Paper Rom

The problem of solid geometry

(Klaus Volkert, University of Cologne and Archives Henri Poincaré Nancy)

Some history

In Euclid's "Elements" solid geometry is studied in books XI to XIII. Here Euclid gives a systematic treatment of geometry in space focussing on polyhedra. He discusses basic notions like "being parallel" and "being orthogonal", gives some simple constructions (like the perpendicular from a point to a plane), discusses the problem of creating a vertex and gives some hints on volume using equidecomposability (in modern terms). Book XIII is devoted to the construction of the five Platonic solids and the proof that there are no more regular polyhedra. Certainly there were some gaps in Euclid's treatment but not much work was done on solid geometry until the times of Kepler. In particular there was no conceptual analysis of solids – that is a decomposition of them into units of lower dimensions: they were constructed but not analyzed. Euler (1750) was very surprised than he noticed that he was the first to do so. In his "Harmonice mundi" Kepler formulated the theory of Archimedean solids, a more or less new class of semi-regular polyhedra. Picturing polyhedra was a favourite occupation of artists since the introduction of perspective but without a theoretical interest.

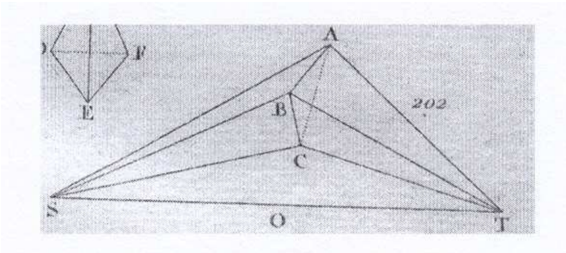
In Euclid's "Elements" there is a strict separation between plane geometry (books I to IV, VI) and solid geometry (books XI to XIII) [of course the results of plane geometry are used also in solid geometry and there are also results of plane geometry in the books dedicated to stereometry – in particular on the regular pentagon] which marked the development afterwards. Many later editions of Euclid's "Elements" contained only the books on plane geometry. The introduction of the basic objects of geometry started with points and straight lines passing over to plane figures and then – perhaps – to solids. A central demand of the reform movement at the end of the 19th century was to remove this separation in order to get a "fusion" of the plane and the solid.

The next important step was done by Euler: Starting from the problem how to classify polyhedra he discovered his theorem $v - e + f = 2$ and gave a list of polyhedra with n vertices (for $n = 4$ to 10). Shortly afterwards he also published a proof of his theorem (1750). The interest in this theorem exploded in the 19th century: on one hand there were several trials to replace Euler's proof by a more rigoruous one (Legendre, Cauchy, von Staudt and other), on the other hand hypotheses were sought which guarantee the truth of the theorem (construction of "monsters" by LHuilier, Hessel and others; cf. Volkert 2006).

In 1794 Legendre published the first edition of his textbook on geometry: "Eléments de géométrie". During his lifetime (he died in 1833) there appeared 13 editions of it in France – so it is obvious that the book was a great success. There were translations in all important languages and its impact on the teaching of geometry were deep (cf. the study by M. Menghini for the situation in Italy). I claim that this book was an important step forward in solid geometry (it is true that the "Eléments" are mainly remembered because of Legendre's work on the parallel problem, but this is only one aspect of it [and I think not the most important one]). Here I list some points:

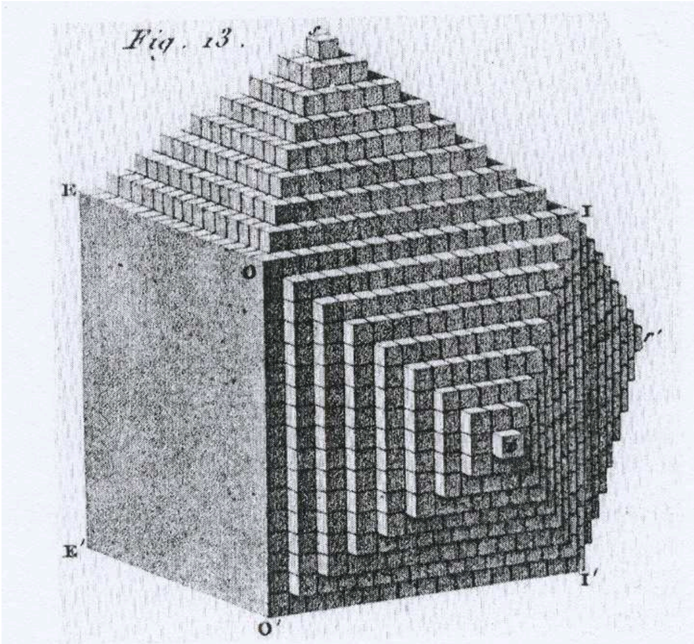
- Legendre integrated spherical geometry (this is not his term) in his system as a link between plane and solid geometry; in particular he used it to proof Euler's theorem.
- Legendre gave a discussion of volume with a clear distinction between the theory of exhaustion and the theory of equidecomposability (this is true for plane geometry too).

