The moves of the game of mathematics

Johann Engelbrecht <jengelbr@up.ac.za>
University of Pretoria

My first question, when receiving the invitation to participate in the activities of this working group, was "what is *disciplinary mathematics*?" In the absence of a formal definition (even after an internet search) I have to assume that it relates to the *discipline* mathematics. Reading through the introductory document that was made available, I realised that the interpretation is wide and I can hardly think of any aspect of mathematics that could be considered as outside the scope of this document.

Disciplinary mathematics

With the latitude offered by the document, I will take the liberty to, rather than generically philosophising about what school mathematics should look like (which I am sure will evolve through our discussions), deliberately misunderstand the terminology and suggest an alternative interpretation of *disciplinary mathematics*. Rather than looking at the content of the *discipline* mathematics, I would like to concentrate on (what I sometimes call) the *disciplinary component* of mathematics. The two interpretations are different but not unrelated.

Following the style of the document, I will not give a formal definition but explain what I have in mind using the analogy of comparing mathematics to the game of chess. To play chess well you need full memory, mastery and deep comprehension of the rules of the game and typical strategic moves. Even the brightest mind would not be a very successful chess player if before you want to move e.g. a rook, you have to spend some time recalling what the possible moves are that the rules of the game allow for a rook. This knowledge has to be intrinsic; it has to be part of your mathematical prowess.

In fact, you need more than a superficial knowledge of the rules. It is not sufficient to only know about the possible moving directions of a rook. To successfully attack or defend with your rook, you must have been exposed to repeated instances where you tried a variety of strategies attacking or defending with the rook. This intrinsic knowledge and fluency in the rules and moves of the game could be called *disciplinary chess.*

There are two extremes. Some chess players remain disciplinary players. To become a truly successful player, you need more than only a disciplinary knowledge of the game. Without building further on this disciplinary knowledge you will always play the game at a low level.

On the other hand, without this disciplinary knowledge, you have no hope of becoming a top class player and will easily be beaten by players with much less potential than you. Without this disciplinary knowledge it becomes impossible to develop further into the delicacies of the game.

Mathematics is not that different from a game of chess. In most topics in mathematics you need some *disciplinary mathematics* – the rules, basic moves and strategies of the game. Without a deep conceptual understanding of this disciplinary knowledge it is impossible to *play* the mathematical topic successfully. Again there is the other extreme, always playing disciplinary mathematics.

Acquiring the disciplinary mathematical skills

In order to master this disciplinary background to the required extent, learning strategies, depending on the nature of the topic, can be followed.

Memorisation

Disciplinary mathematics does require an element of *memorisation* one of the reasons for the use of the term *disciplinary*. Understanding a certain concept well is not sufficient; you have to remember the definition. To be able to *play* calculus or differential equations a student has to know by heart the derivatives and anti-derivatives of basic functions. It is impossible to play fluently if you have to consult a table or textbook every time you need to differentiate or integrate a trig function or polynomial.

Repeated exposure

A number of fine people in the field have made serious attempts to develop frameworks on how students learn mathematics. Conceptual and procedural learning have been addressed by e.g. Piaget (Baker & Czarnocha, 2002) and Anderson (1995). Gray and Tall (1991, 1994) introduced the idea of a *procept* and Dubinsky's (1991) APOS theory is well-known.

A simplistic idea that may be somewhat superficial, is that the process of understanding is an ongoing process converging to full understanding but does not reach the limit of *full understanding.* In this model, the dynamic process of understanding new mathematics takes place in *layers.* With every layer you understand a little deeper. Some people have the capacity to use thick layers. The mathematics community considers these students as *bright*. They *understand* quickly. Unfortunately in many instances some of these people digging in thick layers think they understand fully – which is unlikely. This has as consequence that they do not consider it necessary to visit the topic again. Often, the *not so bright* mathematics students, realising that their layers of new understanding are thinner, deliberately go for repeated exposure to understand deeper. With this repeated exposure, going through a next layer every time, some of these students' understanding grow and they eventually understand deeper than some of the *bright* students. To some extent it is sometimes better to be not too bright – in the end you may have deeper conceptual insight in mathematics and are better equipped to discover/develop new mathematics. This is, of course, if you have the perseverance to expose yourself repeatedly to the particular concept. In my personal experience: I have taught courses in multivariate calculus many years and have to confess that every year that I teach this course, I learn something new. Something that I suddenly realise that I had never really understood properly. Something that makes me realise I am still digging deeper into understanding completely. My layers are getting thinner every year – it feels like a limiting process with definite convergence. But I realise I will never get there.

