What VIEW of Mathematics should we learn at school?

by

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Abstract

I want to suggest that mathematics is more than skills such as addition, solving equations and differentiating basic functions. I believe this because to a research mathematician, mathematics is more than skills. The processes that are required to link skills together in a useful way and to generate new skills are equally as important as the skills themselves. If students are to get a rounded view of what mathematics is they need to see more than skills alone. I illustrate this by means of a problem.

The Sum Problem and Its Conjectures

The Sum Problem: What numbers cannot be expressed as the sum of a string of two or more consecutive positive integers?

The only way to get started on this, that I can think of, is to try a few small numbers to see what can and what can't be done. To help in the discussion, I'll call a number 'nice' if it can be expressed as the sum of a string of two or more consecutive positive intergers. You can see pretty quickly that

$3 = 1 + 2$; $5 = 2 + 3$; $6 = 1 + 2 + 3$; and $9 = 4 + 5$

are nice numbers.

It's clear too that 1 and 2 aren't nice – they're just too small to find positive integers that will add up to them. And 8 isn't nice either. Try a few connective starts. $1 + 2 + 3 = 6$ but $6 + 4 = 10 > 8$; $2 + 3 = 5$ but $5 + 4 = 9 > 8$; $3 + 4 = 7$ but $7 + 5 = 12 > 8$; and $4 + 5 > 8$.

It might be useful drawing up the following table, where we show the numbers less than or equal to 20 and their niceness.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
		nice		nice	nice	nice		nice	nice	nice	nice	nice	nice	nice		nice	nice	nice	nice

At this point we see that only 1, 2, 4, 8 and 16 aren't nice. That should make us sufficiently suspicious to make a conjecture.

Conjecture 1: If $n = 2^\alpha$ for $\alpha = 0, 1, 2, \ldots$, then n is not nice.

But while we're thinking about, that it's worth noting that, in our table, all the other numbers are nice. So we can make a stronger conjecture, which may not of course be true.

Conjecture 2: n is not nice if and only if $n = 2^\alpha$.

Naturally you might prefer the next conjecture, even though it is identical to Conjecture 2.

Conjecture 3: n is nice if and only if $n \neq 2^{\alpha}$.

The thing about Conjecture 1 is that it only requires us to prove one thing. Conjectures 2 and 3 require us to prove two things, one with the "if" and one with the "only if". However these last two conjectures give a much more powerful result, if they're true. They actually tell us **all** numbers that are nice. Conjecture 1 doesn't assure us that there are no other "not nice' numbers.

The Conjectures and Their Proofs

It's by no means clear either that any of our conjectures is true nor how we might go about proving any of them. So maybe we can find a counter-example.

The first power of 2 that we don't have in our table is 32. So let's assume that 32 **is** nice. That means that

$$32 = a + (a + 1) + (a + 2) + \ldots + (a + r - 1)$$
$$= \frac{r(2a + r - 1)}{2}$$

where $a \geq 1$ and $r \geq 2$.

We know then that

$$64 = r(2a + r - 1).$$

What does this tell us? Since r and $2a + r - 1$ are factors of and $r \geq 2$ and $2a + r - 1 \geq 2a \geq 2$, then r and $2a + r - 1$ must both be powers of two. So 32 looks like being nice after all! Until you realize that if r is a power of 2 it's even. So $r - 1$ is odd and $2a + r - 1$ is also odd. No way that r and $2a + r - 1$ can both be powers of two.

But look at what we've done. Where did it matter that we assumed that it was 32 that was nice? Wouldn't the same proof apply equally well to 64, 128, 256, … and any old power of 2 we liked? Well, maybe we'd need to do something slightly different for $2^0 = 1$. Putting that aside though we see that any power of two is not nice. We've actually **proved** Conjecture 1!

Can we go the other way? Is it possible to show that every number that's **not** a power of two **is** nice? Perhaps we should go the same way as we did with 32. How would we show that 367 is nice? We'll assume it is and see what happens.

$$368 = \frac{r(2a + r - 1)}{2}$$
$$\therefore 736 = r(2a + r - 1)$$

What factors does 736 have? Can they give us an r and a $2a + r - 1$? Perhaps we should look for one odd and one even factor? Hello, $8 \times 91 = 736$. So let $r = 8$ and then $2a + 7 = 91$; $a = 42$. That was easy enough!

Sure but how can you convert that into a proof for **all** natural numbers not equal to a power of 2? That needs a little work but the stage has been set.

What Next?

This is one of the fun parts of mathematics – generalizing and extending. What questions could you ask now? Are there any numbers that are 1-nice? That is, are there any numbers that can only be made up of one sequence of consecutive numbers? It's almost certain that 3 is 1-nice. And $9 = 3 + 4 + 5 = 4 + 5$ isn't! What can be said about 1-nice numbers or 2-nice or 3-nice or …?

