SOME THOUGHTS AROUND DISCIPLINARY MATHEMATICS FOR TEACHING

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Reflecting on *disciplinary mathematics* in relation to *school mathematics* leads inevitably to a consideration of the mathematical education provided for those (teachers and instructors) who ultimately will become mediators between these different expressions of mathematics. What kind of approaches to disciplinary mathematics should inform their education, and how might its treatment differ from that designed for intending chemists, engineers, economists etc?
In this paper suggestions are offered from two perspectives.

1. In terms of the kinds of engagement with content we might encourage for those whose future will involve diagnosis and correction of misunderstandings, ability to unpack and explain difficult concepts, and encourage through example the development of skeptical enquiring minds.

2.A teacher cannot be inspiring if they are only ever a purveyor of the ideas of others. How can we foster the ability to develop original teaching examples from the everyday environment?

1. A problematique

With respect to the first perspective of significant importance is the understanding of mathematical fundamentals achieved as a result of school and university experiences as students. Over many years (and the citing of sources encompassing a quarter century is deliberate) studies have reported on the persistence of mathematical misunderstandings among students, even when exposed to rigorous tertiary programs, and how approaches to learning seem so often to default to superficial strategies (e.g Gray, 1975; Tall and Razali, 1993; Galbraith and Haines, 2000; Anderson *et al*, 1998). Characteristics of flawed performance has been variously described, as indicated in the following excerpts:

"...After twelve years of schooling followed by two years of university, they had all but accepted the mindless mathematics that had been thrust upon them...Misconceptions, misguided and underdeveloped methods, unrefined intuition tend to remain assignments, corrections, solutions, tutorials, lectures and examinations notwithstanding." Gray (1975)

"weaker students suffered from the continued misinterpretation that algebra is a menagerie of disconnected rules to do with different contexts." Tall and Razali (1993)

*"In attending module after module, students tended to 'memory dump' rather than to retain and build a coherent knowledge structure. Their presumed examination strategy resulted in such a fragile understanding that reconstructing forgotten knowledge seemed alien to many taking part "*Anderson et al (1998).

Or as expressed by Zorn (2002) in the context of technology aided learning: "*We've all seen students floating untethered in the symbolic ether, blithely manipulating symbols but seldom touching any concrete mathematical ground."*

2. Challenge

So in developing approaches that strive to actively engage the attention and involvement of those who will be teachers rather than chemists, engineers, or economists etc we need to address questions such as the following.

- How to rattle decisively the conceptual cages of intending teachers with respect to fundamental mathematical ideas?
- What tasks will engage basic concepts and procedures so as to confront understanding and misconceptions in ways that are specifically pertinent for those whose profession will be education?
- What is practically feasible? For example how do pressures for dollarship versus scholarship impact on the design of courses for multidisciplinary groups (where future teachers are mingled with those headed for other professions) ?

For a readership such as this, it does not seem useful to use limited space to develop involved arguments based on literature sources. Instead I will simply indicate perspectives that have informed the construction of the material that follows. In addition to basic understandings concerning mathematical learning and content, the following constructs and frameworks have been central: constructivism; deep and surface learning; situated cognition; communities of practice; assessment. For example when considering what activities might be developed to engage future teachers in mathematical habits of thought, and critical reflection, we note comments by (Wenger, 1998: 83) whereby activity in a community of practice:

…includes routines, words, tools, ways of doing things, stories, gestures, symbols, genres, action, or concepts that the community has produced or adopted in the course of its existence, and which have become part of its practice.

The ongoing challenge is illustrated by the unsatisfactory outcomes that have been the subject of interest and analysis for many years, have been noted in the excerpts above, and continue to feature in anecdotal reporting from contemporary programs in a variety of countries.

