"Working like a mathematician" with school mathematics[1]

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As a body of knowledge, school mathematics appears to be a subset of disciplinary mathematics. However, as sets of methods and approaches, school mathematics and disciplinary mathematics are either disjoint or share minimal intersection. This is rather unfortunate and invites change. I assert that teacher education is the potential means to induce this change. That is, guiding prospective teachers through the experience of "working and thinking like a mathematician" may eventually result in instilling these ways of working in students and create a larger intersection between approaches practiced in teaching and learning school mathematics and approaches employed in developing disciplinary mathematics.

It is with trepidation that I dare to talk about "working and thinking like a mathematician". I am not one, and have never been one. Nevertheless, in what follows, I discuss the challenge and the possibility of instilling mathematical ways of thinking in students, starting with their teachers. With the awareness that I may have imagined the work of a mathematician totally wrong, or at least partly wrong, I look forward to the symposium deliberations to clarify and adjust my personal perspective on what developing disciplinary mathematics may entail.

Mathematicians solve problems and prove theorems. But where do they find problems to solve and theorems to prove? Some problems are "out in the air", they have been listed and unsolved for a while and are waiting to be conquered. But most of the problems are not of this kind. They come from making observations regarding connections or relationships and wondering about the scope of the observations made. In mathematics education this activity is referred to as *problem posing*, and was thoroughly elaborated upon as a teaching strategy by Brown and Walter (1983). Furthermore, problems come from varying and extending problems that have been already solved. This strategy is included in what is referred to by Polya (1945/1988) as *looking back*, the fourth step in his classic – and much criticized – four-step approach to problem solving. Following the steps of understanding the problem, making a plan and carrying out the plan, looking back involves – in addition to checking solutions and looking for alternative solutions or approaches – varying the problem and generalizing the solution.

In what follows I present three examples of what I deem as "working like a mathematician", being strongly influenced by "Thinking Mathematically" (Mason, Burton & Stacey, 1982). These examples are taken from engaging prospective teachers in such experiences, whereas the first example also includes my personal engagement with developing and testing conjectures. If teachers' experiences are meaningful and rewarding, then there is a hope that there will be both willingness and expertise to guide students through a similar experience.

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[1] Paper submitted to Working Group 1: Disciplinary Mathematics and School Mathematics at the ICMI Symposium on the Occasion of the 100th anniversary of ICMI

Example 1: Conjecturing in Number Worlds

I would like to start by sharing a conjecture from a personal experience, working with Nathalie Sinclair and Peter Liljedahl on “Number Worlds”, which is a web-based applet developed by Nathalie[2]. Figure 1 shows the opening grid as an array of clickable cells. The user can chose the ‘World’ (Natural, Whole, Even, Odd or Prime) and within each World show/highlight a certain type of numbers (Squares, Evens, Odds, Primes, Factors or Multiples). Further, the chosen multiples can be shifted by an integer to create any arithmetic sequence. The user also has control of the width of the grid (number of columns) and of the starting number. Figure 2 exemplifies the effect of showing and shifting multiples on grids of different width. Sinclair, Liljedahl and Zazkis (2003) provide a detailed account of affordances and mathematical ideas inherent in this applet.

