STUDENT TEACHERS AND *TEACHER EDUCATORS'* PEDAGOGY UNDER CONSTRUCTION

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ABSTRACT

The research results presented in this paper are only a small part of an action research performed with the main aim of improving student teachers' understanding of mathematics. The re-teaching of mathematics was integrated with the teaching of pedagogy by asking student teachers (STs) to perform children's activities which have the potential to develop conceptual understanding of the subject. The data collected indicated that most STs improved their understanding, but some STs needed more time to re-learn certain content in the primary school curriculum. This paper presents some results concerning: (a) STs' previous knowledge of addition of fractions, (b) STs' difficulties in relearning addition of fractions and (c) some practical solutions proposed to ameliorate STs' learning difficulties within the time available.

RELATED LITERATURE

Shulman (1986) identified several knowledge components which teachers may use in order to make decisions for the purpose of teaching and to help them promote understanding on the part of their students. One of these components is subject matter knowledge (*SMK*) which includes both the substantive and syntactic structures of the discipline. The focus of this paper will be on teachers and STs' acquisition of substantive understanding of the mathematics they will teach. According to Shulman, "The substantive structures are the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its facts" (p. 9). The acquisition of pedagogical content knowledge (*PCK*) which includes "the ways of representing and formulating the subject that make it comprehensible to others and ... an understanding of what makes the learning of specific topics easy or difficult" (p. 9) is another focus of this study.

Mathematics is considered a difficult subject to understand (e.g., English and Halford, 1995 and Gustafsson, 2004). It is also a difficult subject to teach. Understanding, particularly in initial school years, requires a detailed and careful teaching approach and so depends on the teachers' acquisition of SMK and PCK. Instructional constraints may, in part, be responsible for students' difficulties in understanding the subject (e.g., Bell et al., 1985; and Hiebert and Carpenter, 1992). Bell et al. (1985) think that some misconceptions may result from new concepts not being strongly connected with the student's previous concepts. Some misconceptions may also result from "the absence of some actually essential detail of the knowledge-scheme which has been overlooked in the design of the teaching material" (p. 2). Adding the numerators and denominators in addition of fractions may be the result of a more limited schema for fractions. The student may see fractions only as a pair of two whole numbers (one written on top of the other) which result from the counting involved in a part whole diagram: (i) the number of shaded pieces, (ii) the total number of pieces and (iii) finally writing the first number on the top of the second. Such a schema for fractions is coherent with adding numerators and denominators in the addition algorithm.

Communication and students' previous schemas seem to influence the learning process. Even experienced teachers with a good mathematical knowledge may not teach conceptually because they hold part of their knowledge in an implicit way (Remillard, 1992). They can make unconscious conceptual leaps in the activities they select for teaching purposes leaving their students with not much understanding. Similarly, Sotto (1994) says that even teachers who have much conceptual understanding may hold part of it in a tacit way and may not be able to translate all they know into activities which can communicate their entire understanding in the classroom. If students are offered restricted mathematics experiences, they will construct restricted internal representations and poor schemas which are difficult to connect to new knowledge (Hiebert and Carpenter, 1992). Certain students' difficulties may be related to "weaknesses in teaching" and so the teacher's knowledge and thinking processes seem to be an important factor affecting students' learning. Therefore, STs should be helped to construct broader mathematical schemas which are easy to connect to their prior knowledge and to new knowledge and which can help them overcome their misconceptions.

Learning operations with rational numbers is not easy (e.g., English and Halford, 1995). For this reason some curriculum developers in Brazil are proposing a reduction in the teaching of operations of fractions at primary school level. Yet research has shown that many school students' (e.g., Ni and Zhou, 2005) and even STs (e.g., Domoney, 2002) see fractions as two separate natural numbers and not a single number and develop a conception of number that is restricted to natural numbers. Ni and Zhou (2005) suggest that the teaching of fractions should start earlier than is it is often recommended by curriculum developers in order to avoid the development of what they call 'whole number bias'. Something similar may be said about misconceptions related to operations with fractions such as the one mentioned before. Besides working only with operations with natural numbers during several school years becomes boring to some students.