- Legendre took up the riddle of symmetric polyhedra which are not congruent (Kant's famous "incongruent counterparts"); he tried to catch this phenomenon by a precise definition using the idea of order (cf. recent work by Hon and Goldstein). Consequently he asked the question: Can we proof that two symmetric but not congruent polyhedra have the same volume? This was a difficult question because in general two congruent polyhedra were supposed to have the same volume by the fact that can be superposed. Exactly this is not true for symmetric but non-congruent polyhedra! This problem was discussed in the correspondence of Gauß with Gerling; its definitive answer came in 1900 with Dehn's work (some weeks before Hilbert had formulated his famous third problem citing the letter written by Gauß).



Legendre's construction of symmetric pyramides
(for us today: reflection in the plane of the base of the pyramide)

The problem of incongruent counterparts was taken up by Möbius who remarked in a footnote in his "Barycentrischer Calcul" (1827) that two symmetric polyhedra could be superposed by a rotation in four space. But he concluded that there is no such space so they can't be superposed. Here we have the starting point of a development which ended in the introduction of higher-dimensional space and its geometry – a difficult and complex process which stretched until the 1870's – and of transformations as the objects of geometric study.
With his axiomatization Hilbert suggested also a solution of the problem of solid geometry because he gave an axiomatic system for it. It is surprising that from the axiomatic point of view the difference between plane and solid geometry is not that big – the main problem is to catch the three dimensions of space by introducing adequate axioms.
Let me end this section by remarking that around 1800 there were some other fields of geometry which raised the question of solid geometry. First there was descriptive geometry explicitly constructed to catch three-dimensional objects by two-dimensional representations (which was thought to be important for the arriving mass production) and analytical geometry which was supplemented at that period by the theory of linear structures in space (lines, planes and so on) by Lagrange, Lacroix, Biot and others. The ideas of solid geometry became very prominent in science starting with Hauy's geometric theory of crystals (1784) and Ampère's speculations on chemical affinity which he wanted to explain by the combination of polyhedra - an idea which became transformed later into the stereo-chemical theory of Le Bel and van't Hoff. Electromagnetism and the polarisation of light provoked questions on space and its orientation.



Constructing a rhombic dodecahedron by Hauy

May be a non-mathematical event was also important for the break-up into space: In 1783 the first flight in a balloon – the famous Pilâtre de Rozier – took place – causing an enormous public interest in all form of going up into the third dimension.
Let me conclude this paragraph by some remarks:

- Until the 19th century solid geometry was the geometry of objects in space not of the space itself;
- During the 19th century "space" became a theme in connection with the following topics: a) dimension b) orientation c) transformations;
- Until the time of Euler solids were not analyzed but constructed; this is holistic point of view
- Logical order was important (from the simple to the complex) not epistemological order (what comes first to us?)

Solid geometry in teaching

I must confess that I do not know much about the teaching of solid geometry before 1900. May be this is so because there was simply no such teaching. At least in the German gymnasium geometry was taught with the explicit idea to train the ability of thinking logically (the so-called "formale Bildung"). This was done by teaching plane geometry, in particular the idea of proofing and the techniques of construction. In the second half of that century some people demanded a training in descriptive geometry (which was certainly done in France at the Ecole polytechnique since its foundation [1794]) because this is important for technical applications and for the development of spatial intuition. This demand was often combined with a promotion of projective geometry yielding the "new" geometry (not very well defined but a good slogan).
The way in which geometry was taught at the Gymnasium was strongly influenced by the model of Euclid's geometry. In particular one started with the introduction of points, straight lines and circles to go on to plane geometry. Here is a description given by Peter Treutlein in 1911:

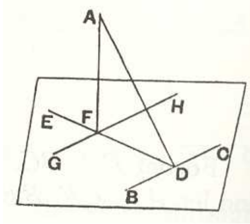
Der geometrische Unterricht dieser Schulen war früher rein euklidisch nach Form und Stoffanordnung. Im Verlaufe etwa des letzten halben Jahrhunderts ist, als Wirkung durch Jahrhunderte fortgesetzter zahlreicher, wohlbegründeter Angriffe und Verbesserungsvorschläge, mit Recht wohl fast durchweg an Stelle der dogmatischen Darstellung und Lehrform das sog. genetische, d. h. das entwickelnde Verfahren getreten, und zugleich hat man vielerorts auch den Betrachtungsweisen der sog. neueren Geometrie Eingang gewährt. Nicht in gleicher Weise ist bis heute eine Änderung eingetreten in bezug auf die Anordnung, auf die Reihenfolge des an die Schüler zu übermittelnden Stoffes: man folgt auch heute noch im wesentlichen dem Gang des großen Alexandriners, der mit Punkt und Gerade beginnt und spät erst zum Studium der Körper gelangt. Zwar ist vielfach, freilich seltener auf Anordnung der Behörden, sondern meist gemäß dem Gefühl und der Einsicht der Lehrer, dem strengen euklidischen Lehrgang als Einführung in diesen ein kurzer, auf wenige Stunden oder Wochen sich beschränkender Vorkurs vorgelegt worden, der in Anlehnung an die Betrachtung einiger, meist weniger Körperformen aus diesen die Grundgebilde der Geometrie ableitet — ein solcher einigermaßen anschaulicher Unterrichtsbeginn ist aber auch heute noch nicht überall in den Lehrplänen verlangt. Und wie nötig wäre er doch!

If one accepts the idea that teaching should be genetic then it is clear that Euclid's way of building up geometry is not the best. What we know – and what pupils know – is basically the three-dimensional world with its objects (that is solids). Consequently Treutlein proposed to start with a course in "intuitive geometry" which treats the phenomena of the surrounding world like the cube, the prisma, the ball and so on. Straight lines and squares are obtained by considering the cube and so on. So Euclid's way is completely reversed. But it must be noted that Treutlein didn't vote for a systematic treatment of solid geometry at this state (the "propädeutische Raumlehre"); for him this is only the way to connect the experiences of the pupils with the basic facts needed to do geometry.
But since the days of the Meran conference (1905) of the GdNÄ there was the demand to develop spatial intuition (Raumanschauung) by teaching geometry. This couldn't be done only by a propedeutic teaching in the way Treutlein proposed, but should propose also a systematic treatment of solid geometry in later classes. There was one topic in the classic curriculum which had a certain relation to solid geometry: the theory of conic sections (think of Dandelin's spheres e.g.).

Let me note some intrinsic difficulties of solid geometry which hindered certainly the introduction of its systematic teaching:

1. Solid geometry is much more complicated than its plane counterpart. Whereas in plane geometry there is one sort of angles in solid geometry you have three, in plane geometry the relation "being orthogonal" is defined for lines, in solid geometry it is defined for lines, lines and planes and planes. And so on. There is a factor of about three!
2. The objects of solid geometry are in generally represented on paper, but our "paper-tools" (U. Klein) are essentially two-dimensional. The representations are therefore difficult to understand, the resemblance to the objects represented is much weaker than in plane geometry.

   Here is an illustration (XI, 11: From a given elevated point to draw a straight line perpendicular to a given plane, that is the construction of the perpendicular to a given plane):