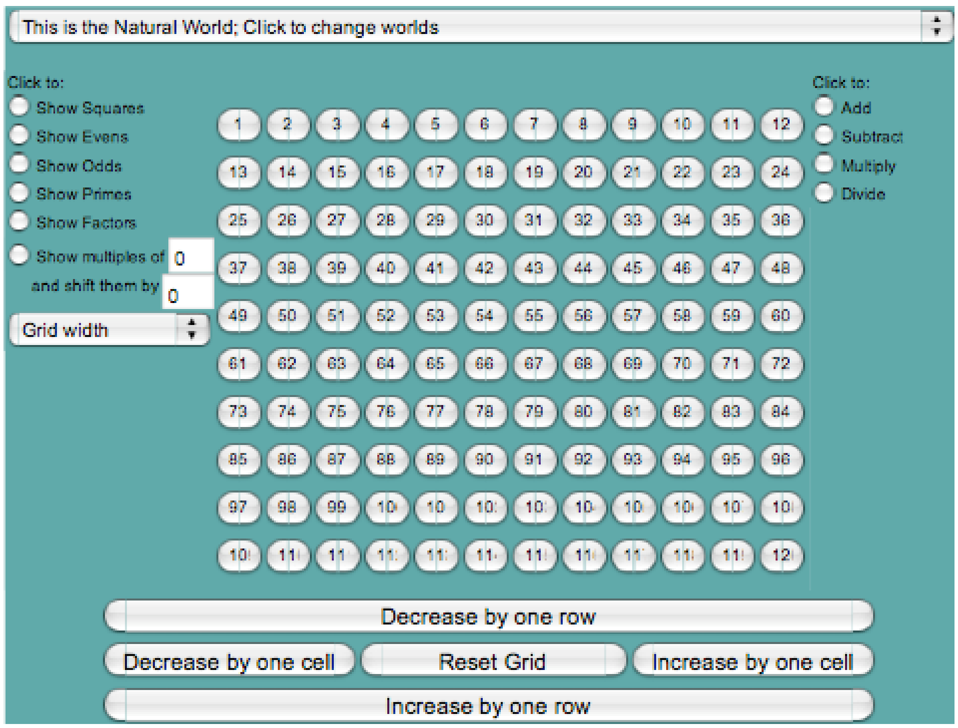


Figure 1: The Number Worlds Microworld

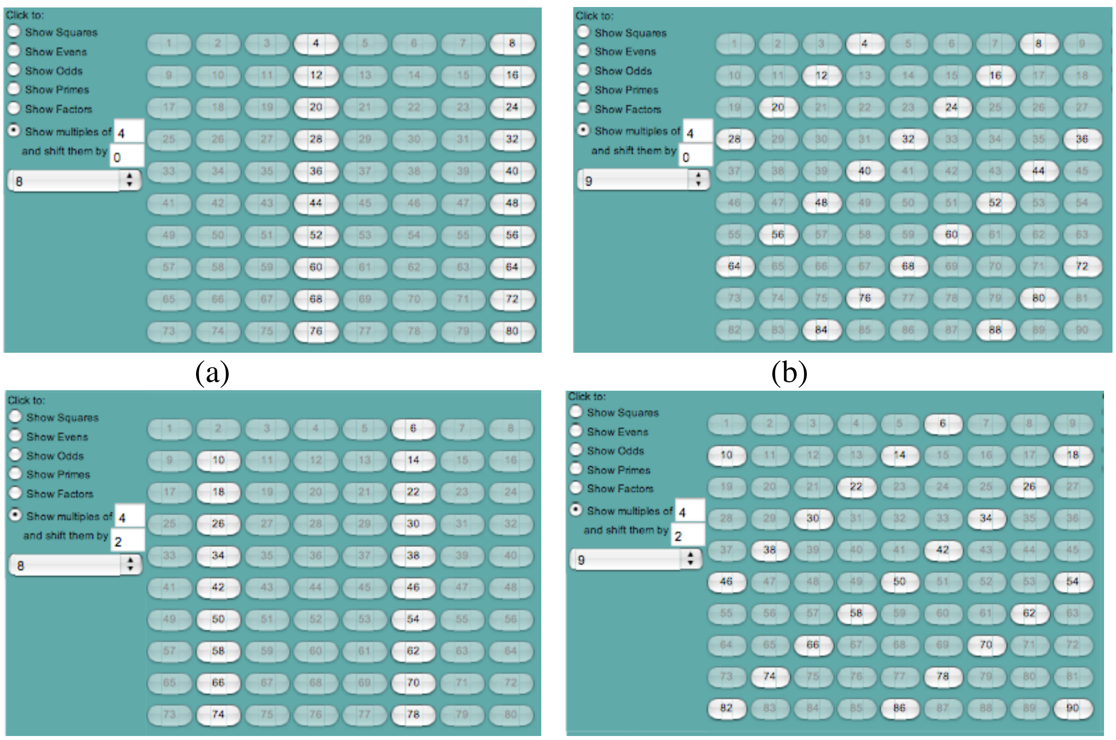
Our work with Number Worlds (Sinclair, Liljedahl & Zazkis, 2003; Zazkis, Sinclair & Liljedah, 2006) describes participants’ (prospective elementary school teachers) engagement with this computer supported environment, that provides instant feedback for one’s conjectures. The participants developed a reasonable understanding of multiples, of primes and factors, as described in detail in Sinclair et. al (2003). For example, in prior research the property that “every n^{th} number is a multiple of n (starting with n), was noted as a challenging and described as “overlooked relationship” (Zazkis and Liljedahl, 2002). For participants working with Number Worlds this property was observed by its visual

[2] Available online: http://www.mth.msu.edu/~nathsinc/java/NumberWorlds/

equivalent, noting that multiples appear with the "constant spread" or "same amount of space in between".