In order to provide school students with an early start, teachers must develop themselves a deep understanding of fraction concepts and operations which, among many other things, involves the ability to differentiate and integrate the concepts and operations with natural numbers and fractions. Teachers should also have enough PCK to be able to teach young students an introduction to operations with fractions in a more informal and delicate way with concrete and iconic representations. The teaching focus could be on early notions related to different denominators such as "pieces of different sizes can not be added or subtracted". This notion can also be useful in preventing errors related to adding and subtracting metric units of length (e.g., 3 m + 7 cm = 37 cm instead of 3 m + 7 cm = 300 cm + 70 cm = 370 cm) and help students avoid the misalignment of the decimal point when adding and subtracting decimals (e.g., 21.75 + 0.4 = 21.79 instead of 21.75 + 0.4 = 22.15). On the other hand, studies of primary school STs' knowledge of fractions tend to show that it comprises mainly of remembering a large repertoire of rules and algorithms with not much understanding of the underlying mathematical concepts and relationships (e.g., Sowder et al., 1993). Understanding of algebra algorithms is said to be dependent on the understanding of number relationships expressed on written arithmetic algorithms (e.g., English and Halford, 1995). Mathematics is not only beautiful and useful in everyday life but it is also the language of science. Conceptual understanding of algebra can be empowering and help to progress in science subjects.

Hiebert and Carpenter (1992) argue that constructing relationships within a representation mode helps to increase the cohesion and structure of the schema. The Brazilian conventional algorithm for addition and subtraction of mixed numbers appear unreasonable and lacks cohesion. First mixed are transformed into improper fractions (breaking all the wholes into pieces) and then the result is converted to a mixed number (gluing the pieces to make wholes again). In this case, the vertical algorithms (e.g., English and Halford, 1995) which are similar to the conventional ones used for natural numbers and decimals may help students think of fractions as extensions to the number system. In this way fractions are added in similar manner to that of natural and decimal numbers and the "carrying" and "borrowing" processes can be extended to fractions in a way that reinforces the relation between fractions of the type n/n and the number 1 (Amato, 2005b and 2006). The vertical algorithms for fractions also make easier to record the renaming when later fractions with different denominators are added or subtracted. Visually such recording also seems to make clearer the use of equivalence in the two operations because the equivalent fractions and mixed numbers are written side by side as will have been done previously.

Ball (1990) argues that the mathematics curriculum studied at school presents the subject as discrete pieces of procedural knowledge. Representing the vertical algorithm in multiple ways for fractions and mixed numbers is a way of connecting addition of fractions to an addition procedure which is part of an existing schema for place value with natural numbers. If the relationship is made, not only is addition of fractions understood, but the place value schema is also enriched. In order to develop a conceptual knowledge of operations with rational numbers, students and teachers should be able to both differentiate and integrate operations with natural numbers and fractions.

Teachers have the social responsibility of helping students learn mathematics. They must develop the ability to work backwards from their symbolic ways of representing mathematics to more informal ways of representing the subject (Ball and Bass, 2000). Otherwise they may lose precious opportunities of using representations in unpredicted moments and helping students construct further relationships. According to Rowland et al. (2005), it is part of a teacher's job to execute "*contingent actions*". This means that they must be ready to "respond to children's ideas ... and to deviate from an agenda set out when the lesson was prepared" (p. 7). The ability to translate SMK into representations is considered a fundamental part of teachers' PCK (Shulman, 1986).

Lesh et al. (1987b) describe five types of representation they identified in mathematical learning and problem solving: (a) real world contexts, (b) concrete materials, (c) pictures and diagrams (d) spoken languages and (e) written symbols (p. 34). According to Cramer (2003), a deep understanding of mathematics can be achieved by involving students and teachers in "activities that embed the mathematical ideas to be learned in five different modes of representation with an emphasis on translations within and between modes" (p. 462). Yet there is also some research evidence which shows that some teachers, especially novice primary school teachers, do not have a good knowledge of mathematical representations (e.g., Ball, 1990). Therefore, it seems important to strengthen or remediate STs' ability to work with several types of representation within each system, translations among them and transformations within them.