In the mathematical community we often measure talent in mathematics by the thickness of the student's understanding layers. Students with thick layers of understanding, that can keep up with the rapid pace at which we introduce new mathematical concepts, theories and ideas, are called *bright*. When some of these students lose interest in mathematics or suddenly start performing badly in our assessment, we wonder about what went wrong.

This process of understanding is also misleading. Mathematics can be the easiest subject in the world, if you watch somebody else do it. A good teacher explains difficult mathematics so well that you are convinced that you understand everything – s/he makes it look so simple. It is only when you try to do it yourself that you suddenly are confronted by the intricacies, the cognitive depth of a concept. It is only then that you realise how thin your first layer of understanding really was. So for some students it may be better not to have a good teacher. They realise that they are not going to learn from the teacher, they have to do it themselves.

Rote, repeated exposure to mathematical problems have been condemned by educationists over the last few decades, in favour of a constructive, experimental approach in which every student has to discover or construct new concepts and develop his/her own algorithms. I share this view only partially. In my opinion, learning mathematics consists of two processes, the first-time exposure and the consolidation process. Granted, in the first-time exposure to a new idea or concept it is much more pleasing and sensible to discover or construct it yourself than when you are merely informed by the teacher and this approach will probably enhance understanding and longer-term retention. However, deeper understanding can only grow with repeated exposure. This implies that drill-and-practise exercises do have a place – doing more problems of a certain type rather than only one brings repeated exposure and deeper eventual understanding. Being exposed to the moves of mathematics repeatedly, one acquires the disciplinary mathematics required for conducting mathematics on a deeper level.

Example Let us consider an example from elementary topology that would often form part of a second or third year undergraduate course in Real Analysis. I like to guide students in acquiring the disciplinary background in this topic, following the following strategy.

For each new concept (definition) or result (theorem) students have to be able to understand/explain the concept or result in eight different ways.

1. Verbal, hand-waiving, informal explanation in English (or whatever language they speak)
2. Visually (drawing a picture)
3. Formal mathematical symbolism
4. How would you begin proving that this concept is valid, or that the result is true?

The same four ways of understanding should be done for the negation of the concept or converse of the result.

To be more specific, consider the idea of an open set in a metric space. We call a set A *open* in a space X if for each point $a \in A$ there is a neighbourhood N around a such that $N \subset A$.

An informal way of explaining this idea is to say that all points have to be inside (interior), i.e. that the boundary is excluded. *A* visual picture is obvious and the formal definition is as above. With the fourth requirement, deeper understanding is required – since something has to be true for *each* point in *A*, you will *choose* an arbitrary point of *A*, around which you have to *construct* a neighbourhood that is still inside *A.* On the other hand, to begin proving that a set is *not* open (number eight on the list above), you will have to *find* (construct) a point that is not interior. Taking this further, a point is not interior if you cannot find a neighbourhood that is entirely inside the set, i.e. every (choose an arbitrary) neighbourhood around the point contains at least one point (construct) that lies outside the set *A.*

This argument clearly requires a huge cognitive leap from the student. Deeper understanding of the concept can only develop once a student has developed fluency in all eight of the mentioned ways of understanding and explaining a concept or result. Disciplinary mathematics entails the ability to employ these tools fluently, content related or non-content related.

This logical thinking process that goes with advanced mathematical thinking is experienced by students as probably the most difficult component in this transition process to conducting formal mathematics and requires fluent proficiency in their disciplinary mathematics.

Finding a balance

As mentioned earlier, there are two extremes. Perhaps we can call the *playing* part of mathematics *creative mathematics* or *intuitive mathematics*. The message of this writing is that a balance should be established between disciplinary and creative mathematics. On the one hand there is a danger that the emphasis stays on disciplinary mathematics and that our students never get the opportunity of *playing* the game, of getting involved in creative mathematics. On the other hand, trying to play the game without the disciplinary background is as dangerous.

We recently conducted an international study (Petocz et al., 2007) investigating undergraduate mathematics students' conceptions of mathematics and their notions of how mathematics will contribute to their further studies and professional work in the mathematical sciences. About 1200 undergraduate students in five countries answered three open-ended questions, expressing their views of mathematics and its role in their future studies and planned professions. Responses were analysed using a framework developed from a phenomenographic approach. We classified students' conceptions of mathematics by arranging these conceptions from the narrowest view as a focus on calculations with numbers, through a notion of mathematics as a focus on models or abstract structures, to the broadest view of mathematics as an approach to life and a way of thinking.

Results on the first question about their conception about mathematics, were as follows: About 10% of the students consider mathematics to be connected with *numbers* and calculations with no essential advance beyond elementary arithmetic. 45% of the respondents regard mathematics as a *toolbox* to be dipped into when necessary to solve a problem. The emphasis on mathematics as a *logical system or structure*, perhaps even a kind of game of the mind was the view of 14% of the respondents, 20% view mathematics as *modelling* the physical world and 7% of the respondents see mathematics as an integral part of *life* and a way of thinking.