Maybe there are other things that can be done with this problem. It's time to use the imagination.

What's It All About Then?

Let's go back over the steps we've just been through. If we do, and there is license to imagine other steps, we find the following diagram.
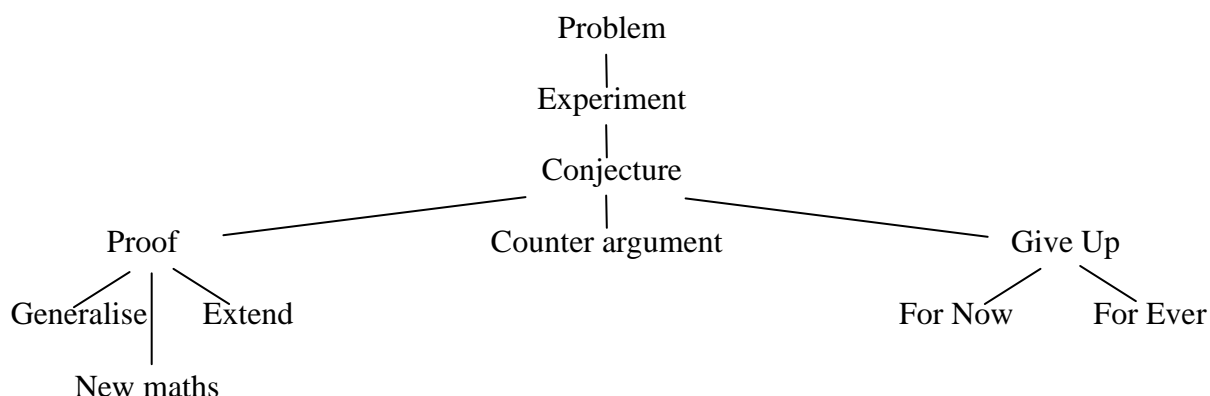
All maths starts with a problem, whether it's a long known and revered one or something someone just made up. But a problem is something that you can't solve immediately. You have do scratch around for a while, experiment in some way. Here we experimented by seeing what nice numbers we could found.

This experimentation led us to a guess, a conjecture. Now these are not always good guesses, research mathematicians especially get them wrong all the time. But guessing itself is good if the problem is hard, as it enables us to whittle the problem away and get successively better conjectures.

When we've got a pretty decent conjecture we might think of trying to prove it. How would we do that? It's not easy, especially at first. It definitely needs practice. At this point, then, we either find a proof or we construct a counter-argument. In the latter case it's back to experimenting and onto yet another conjectures. In the former

case there's hopefully a proof and a theorem. Out has possibly popped some new maths. And then it's time to think about generalizing or extending the original problem and so you are back at a new problem and the whole process starts again.

Why Do this?

First of all it's important to say that something like the process described in the last section is gone though by pure mathematicians when they are doing research. There's a fundamental series of steps that are gone through, though generally in a very haphazard manner – this is no linear process.

Second we should say that students of all ages can go through this process and experience the joy of discovery. At the same time they are experiencing an important part of mathematics that is often hidden away both by text book writers and by research mathematicians when they are writing up their papers.

Further, I believe that it's possible to go through many of the individual steps with school students – of all ages. There are problems of varying difficulty for varying ability. One of the individual steps is proof. It's a fundamental part of mathematics. I think it should be introduced to all children in all mathematics right from the start. It's very important for children to be able to justify things if for no other reason than the power it gives them over the content of the subject.

Now I have taken an interesting and none too easy a problem here. But it is artificial in some sense – there is no obvious link with the normal school curriculum. However, there are certainly things in the secondary curriculum that can be posed as problems that students can work through in the way I've suggested here. For instance, after students have explored Pythagoras' Theorem they can be asked to consider what happens if the triangle doesn't have a right angle. This can lead them to the Cosine Rule.

Finally I want to say that it may not be possible to do this kind of thing with every piece of mathematics at every level of the curriculum. First there may not be time. You may need to cover more material than you can reach by allowing the students to explore. Second, some result/skills at a given level are too hard to be discovered at that level. Much of what is in the secondary syllabus took years to develop after a great deal of effort by the finest minds. And then, of course, students should begin to learn how to learn to use books and people as sources.

But at least **some** of every course should be approached this way, otherwise the students are being short-changed. They are not seeing for themselves part of the mathematical process and worst of all, they are being denied the joy of discovery.

So I strongly advocate a wider view of mathematics in school (and possibly even at university too) that encompasses more than simply skills. As a step in this direction I would propose that examinations be set that are not mere repetitions of learned algorithms. That way students might be tempted to learn about a wider and more interesting view of mathematics.