METHODOLOGY

I carried out an action research at University of Brasília through a mathematics teaching course component (MTCC) in pre-service teacher education (Amato, 2004b). The component consists of one semester (80 hours) in which both theory related to the teaching of mathematics and strategies for teaching the content in the primary school curriculum must be discussed. This is the only compulsory component related to mathematics offered to primary school STs at University of Brasília. There were two main action steps and each had the duration of one semester, thus each action step took place with a different cohort of STs. As the third and subsequent action steps were less formal in nature and involved less data collection, not many results will be reported from the latter. The main research question of the study was: "In what ways can primary school STs be helped to improve their conceptual understanding of the mathematical content they will be expected to teach?".

A new teaching programme was designed with the aims of improving STs' conceptual understanding of the content they would be expected to teach in the future. In the action steps of the research, the re-teaching of mathematics (SMK) was integrated with the teaching of pedagogical content knowledge (PCK) by asking the STs to perform children's activities which have the potential to develop conceptual understanding for most of the contents in the primary school curriculum. About 90% of the new teaching program became children's activities. The children's activities performed by the STs had four more specific aims in mind: (a) promote STs' familiarity with multiple modes of representation for most concepts and operations in the primary school curriculum; (b) expose STs to several ways of performing operations with concrete materials; (c) help STs to construct relationships among concepts and operations through the use of versatile representations (Amato, 2005b and 2006); and (d) facilitate STs' transition from concrete to symbolic mathematics. In the teaching programme, the translation model put forth by Lesh et al. (1987b) was used to organise a sequence of children's activities. In the case of addition and subtraction of fractions, the activities progress from very informal activities focused on the manipulation of three or more types of concrete materials to exercises involving translations from pictures of the concrete materials and different part-whole diagrams to symbols. Finally, more formal activities are presented with only translations within written symbols with the purpose of generalization (Presmeg, 2006). A summary of the main activities in the teaching program can be found in Amato (2004b).

Four data collection instruments were used to monitor the effects of the strategic actions: (a) researcher's daily diary; (b) middle and end of semester interviews; (c) beginning, middle and end of semester questionnaires; and (d) pre- and post-tests. The daily diary was a way of keeping a record of my own thinking and of observations made inside and outside the classroom concerning the research question, the strategic actions and the problems encountered during the action steps of the research. The questions in the questionnaires and interviews focused on STs' (i) perceptions about their own understanding of mathematics and their attitudes towards mathematics before and after experiencing the activities in the teaching programme, and (ii) evaluation of the activities in the teaching programme. The tests involved open-ended questions in such a way that conceptual understanding could be probed through a context of teaching children. Each page of the tests contained three questions. The same heading was used for all the pages in the tests: "Answer the following questions as if you were introducing the concepts involved to primary school children. Describe briefly what you would do and say in each situation. Whenever possible draw pictures to illustrate your ideas." Question F4 of the pre-test about fractions was about addition of fractions with different denominators: "How would you explain the reason for the result of 1/2 + 1/4 (2/3 + 1/6 in the post-test)?". The data analysis was mostly qualitative, but a simple quantitative analysis (frequency and percentages) was also used to describe some of the results. Much information was produced by the data collection instruments but, because of the limitations of space, only some STs' responses related to their use of children's activities for addition of fractions will be reported here.

RESULTS

(a) STs' previous knowledge of addition of fractions – Only the second semester tests were analysed in great detail. The pre- and post-tests responses of each ST were compared to investigate any changes in conceptual understanding which could be attributed to the teaching programme. There were 42 STs in the first semester class and 44 STs in the second semester class. An example of what I considered to be an improvement in conceptual understanding for question F4 is:

> [Pre-test ST203] I learned only as a rule (find the least common denominator, etc.).
>
> [Post-test ST203] Before [before the MTCC]: Find the LCM of the denominators and divide it by the previous denominators and then multiply by the numerators: 2/3+1/6 = 4/6+1/6 = 5/6. Today [after the MTCC]: What happens is a transformation into slices of equal sizes. [Drew a picture of a unit divided into thirds with full lines and shaded 2/3. In the same picture she subdivided the thirds into sixths with dotted lines. She also drew a picture of 1/6. The units were of similar size.]