As one student commented, "the program allowed me to test potential theories as they entered the mind, and quickly enough so that the thought was not lost; the program did the time-consuming work" (Sinclair, Zazkis & Liljedahl, 2003, p. 257). Students' conjectures included, for example, the "constant spread" in the sequences of multiples, (as well as in the sequences of non-multiples or "multiples-shifted") and that a factor of a number is never larger than half the number being factored, not including the number itself.

Sinclair et. al. (2003) attribute students' success to two main aspects: novel visual representation and the ability to experiment. This ability to experiment in mathematics is strongly enhanced by the available technology. This is true for experiments at all levels: from elementary school students to research mathematicians.

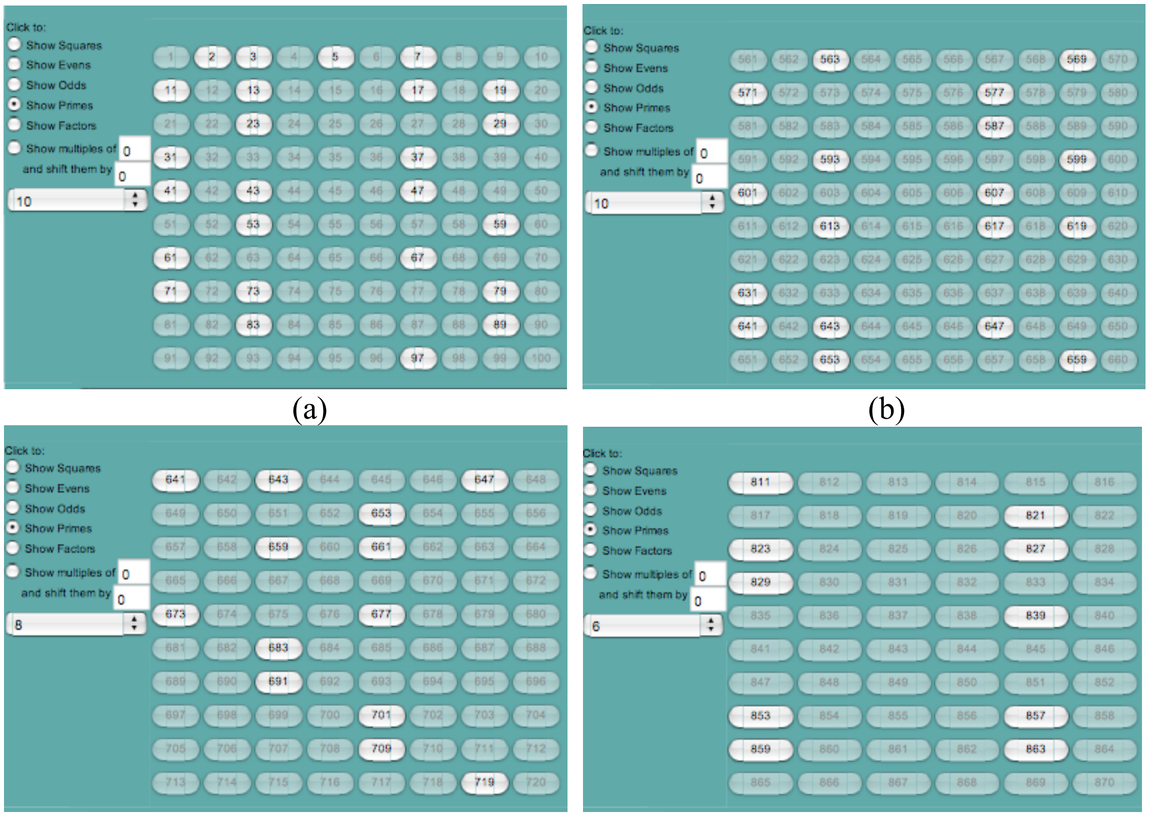


(a) (b) (c) (d)

Figure 2: Number Worlds screen showing Multiples of 4:
(a) with grid width 8; (b) with grid width 9; (c) with grid width 8, shifted by 2;
(d) with grid width 9 shifted by 2.

