THE RELATIONSHIP BETWEEN TEACHERS' MATHEMATICAL KNOWLEDGE AND TEACHING IN CALCULUS

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In this paper we address some issues about the nature of secondary teacher mathematical knowledge in the area of Calculus. Our research data comes from observing classroom teaching of nine teachers and from interviews with them. In particular, we indicate some qualitative characteristics of the specialised mathematical knowledge for teaching that allow the teacher to create a rich mathematical communication in the classroom. These characteristics include teacher's ability to make connections between different mathematical areas; to be aware of the role of these connections in the discipline of mathematics; to extend these connections to other disciplines; to blend mathematical and pedagogical dimensions.

The last two decades a number of studies have attempted to define the nature and the components of teacher knowledge that is necessary for mathematics teaching. In most of these studies the investigation of this knowledge is based on the analysis of data that come from the actual teaching practice. It seems that the teacher knowledge required for teaching is rooted in the mathematical demands of teaching itself and it differs from the knowledge that a teacher has acquired in the formal education (Ball and Bass, 2003; Cooney and Wiegel, 2003). An initial characterization of teacher knowledge comes from Shulman's work (Shulman, 1986; 1987) who distinguishes three categories that describe teacher content knowledge, the *subject matter content knowledge*, the *pedagogical content knowledge* and the *curricular knowledge*. Although these categories are not specific to mathematics teaching, many researchers in mathematics education have used these as a framework of their work. In particular, the notion of pedagogical content knowledge which identifies the special kind of teacher knowledge that links content and pedagogy is used as the basis to define the specialized mathematical knowledge that the teacher needs. Ball and Bass (2000) used the term *mathematics knowledge for teaching* to capture the complex relationship between mathematics content knowledge and teaching. They also distinguished in this knowledge two key elements: "common" knowledge of mathematics that any well-educated adult should have and "specialized" mathematical knowledge that only teachers need to know (Ball, Hill and Bass, 2005). Rowland, Huckstep and Thwaites (2005) also constructed a framework, the *knowledge quartet*, to describe mathematics content knowledge of prospective primary school teachers. Mason (1998) elaborated further the notion of teacher knowledge and talked about three levels of awareness, awareness in action, in discipline and in counsel both in mathematics and in mathematics teaching.

There is a rather small number of studies that focus on mathematics secondary school teacher knowledge. For example, Even and Tirosh (1995) studied teachers' subject matter knowledge and knowledge about students in the case of function concept, discriminating each of these in terms of "knowing that" and "knowing why". An, Kulm and Wu (2004) also considered a network of pedagogical content knowledge and investigated the differences in teachers' pedagogical content knowledge between middle school mathematics teachers in China and the United States. Chinnappan and Lawson (2005) examined the connectedness of secondary school teachers' geometric knowledge by using as an analytic tool, the concept map. However, more studies are needed investigating teacher knowledge on topics included in the high-school curriculum and also identifying what constitutes this knowledge. In this case, the content knowledge is rather complex so we need more specific frameworks to describe this knowledge. Mason and Spence (1999) consider that mathematical and pedagogical knowledge constitutes not only knowing-that, knowing-how, knowing-why but also knowing to act and knowing to act in the moment. The last two elements are related to teacher's ability to recognise the appropriate method of approaching a particular mathematical situation, to evaluate teaching situations and to make on the spot decisions. Killpatric (2001) described five strands that define mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. Boaler (2003) talked about elements of a rich mathematical activity such as creativity, inquisitiveness, making connections, viewing mathematical representations dynamically.

