the web as in the case of Study 16 (see http://www.amt.canberra.edu.au/icmis16participantpapers.html) so that they can be available to participants before the conference and released to a web audience after the conference. Some of these papers will go directly into the study volume (this is more likely for the papers of plenary speakers) but generally they will be present more in substance as part of an argument in a particular chapter.

We note that some of the earlier studies (Study 2 and Study 4) did not hold a conference. Further the LOCs of Study 17 and Study 18 held special meetings for local teachers alongside the study conference to enable them to benefit from the presence of the experts that were assembled for the conference. This follows an idea started with Study 5, where a large exhibition was held alongside the conference and attracted significant numbers of adults and schoolchildren. The public was also able to hear a series of popular lectures. (See [4] for a more detailed discussion of this exhibition.)

3.3 The Study Volume. The aim of the study volume is to provide a state-of-the-art compendium of the study area. It is not simply the proceedings of the study conference. A recent principle enunciated by the EC, says that "Study volumes should contain a balance of papers that specifically critique, or develop, or apply theories or frameworks identified as particularly relevant to the topic area; noting that the balance may vary with the nature of the topic." (See [5].) In that same paper, the EC is reported as saying that "ICMI studies are closer to reviews of research in certain focus areas, then to reports of particular projects."

The actual format of the study volume varies from a collection of selected and edited papers (see Study 11) to a comprehensive monographic exposition (see Study 12).

It should be emphasised that the study volume doesn't magically appear at a study conference. After the conference is over there is still a great deal of work that has to be done. For instance, in the case of Study 11, group leaders were asked to put together the deliberations of their groups, papers were asked from plenary speakers, and some participants were asked to develop themes that arose from the conference. In that study it was felt that the conference hadn't completely covered all of the issues involved and some authors who had not even attended the conference were asked for material.

In the case of Study 16, the various chapters are being developed from the different working groups. The study volume will have an introductory chapter prepared by one of the Chairs of the IPC with contributions from a few participants.

But the study volume may require a significant effort after the conference by IPC chairs. This effort is required to both gather and edit the material for the final published book.

3.4 Optional Extras. After most studies the only publication is the study volume but there are variations. For example, after Study 3 and Study 11, selected papers were published. And after Study 6, two study volumes appeared.

In more recent times, invitations have been given from various conferences for panels to be given on ICMI Studies. So representatives from Study 11 gave a panel on their work at the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level) held in Crete in 2002 as well as at ICME 9 in Japan. Often Topic Study Groups or Working Groups on a study theme or one closely akin to a Study theme have been held at an ICME.

Invitations to publish papers from the study conference have also been given. For instance, the American Mathematical Notices published similar papers to the chapters of Artigue [8] and Schoenfeld [9] in the Study 11 Volume.

4. Impacts of Studies

It would seem very difficult to gauge the effect of a particular study. The number of study volumes sold and the number of times volumes and chapters have been cited will give only part of the story. The simple fact of their existence must be important.

In the first instance a relatively large group of people spend a significant amount of time thinking and writing about a study topic and this must influence the way they act and the work that they publish. Surely studies stimulate action stimulated reflection and action but to what extent they influence areas of mathematics education is very difficult to gauge. As Hodgson said, [4],

> "le symposium aura été pour ses participants l'occasion d'une expérience professionnelle des plus exceptionnelles: la petitesse relative du groupe (habituellement moins de 70 participants), la durée de la rencontre (5 à 6 jours) et la densité du programme font d'un tel événement une occasion incomparable d'échanges et d'interactions."[3]

In [5], Galbraith will report on a review of the scholarly impact of ICMI Studies 6 to 11 but this is more concerned with the form of article in the studies than the way they have produced change in teaching or research. It would be valuable for ICMI to undertake a review in this area though it would not be any easy task.

5. Reflections

It seems to me that the STUDY idea is a very good one and one that should be continued. It brings experts in an area together to produce a state-of-the-art publication that can serve for many years both as a stimulus for work in the area and as a valuable reference. This final section consists of personal reflections on my experience of being involved with two Studies, Study 11 and Study 16. I have written them in no particular order.

It is interesting that the five points (Section 1) that were developed in the early 1980s for the first Study have proved so durable.

5.1 Study Volume. I have to say that both studies I have been involved in seemed to be hard work in getting the ideas from the study conference into print in the study volume. It is very hard for the chair(s) of the IPC to get work from individuals who are generally busy people. Indeed it is very hard for the chair(s) to find the time for the post-conference work that is necessary. Is there a way to facilitate the publication process?