The first two categories, *numbers* and merely a *toolbox* can be considered as *disciplinary* (in my terminology) and are the conception that 55% of the 1200 respondents have about mathematics. These data seem to indicate that the majority of our students consider mathematics as disciplinary.

There is the danger that mathematics is taught in such a way that only the disciplinary component is addressed. Many schoolteachers do not move beyond teaching the disciplinary part of mathematics. The only way in which teachers move away from disciplinary mathematics is often purely technical, using predetermined algorithms that are triggered by key words. This is almost common practice in many high school mathematics classes and also the case with many lower level calculus classes at universities. This debate has been running for a few decades now and I will not open it up again. Suffice it to say that students, who do not get the opportunity to play real mathematics, miss out.

In more advanced university mathematics courses, staying with the disciplinary component comes down to staying only with the formal part, the definitions, theorems and proofs that are often merely memorised by students with little attempt to understand. Presenting mathematics too rigorously does not contribute to the process of understanding. Quoting Rota (1997),

> An axiomatic presentation of a mathematical fact differs from the fact that is being presented as medicine differs from food. It is true that this particular medicine is necessary to keep mathematicians from self-delusions of the mind. Nonetheless, understanding mathematics means being able to forget the medicine and enjoy the food (p. 96).

The most serious danger in staying with disciplinary mathematics is that students get an entirely incorrect (or at least one-sided) idea of what mathematics really is. High school children and junior students think that mathematics is fluency in algebraic manipulations. Senior students think that mathematics is theorems and proofs, many of which are memorised to be reproduced in assessment.

On the other hand, trying there is the danger of trying to play the game without properly knowing the rules. The message conveyed by the reform movement in mathematics, that we will acquire the necessary skills when we need them, is interpreted as conducting creative mathematics without knowing the rules, by people opposed to this approach in teaching mathematics. This has been a point of criticism from many mathematicians. Colleagues in my department complain constantly that our current students cannot handle the basic algebra and calculus manipulations.

Playing mathematics without proper exposure to the disciplinary component can in some instances (as sometimes in engineering mathematics) result in, what is sometimes described as *mindless symbolic manipulation.*

Quoting Stephen Boyd (cited by Shaw),

> I can't say too strongly how unimportant symbolic manipulation is in engineering. We see the effects of MSM (mindless symbolic manipulation) every day: Students who can integrate t^3 Cos t but have no idea what they are doing and why.... Not only is the material of an MSM course useless and outdated, but the message it sends to students is bad. It basically suggests that learning math is mastering a certain list of stimulus-response behaviors.....No wonder we in EE and CS and other engineering fields rely less and less on people in math to deliver the basic training needed in our fields.

On an advanced level, conducting research in mathematics could surely be described as conducting creative mathematics. This can not happen successfully without thorough exposure to disciplinary mathematics. So we have to create a balance between these two extremes in our teaching. With the difference in approaches followed by mathematicians and mathematics educationists through the past few decades we moved from the one to the other extreme. One could consider this constant change in teaching approaches *exciting* or *dynamic.* Perhaps it is. But perhaps we should try to reduce the amplitude of these oscillations. Perhaps we should try to move closer to some stability. Or will that be too boring?

References

Anderson, J. R. (1995). *Cognitive Psychology and its Implications*, 4th edition, W. H. Freeman.

Baker, W. & Czarnocha, B. (2002). Written meta- cognition and procedural knowledge. Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the undergraduate level) University of Crete, July 2002, Hersonissos, Crete.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), *Advanced Mathematical Thinking*, (pp. 95–123). Dordrecht, Kluwer.

Gray, E. M. & Tall, D. O. (1991). Duality, ambiguity & flexibility in successful mathematical thinking, Proceedings of PME XV, Assisi, Italy, II, 72–79.

Gray, E. M. & Tall, D. O. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal of Research in Mathematics Education*, 26(2), 115–141.

Petocz, P., Reid, A., Wood, L., Smith, G., Mather, G., Harding, A., Engelbrecht, J., Houston, K. Hillel, J., & Perrett, G. (2007). Undergraduate students' conceptions of mathematics: An international study. *International Journal of Science and Mathematics Education* 5(3), 439-459.

Rota, G-C. (1997). *Indiscrete thoughts*. Boston: Birkhauser.

Shaw, D. (Current). *Jerry Uhl's favorite quotations*. Retrieved June 2007 from http://www.dougshaw.com/uhl/