The results of the pre-tests indicated that most STs did not have a conceptual understanding of rational numbers concepts and operations. The findings are consistent with previous studies (e.g., Ball, 1990; Sowder et al., 1993; and Herman et al., 2004). Some insecurity about the teaching of rational numbers was expressed by a few STs in the pre-tests: "ST118 I have great difficulty in transmitting fractional numbers to children". Most STs were unable to explain the reasons behind the steps in the algorithms they used for adding with fractions. Only six STs gave indications in the pre-tests that they could add fractions conceptually. They all appeared to have relied on their part-whole diagrams to conclude that 1/2 = 2/4 and to explain the result of the addition. A part-whole diagram was considered useful when the ST drew 3/4 (or 5/6) in a way that it made clear its relationship to the addition of the initial fractions (1/2+1/4 or 2/3+1/6). Yet some STs used part-whole diagrams in a way that was thought to be unhelpful and sometimes even misleading.

Eight STs drew part-whole diagrams to represent the two initial fractions and the result, but did not relate the diagrams in any visible way to the result 3/4. They appeared to have found the result using the written algorithm and decided to represent the three numbers with diagrams. This result is consistent with Herman et al. (2004). These diagrams were thought to be unhelpful as they did not to provide explicit visual clues about what was behind the addition algorithm. A few STs used different units to represent each of the fractions and in some cases those representations affected the correctness of their responses when adding fractions. ST243 made the common error adding numerators and denominators. Similarly to school students, she justified her conclusion by combining her two part-whole diagrams. In the pre-tests only eight STs provided useful diagrams to represent the equivalence of fractions involved in the addition algorithm. In the post-test a few STs also did not provide good diagrams. As in the pre-test, six STs only drew part-whole diagrams to represent the two initial fractions (2/3 and 1/6), but did not make any visual connections between them and the result. However, the number of useful diagrams increased from 8 to 26 in the post-tests.

More than half of the STs who answered the pre-test wrote the correct result. A few of them just used part-whole diagrams to reach the result, but the majority performed correct variations of the conventional algorithm. Yet they could not explain why they found a common denominator. The vertical algorithm is not presented by the majority of Brazilian textbooks and so it did not seem to be known by any of the STs in the pre-test. There was also a number of incorrect algorithms in the pre-test. Five STs described the algorithms in ways that sounded confusing and not helpful even to develop a procedural knowledge of addition of fractions. Some of them misapplied portions of algorithm they had previously memorised: "ST216 [pre] We have to find the HCF [highest common factor] and then the addition will be automatic. (1+2)/2 = 3/2".

Only six STs gave indications in the pre-tests that they could add fractions conceptually by using equivalence of fractions. The number of STs using equivalence increased to 35 in the post-test. Most of them wrote correct algorithms and/or made explicit in their verbal representations the use of equivalence in performing them. Twelve STs used the vertical algorithm they had learned in the MTCC either alone or together with a conventional algorithm. In the pre-test six STs used the Brazilian conventional algorithm called multiple factorisation for finding the least common multiple (LCM). These algorithms were considered very formal to teach young students. More informal methods for finding a common multiple with concrete materials and diagrams were discussed in the MTCC and, probably for this reason, the LCM conventional algorithms were abandoned in the post-test. Seven STs showed very little change in their conceptual understanding of addition of fractions and continued to give weak or confusing explanations in the post-tests. Eighteen STs showed good improvements in their understanding of addition of fractions. Six STs were thought to have had small improvements in their understanding. They either provided better diagrams or more clear and rich written explanations about equivalence in the post-test.

(b) STs' difficulties in relearning addition of fractions – Making relationships between mathematical concepts and operations is the basis for conceptual understanding. The use of the same concrete materials for most of the operations in the primary school curriculum was found to be beneficial to STs' relearning of addition of fractions. Some STs mentioned that the idea helped in making relationships among the operations and that it was an important pedagogical aspect:

> Int22(4)(b) ST207 ... At the beginning we found it strange to do the operations with fractions on the PVB [Place Value Board, Amato (2006)]. The same materials could be used for working with all operations, not only for natural numbers but also for fractions. You can construct the materials once and use them for everything. This is an important pedagogical aspect to teach children. ... The child needs to work with what she knows, with what she has already manipulated.