AD is orthogonal to BD,
AF is orthogonal to GH and to ED

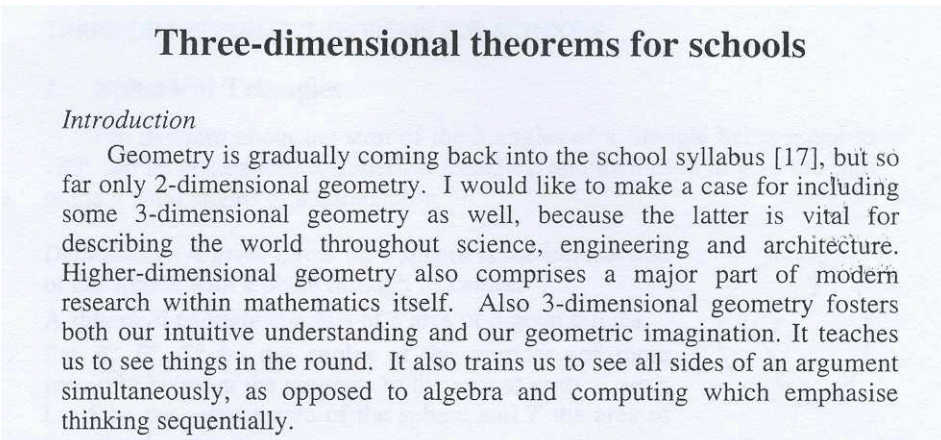
3. The problem of intuition and evidence is virulent in solid geometry. Because the situation is that complex it is not easy to grasp it by "pure" thinking completely. Often there are implicit suppositions. The history of the Euler theorem is a very nice illustration for this topic. Another example is given by Euclid's definition of the "congruence" of polyhedra ("Equal and similar solid figures are those contained by similar planes equal in multitude and magnitude" [XI, def. 10]), which had to be corrected more than once. As a last example I cite the missing "sixth" postulate in Euclid: It is always possible to draw a plane through three given points if they are not on a straight. This is used by Euclid in books XI to XIII time and again but he didn't mention it at all.

In concluding we may state the following conviction which was seemingly widespread: Because the teaching of geometry in school should be rigorous solid geometry is a dangerous theme. Otherwise said: the needs of teaching tend to petrify its contends!

There were at least two aspects in which the teaching of solid geometry (or should we say: the non-teaching?) was influenced by the development of the science itself:

a) Euclid's decision for rigour and the resulting ordering of themes put solid geometry at the very end of the themes treated.
b) Following Euclid there was a strict separation between plane and solid geometry.
c) The need of rigour made it suspicious to treat with solid geometry because of its logical and intuitive complexity.

At the end of this short overview I want to cite a recent plea for solid geometry by Chr. Zeeman (2005):



Three-dimensional theorems for schools

Introduction

Geometry is gradually coming back into the school syllabus [17], but so far only 2-dimensional geometry. I would like to make a case for including some 3-dimensional geometry as well, because the latter is vital for describing the world throughout science, engineering and architecture. Higher-dimensional geometry also comprises a major part of modern research within mathematics itself. Also 3-dimensional geometry fosters both our intuitive understanding and our geometric imagination. It teaches us to see things in the round. It also trains us to see all sides of an argument simultaneously, as opposed to algebra and computing which emphasise thinking sequentially.

Zeeman uses more or less the traditional arguments in favour of solid geometry (as we have met them above). The idea that solid geometry gives us an over-all view of the things is new –

it reminds us to the big discussions between the analytic and the synthetic point of view (in particular in France during the 19th century).
Here is Zeeman's list of themes he proposed to treat in school:

The theorems will be grouped under the following topics:

1. Spherical triangles
2. Angles in a tetrahedron
3. Concurrencies in a tetrahedron
4. Perspective
5. Desargues' theorem
6. Regular polyhedra
7. Rotation groups
8. Tessellations and sphere-packings
9. Conics
10. Inversion
11. Cross-ratios
12. Rings of spheres
13. Areas of spheres and volumes of balls
14. Map projections
15. Knotting
16. Linking.

It is remarkable that there are only few themes in this list which are classic (let's say Euclidean); once again we see that solid geometry was (re-)born in modern times.

Bibliography

Hon, Giora/Goldstein, Bernhard R.: Legendre's Revolution (1794): The Definition of Symmetry in Solid Geometry (Archive for the History of Exact Sciences 59 (2005), 107 – 155).

Klein, Ursula: Experiments, Models, paper tools (Stanford: Standford University Press, 2003).

Menghini, Marta: Die euklidische Methode im italienischen Geometrieunterricht seit 1867. In: Der Wandel im Lehren und Lernen von Matheamtik und Naturwissenschaften. Band I: Mathematik, ed. by J. Schönbeck, H. Struve and K. Volkert (Weinheim: Deutscher Studienverlag,1994), 138 – 151.

Treutlein, Peter: Der geometrische Anschauungsunterricht (Leipzig und Berlin: B. G. Teubner, 1911).

Faut-il étudier la tératologie ? (Annales de didactique des mathématiques **11** (2006), 217 – 228).

Zeeman, Sir Christopher: Three-dimensional theorems for schools (The Mathematical Gazette. Special Issue March 2005)

See also my lecture notes "Der Raum und seine Geometrie in der Geschichte der Mathematik": http://www.uni-koeln.de/ew-fak/Mathe/volkert/Vorlesung%20WS%2004-05-Version%202.pdf

k.volkert@uni-koeln.de