Mathematical Knowledge, Mathematical Culture, and Mathematics Teacher Education

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One specific problem in the professional formation of teachers resides in the tension in the dichotomy between the mathematics content and the pedagogical approaches that teachers (including prospective teachers) are immersed in. One source of this conflict of mathematics content knowledge and pedagogical knowledge appears to be situated in the (misaligned) nature of the mathematics content offered to teachers, which are courses in academic mathematics offered by mathematicians for future mathematicians. While these courses are important for future mathematicians, it is to be questioned if they are of value to mathematics teachers.

Mathematical content and form

The mathematical practices enacted through the academic mathematics courses appear to have negative repercussions for (future) *teachers*. Significant critiques in the research literature in mathematics education problematize this situation. One of these critique concerns the form or content of the mathematics taught in these courses. Some researchers have expressed that the advanced and formal nature of the mathematics worked on in most of these courses could have the detrimental effect of reinforcing the abstract, technical and formal side/level of mathematics in teachers' understanding of mathematics and also in its teaching (Ball, Lubienski & Mewborn, 2001; Cooney & Wiegel, 2003; Gattuso, 2000), potentially leading to particular difficulties in teachers' pedagogical approaches.

Two studies appear supportive, to some extent, of these hypotheses. The first one is from Thompson and Thompson (1994, 1996) who studied a teacher, Bill, who had a robust understanding of the concept of rate and speed. Bill's understandings of the concepts of rate and speed were rich with plenty of connections between ideas, but were so tightly woven and hidden under calculations and operations that it made him unable to articulate clearly these understandings to his student in order to make the notions accessible and conceptually sound for her – even to the point that she could not make sense of his questions and felt immensely confused. Thompson and Thompson theorize that this teacher's strong knowledge base of the notion, at a formal and higher mathematical level, led him to perceive the connections and meanings as obvious, where he consistently used operations and calculations that *for him* made the connections explicit but left them opaque for his student, creating a gap between him and his student. Concepts of rate and speed were so obvious for Bill, and his understandings of them was so tightly encapsulated and hidden within calculations and operations and formalized ways of operating, that it made them far from transparent for his student.

The second study is from Nathan and Koedinger (2000). They administered questionnaires to teachers requiring them to rate a list of algebraic problems by order of predicted difficulty for their students. The correlation between the lived difficulties of students and the teachers' predictions was very low, where teachers had overestimated the facility that students would have with formalisms and symbolism manipulations. The researchers conjectured that teachers' own facility with symbolic manipulations led them to undervalue the difficulties that these abstract forms could had on students. Discussing the study, the National Research Council (2001) expressed that university-level mathematics content knowledge "by itself may be detrimental to good teaching" (p. 399).

In addition to the formal aspects of academic mathematics, the nature of the concepts taught in academic mathematics courses appears at odds with the ones teachers enact when teaching. It is in fact the very nature and strength of academic mathematics to make mathematical understandings, concepts and ideas "compact" and "compressed" so that they are more efficient, powerful, and easier to handle

[1] The ideas reported in this paper represent a work in progress, and are inspired by a research project currently being designed and undertaken by myself and Nadine Bednarz (*Université du Québec à Montréal*, Canada).

and utilize (Adler & Davis, 2006; Ball & Bass, 2003; Moreira & David, 2005). However, as these previous researchers mention, it is the exact opposite that appears relevant and suitable for teaching mathematics concepts in an efficient way to students. A teacher has to be able to untangle, unpack, dismantle, and decompress the mathematical concepts to make emerge and unearth the meanings, relations, subtleties and nuances hidden behind their compact structure in order to foster and promote students' mathematical understandings.

Additionally, given the research literature that fails to demonstrate a significant relationship between the number of academic mathematics courses taken by a teacher and his or her students' performances (Begle, 1979; Monk, 1994), we do not have a very strong case for requiring that teachers take academic mathematics courses in their mathematics teacher education (or through professional development activities). As Bauersfeld (1998) expresses, mathematics teacher education models simply overestimate the positive effects of the academic training in mathematics on teachers.