Allied with this is the delay problem. Recent study volumes seem to be appearing from 2.5 to 3 years after the conference. In the process some topicality is lost. It seems to me that it would be good if this time delay could be minimised. But that would mean that a number of people would have to commit a greater time nearer the conference. Is it possible that the working group chairs who are responsible for writing chapters could remain at the conference venue for a further week of solid writing? Could most of their work be done in this short time frame when they had no lectures to give or meetings to attend?

In view of the comments in [5], it would be good to widen the writing base of ICMI studies. Though the EC had put forward a Principle on this theme, and was

[3] the symposium will have been for its participants the occasion of an exceptional professional experience: the relative smallness of the group (usually less than 70 participants), the length of the meeting (5 to 6 days) and the density of the programme have as a result provided an incomparable opportunity for exchanges and interactions.

sympathetic towards it, it was not able to support it. This was because the distribution of expertise and the undertaking of research are not uniformly distributed. They did, however, recommend that the matter be considered on a study-by-study basis. There are two things that might be added here and they both have something to do with language. While I understand the reasons for it, the language of ICMI studies continues to be English. It is very hard for many active practitioners and researchers to participate fully in both the conference discussions and the study volume writing when this is the case (see [5], a note after Principle 6). Perhaps some effort could be put into allowing participants to submit articles in their own language and have them translated into English later.

Another argument for advocating translations, even if only from English to other languages, is that this might help developing countries to develop expertise in areas that are important both to mathematical education in general and to their countries of origin in particular.

5.2 Web Site. It has always seemed to me to be a pity that the study seems to end with the production of the study volume (except for Study 1, where UNESCO published a follow up in 1992 [10]). I guess that this is not totally true in the case where study themes are continued at a following ICME. Would it be possible though to keep some continuity by using the web and updating the material regularly?

5.3 Cost. I understand that if Study participants cannot find support from their own home institution then assistance can be found for them. However, the actual cost borne by individuals and institutions is significant. Is it possible for sponsorship to be found to defray more of the expenses than are currently possible?

Actually one part of the last bullet point in Section 1 is a cause for concern here - the cost of study volumes. Certainly discounts (of 60%) can be obtained by members of the ICMI community but still the volumes are rather expensive. It would be good to both reduce this cost and to consequently make the study volumes available to a wider audience. If there was a parallel publication on the web, part of this problem could be overcome. Since for all languages other than English the copyrights belong to ICMI, it might also be simultaneously possible to ensure that the volume could be translated into several languages.

Thinking more about the mathematical education audience, it seems to me that it might be worthwhile to publish some studies in two volumes to suit two audiences. While it is not always true that there is a research community and a practicing community, this is often the case (see [5]). What's more, each community does not always want to read material that is aimed at the other. Dividing the study publication in two might also reduce costs.

Thus division could well have been profitable in Study 11. In Study 16 on Challenge, whose study volume is yet to appear, there are also potentially two audiences. One of these is classroom teachers who want challenges that they can cope with, understand, and use in their regular classroom. On the other hand there are academics, who are more interested in challenges whose the solutions are quite difficult. It is possible that separating these two audiences might make two volumes that would be better than one as well as satisfying two audiences where one volume might not satisfy either.

5.4 Publicity. One final thing, I like the idea of having more popular events alongside conferences. Mathematics needs to take every chance to advertise itself to a general audience. I would encourage future studies to hold exhibitions and open lectures for this purpose. It would also be valuable, where appropriate, for local teachers to be involved in some way.

Acknowledgements

I am grateful to Jean-Pierre Kahane, Geoffrey Howson and Bernard Hodgson for their valuable comments on the origins and history of the ICMI studies. (Geoffrey, in particular, took the time to write several pages of notes that would make an interesting historical paper and he and Bernard helped to refine drafts of this paper.) I am also grateful to Michèle Artigue, Lee Peng Yee, and Mogens Niss for their comments on the processes involved in the more recent Studies:

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COMMISSION INTERNATIONALE DE L'ENSEIGNEMENT MATHÉMATIQUE

(THE INTERNATIONAL COMMISSION ON MATHEMATICAL INSTRUCTION)

PERSONAL THOUGHTS ON AN ICMI STUDY: "THE TEACHING AND LEARNING OF MATHEMATICS AT UNIVERSITY LEVEL"

by Derek HOLTON

ABSTRACT. In 1997 the International Commission on Mathematical Instruction decided to organise a study into the teaching and learning of mathematics at university level. I have collected here some 'thoughts' that have occurred to me since that study. They are based on notes that I wrote at the time. I look at some personal highlights and concerns from the Study.

