About sequences and developmental stages in the teaching of mathematics. Some contributions.

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Our contribution to Working Group 1 refers to some questions pertaining to the topic *The development of mathematical ideas*. However, it has been impossible for us to resist the temptation to open this paper with some thoughts about the stimulating general question presented by Bill Barton and Fréderic Gourdeau: *What sort of mathematics do we learn at school?* Not only does this question suggest to us some ideas for starting out, but the content of the text which follows it also suggests to us some starting ideas as fundamentals for our discussion of the selected topic, as follows:

- The mathematics which is taught at any school institution essentially bears some differences in relation to so called mathematical science. These differences are supposed to be due to the different goals of these two forms of "mathematics": on the one hand, the communication and/or the diffusion of a specific disciplinary knowledge in order to be learned by individuals of different ages and backgrounds; on the other hand, the questioning of the mathematical scientific knowledge available in order to make it advance scientifically, the task of the mathematicians.

- Nevertheless, and despite the differences in goal, in destiny, school mathematics (even that taught at Elementary School) not only may, but also must, be understood in terms of a process of abstraction, symbolization and axiomatization, since it must follow that scientific knowledge in their constant changes because it must retain its mathematical identity. Therefore, school mathematics would not essentially be different in nature from mathematics as a science.

Based on the ideas expressed above, we now enter the discussion about the content initially stated in the topic concerning the development of mathematical ideas: *the way fundamental mathematical ideas are presented to learners of mathematics will, by necessity, be shadow versions of the concept as held by mathematicians*. *It is hoped that these shadows will develop more substance as the learning becomes more sophisticated*.

It is our view that this statement goes exactly in the same direction of the ideas we have outlined above concerning school mathematics as it is, as a matter of fact, a knowledge, a cultural production to be understood, comprehended by students in a specific social institution - the school - besides the particular goals of this institution in the society in which it is inserted. However that may be, there, it is still mathematics.

School mathematics is a product of this science. But it is not only that: many mathematical ideas which are present in school institutions go beyond the particular scientific content of the subject (as validated knowledge) because, derived from a certain culture, they are representations of the varied ways of interpreting the world of the human actors engaged therein (Pedra, 2002).

Having made this clear, we pass on to some ideas in order to boost the discussion concerning the following questions pertaining to the topic: *Is there a best developmental sequence for any particular mathematical idea? Are there stages of students' development at which particular concepts are best taught?*

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The content of the two aforementioned questions plus the general arguments expressed beforehand require, in our view, that attention be paid to some assumptions originating from at least two fields in order to consider the issue: that of curriculum theories and that of psychology.

From curriculum theories arise the classic rationally based assumptions that a curriculum for mathematics (like curricula of any other field) must be organized in and along a certain time, according to criteria whose employment takes for granted its continuity, its integration and a kind of progressive sequence in order to obtain, at the end, a successful learning of certain students (Posner, 1974; Taba, 1974, Tyler, 1975, *e.g.*). In terms of curriculum organization, we recall, these measures correspond to the *logical order* of the "materials" (Taba, 1974). In this case, would an ensemble of mathematical content with its cumulative and hierarchical structure be the best way to organize a teaching sequence for the learning of some mathematical idea?

It is quite improbable that any answer to this question could be supported or infirmed in absolute terms, at least for one fundamental reason: to be the best sequence depends on the relationship between learning and a series of other factors playing their role in its accomplishment, in addition to the logical order of the materials or the ideas already developed by the student, as suggested by the question. These factors concern: the learner ("Who should learn mathematics?"); the context or learning conditions ("Where…?", "When…?"); and the teacher as the interpreter and agent of the curriculum achievement when he/she plans his/her teaching in order to achieve the intended learning.

Furthermore, this discussion must deal with the proposition concerning the relativity of any teaching sequence for its relationship to psychological phenomena. That is, once again matching some classical propositions in curriculum organization, what is named the *psychological order* (Taba, 1974), without leaving aside other compulsory dimensions of curricula decisions, among them the social-cultural, which brings about a *sociological order* pertinent to the demands that any society makes on its educational institutions, in the circumstance that these institutions deal with knowledge, cultural production in constant change.

Then, in order to support this reflection, we have psychological assumptions, besides those coming from other different fields. And from psychology we specially have those related to the existence of a relationship between the phenomenon of psychological development, at least in its cognitive dimension, and the phenomenon of school learning, a classical issue in the field of educational psychology (Ausubel, Novak & Hanesian, 1978; Bruner, 1972; 2001; Piaget, 1964; Vigotskii, 1933).

Viewed in the literature as a field with strong influence in the history of mathematical education (Kilpatrick, 1992), first of all, psychology (educational psychology) brings to light the perspective of the learner as a human being whose psychological development, particularly the cognitive, is one of the learning factors.

Secondly, it provides knowledge about the learning phenomenon on its own, and its complex peculiarities when school learning is in focus. Moreover, all the attention must be put to the learner's form of apprehending a specific content, what will have as a consequence the necessity of taking into account, in the curriculum organization, not only the psychological order but also the logical order of the field to be taught.