Further, discussing and analyzing students' conjecturing turned into a conjecture of our own: For any even number k, there are infinitely many pairs of primes of the form (p, p+k). The famous prime twins (or twin primes) conjecture is a special case, where k=2, of the extended conjecture. This conjecture was based on the observation that when "Show Primes" command is used in Number Worlds on grids of different width, there always appear to be highlighted numbers one on top of another. For example, Figure 3 shows (a) primes on a grid of width 10, in among the fist 100 numbers, (b) in the interval from 561 to 660, (c) on a grid of width 8, starting with 641, and (d) on a grid of width 6, starting with 811. The pairs of the form (p, p+10), (p, p+8) or (p, p+6) are clearly shown one underneath the other.

We later found out that our conjecture is already known to the field of mathematics, and in fact pairs of the form (p, p+4) are called "cousin primes" and pairs of the form (p, p+6) are called "sexy primes" for "sex" is a Latin word for "six" (Weisstein, 2007). And the proof of this conjecture awaits new Wileses and Perelmans.



(a) (b) (c) (d)

Figure 3: 'Show Primes' with different grid width

Moreover, watching carefully the spread of primes on the grid with the width of 6, we note repeating appearances of triples of highlighted numbers above each other. Triples (31, 37, 43), (97, 103, 109), (151, 157, 163) are just a few examples. With this observation, I wonder, is there a Sexy Prime Trio's conjecture, stating that there are infinitely many triples of the form (p, p+6, p+12)? Further, are there infinitely many Sexy Prime Quartets, such as (41, 47, 53, 59) or (61, 67, 73, 79), or (251, 257, 263, 269)? Can a new-Euler predict in what range the next such quartet will be found?

However, similar trios and quartets are not observed – within the first 1000 numbers – when the grid width is 8 or 10. I leave for the reader the opportunity to wonder about this. While some of our observations can be proved with a variable amount of effort, others leave us wondering. "Working like a mathematician" will give the same opportunity – to either become convinced or to keep wondering – for students. The impression that every problem has a solution and an algorithm to find this solution is, I believe, one of the worst outcomes of current traditional practices in school mathematics.

<u>Example 2: Exploring tridians</u>

Similarly to a median, we define a *tridian* as a segment that connects a vertex of a triangle to a point on the opposite side that is marking 1/3 of the side's length. As shown in Figure 4, BX, BY, AW, AZ, CS and CT are tridians in triangle ABC.

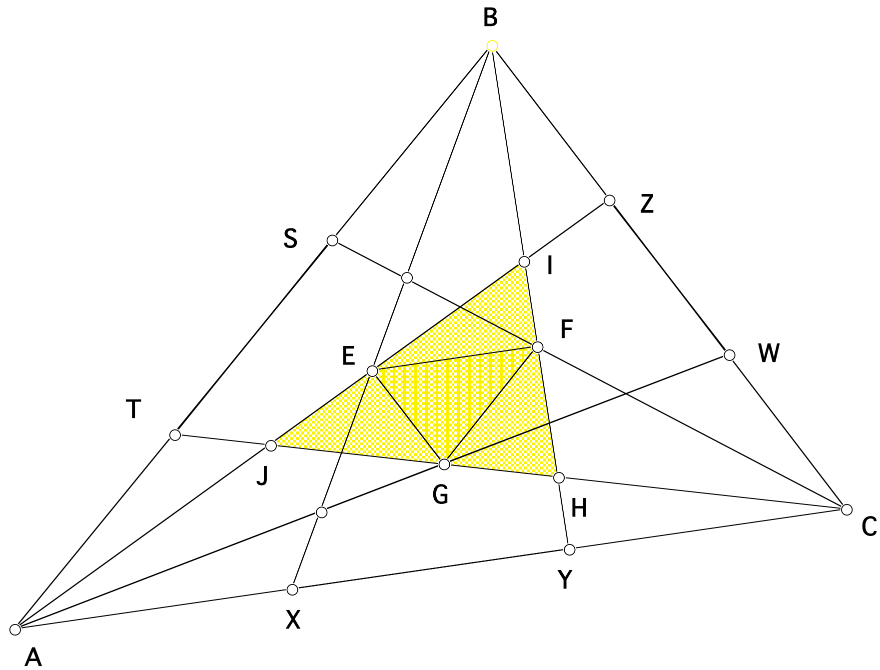


Figure 4: Tridians in a triangle

The task I present to prospective secondary teachers is as follows:

> Tridians generate all kinds of interesting relationships at various points of intersection. They also generate interesting relationships of various areas they cut. Compare for example the areas of triangles ABC, AZW, IJH and BIZ. With the help of the Geometer's Sketchpad, make several conjectures about tridians in a triangle. Formulate and prove at least 2 of your conjectures.
> What if not? What if not a triangle? What if not tridians? What if... Explore. Present at least one variation on a theme of tridians in a triangle. Prove at least some of your observations.