1. THE STUDY

As with all studies of the International Commission on Mathematical Instruction [1]) (ICMI), the theme of the 11$^{\text{th}}$ ICMI Study, on the teaching and learning of mathematics at university level, was decided by the Executive Committee of ICMI. It then appointed the International Program Committee, IPC, that took over the responsibility for conducting the Study. The first job of the IPC was to produce a Discussion Document [3] that laid out the rationale for the Study and ended with a call of interest for the Study conference. The second job was to organise a conference. This was duly held in December 1998 in Singapore. Along with plenary speakers, around 80 participants were invited from among those who responded to the Discussion Document. On the

[1]) More information on the International Commission on Mathematical Instruction can be found in [1] and [2].

basis of the papers presented at this conference, a special issue of the *International Journal of Mathematics Education in Science and Technology* [4] was published early in 2000. In 2001 the Study volume [5], consisting of plenary talks and working group reports from the conference, along with commissioned papers to cover perceived gaps, was published.

2. Personal background

I think it is important to say that what I now want to talk about are the highlights and concerns of the Study, from my own perspective. Other participants and observers will likely see different aspects as being of importance.

Since this is a personal view, the reader should know the biases that I brought to this Study. First I am by training and inclination a pure mathematician and I am still publishing in the area of Discrete Mathematics. However, I trained (but never practised) as a secondary mathematics teacher and then entered academia almost by accident. As a result, I suppose that I have always been interested in teaching as opposed to lecturing [6]. For a long time I have been aware of the shortcomings of the traditional lecture format used in many countries and I have tried to find ways to overcome its deficiencies. I am now in a department of mathematics and statistics where many of the staff are also concerned but are unsure as to how they might change things. Like many such departments we were happier in the "good old days" when fewer, but more 'serious', students studied our subject.

3. Why worry ?

The Study Discussion Document identified five changes that have had a profound effect on mathematics teaching at university level. These are

(i) the increase in the number of students who are now attending tertiary institutions;

(ii) the major pedagogical and curriculum changes that have taken place at pre-university level;

(iii) the increasing differences between secondary and tertiary mathematics education regarding the purpose, goals, teaching approaches and methods;

(iv) the rapid development of technology; and

(v) demands on universities to be publicly accountable.

In some universities, although there has been a doubling of the total number of students enrolled, there has either not been a similar rise in the number of students taking mathematics or this rise has not affected the number majoring or taking honours degrees in the subject. Where universities have been under financial pressure this has led to staff reductions. In some places it has also led to the closure of mathematics departments. This has been a cause of concern and worry to many academics.

At the same time, there has been a demonstrable change in the average quality of first-year students. Many staff have given up the struggle to raise them to the graduating level of the 1970s or early 1980s. But they are worried about what they see as a downward spiral in mathematics in the community. This is especially so where secondary teachers but more especially where primary teachers are concerned. In many countries, students entering primary teacher education have very weak mathematical backgrounds and it seems that a smaller percentage of majors goes on to secondary teaching every year. There seems to be cause for concern for society at large given that so much is based on such a potentially poor mathematical foundation.

Another worry in this context in some universities is that an ageing department does not understand the basic philosophies and motivation of many undergraduates and, accordingly, finds it increasingly difficult to teach them.

There is a genuine concern for the students in our courses, for their needs and society's future needs, for mathematics itself and its future and, consequently, for what to teach and how to teach it.

4. Highlights of the Study

I now want to concentrate on three personal highlights of the Study. These are

(i) the efforts being made to teach mathematics;

(ii) that valuable research does not have to be big research; and

(iii) the discovery of a theory of learning.

Under the first heading I want to emphasise both the word "teach" and the word "mathematics". By "teach" I don't necessarily mean the transmission of material in the manner of a traditional lecture. Rather I mean a delivery that is based upon an awareness of the student's knowledge and needs and allows for some dialogue to take place between student and lecturer. In the process the lecturer recognises, and tries to employ, current learning theories.

Mathematics to me means more than a set of skills or a chain of theorems. I believe that we sell our subject short if we hide the human, processes, side from our students (see [6]). So I feel that some attempt needs to be made to encourage students to discover *some* mathematical results for themselves. The aim is to let students see that mathematics is more than the set skills/algorithms that we have traditionally asked them to regurgitate in examinations. But I deliberately underlined 'some' because there is not time to cover all that we want to cover by discovery alone.