So a continuing issue remains how to engage students in habits of thought that continually enforce confrontation with understanding, using methods and approaches that draw upon the stock in trade of education professionals and are practically feasible in terms of institutional pressures, whereby for example, a single mathematics course is designed for students destined for a variety of vocations. Below are nine examples of teaching or assessment items designed to provoke mathematical reflection, and to provide contexts in which mathematical argument involving symbolic, graphical, logical, disputational, and communication skills are variously required. While the classroom differs in important ways from the research seminar, such attributes are common to both, and their intrinsic incorporation in coursework provides a bridge across which characteristics valued by the mathematical community may be conveyed to learners.

1. RESOLVE CONFLICT

1. Find the oblique asymptote of $y = \frac{(x^2 - x - 2)}{x+2}$.

Student A: By division $y = (x-3) + \frac{4}{(x+2)}$

As x → ∝ $\frac{4}{(x+2)} \to 0$ so the asymptote is y = x - 3;

Student B: Dividing Num and Den by x, $y = \frac{(x^2 - x - 2)}{x+2} = \frac{(x - 1 - \frac{1}{x})}{(1 + \frac{2}{x})}$

As x → ∝ $\frac{1}{x} \to 0$, so the asymptote is y = x – 1.

Compare and **resolve the conflict** between the respective arguments?

2.EXPLAIN
Explain, what (-∞ , 3] and (-∞ , 3) mean and the essential difference between them?

3.DEFINE
Define a logarithmic function and ***explain*** its relationship with an exponential function

4. COMPARE/CONTRAST
Compare/contrast the functions given by f(x) = sin |x| and g(x)= |sin x| for $-2\pi \le x \le 2\pi$

5. JUSTIFY
Justify fully whether the following are true or false

(a) If f is continuous at x = a then f is differentiable at x = a.
(b) If f(c) is a local maximum of f then $f^1(c) = 0$.
(c) If Σa_n converges then $a_n \to 0$

6. CREATE
Create a function f whose graph has the following properties.

(a) Cuts the x-axis once

(b) Has 3 vertical asymptotes
(c) Has y = 1 as a horizontal asymptote

7. IDENTIFY/CORRECT

Consider the following calculation:

$$\int_{-2}^{-1} \frac{dx}{x} = [\log x]_{-2}^{-1}$$
$$= \log(-1) - \log(-2)$$
$$= \log(-1/-2)$$
$$= \log(1/2)$$
$$= -\log(2)$$

Identify any errors in this example and ***complete*** the calculation correctly.

8. MOVE BETWEEN REPRESENTATIONS

A graph of a continuous function 'f' on the interval [-3,5] has the following properties:

(a) passes through the point (0,2)
(b) has a single turning point at (1,3)
(c) $f^{1}(-1) = f^{11}(-1) = 0$
(d) there are endpoint minima at the ends of the given interval
(e) f has positive values throughout the interval.

Draw a graph representing f on [-1,3]

9. DEFEND

$$A = \begin{bmatrix} 1/8 & 1/8 \\ 1/8 & 1/8 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Khan : A and B are multiplicative inverses within the set of matrices of the form

$\begin{bmatrix} x & x \\ x & x \end{bmatrix}$ *where* $x \neq 0$ and the usual laws of addition and multiplication apply

George: That is not correct as singular matrices do not have multiplicative inverses.

Decide which statement you agree with and ***defend*** your choice by mathematical argument

It is not suggested that these are optimum examples of *ideal types*, nor that the types canvassed are exhaustive, or that a course should be comprised entirely of such examples. What is argued is the need for disciplinary mathematics courses taken by future teachers to include examples that force confrontation with understanding of mathematical specifics, using approaches that provide practice in the very attributes of argument and communication that are the lifeblood of vigorous didactics. Such a view is at odds with beliefs that mathematics can be first learned, and teaching skills somehow layered on top; for the most elementary application of constructivist principles indicates that what happens in the latter will be profoundly influenced by experiences within the former.