A third, but not less important, point is derived from the assumption that if we have to think about a hypothesis concerning sequences to teach at school, some mathematical idea (relative sequences, always), we must think about the relationships between cognitive development and the learning phenomenon in the terrain of a particular learning, in this case, the acquisition of mathematical knowledge. Therefore, we are faced with the classic controversy between the hypothesis of a cognitive development in general or one of a cognitive development specific to each field and/or, indeed, of a relationship between both (Castorina, 1997; Zaslavsky,1999, *e.g.*).

Nevertheless, it should be remembered, how to learn implies how to teach a particular content. Therefore, teaching, the teaching behaviour of a teacher is another indispensable "piece" in the process of selection and organization of mathematical contents at school. By excellence, an object of investigation of educational psychology as well (although not exclusive to it), the literature about teaching behaviour already offers elements referring to how relevant a teacher's decision about what to teach from mathematics is, about when and how to perform his/her teaching to particular students (Mansfield, 1996), his/her views of mathematics (Becker, 1995; Dossey, 1992), and also his/her own comprehension of the content he/she teaches (Soares, 2000; Thompson & Thompson, 1996).

Then, in order to deal with the selected questions, it is imperative to consider, as background, the inter-relationships among the following dimensions: the student and his/her learning process, the teacher and his/her teaching, and the learning contexts (English, 2002; Kilpatrick, 1992); or, in other terms, the "addresses" of school curriculum; his/her social-cultural reality; the state of the knowledge to be communicated to those students (Tyler, 1976).

The role of these particular aspects, and the nature of their relationships, may be researched, conceived and elaborated according to different epistemological paradigms and different theoretical positions. Many contributions concerning these positions are already available, but many gaps concerning theoretical orientations in the field of mathematical education are still present as a means to integrate what we already have as knowledge, and to avoid the divorce between scientific production and teaching practice (Kilpatrick, 1992). So, for example: Is it the curriculum content, in its logical or sociological order, that should be imposed and model the learner's cognitive functions? Is the teachers' role simply "to activate" the curriculum offered or is it his/her duty to interpret, to adapt it to his/her students according to his/her way of seeing school mathematics? Are the students' cognitive potentialities (their conditions or readiness for learning) what must conduct the teaching staff options, the school options? In this picture, what mathematics is in scene? Or which mathematics should be dealt with?

In short, in answering the first question, we deny the hypothesis that it is possible to have, in absolute terms, a best sequence to teach/learn any mathematical idea owing to the diversity of factors involved in that process, unless an exclusive privilege is attributed to one or another of the aforementioned poles.

In order to answer the second question - concerning the existence of *stages of students' development where particular concepts are best taught* – in addition to the assumptions and ideas exposed above, other factors should be remembered, as follows:

- There are stages of learners' cognitive development to be considered in the teaching of mathematics, and they act as one of the factors in the acquisition of mathematical knowledge at school. In consequence, there is the hypothesis that some mathematical concepts should be taught at a particular moment of the learner's life and not at another, a sort of proposition that feeds the psychological order of the curriculum content, either viewed according to the hypothesis of the critical moments of the cognitive development

(Ausubel, Novak & Hanesian, 1980, *e.g.*), or according to the more recent hypothesis about the neural plasticity or the cognitive inhibition (Houdé & Meljac, 2000*, e.g.*).

Therefore, any possible response to the question in focus claims to take into account, as a basis for reflection, at least, the following dimensions, as well as their interrelationships: cognitive development, school learning, the nature of mathematical knowledge and the forms it undertakes as curriculum content.

From their recent history, psychology and mathematical education provide us with numerous theoretical and epistemological interpretations concerning each one of those axes or dimensions, as well as their interrelationships.

This fact constrains us to foresee how hypothesis about a better teaching of particular mathematical contents as linked to stages of cognitive development should have different approaches, different responses, according to the perspective adopted by their authors concerning to the axes mentioned above. It also makes us anticipate that to figure out a direct, linear or fixed relationship between cognitive development and mathematical contents throughout schooling seems hard to be accepted.

It is not the occasion, in this paper, neither to discuss, nor to enumerate all the patrimony of theoretical alternatives pertaining to cognitive development, school learning, or features that mathematics can undertake when transposed to the classroom as stated by mathematicians, didactic specialists and teachers.

It is only important to remember that, as a support for theoretical contributions, there are always epistemological presuppositions concerning the human being as someone who possesses and produces knowledge, in this case, mathematical knowledge.

Only to illustrate the complexity of this aspect and the multiple different responses it can bring about, let us consider here the relevant issue of the relationship between cognitive development and school learning. In the recent history of psychology, this relationship is still present in the persistent and fundamental debate over the value of models concerning it, for instance: those of the prevalence either of cognitive development in learning, either from learning in cognitive development; or even, those which propose the absence of any prevalent effect from one or another of those phenomena, in favour of the occurrence of a complex and interactive combination of effects between both phenomena in school learning.

In short, we support the idea that any plausible response to these questions should have its scientific value appraised and its viability for application evaluated, only if it takes into consideration the theoretical and epistemological fundamentals that it is adopting concerning the nature of the knowledge to be learned/taught at school, the learner's role concerning that knowledge, the conceptions involved in the psychological development and learning of human beings and the particularities of the phenomenon of cultural transmission of knowledge in a school inserted in a particular society.

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