It was a pleasant feeling to find that there were many people at the ICMI Study conference who not only agreed with that perspective but had gone a long way down the implementation path. Naturally I had known of the reform movement in the United States but I didn't have any idea of the extent and impact of programmes, and their research support, that were being undertaken in, for instance, France. The references [7], [8], [9], [10], [11] and [12] are in no way meant to be exhaustive but rather are meant to act as pointers into this literature.

Two aspects of the French scene that appealed to me were *la situation didactique* and *le débat scientifique.* The former involves prior analysis of students' past experiences, of what they bring to the session, and of what sorts of difficulties, pre-conceptions and misapprehensions they are likely to have or to form. Examples of tasks for the students, in order to challenge their conceptions, are chosen using what has been learned from research. So in a tutorial situation one might ask "if f is continuous on $[0,\infty)$, is strictly positive and has a limit of zero at ∞, does f necessarily have a maximum value ?" Most students are likely to answer this question positively. But then

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they are required to provide a proof. In the ensuing discussion, hopefully learning and understanding take place.

Under *le débat scientifique*, students are seen as participants in a scientific community whose methods of development include conjectures, proofs and regulations. Scientific debates can arise spontaneously, as when a student asks a question, or can be intentionally provoked. The guiding principles for scientific debate include:

disturbance	– students must encounter and deal with conflict;
inclusiveness	– everyone should have an opportunity to understand what we try to teach; and
collectivity	– collective resolution of issues shows how to work with contradictions and to respect the views of others.

Turning now to research, it had always seemed to me that the problems involved in mathematics education were major ones and that worthwhile results could only be obtained by large studies over long periods of time. And how could you prove anything in education anyway? Perhaps this was just the naïve opinion of an external observer. There is no doubt there are big issues that will not be resolved quickly. These must surely exist in all disciplines. However, it is possible to gain insight into some worthwhile areas of mathematics education without a huge budget. And these insights can be extremely useful for teaching, especially at a local level. What you discover for yourself about your students can lead to the improvement of your own teaching.

Simple questionnaires, for instance, can provide valuable feedback. Anthony [13] used a questionnaire plus interview methodology to identify factors which are seen as having an important influence on first-year students' success in mathematics. Her sample consisted of 65 students and 22 teaching staff. Both lecturers and students agreed that self-motivation and study were important for success. They also agreed that lack of effort led to failure. However, in some ways the places where they significantly disagreed might be more illuminating. For instance, students placed availability of help much higher on their list of things contributing to success than did lecturers. The same was, perhaps surprisingly, true for regular attendance at lectures and taking notes during lectures. Poor study techniques were listed more highly by lecturers than by students as a factor influencing failure.

Lithner [14] worked with four students who "talked aloud" as they completed two mathematical tasks. Their work was captured on videotape. He found that when they ran into difficulties they tended to revert to what was remembered and familiar, even if inappropriate, rather than to apply mathematical reasoning to the task.

While I had some knowledge of David Tall's work in developing a theory of learning for mathematics, I had not encountered APOS theory before the conference. In [15], Gray and Tall call the combination of process and concept produced by the process that may be evoked by the same symbol, a procept. The work of Dubinsky and others, [16], links Action, Process, Object and Schema. The ideas involved in both procepts and APOS theory seemed to fit well with my understanding of mathematical learning. Clearly neither of these presents a complete picture but they have provided me with a stimulus to further reading and thinking.

5. Concerns

In the process of being involved in the current Study I have become concerned about a number of issues. These are

(i) the mathematician/mathematics educator divide;

(ii) technology, and

(iii) culture.

There does appear to be a feeling among some mathematicians that mathematics educators have little to contribute to lecturing or teaching at the tertiary level (and possibly at other levels too). This is not the case. However, in some places the gulf between these two groups is wide and it is not immediately clear how to bring them together.

Two schemes, one in England and Wales[2]) [17] and one in France [18], may help to ameliorate this in the long term. They are both attempts to recognise the fact that experts in a subject are not automatically even competent teachers. Both schemes provide opportunities for instruction for teaching at university level across all disciplines. The one in England and Wales is largely an in-service scheme while that in France is a pre-service one. Hopefully initiatives of this kind, both of which are backed by government, will improve the quality of teaching in universities in their respective counties. Possibly they will serve as models for other countries. With any luck they may become a catalyst to get mathematicians and mathematics educators talking to each other.

Through technology we have an ever-increasing set of tools to aid our research. Technology should also be providing us with a range of tools to aid our teaching. There is no doubt that these tools are being used but it is not clear to me that, at this moment, we are using them in ways that promote learning as efficiently as we might.