Yet a few STs experienced some difficulties in translating from operations with concrete materials to operations with symbols. The STs' well memorised algorithms were thought to interfere in the learning of new connections between procedural and conceptual knowledge (Amato, 2005a). Some STs expressed their problems in translating from concrete to symbolic and many STs suggested increasing the teaching time for operations with rational numbers because fractions and decimals were much more difficult for them than place value and operations with natural numbers:

> Int22(1) ST243 When the concrete materials are taken away and I have to work symbolically I get lost. ... They seem to be two different things. … For natural numbers and decimals there are no problems. Sometimes I do not even need to use the concrete materials. … We deal a lot with decimals in money. … Fractions are alright when you deal with the pieces

in the concrete materials, but when you start comparing them and finding common denominators I cannot visualise the ideas.

Int22(2) ST231 I think you should dedicate more time to fractions than to natural numbers. Everybody has a greater knowledge base about natural numbers. Not with respect to teaching children, but with respect to our class. With the child you have to work very well with natural numbers. However, the class has a lot more difficulties with fractions so we need more time on fractions.

(c) Some practical solutions proposed to ameliorate STs' learning difficulties – After the first semester finished, all operations with rational numbers were thought to need greater emphasis in the programme. I also decided to make changes in the distribution of the content within the semester. In second and subsequent semesters the idea of a spiral curriculum was gradually improved in my practice. The activities related to more difficult content were spread along the semester providing the STs with several opportunities for accommodating previous content through activities involving extensions of the content and relationships with other content. The number of practical activities and games for fractions and decimals was greatly increased in the third and subsequent semesters. For this reason, the number of activities for place value and operations with natural numbers alone was reduced. However, there were still many activities about operations with rational numbers which included a natural number part. Through operations with mixed numbers and decimals (e.g., 35¾+26¼ or 24.75-12.53) with the use of versatile representations (Amato, 2005b and 2006), STs experienced further activities related to operations with natural numbers and had the opportunity to make important relationships between operations with natural numbers and operations with fractions and decimals. These changes proved to be quite effective in helping other classes of STs overcome their difficulties in relearning addition of rational numbers conceptually within the time available.

After the implementation of the changes above, the subsequent classes did not seem to have many problems in the manipulation of the concrete materials, in working with diagrams and in performing the vertical written algorithm. However, some STs said that they wished to understand why LCM algorithm worked. I told them to use the multiplication of the two denominators as the result would always be a common denominator. A few STs were not happy using the product method as they noticed that the common denominator became too big in some cases and they wished to learn how to teach the LCM they had memorised. Yet I did not know how to present the algorithm in an easy and conceptual way, based on the students' prior knowledge. As a school student I had only memorised the conventional algorithm and I could not explain it worked. Depending on the numbers, the more informal method involving writing two sets of multiples could become very long lists of numbers to be compared.

Only after around 18 semesters, I discovered a way of presenting the LCM that could be easily understood by young students. It is based on the LCM algorithm used for algebraic fractions and young students only need to know how to write a multiplication sum for simple numbers (e.g., 36 = 4 x 9 or 36 = 6 x 6). The "the times table method" also became an important way of consolidating the multiplication facts. Initially the students are asked to write a multiplication operation for each denominator (e.g., 36 = 4 x 9 and 30 = 2 x 15). Then they are requested to "break" again the two numbers involved in each multiplication into further multiplication sums until there are no numbers to be broken (e.g., 36 = 4 x 9 = 2 x 2 x 3 x 3 and 30 = 2 x 15 = 2 x 3 x 5). The idea of relating the algorithms for operations with fractions to the algorithms for operations with natural numbers and algebraic fractions is, therefore, seen as relating new content to previous learned content and so to the acquisition of meaningful learning (Ausubel, 2000).

The school students, STs and teachers usually say they enjoy having started from different sums and, in the end noticing, that they get the same (prime) numbers in a different order (e.g., 36 = 6 x 6 = 2 x 3 x 2 x 3 and 30 = 5 x 6 = 5 x 2 x 3). Finally they are asked to write inside brackets the numbers that are missing in one multiplication but which exists in the other multiplication in order to make the two multiplications equal (e.g., 36 = 2 x 2 x 3 x 3, [x 5] and 30 = 2 x 3 x 5, [x 2 x 3]). The numbers inside the brackets are the numbers which should be used to multiply the numerator and denominator of the initial fractions in order to find a pair of equivalent fractions with a common denominator. Some STs usually reveal to the rest of the class that they have enjoyed leaning "the times table method". For example, in the second semester of this year (2007), a ST has spontaneously mentioned in the classroom: "It is such simple way of presenting the LCM to children".