3. Towards creativity

A teacher of mathematics who only ever acts as a conduit for ideas and material produced by others stands to be uninspiring to both themselves and their students. One significant attribute is the ability to identify potential mathematical tasks that are embedded in the environment, and to develop these as classroom activities. This involves incorporation of aspects of disciplinary mathematics directly into school mathematics, and may be encompassed within the descriptor *Mathematical Modelling*.

Mathematical Modelling in Education is comprehensively treated in the recent volume (Blum, et. al., 2007) as the Proceedings of ICMI Study 14, in which various interpretations of the term are described.

In summary two of relevance are those described by Julie (2002), as modelling as *vehicle* and modelling as *content.* When used as a *vehicle*, modelling contexts are chosen so that the mathematics of interest is deliberately embedded in the associated examples – that is the principal driving force in defining the activity is pre-specified curriculum content. In 'modelling as content' the intention is for students to develop and apply skills appropriate to obtaining a mathematically productive outcome for a problem with genuine real-world connections. Here the solution to a problem must take seriously the

context outside the mathematics classroom, within which the problem is located, and the mathematics required may not be overtly obvious ahead of formulation.
The significance of using the environment is two-fold. Firstly the modelling of scientific phenomena requires discipline knowledge not only of mathematics, but also of the discipline – physics, chemistry, economics etc within which the problem is located. This is often a tall order. The environment provides a rich source of problems that can be tackled using everyday knowledge of the context, in addition to mathematics. Secondly there is opportunity to engage with topical issues, as illustrated below.
In enabling students to access their disciplinary mathematical knowledge for addressing problems relevant to their world, another belief is being pursued. This is the conviction that it is unsatisfactory for students learn to 'bank' mathematical knowledge for 10, 12, or 15 years and yet be unable to 'withdraw the funds' for purposes other than answering standard questions, and performing on formal tests. The following criteria for development of modelling from this perspective have been developed.

- There is some genuine link with the real world of the students. (RELEVANCE AND MOTIVATION)
- It is possible to identify and specify mathematically tractable questions from a general problem statement. (ACCESSIBILITY)
- Formulation of a solution process is possible, involving:
  - the use of mathematics available to students
  - the making of necessary assumptions
  - the assembly of necessary data. (FEASIBILITY OF APPROACH)
- Solution of the mathematics for a basic problem is possible for the students, together with interpretation. (FEASIBILITY OF OUTCOME)
- An evaluation procedure is available that enables solution(s) to be checked:
  - for mathematical accuracy
  - for appropriateness with respect to the contextual setting. (VALIDITY)
- The problem may be structured into sequential questions that retain the integrity of the real situation. (DIDACTICAL FLEXIBILITY)*

* The final criterion allows for a distinction between *open* and *structured* approaches. In the former only a general question is posed. The latter approach is structured by the provision of staged questions. (In this context of this paper we are considering the development in terms of teachers rather than students – so that the desirable outcome is a teacher who can both undertake successful modelling, and thence pose or structure original examples in ways most appropriate for her/his teaching needs.)

Example: The problem of obesity/diet/exercise is a much-publicised issue in a number of countries. In some places legislation has been enacted that is directed among other things at the kind of foods available in school tuckshops, and the amount of exercise required on a daily or weekly basis. A focus for an approach to this issue through mathematical modelling is provided by the Morgan Spurlock 2004 documentary/film SUPERSIZE ME in which he described his experiment of eating fast food.

Rules of the experiment

- For 30 consecutive days Morgan Spurlock ate three meals consisting of nothing but McDonalds food and beverages. (consuming approximately 5000 calories daily).
- If offered an 'upsize' he must take it.
- Ate everything on the McDonalds menu at least once.
- Limited daily exercise to that of the average American office worker.

Consequence
Went from 84 kg to 95.5 kg - there were other effects such as a dangerously high cholesterol level.

