Non-standard mathematical structures in mathematics teacher training

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Formality of knowledge, understanding, teacher training, mathematical structures

> A teacher in school should develop his/her students' know-how, their ability to reason as well as encourage their creative thinking. (G. Polya, 1966).

Abstract

This report is a part of ongoing research on prospective teachers' mathematical content knowledge. One of the main objectives of teacher training is to determine the balance between theoretical and practical knowledge and skills, i.e. the knowledge of mathematics (mathematical concepts and procedures, methodology, relationship with other areas etc.) and the knowledge of learning/teaching mathematics, beliefs and attitudes towards mathematics and practical skills.

This report deals with the following question: "Should a future mathematics teacher meet during his/her professional training non-standard mathematical structures which he/she will never use in school practice?'. We claim that the answer is positive; the reasons are illustrated by examples of non-standard structures.

Introduction

Learning to teach (whether in pre- or in-service) requires the balance between teachers' theoretical and practical knowledge and skills: specific knowledge (knowledge of mathematics, psychological-pedagogical knowledge and knowledge of learning/teaching mathematics); knowledge, beliefs and attitudes towards mathematics; practical skills (see e.g. Nieto, 1996). These components are only general, they do not answer the basic question about the content and extent of knowledge required from future teachers.

Future teachers entering faculties of education were taught mathematics at primary and secondary schools. Their knowledge of mathematical concepts and skills is at different levels and they also have different personal experience of how mathematics was taught. We assume that the student – future mathematics teacher – has positive attitude towards this subject. Unfortunately, this attitude is not always accompanied by the student's experience with any teaching strategy other than instructive teaching. Mathematics is often taught as an isolated school subject connected with other subjects or real life problems only in a very formal way. Various teaching methods are studied in many papers; for example (Littler & Taylor, 1995).

The view of mathematics which the student has built up during their school career survives long after he/she leaves secondary school. If we do not change misconceptions which the students might have during their teacher training at the faculty, these misconceptions will return with the teacher back to schools. The situation where mathematics is taught only as a set of precepts and instructions which have to be learnt leads to ever deeper formalism in the teaching of mathematics; it results in lack of understanding of the conceptual structure of the subject and inability to use mathematics meaningfully when solving real problems.

The influence of the student's previous experience from his/her home, school and society on test results, acquiring knowledge and its linking together into schemes is discussed in (Pasch, 1995). Similarly, a teacher's previous experience can significantly influence his/her ability to get an insight into cognitive processes of a student, who meets new, for him/her often surprising concepts, properties and relations. (For example, order in positive fractions; in the case of fractions with the same numerator, the fraction with bigger denominator is the smaller fraction. This is in contradiction with the student's previous experience with the order of natural numbers.)

One of the ways for improving the above described alarming situation is to expose the students to non-standard mathematical situations that contradict their longitudinal experience and force them to look for the explanations of surprising behaviours of mathematical objects and structures.

Non-standard structures in future mathematics teachers training

When a teacher neglects the development of a student's thinking during teaching and concentrates only on teaching prescribed knowledge and skills, the result is often nothing but formal knowledge. How do we discover lack of understanding which such formal teaching produces? Everything seems to be all right; the student defines concepts correctly and describes their properties, and calculates without mistakes. Long-lasting observations of future teachers during their training and later in their practice show that cases where the future teachers' knowledge is purely formal are not rare (Novotná, Stehlíková, Hoch, 2006). Different possibilities to improve the situation in teacher training and teachers' attitudes are studied in many articles devoted to mathematics education.

In the following text we will try to answer the question 'Should a future mathematics teacher meet during his/her professional training non-standard mathematical structures which he/she will never use in school practice?'

Our answer will be demonstrated on two examples from the university algebra courses in mathematics teacher training at the Faculty of Education of Charles University in Prague. In algebra courses our students learn definitions and theorems (even with their proofs) often without deeper understanding, only by memorising them. Moreover, the domain of algebra is available in many resources; students take many facts automatically without analysing their validity and adequacy.

Both activities presented bellow have a common feature: They both lead to a cognitive conflict, i.e. the conflict between the learner's experience with work in some context and the new environment; it is invoked when a learner is faced with contradiction or inconsistency in his or her ideas.

Other examples of activities breaking the mechanistic nature of students' grasping of mathematical concepts are presented in (Novotná, 2000): algorithms for numerical operations in non-decimal bases and criteria for divisibility in them.

Functional definition of a polynomial

<u>Activity</u>: Already at the lower secondary levels student learn to solve linear and quadratic functions. Higher degree polynomials are an important component of upper secondary mathematics courses. The infinite number sets (rational, real, possibly complex numbers) are always used.

Students entering the Faculty of Education of Charles University should know that two polynomials are equal when the coefficients for the same powers of the variable are the same. They should also know that the product of two non-zero polynomials is always a non-zero polynomial, what a polynomial degree is etc. The course Polynomial Algebra (Novotná, Trch, 1993) contains work with polynomials in finite domains of integrity, in which the previous statements made for infinite number sets are not true; e.g.:

- There exist polynomials with different coefficients for the same powers of the variable which are equal.
- There exist non-zero polynomials whose product is the zero polynomial.
- It is not possible to define the polynomial degree by defining it as the highest power of the variable with a non-zero coefficient, since the degree would not be unique.

This situation is in contradiction with students' previous experience and it is often difficult for them to grasp it.