However, and this is important, the critiques put forward here are *not* addressed toward mathematicians or the courses themselves, but are made toward the fact that teachers are required to take these courses. As Cooney and Wiegel (2003) explain, mathematicians teach academic mathematics to and for future mathematicians, this is what they do and should be doing. The issue does not lie with the way mathematicians teach their academic mathematics courses or educate future mathematicians, but to the problematic feat of placing our teachers in these courses. It is the fact that these experiences are not relevant, and appear even detrimental, to the work of mathematics teachers, who need something different in their education. But, the question still remains as to what are these different mathematical experiences to provide teachers with?

Knowledge of school mathematics

Usiskin (2001) points to an important issue in regard to mathematics teachers' knowledge of the school curriculum mathematical content:

> Often the more mathematics courses a teacher takes, the wider the gap between the mathematics the teacher studies and the mathematics the teacher teaches. The result of the mismatch is that teachers are often no better prepared in the content they will teach than when they were students taking that content. A beginning teacher may know little more about logarithms or factoring trinomials or congruent triangles or volumes of cones than is found in a good high school text. (p. 2)

There is indeed some research that points to secondary teachers' difficulties with aspects and concepts of the mathematics they teach. For example, studies from Ball (1990) and Bryan (1999) have illustrated that the secondary mathematics teachers that they studied made few if any mistakes in their usage of mathematical procedures; but, that teachers experienced significant difficulties to provide sound meaning and explanations of the mathematical rationales lying behind the same procedures. Quoting from Mewborn (2003): "By and large, teachers have a strong command of the procedural knowledge of mathematics, but they lack a conceptual understanding of the ideas that underpin the procedures" (p. 47). Other studies have also highlighted difficulties of another order, for example, concerning secondary teachers' unfamiliarity with the meaning of concepts themselves (definitions, conjectures, relationships within concepts, etc.). For example, Even (1993) and Hitt-Espinosa (1998) observed that many teachers possessed the "old" definition of a function as a continuous graph, which prevented them from recognizing or accepting alternative drawings as representative of a function. This also led them to transform or treat any discrete function as continuous. Also, Schmidt and Bednarz (1997) and Van Dooren, Verschaffel and Onghena (2003) have reported on secondary teachers' difficulties to appreciate arithmetical procedures as valid solutions to traditional algebraic problems.

While these types of studies have been criticized as presenting a "deficit model" of teacher knowledge, and moreover cannot necessarily be generalized to all teachers, these studies offer significant and insightful information. In that sense, it is not about what teachers know or don't know, but about how we inform ourselves and learn from these studies. What do these results tell us about what we could or should offer teachers as mathematical experiences in our teacher education initiatives?

Therefore, what is significant in these studies is mostly that there seems to be a need for providing mathematical experiences to teachers to study and explore school mathematics concepts through teacher education practices – rather than providing mathematics teachers with more higher-level or formal academic mathematics experiences. As Bryan (1999) suggests, there is a need to offer teachers opportunities “to deepen their conceptual understandings of the *content of the school mathematics curriculum*” (pp. 8-9, my emphasis).

A teacher education project focused on the exploration of school mathematics

A professional development (PD) initiative focusing on secondary mathematics teachers’ exploration of school mathematics content was recently undertaken by myself in a research study (Proulx, 2007). Working with teachers that had important procedural skills (similar to teachers reported in Ball, 1990, or Bryan, 1999), the intention was to build on their knowledge and enhance it, through having them explore school mathematical concepts beyond procedures. Put explicitly, one main goal of the PD was to have teachers develop and enrich their understanding of the procedures and concepts of the school mathematics curriculum, mainly through explorations and discussions.

Space constraints do not allow for much elaboration, but one important aspect stemming out of the research is that teachers learned a lot of/about the mathematics (that they teach) through their explorations of school mathematics content. The explorations opened new ways of making sense of the mathematical concepts for the teachers, ways that they explained they had never thought of or learned about before. In addition, these mathematical explorations led teachers to engage in important pedagogical discussions and considerations. The new mathematical understandings that teachers developed appeared to open new possibilities for approaching the concepts in teaching. By enlarging their understanding of the mathematical concepts they teach, teachers also enlarged their understanding of what teaching these mathematics implied. Teachers recognized that their mathematical understanding was largely instrumental and oriented toward procedures, algorithms, and formulas. In addition, they also recognized that their teaching was along this orientation. Hence, the explorations afforded teachers with new knowledge, perspectives, insights and ways of understanding school mathematics concepts, and in return of approaching these concepts in their teaching – ways that went deeper than a procedural or mechanical treatment. These practices of teacher education appeared of relevance since, as the truism says, “one cannot teach what one does not know about,” but also and mostly because this sort of work offered teachers new possibilities for mathematics and its teaching that they simply did not have before.