In Celia Hoyle's panel presentation at the ICMI Study conference she made the point that there was much for tertiary educators to learn from the secondary arena in the area of technology. She said that there was considerable evidence to support the computer's potential to

❄ foster more active learning using experimental approaches;

❄ provoke constructionist approaches to learning mathematics;

❄ motivate explanations in the face of "surprise" feedback;

❄ foster cooperative work, encouraging discussion of different solutions and strategies; and

❄ open a window on student thought processes.

But she said there was also evidence that we need to rethink

- the content and sequence of courses;
- teaching approaches; and
- the relationship between 'computer maths' and pencil and paper maths.

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[2]) This has developed in two directions. For information see the Higher Education Academy (http://www.heacademy.ac.uk/) and the Maths, Stats and OR Network (http://www.mathstore.ac.uk/).

On the other hand, she noted that there are many obstacles to using computers. Among these are

◊ access to hardware;

◊ time for familiarisation with systems;

◊ focussing on calculation rather than reflection; and

◊ the potential disadvantaging of certain groups.

It seems to me that we have a long way to go before we are able to use technology for better understanding of mathematics. We need to experiment and discuss the results of our experimentation to find the best ways to use the tools that we have. I am not convinced that just giving computer packages to students will promote learning and understanding. I am also concerned about the blind rush of some administrators to the use of web sites for teaching. If those sites are no more than text-books we have made no progress.

The last point above leads me into my final concern, that of culture. At one level I feel that the Study I was involved in has, to a large extent, reflected only Western Culture. For various reasons it has not been possible to extend the papers in the Study volume as far as I would have liked. But the more I think about it the more concerned I become about the various cultures, or maybe subcultures that exist in the classroom. Certain of these are obvious. Just by looking at the students I can see that they were not all brought up in the same mainstream culture that I was. I clearly don't understand all the cultures that are represented. A couple of years ago, in an attempt to help a student from another culture, I inadvertently caused the student to withdraw from the class.

But there are less obvious cultural differences. In my classroom, middle-class culture, middle-aged culture and mathematics culture prevail. Hence I use language in ways that are different from those of many of the students I teach. I also have different views on life and learning which are not necessarily shared by all of my students. These issues are taken up in the paper by Robyn Zevenbergen [19]. It is something that I need to be more aware of. In the past when we had fewer students and many fewer overseas students, this cultural "complication" hardly existed. This is just another thing that has changed in the last 15 years or so.

6. Outcomes

It is very difficult to quantify the influence of a study such as this one. The Study volume and the articles in the special issue of the *International Journal of Mathematics Education in Science and Technology* have been cited in numerous publications. At ICME-9, held in Japan in 2000, there was a Working Group on Mathematics Education in Universities as well as a panel where issues related to the Study were aired further (see [20] and [21]).

The influence can also be seen at several other conferences where presentations directly related to the Study were given. For instance, at the 2^{nd} International Conference on the Teaching of Mathematics (at the undergraduate level) held in Crete in 2002 a similar panel to that that took place at ICMI-9 was held. And the theme was maintained at ICME-10 in Denmark in 2004 with the Topic Study Group on New Trends and Developments in Tertiary Education. (Papers from this conference were also published in the *International Journal of Mathematics Education in Science and Technology* [22].)

The International Conference on the Teaching of Mathematics is now well established, as is the Delta series of conferences that are held in the Southern Hemisphere in odd years. It is not clear how much the ICMI Study has influenced these developments or is just part of a general movement towards interest in tertiary mathematics education.

7. Where to now ?

So, given the circumstances in which universities find themselves, and given the interests and abilities of staff, and given the students who hopefully will continue to come to us for accreditation if not always for learning, and given the constraints of society, what should we teach and how should we teach for optimum learning ? The current ICMI Study does not have the definitive answer to that question. Indeed, such an answer may not exist. But it does have perspectives, suggestions and pointers. Hopefully it has ideas that will stimulate its readers to enhance practice over the next several years. This will require a knowledge of

- the mathematical needs of society ;
- the learning capabilities of our students ;
- mathematics and especially its difficulties ;
- the pedagogical use of technology ; and
- how to teach mathematics so that it becomes more accessible and less aimed at an exclusive market.

To me that doesn't seem to be a very easy challenge.

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(Reçu le 22 octobre 2007)

Derek Holton

Department of Mathematics and Statistics
University of Otago
New Zealand
e-mail : dholton@maths.otago.ac.nz