There is a tremendous number of possible relationships. Some are quite obvious and easy to prove from the definition of a triangle, such as the ratio of areas ABX:ABC = 1:3. Others are rather challenging, such as the ratio of areas IJH:ABC = 1:7 (When one proves this using linear algebra, vectors or affine coordinates, the challenge is to prove it using only methods of Euclidian geometry).

This task is an example of engaging learners in "working like a mathematician" on several different counts. First, there is an invitation to observe relationships and formulate and test conjectures. The software helps in exploring and verifying conjectures: Preservation of relationship when "dragging", while not a proof, is a very good pointer to what is there to be proven. This is similar to the work of mathematicians in constructive mathematics: the computer is used not to prove theorems but to find out what is there to be proven. Second, there are choices. Working with students at different levels of mathematical sophistication, these choices, while limited, are extremely important: after all, mathematicians chose for themselves what would be their next investigation or their next proof. And third, there is an incentive in the task, that would hopefully develop into a habit, not to leave the problem when it is solved, but to develop a variation and possibly a generalization.

Example 3: Looking at a sequence: The art of noticing.

The previous two examples involved conjecturing using computer software. However, while computer feedback presents wonderful encouragement for testing conjectures, it is not a necessary component of "working like a mathematician". The task in the next example can be explored without any technological support, though a hand-held calculator can be helpful in some of the observations.

Consider the following sequence:

2, 9, 16, 23, 30, 37, 44, 51, 58, 65, 72, 79, …

Notice something about the sequence, and ask a question about this observation. Examples of noticing and qustioning may include the following:

- Even and odd numbers alternate. Will this pattern continue? Why?
- All the listed elements leave a remainder of 2 in division by 7. Will this property hold as more elements are listed?
- There appear to be repeating pattern in the last digit. Will this pattern continue? Why?
- All the digits occur in the unit's place. Will all digits occur in the tens place? Is there a pattern?
- There are two square numbers among the listed elements. Are there other square numbers in the sequence? If so, squares of what numbers are in the sequence?

While the first 4 examples can be approached at very elementary level, the last one is not trivial. If a square of n is in this sequence, it is of the form $n^2 = 7k+2$ (for $k \in N$). To find such n it is helpful to notice that the next square numbers in the sequence are 100 and 121, which are squares of 10 and 11 respectively. Then, acknowledging the first pair of square numbers, 9 and 16, a natural question is, are square numbers in this sequence squares of consecutive numbers? To answer this question, it is useful to point out that 10= 3+7 and 11= 4+7, that relates the first pair of squares to the second.
From here,

$(3+7k)^2 = 9 + 42k + 49k^2 \equiv 2 \pmod 7$

$(4+7k)^2 = 16 + 56k + 49k^2 \equiv 2 \pmod 7$

Which shows that squares of numbers of the form 3+7k and 4+7k (k= 0, 1, 2, …) are elements in the given sequence.

The above provides a complete answer to the question posed previously, that is, identifies the numbers for which their squares are elements in the given sequence. However, mathematical thinking generates further questions: Are there other arithmetic sequences that include squares of consecutive integers? Is this true for all the arithmetic sequences? If not, how can we describe sequences that have this property?

This task exemplifies several important features of "thinking like a mathematician". First, there is observing and wondering. At the elementary level conjectures take a form of questions. Second, there are choices of engagement. Though the task directs learners to one particular sequence, different individuals will observe different properties and will wonder about different patterns. As such, different students will prove or justify different properties. This variety also provides different difficulty for the level of engagement. Third, once the initial question is answered, the task is not left to rest. Answers generate further questions, that involve either variation or generalization of the previous ones, where new observations and conjectures will emerge.

Summary and conclusion

Disciplinary mathematics – I conjecture – is developed by observing relationships and proving their scope of existence. A proof is preceded by a conjecture that formulates the problem, and is followed by a conjecture that varies or generalizes the proved proposition. Some proofs take hours or days, other take centuries, and there are a few propositions that remain as unproved-yet conjectures.

Similar engagements do not require one to be working at the frontiers of the discipline; they can be presented at any level of difficulty. The advantage of this practice is in introducing mathematics as a dynamic human endeavor, rather than a static body of facts. "Working like a mathematician" experiences in teacher education open the gate for similar experiences for students.

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