Some epistemological question addressed by school mathematics to mathematicians

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Abstract

I will start from the main question addressed in the draft of the working group:

> *"How do (should) the goals of school mathematics reflect the nature of disciplinary thinking and practice?"*

Then I will propose to move from this question to another one, which is:

> *"How school mathematics reveal or enlighten some mathematical practises that have to be overcame together by both students and teachers in mathematical classroom?"*

I will illustrate this point of view through logical questions concerning truth, validity, certainty and contingency. Mainly, I will show that some 'expert mathematical practises' may lead to deep misunderstandings with students, due in particular to the fact that mathematicians as teachers control proofs by their mathematical knowledge, while for students as learners, proving and mastering mathematical contents are closely intertwined. Finally, I hope this will support the thesis that school mathematics may address epistemological questions to mathematicians themselves.

Introduction

It is generally admitted that school mathematics ought to remain as near as possible of mathematics. Nevertheless, many authors claim that there is an incompressible distance between them, and even that it is necessary to operate a transposition from mathematics to school mathematics (Chevallard, 1985, 1991). On another hand, some authors, referring in particular to Lakatos (1984), try to recreate in classroom the conditions for practising mathematics as professional mathematicians do (for example, in France: Arsac and al.1992, Balacheff, 1987, Legrand, 1993; Grenier & Payan, 1998). So the main question addressed in the draft of the working group *"How do (should) the goals of school mathematics reflect the nature of disciplinary thinking and practice?"* is an ancient question with rather different answers. Certainly, the questions concerning proof and proving are crucial in this perspective: which is an acceptable proof in a given level at school? Which role for logic in proof and proving? Which level of rigor is due, from students? from teachers? How can students develop abilities in proof and proving? When and why introducing formalism in mathematics curriculum?

Among these questions, I will principally discuss the first one "*Which is an acceptable proof in a given level at school?*" that I will examine at the very beginning of tertiary level. On an example, I ought to show that this question is not only addressed to students, but also to teachers, and I will stress the fact that in classroom, ordinary mathematical practises may introduce didactical obstacles for an adequate development of abilities in proof and proving. The analysis I will propose rely on the distinction made in logic between truth in an interpretation and logical validity in a semantic point of view.

A rather elliptic proof

The theorem to prove concerns the limit of a sum of two functions:

"Given two functions f and g define in a subset A of the set of real number, and a an adherent element of A, if $f(t)$ and $g(t)$ have h and k respectively for limit as t tends to a remaining in A, then $f+g$ has $h+k$ for limit in a".

The proof proposed is the following one[1]:

" By hypothesis, for all $\varepsilon>0$, there exists $\eta>0$ such that $t \in \mathrm{A}$ et $|t–\mathrm{a}| \leq \eta$ imply $|f(t) - h| \leq \varepsilon$ and $|g(t) - k| \leq \varepsilon$; then we have

$|f(t) + g(t) – (h + k)| = |f(t) – h + g(t) - k| \leq |f(t) - h|+|g(t) – k| \leq 2\varepsilon$"

This proof can be considered as an elegant one, rather concise, and seems to suits the standards for proof in French school.

Try now to imagine in which context this proof is given: produced by a students during a mathematical course; produced by a student in a personal work; produced by a student during an exam; produced by a teacher during a course; produce by a teacher as a correction; written in a textbook?

In which cases, if there are, would you consider that this proof is acceptable?

Coming back to the proof, we can read at the beginning "By hypothesis". A priori, there are two hypotheses that can be formalized by:

For all $\varepsilon>0$, there exists $\eta>0$ such that $t \in \mathrm{A}$ et $|t–\mathrm{a}| \leq \eta$ imply $|f(t) - h| \leq \varepsilon$ (1)

For all $\varepsilon>0$, there exists $\eta>0$ such that $t \in \mathrm{A}$ et $|t–\mathrm{a}| \leq \eta$ imply $g(t) - k| \leq \varepsilon$ (2)

Following the author of the proof, theses two hypothesis can be collapsed in a unique one:

For all $\varepsilon>0$, there exists $\eta>0$ such that $t \in \mathrm{A}$ et $|t–\mathrm{a}| \leq \eta$ imply $|f(t) - h| \leq \varepsilon$ and $|g(t) - k| \leq \varepsilon$ (3)

[1] Our translation

In this precise example, (1), (2) and (3) are true. Does the author suggest that (3) is a logical consequence of (1) and (2) ; that means that as soon as “for all x there exists y F(x,y)”, and “for all x, there exists y G(x, y)”, then we can assert that “for all x there exist y F(x,y) and G(x, y)”. ? Does he believe that this is a general rule or does he know why in this particular case it is possible to assert (3). Does he know in which cases it is possible to assert (3) from (1) and (2)? Could you imagine that in some cases, following this rule, the author could produce an incorrect proof for a wrong statement, or an incorrect proof for a true statement?

Knowing now that this proof is in a French textbook (Houzel, 1996, p.27) for beginners, you can answer no for some questions, and yes for others, and you can imagine that the author knows that given a real strictly positive number ε, assertions (1) and (2) ought to consider two real numbers η_1 and η_2, on which it is possible to build a third real number η_3 such that $F(\varepsilon,\eta_3)$ be true. Doing this, it is necessary to get rid of quantifiers: given ε, we can assert “there exists y $F(\varepsilon,y)$” and “there exists y $G(\varepsilon,y)$” ; that means, it is possible to consider η_1 and η_2 such that $F(\varepsilon,\eta_1)$ and $G(\varepsilon,\eta_2)$ are both true. At this point, nothing tells us that it is possible to go further. Logically, it is not possible to assert (3). What allows us to conclude is a specific property of real number ordered. So we have here closely intertwined a logical argument and a mathematical argument.

Truth in an interpretation and logical validity

Some authors are Dieudonné (1987) or Thurston (1994), claim that logic is seemingly useless for mathematicians. Opposite, authors developing a model theoretic point of view as initiated by Tarski (1969) show that methods of logic may be fruitful for various mathematics fields. This is developed in Sinaceur (1991) who shows that the model theoretic point of view has contributed deeply to the development of algebraic geometry. This author assumes that, in this model theoretic perspective, logic is a relevant tool for understanding mathematics. A main distinction made by Tarski (1933) for quantified logic, and before him by Wittgenstein (1921) for propositional calculus, is between truth in an interpretation (that means for example a mathematical theory) and logical validity[2]. Tarski (1933) said that he wants to propose a definition of truth in formalized languages formally correct and materially adequate. For him, an interpretation is *a model* of the formula if the sentence that interprets it is true. Given a formula in predicate calculus, we can interpret it in various fields by defining

[2] Aristotle already did this distinction.

the discourse universe and the extension of the predicates involved in the formula. For example, a formula as '$\forall x\ (p(x) \quad q(x))$" ($F$) can be interpreted in standard elementary arithmetic : the universe of discourse is the integer set ; "p" is interpreted by "to be divisible by 2" and "q" by "to have a primary number as successor" ; that means that F is interpreted " For all x, if x is divisible by 2, then its successor is a primary number". This sentence is false; it is a mathematical result; as told Wittgenstein, it is not to logic to state if the sentence that interprets a formula is true or false. Consequently, this interpretation is not a model of F. Most formulae are true under some interpretations and false under others. Some of them are false under any relevant interpretation (often named contradiction), others are true under any relevant interpretation; Wittgenstein (1921) named them *tautologies* in propositional calculus; Quine (1950) said that they are "universally valid". Theses particular formulae play a prominent role in the logical system: they are the logical theorem; these formulae are interpreted by sentences that are true because of there particular form. More over, those in form "$F \quad G$" where F and G are formulae are related with most of the classical inference rule (Quine 1950). For example "$(((\forall x\ (p(x) \quad q(x))) \wedge p(y)) \quad q(y))$" is logically valid. It is associated to the most common inference rule in mathematics: given a universal theorem on a domain and an element of this domain that satisfies the antecedent of the theorem, then this element satisfies the consequent. In case you are in presence of such a universally valid formula, you can make a deduction as soon as premises are true. In other cases, you cannot.

