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Working Group 1: Disciplinary Mathematics and School Mathematics

Why are Disciplinary Practices in Mathematics Important as Learning Practices in School Mathematics?

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*In this paper, we discuss why it is important that the practices of school mathematics reflect the way mathematicians do mathematics; that is the thinking and practices in mathematics. One way to view the differences between mathematics and school mathematics is to describe them as disciplinary practices and learning practices following Cohen and Ball (2001). Consider for example justification and argumentation, these are disciplinary practices in mathematics, but in school mathematics these are learning practices. In mathematics justification and argumentation are disciplinary practices because they are the means by which mathematicians validate new mathematics. In school mathematics argumentation and justification are learning practices because they are the means by which students enhance their understanding of mathematics and their proficiency at doing mathematics. Thus, these practices are not just a desirable end product or outcome of a mathematics education; they are a means by which to learn and do mathematics. It is this aspect that we will take into consideration as we discuss the relationship between disciplinary mathematics and school mathematics.*

One of the overarching questions in the overview proposal for this ICMI Working Group is, “How do (should) the goals of school mathematics reflect the nature of disciplinary thinking and practice?” Further, “the way in which mathematicians view their subject area is deeply rooted in the way they do mathematics”. In this paper, we would like to extend this question and statement to discuss *why* it is important that the practices of school mathematics reflect the way mathematicians do mathematics; that is the thinking and practices in mathematics. With this in mind, it is important to remember that are important differences between disciplinary mathematics and school mathematics. Mathematics is a product of our culture that is constantly undergoing change. This change comes about from the thinking in the discipline of mathematics, and the disciplinary practices that support this goal. However, in school mathematics the goal is for students to learn the mathematics of the culture and the question is what practices support this goal. One way we might view these differences is to describe them as *disciplinary practices* and *learning practices* following Cohen and Ball (2001).

Consider for example justification and argumentation, these are disciplinary practices in mathematics, but in school mathematics these are learning practices. In mathematics justification and argumentation are disciplinary practices because they are the *means by which* mathematicians validate new mathematics. In school mathematics argumentation and justification are learning practices because they are the *means by which* students enhance their understanding of mathematics and their proficiency at doing mathematics. Thus, these practices are not just a desirable end product or outcome of a mathematics education; they are a means by which to learn and do mathematics. It is this aspect that we will take into consideration as we discuss the relationship between disciplinary mathematics and school mathematics. Certainly, there is a need to make connections among these two communities of practice, but at the same time, it is essential to acknowledge that they serve different purposes. In this paper we consider the *social aspect* to discuss why it is important that learning practices should reflect the disciplinary practices in mathematics. First, we will briefly review the research on classrooms to establish what we know about these learning practices and to describe what happens in school mathematics when disciplinary practices are transformed as learning practices? Next, we continue with a discussion about *how we learn* is linked to *how we see mathematics* in school. Finally, we will discuss what is still missing and how we might go about investigating the connection between disciplinary practices and learning practices.

#### *Research on Mathematics Classrooms*

An underlying theoretical assumption of a cognitive position on learning in school is the importance of mathematical thinking and reasoning in the development of conceptual understanding and the central role abstraction and generalization play in the disciplinary practices in mathematics. Moreover, it is widely believed that when students learn mathematics through thinking and reasoning that resembles thought in mathematics, they understand conceptually. In this research, differences that exist in the nature of and goals for school mathematics highlight how ways of thinking in mathematics are transformed to meet goals for learning.

Another widely accepted theoretical assumption which takes a social perspective states that in order to develop new learning practices for learning mathematics requires the development of different classroom practices or environments (e.g., Cobb, Wood, Yackel, & McNeal, 1992; Lampert, 1990), different social norms for students' actions (e.g., Wood, 1999) and specific sociomathematical norms (Yackel & Cobb, 1996). This perspective on classroom practices generally assumes that these specific social and sociomathematical norms and pedagogical tools influence not only the quality of student participation but also the mathematical thinking and thus the learning of mathematics (e.g., Kazemi & Stipek, 2001; Rasmussen & Marrongelle, 2006; Wood & Turner-Vorbeck, 2001; Wood, Williams, & McNeal, 2006). Further, theoretically, it is generally assumed that there is a link between social interaction, the development of human thought, and the construction of cultural knowledge (e.g., Bruner, 1996; Hobson, 2004; Tomasello, 1999, 2001).

Extending the social perspective beyond that discussed above, is a view that draws from two different sources, one is mathematics (e.g., Lakatos, 1976) and the other cultural psychology (e.g., Lave (1996) and Lave and Wenger (1991) to emphasize the importance of disciplinary and learning practices and the differences between them. It is these theoretical perspectives that also

underlie the assumption that school mathematics should reflect the practices in mathematics that brings us to the discussion about “*how we learn* is linked to *how we see mathematics*” in school.