CONCLUSIONS

Students' facility or difficulty in learning certain content is a function of the quality and quantity of their previous experiences. The STs' previous symbolic and procedural ways of thinking appeared to have interfered with their learning of more informal representations and with their acquisition of conceptual understanding (Amato, 2005a). Some STs also seemed to need a greater time to make the transition from concrete to symbolic and to deal with alternative symbolic algorithms. The representations used in the programme, in particular the practical work with concrete materials, are not simply a teaching strategy which could be easily replaced by another strategy. They were considered the most basic form of PCK as other teaching strategies were seen to be dependent on such knowledge.

Orton (1987) argues that statements in which absolute levels of difficulty are assigned to particular mathematical concepts can be unhelpful. Teachers must try to find simple ways of teaching those concepts. If difficulties are inherent in particular topics, it is the teacher's duty to provide more experiences (quantity) with the potential to improve understanding (quality). The same can be said about certain representations and content that caused difficulties to certain STs. Instructional constraints may, in part, be responsible for students' lack of connections and weak understanding. STs' difficulties were also related to "weaknesses in my teaching" and so my own PCK and thinking processes were important social factors affecting their re-learning of mathematics and pedagogy. It must also be said that useful ideas for ameliorating underlying and unanticipated problems did not come to my mind immediately after observing these problems. The literature about teaching and learning mathematics does not always present solutions to very specific problems. Discovering weaknesses in my own teaching proved to be a slow process. In some cases insight only came after much thinking, effort and time on my part. Like STs, teacher educators' mathematical and pedagogical knowledge is still under construction and also presents weaknesses that are transferred to their teaching.

Some teacher educators seem to believe that working towards developing teachers who are autonomous, and who seek study groups and other means of learning and growth, is incompatible with the idea of learning about SMK and PCK through formal instruction in pre-service teacher education. On the contrary, my own experiences as a novice mathematics teacher (Amato, 2004a) led me to think that STs' acquisition of SMK and PCK in pre-service teacher education is an important precondition for their future autonomy as teachers. My professional autonomy as a novice mathematics teacher was, in many moments, hindered by my procedural knowledge and by my insufficient knowledge of appropriate representations to deal with my students' difficulties.

Novice teachers have to face many constraints and challenges at the beginning of their careers. Natural classroom settings can be quite stressful for novice teachers whose pedagogical thinking appears to be dominated by concerns of classroom management. I think that artificially constructed environments in teacher education may help STs focus more on their learning by avoiding the complexity and stress associated with whole classes. Learning some SMK and PCK from my own teaching experiences and from other teachers proved to be a very slow process. It took me a long time and a great effort to acquire some conceptual understanding and PCK while teaching several large classes simultaneously. Learning mathematics from teaching also seems to be a slow process for primary school teachers, as they have to teach several subjects simultaneously.

Therefore, STs must acquire in pre-service teacher education enough knowledge to face the responsibility of providing effective learning experiences to all school students since the beginning of their careers. When teachers find the time to work together in study groups, they should be discussing complex problems related to their practice and not dedicate their time trying to acquire professional knowledge which is the responsibility of teacher education. An initial knowledge base, which I think it is a combination of a strong conceptual understanding of mathematics (SMK) and knowledge of a repertoire of representations (PCK), must be available to STs in pre-service teacher education. Otherwise their first students may well be led to think that mathematics is a complicated and unreachable form of knowledge because teachers have not yet learned ways of communicating the subject in a conceptual way. Although it is a very basic form of PCK, knowledge of representations was also thought to be the most adequate knowledge about teaching in order to foster STs' initial feelings of success that would be needed to continue their learning from teaching mathematics. With time and teaching experience STs would be more able to use such knowledge in combination with more sophisticated teaching strategies.

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