Back to our example

We have said that the proof we discuss could be considered as an application of the rule "for all x there exists y $F(x,y)$", and "for all x, there exists y $G(x, y)$", then we can deduce that "for all x there exist y $F(x,y)$ and $G(x, y)$". But of course, this rule is not a valid inference rule; the formula $((\forall x \exists y\ F(x, y)) \wedge (\forall x \exists y\ G(x, y)) \quad (\forall x \exists y\ (F(x, y) \wedge G(x, y)))$ is not universally valid. It is true in some interpretation and false in others. So, the fact that the conclusion is true is contingent. The certainty, for the teacher, comes from his mathematical knowledge. At the same time students encounter this elliptic proof, they may be asked to proof the *Cauchy mean value theorem* that asserts that "If f and g are differentiable in an interval (a, b) and continuous on [a ; b], then :

$$f'(c)\ [g(b) - g(a)] = g'(c)\ [f(b) - f(a)]$$

A very common proof produced by students consists precisely in deducing implicitly from the premise that

"There exist an element c such that $f(b) – f(a) = f'(c) (b-a)$ and $g(b) – g(a) = g'(c) (b-a)$"

Opposite with the previous example, this sentence is false except for some particular functions such as polynomials under degree two. Due to the fact that there is no obvious way to prove this result, many students discovering that for some pair of functions it was impossible to get a common element conclude that the sentence was false; it is then an occasion to recall some properties of quotients.

This offers also the opportunity to clarify with students the specificity of mathematical reasoning: that means that the deduction from a true premise to a true conclusion is valid if and only if we are in a case where the conditional relying the premises and the conclusion is an interpretation of a formula universally valid. Following Aristotle, in that case the truth of the conclusion is necessary. It might happens that the premise and the conclusion are both true and the corresponding formula is not universally valid, as it is the case for the elliptic proof. In that case, the truth of the conclusion does not rely only on the premises; it is contingent in the sense that in another interpretation, we could have true premises and false conclusion, as it is the case with Cauchy mean value theorem. In that case (the truth is contingent), asserting directly the conclusion from the premise does not provide an acceptable proof; it is necessary to complete the proof, generally by adding premises. In our case, it is necessary to add two premises: the first one is that if an element satisfies the condition, then any inferior number satisfies it also; the second one is that *less than* is a total ordering on the real numbers set. We can see here a fundamental difference between novices and experts: an expert is able to control the use of "invalid" rules by its mathematical knowledge. He knows that in the first case, it is mathematically possible to complete the proof, but not in the second case. More over, he might consider that it is no use to explicit completely the mathematical argument, and of course, for a mathematician or an advanced student, this seems rather reasonable. He also knows that in the second case, applying this invalid rule might lead to a false intermediate result, on which if it were true, it would be possible to conclude to the true conclusion. Consequently, he would not accept such a proof. For a novice, that means here a student, who does not master the mathematical knowledge involved in the proof, it might be difficult to understand why in one case, it seems to be correct to apply (implicitly) the invalid rule, while in the other case, it is not.

Conclusion

In Durand-Guerrier and Arsac 2003 and 2005, we show from an empirical enquiry addressed to university teachers that they use preferably contextualised mathematical rules rather than logical valid rules. In particular concerning the statement in form “for all, there exists”, we have shown the prominence of the dependence rule and the introduction of a notation for recalling dependence when it would be dangerous to forgot it; very few teachers consider that logical tools might help students to overcome their difficulties, and finally our results confirm the importance of mathematical knowledge for controlling the validity of a proof, especially with respect to the availability of relevant examples and counter-examples (Durand-Guerrier and Arsac, 2005, p.). Considering the persistent difficulties faced by students at tertiary level, we share with other authors the thesis that it would be necessary to reconsider the place of logic in proof and proving, and hence in elaborating mathematics knowledge (for example: Dubinsky & Yparaki 2000, Selden and Selden 1995, Rogalski and Rogalski 2003, Epp 2004, Chellougui 2003). Due to the fact that, at tertiary level, most teachers are professional mathematicians, this epistemological question is addressed from school mathematics to mathematicians.

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