In the beginning a vision of a connection between disciplinary practices was put forth by Lampert (1990) (followed by Ball) who saw a new possibility for school mathematics as a replication the ways of doing mathematics in the mathematics community. Her work provided an existence proof of school mathematics as authentic disciplinary practice. She drew from Lakatos to describe a view of doing mathematics. Lampert (1990) states:

This essay is a description of a research and development project in mathematics teaching designed to explore whether it might be possible to produce lessons in which public school students would exhibit – in the classroom – the qualities of mind and morality that Lakatos and Polya associate with doing mathematics. (p. 35).

Ball and Bass (2000) continued this line of thinking and contended that mathematics practices were fundamentally important and should be considered when investigating classrooms. Ball and Bass defined these mathematical practices as the ways in which inquiry and validation of mathematical knowledge occurred in the classroom. They argued that mathematical practices were encapsulated in mathematics and were not derived from or viewed as psychological or social practices. It can be said that both Lampert and Ball looked at the classroom from their role as teachers describing the practice and the pedagogical situations they encountered. Although Lambert and Ball described student actions and discourse, they did not consider the practice that evolved from the perspective as learning practices for students.

It was Boaler (1999) who took the notion that school mathematics should reflect the practices in mathematics in a different direction that we believe gets at the heart of the matter. Boaler examined the effect of differences in learning practices in two different school mathematics settings. Her findings showed that students who participated in learning practices that reflected disciplinary practices learned more, had a better disposition toward mathematics than those in typical school mathematics classes. In examining the classrooms, Boaler noticed that there were gender and class differences in the typical classroom but not in the class which reflected disciplinary practices. She concluded that learning practices derived from disciplinary practices played an essential role in these differences. Boaler concluded from these results that the importance of access to participation in mathematics practices that reflected disciplinary practices to resolve equity issues was grounded in cultural psychology, specifically in the work of Lave (1996). She argued the focus on mathematics as a discipline (which is a product of our culture) and the practices associated with ‘doing’ mathematics transforms school mathematics such that the focus of teaching is on mathematics as a set of practices and how to support students’ increasingly sophisticated participation in those practices. Thus, learning happens through participation. What is learned is not just how to do something in mathematics, but the cultural meanings – what is the purpose of a particular practice and how does this component fit in with the larger whole. Therefore, the learning practices, such as argumentation and justification, students’ participation influences not simply if they know what to do to justify but whether they know the meaning of this practice as a disciplinary practice of mathematics. Whether students have access to the culture of mathematics depends on knowing the meaning of the disciplinary practices as well as knowing how to act in the mathematical ways of the discipline.

If we take the view that mathematics is considered as a set of practices, then students need to participate in these practices in order to become more proficient in mathematics. In school the focus is on mathematics as a set of practices and how to support students' increasingly sophisticated participation in those practices. Teaching is engaging students in core mathematics practices and knowing what these core mathematics practices are. Equity then becomes an important theme. There is some evidence that students of different backgrounds participate differently in mathematics classes. It is the teacher's role to find ways to support *all* students in participating in mathematical practices and developing proficiency (Boaler & Staples, in press).

What is needed now is research that moves beyond this widely accepted knowledge base of school mathematics to carefully examine the development of more complex forms of learning practices to examine how these conceptions further impact on student learning. Let's take as an example the central role that argumentation, justification and proof, play in the discipline of mathematics (Lakatos, 1976); the importance to mathematical argumentation of making claims, examining evidence for claims, and making bridging statements (Toulmin, 1958); and the difficulty of students to do so (Martin & Harel, 1989; Stylianides, 2007) to address how the connection of disciplinary practices in mathematics to specific learning practices in school mathematics might occur (Larsen & Zandieh, in review).

*How might this be accomplished?*

We might ask, what is the role of justification and argumentation in, for example, middle grades (11-13 years) algebra, and how can justification and argumentation as learning practices support gaining knowledge in algebra? How can these practices become instituted in classrooms as a way for students to evaluate their own justifications?

*We propose:*

#### *Seminars*

Seminars could be developed involving mathematicians, mathematics educators in discourse, for example, about justification in cutting-edge mathematical research (e.g., Thurston, 1994; Jaffe, 1997) and in school mathematics classrooms. In the mathematics education literature, Harel describes a categorization of proof schemes that fall into three main types, 1) external conviction proof schemes, 2) empirical proof schemes, and 3) deductive proof schemes. A proof scheme is what "constitutes ascertaining and persuading for that person (or community)," where ascertaining refers to the process of removing one's own doubts regarding the truth of an assertion and persuading refers to the process of removing the doubts of others (Harel & Sowder, in press). External conviction proof schemes involve some appeal to an external authority (e.g., textbook, teacher) while empirical proof schemes rely on evidence from examples or perception. However, neither external conviction proof schemes nor empirical proof schemes are consistent with the standards of justification held by the mathematics community. Yet, both are commonly observed among students in school mathematics (Harel & Sowder, 1998, 2007; Martin & Harel, 1989).