The research also points to and reminds us of fundamental aspects concerning secondary-level mathematics teachers. First, secondary mathematics teachers possess important knowledge of mathematics; even if some of this knowledge is very technical or procedural (knowledge of procedures is of great importance in mathematics). This knowledge needs to be built on and be used as a springboard to enrich, enhance, re-elaborate and deepen teachers’ mathematical knowledge. Second, secondary teachers demonstrate a strong interest in knowing mathematics and are very curious to learn more. Simply said, they enjoy mathematics a lot: they have had, for the most part, a lot of success in school mathematics as students and they have in fact chosen to teach it fulltime! Therefore, working on exploring school mathematics concepts is seen positively for them: they welcome and “enter” well into the study and deep exploration of it. An entry through “mathematics” appears as a fruitful way to engage secondary teachers (Cooney & Wiegel, 2003). Third, even if obvious, one is led to realize that secondary mathematics teachers *can* learn a lot of mathematics. Hence, it is not about what they know or don’t know, but about the fact that their orientation, appreciation and knowledge and understandings of mathematics leads them to be able to know and learn more and make new sense of mathematical concepts (at a deeper level than *only* procedural). There appears consequently to be a need for us, as mathematics teacher educators, to seriously realize, harness and take advantage of this context in order to participate in the continuing growth of mathematics teachers mathematically, and pedagogically.

A last point: developing a practice and a culture of doing mathematics

Another critique to the issue of having teachers taking academic mathematics courses concerns the way in which these course are taught. As Bauersfeld (1998) explains, the usual way academic mathematics courses are taught is through modes of lecturing and exposing of mathematical knowledge. The ways of doing and habits developed in these courses are therefore more about "standardized knowledge" than about a participation in a *process of learning* that reflects teacher's (future) practices. For Bauersfeld, it is important to immerse teachers and have them participate in a culture of mathematics, rather than introducing them to a body of objective knowledge (where mathematics is an epistemological absolute). Participating in a culture of mathematics is participating in a culture that uses mathematics, that negotiate its meaning, that establishes norms and ways of doing in mathematics, and so on; a culture that enacts mathematics as a practice[2].

Hence, in addition to the importance of offering teachers opportunities to explore and make more sense and develop further knowledge of school mathematics concepts through mathematics teacher education practices, the *practices* in which teachers are embedded are fundamental. At the core of creating a mathematical culture is the idea of teachers becoming authors and producers of mathematical knowledge and understandings. Teachers need to be engaged in the practice of doing mathematics, in a culture of mathematics where concepts, notions, ideas and issues are explored and worked on: where teachers are encouraged to probe and dig into the concepts and their meaning; to generate ideas, questions and interrogations; to offer and share the understandings they have (and have developed); to offer explanations and to develop valid argumentations to support their points and reasoning; to share and develop different ways of understanding, theories, solutions, strategies, newly invented procedures or symbolisms, etc.; to negotiate meanings, between them and the other teachers and with the teacher educator; to assess and validate of other's understandings and explanations; and so on.

Concluding remarks

For us, these aspects bring a rethinking of the sorts of mathematical experiences and practices that teachers could/should be exposed to through mathematics teacher education practices. Our research interests and efforts are therefore invested in creating and offering alternative approaches that attempt to offer teachers these experiences of entering in a mathematical culture and developing and enriching their understanding and knowledge of school mathematics – an important shift, we believe, concerning the mathematical learning opportunities offered to teachers in mathematics teacher education practices. It is through this shift and alternative experiences that we feel that teachers will have the opportunity to continue growing in mathematics and enhance their ways of knowing and doing these mathematics, and consequently of developing alternative orientations toward mathematics and its teaching.

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[2] One could also be referred to studies on the culture of the mathematics classroom (e.g. Seeger, Voigt & Waschescio, 1998).

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