Goal: We claim that if a student teacher or teacher is to gain the experience to understand the attitudes and feelings of a student facing a new mathematical structure that "contradicts" his/her previous experience, the teacher must have been placed in a similar situation. Only few of us can remember clearly our own feelings from our days when we went to school, when we were in a similar situation (e.g. passing from natural numbers to fractions or negative numbers).

Restricted arithmetic

Activity (Stehlíková, 2004):

Notation: $\boldsymbol{N}$ is the set of natural numbers, $\boldsymbol{Z}$ is the set of integers, $\boldsymbol{R}$ is the set of real numbers. The mapping []: $\boldsymbol{R} \to \boldsymbol{Z}$, $x \mapsto [x]$ is called the integer part ($[x]$ is the integer such that $x - 1 < [x] \le x$).

Theoretical definition: Let $A_2 = \{1, 2, 3, \ldots, 99\}$. Let us call its elements z-numbers. The mapping r: $\boldsymbol{Z} \to \boldsymbol{Z}$, $n \mapsto n - 99.\left[\frac{n-1}{99}\right]$ is said to be the reduction. It is easy to prove that:

- The range of r is A_2.
- For any $n \in \boldsymbol{Z}$, we have $r(n) = n$ if and only if $n \in A_2$.
- For any $n \in \boldsymbol{Z}$, $r(r(n)) = r(n)$.
- For any $n, k \in \boldsymbol{Z}$, $r(n + 99k) = r(n)$.
- For any $m, n \in \boldsymbol{Z}$, $r(m) = r(n)$ *if and only if there exists* $k \in \boldsymbol{Z}$ *such that* $m = n + 99k$.
- For any z-number n we have $r^{-1}(n) = \{n + 99k; k \in \boldsymbol{Z}\}$.

Two binary operations $\oplus$ and $\otimes$ called z-addition and z-multiplication are defined as follows:

$$\oplus : A_2 \times A_2 \to A_2, (m, n) \mapsto m \oplus n = r(m + n),$$

$$\otimes : A_2 \times A_2 \to A_2, (m, n) \mapsto m \otimes n = r(m \,.\, n).$$

$(A_2, \oplus, \otimes)$ is a commutative ring with a unity.

Note: There is an isomorphism between $\boldsymbol{Z}_{99}$ and $(A_2, \oplus, \otimes)$.

Presentation to students (Novotná, Stehlíková, 2000): The structure $\underline{A}_2 = (A_2, \oplus, \otimes)$ consists of the set $A_2 = \{1, 2, 3, \ldots, 99\}$($z$-numbers) and two binary operations z-addition $\oplus$ and z-multiplication $\otimes$ defined as follows: $\forall x, y \in A_2$, $x \oplus y = r(x + y)$, $x \otimes y = r(x \,.\, y)$; the operation r is called reduction and we define it for three- and four-digit numbers ABC, $ABCD$ as follows (for numbers with more digits, the definition is analogous):

$r(100A + 10B + C) = A + (10B + C)$,

$r(1\,000A + 100B + 10C + D) = (10A + B) + (10C + D)$.

The reduction is repeated as long as the result is a number from A_2.

Students have to discover all properties of the structure on their own. Not every student proceeds in the same order when investigating the properties of the structure. Each new discovery opens new directions for the work.

Examples of questions for discovering properties:

- What are the numbers whose reduction equals 6?
- Propose a graphical representation of *z*-numbers.
- Solve linear equations; find linear equations with one, two, three, …, no solutions.
- Find algebraic properties of the structure (A_2, $\oplus$, $\otimes$) (identity, inverses for both operations, …).

The structure is very rich; other concepts that can be studied are e.g. properties of divisibility, solving quadratic equations etc.

Goal: In order to make students construct and to deepen their knowledge of abstract algebraic notions and their properties, (A_2, $\oplus$, $\otimes$) was chosen as a suitable structure because it is not a ready-made product that can be simply learned by memorising of published knowledge. As students do not know about its isomorphism with $\boldsymbol{Z}_{99}$ they cannot rely on their experience with working in standard number sets or $\boldsymbol{Z}_{n}$. They work in a non-standard structure whose properties are not immediately transparent although the elements of the set are numbers; for discovering the properties, students have to do their own piece of mathematics in a way similar to the work of a mathematician. We claim that such an experience deepens their understanding of mathematics. “The process of looking for results might be more important than the results themselves no matter what they are.” (Stehlíková, 2004, 66).

The structure (A_2, $\oplus$, $\otimes$) is the source of a variety of problems that students can formulate themselves using their experience from standard arithmetic and from their progressive discoveries. The properties that they take as granted from their previous experience (e.g. 0 as the identity for addition of numbers) are not valid and the work in the structure asks for using theoretical definitions in a new situation. Our experience from the courses at Charles University indicates that our students’ understanding of the basic concepts of abstract algebra became deeper and long-lasting.

Conclusions

We must point out that the students will not use similar structures in their school practice. This brings us back to the question whether it is necessary to present students with a structure that goes significantly beyond the scope of primary and secondary mathematics which they will teach. As we have already expressed above, we consider this aspect of mathematics education important for future teachers.

It is our belief, and also students who have already graduated and are teaching mathematics in schools confirm it, that reflection on one’s own experience helps the teacher understand cognitive processes of problem solvers better. To make mathematics education an “active activity” for students, teachers must have experience of constructive approaches to mathematics teaching in their training and be aware of the danger of formalism hidden in the use of purely instructive teaching methods. In this, we see the importance of the work with non-standard mathematical structures for future teachers of mathematics.

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