Deductive proof schemes, which are consistent with the standards of justification held by the mathematics community, include both *axiomatic* and *transformational* proof schemes. The three primary characteristics of transformational proof schemes are 1) generality, 2) operational

thought, and 3) logical inference. Generality refers to the individual's (or community's) understanding that the assertion needs to be shown true for all cases. Operational thought refers to the forming of goals and sub goals and attempts to anticipate their outcomes during the evidencing process. Logical inference refers to the understanding that justification must be based on mathematical rules of logical inference. These three characteristics could be focal points for discussion about the relationship of disciplinary practices to learning practices in the seminar. Perhaps this is not the way mathematicians would see this as a disciplinary practice; but by “unpacking” the disciplinary practice of justification during these seminars, questions could be discussed such as, what constitutes a mathematical justification as a learning practice?

### *Intellectual Roles for Students*

Another critical feature of the transformation process of disciplinary practices to learning practices that the social aspect draws to our attention is the role of *tools*. A tool serves as “a resource for the negotiation of meaning” (Wenger, 1998), prompting community members to develop shared understanding of the values, purposes and meaning of particular objects or ideas. One such tool to support the development of disciplinary-based learning practices is efforts to enact new forms of participation in their classrooms draws on the research of Palincsar and Herrenkohl (1999) on the intellectual roles for students in collaborative group work.

Palincsar and Herrenkohl assert that developing an environment that consists of collaborative group work and productive class discussion derives from “cognitive-role taking” that relies on situations in which students assume intellectual engagement roles for participation. To advance student collaboration in mathematics classrooms, teachers could establish not only social but mathematic-specific norms and intellectual roles. These roles are designed to focus individuals' attention on the need for a justification to: 1) be valid; 2) use appropriate notation and representations; and 3) promote understanding. These intellectual roles could serve as tools for students in two ways. First, they offer a framework for students to use in analyzing a justification to determine whether and how it might be improved. Second, these roles can be instituted in classrooms as a learning practice, a way for students to evaluate their own justifications and to establish classroom norms for what constitutes a valid and useful justification.

A social aspect helps us focus on the idea that two communities of practice exist-- mathematics and school mathematics. A core challenge for mathematicians and mathematics educators will be rethinking not only how disciplinary practices can be transformed as learning practices in school mathematics, but how the transformation supports students' participation in these practices and in ways that acknowledge the dual role of as disciplinary and learning practices as sources of access for all students to mathematics.

## References

- Ball, D., & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. Phillips (Ed.), *Yearbook of the National Society for the Study of Education, Constructivism in education* (pp. 193–224). Chicago: University of Chicago Press.
- Boaler, J. (1999). *Experiencing school mathematics: Traditional and reform approaches to teaching and their impact on student learning*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Boaler, J., & Staples, M. (in press). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers College Press*.
- Bruner, J. (1996). *The culture of education*. Cambridge, MA: Harvard University Press.
- Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal*, 29, 573–604.
- Cohen, D. K., & Ball, D. (2001). Making change: Instruction and its improvement. *Phi Delta Kappan*, 73-77
- Jaffe, A. (1997). Proof and the evolution of mathematics. *Synthese*, 111, 133-146.
- Harel, G., & Sowder, L. (2007). Toward a comprehensive perspective on proof., In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*, (pp.805-842). Charlotte, NC: Information Age Publishing National Council of Teachers of Mathematics.
- Hobson, P. (2004). *The cradle of thought: Exploring the origins of thinking*. New York: Oxford University Press.
- Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *The Elementary School Journal*, 102, 59–80.
- Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge: Cambridge University Press.
- Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, 27, 29-63.
- Larsen, S., & Zandieh, M. (in review) Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics*.
- Lave, J. (1996). Teaching, as learning, in practice. *Mind, Culture and Activity* 3(3), 149-164.
- Lave, J. & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. New York: Cambridge University Press.
- Martin, W.G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education*, 20, 41-51.
- Palincsar, A., & Herrenkohl, L. (2002). Designing collaborative learning contexts. *Theory into Practice*, 40(1), 26–32.
- Rasmussen, C., & Marrongelle, K. (2006). Pedagogical content tools: Integrating student reasoning and mathematics instruction. *Journal for Research in Mathematics Education*, 37(5), 388-420.
- Thurston, W. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society*, 30(2), 161-177.
- Tomasello, M. (2001). Bruner on language acquisition. In D. Bakhurst & S. Shanker (Eds.), *Jerome Bruner: Language, culture, self* (pp. 31–49), London: Sage Publications.
- Tomasello, M. (1999). *The cultural origins of cognition*. Boston, MA: Harvard University Press.
- Toulmin, S.E. (1958). *The uses of argument*. Cambridge: Cambridge University Press.
- Stylianides, A. (2007). The notion of proof in the context of elementary school mathematics. *Journal for Research in Mathematics Education*, 39, 289-321.
- Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge: Cambridge University Press.
- Wood, T. (1999). Creating a context for argument in mathematics class. *Journal for Research in Mathematics Education*, 30, 171-191.
- Wood, T., & Turner-Vorbeck, T. (2001). Extending the conception of mathematics teaching. In T. Wood, B.S. Nelson, & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 185-208). Mahwah, NJ: Lawrence Erlbaum Associates.
- Wood, T., Williams, G., & McNeal, B. (2006). Children’s mathematical thinking in different classroom cultures. *Journal for Research in Mathematics Education*,
- Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27, 458–477.