AN OPEN PROBLEM
Build and *test* a model to describe Morgan Spurlock's weight gain

ASSUMPTIONS

- inactive lifestyle
- average food intake and 'exercise' is constant from day to day
- OTHER DATA (identified as necessary – energy conversion)

Many websites offer information on diet, exercise, and the energy value of foods. These differ a little on specifics but tell the same general story. Here is one simple method given for estimating the daily calorie intake to maintain body weight according to metabolic needs.

Males: Weight (kg) x 24 x Activity Factor = Daily calorie needs
Females: Weight (kg) x 24 x 0.9 x Activity Factor = Daily calorie needs

Very active	1.4 - 1.5	(Daily Intense exercise + construction work most of day)
Active	1.3 - 1.4	(Daily exercise + work on feet most of day)
Light active	1.1 - 1.2	(Regular Exercise 3 times/week + desk job or at home most of day)
Sedentary	1	(No exercise + desk job or at home most of day)

- Energy-weight conversion: Nutritionists estimate a weight gain of roughly 1 kg for every 7800 extra calories.

POSSIBLE SOLUTION

Assume average daily food intake (I) = 5000 calories (approx), and let w_n be weight after day n - Spurlock's original weight was $w_0 = 84$ (kg)
weight $_{today}$ = weight $_{yesterday}$+[(energy intake $_{today}$ - energy used $_{today}$]/7800
So $w_n = w_{n-1}$ + [calorie intake (day n) - calories used (day n)]/7800
$w_1 = w_0 + (I - 24x1x\ w_0)/7800 = I/7800 + (1-24/7800)\ w_0$ as the lifestyle is sedentary.
Hence $w_1 = aI + bw_0$ where a = 1/7800 = 0.000128, and b =(1- 24/7800) = 0.997.
Similarly $w_2 = aI + bw_1 \dots w_{30} = aI + bw_{29.}$ This leads readily to *Spreadsheet* solution: $w_{30} = 95.148$.

Geometric Series Solution
Alternatively continue the pattern: $w_2 = aI + bw_1 = aI(1+b) + b^2w_0$ etc leading to:
$w_{30} = aI(1+ b + b^2 + \dots + b^{29}) + b^{30}w_0$
Hence $w_{30} = aI(1-b^{30})/(1-b) + b^{30}w_0 = 95.15$(approx), and so the predicted weight gain ≈ 11.15 kg.
This is respectably in agreement with the actual figure of 11.5 kg. Given the estimates involved we should not claim too much, but the model at least provides an approximation that suggests the approach and assumptions are reasonable. A number of 'what if' scenarios can now be explored.

PROBLEM EXPRESSED USING A STRUCTURED FORMAT

1. In general weight (today) = weight (yesterday) + excess calories consumed today converted to weight. If 'I' represents the average daily intake of calories, show that Morton Spurlock's weight after day 1 of his 'diet' is estimated by: $w_1 = w_0 + (I - 24\ w_0)/7800$.
2. Express this in the form $w_1 = aI + bw_0$, and give the numerical values of 'a' and 'b'.
3. Design a spreadsheet that will calculate weight w_n for each successive day from n = 1 to 30, and use it to estimate his weight after 30 days. Comment on the result.
4. Show that in general $w_n = aI(1+ b + b^2 + \dots + b^{n-1}) + b^nw_0 = aI(1-b^n)/(1-b) + b^nw_0$, and use it to check the result obtained in 2.

Structuring may be used in more than one way. Within an open modelling approach the detail can be withheld and released selectively as hints when deemed necessary by the teacher. Alternatively, and particularly when curriculum restraints apply, a structured format may enable creative applications of mathematics to be introduced that otherwise would never occur.
For present purposes we note the bridge provided between disciplinary mathematics and school mathematics provided by modelling, a bridge that can be constructed in a variety of ways.
For both themes addressed in this paper it is argued that disciplinary mathematics for teaching needs to be viewed, at least in part, as a branch of *applied mathematics* where the field of application is education. This implies attention to specific needs as much as any application in the sciences.

4. References

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