Mason's, Killpatric's and Boaler's work attempt to define what we consider as quality of mathematics knowledge which goes beyond procedural and conceptual knowledge and requires deep understanding of connections in mathematics itself and between mathematics and other situations. In our study, we draw elements from the above characterisation to form a framework for analyzing the quality of teachers' mathematical knowledge concerning Calculus. In this paper, we will discuss further some of the issues that emerged from our data about the nature of this knowledge and its relation to teachers' practice and have been presented in Potari et al (2007)

METHODOLOGY

Our data was comprised of classroom observations, informal discussions before and after teaching and audiotaped semi-structured interviews. The researchers observed and took field notes from three teaching sessions on derivative conducted by each of nine teachers. The interviews focused on teachers' experience concerning mathematics and mathematics teaching; teachers' views about teaching and learning mathematics in general and calculus and derivative in particular; and teachers' interpretations of specific pedagogical actions that were identified during the observations. The analysis of the classroom data aimed at identifying elements of teachers' knowledge as they emerged from their practice. These elements were discussed in terms of our conceptual framework of teachers' mathematical knowledge. The analysis of the transcribed interviews was initially done vertically for each teacher and then horizontally across the nine teachers in order to identify general patterns and relations among the different elements of the knowledge.

RESULTS

The elements of teachers' mathematical knowledge that were identified from our analysis concerned their conceptual understanding, their procedural fluency, their ability to make connections, to prove and justify, their realization of the role of symbols and their ability to reflect and extend the mathematical activity. Two of these elements will be discussed in this paper and we will attempt to see what goes beyond the formal mathematical knowledge that someone with a mathematics degree should have and belongs to the specialised knowledge of teaching (Ball, Hill and Bass, 2005)

The quality of conceptual understanding: Teachers' conceptual understanding is necessary for teaching. However, the nature of this understanding needs further clarification. For example, understanding the tangent of a curve, a central concept that was considered in the observed lessons, goes beyond the formal definition and it includes different qualitative characteristics. Most teachers in our study did not seem to realize that the tangent of the circle is a particular case of the tangent of the curve. They gave as a starting example the tangent of the circle, an approach that was suggested in the school textbook, for reminding the students where they had met before this concept. Then, they introduced the tangent of a curve, they wrote its equation on the board without any further reference to the circle's tangent. In the interview, a teacher (teacher A) distinguished two types of tangents, one that has only one common point with the curve which is called "tangent of the curve" and another which has more than one common point and is called "tangent of the curve at a point". On the contrary, another teacher (teacher B) seemed to believe that there is possibly a relation between the tangent of the circle and the general notion of the tangent of a curve but she could not identify it. In the interview, she wondered: "Can we give a global definition for the tangent of a curve like in the case of circle? I have looked to find a definition as we say this is… but I haven't found it in the textbooks…". In the case of the tangent of the circle she recognized a global characteristic property- exactly one common point – while in the general case of the curve the definition is of a local nature as it refers to a specific point. These teachers seemed to have a rather fragmented view of the concept of tangent and they could not identify relations between the tangent of circle and of curve. This is an indication that their mathematical awareness has not reached the level of awareness in discipline. Teachers' realisation when a property is local or global and that a local property could become global in some special cases, as in the tangent of the curve, seemed to be crucial for the teaching of this concept and it is an example of the specialised mathematics knowledge for teaching.

The integration of mathematical and pedagogical knowledge: In most of the observed teaching sessions the mathematical activity remained at the level of action without further explorations at a meta-level. However, there were few cases where such explorations occurred. For example, a teacher (teacher C) often asked "how" and "why" questions in his teaching. He also encouraged the students to make conjectures and think about some uncommon cases. In one episode, the teacher asked the students to find the equations of tangents of a parabola in two different points. The slope of one of these tangents by using the derivative was found to be zero. The teacher asked the students to interpret this: "A lot of you tend to believe that when you find the slope to be zero something is wrong… What is the special for a straight line with slope zero?" The students made conjectures and the teacher asked them to think further about their validity. He also encouraged them to think and interpret what is happening when the slope becomes infinity and he used the metaphor as a way to motivate them that "this is food for thought". In the above example, teacher's mathematical knowledge is integrated to his pedagogical knowledge about his students' understanding. However, his teaching behaviour indicates teacher's ability to identify critical points in mathematics and also a problem solving view of mathematics that the teacher has. It appears that teachers' mathematical and pedagogical experiences cannot be two distinct forms of knowledge. The specialised mathematics knowledge for teaching allows for such integration.

Teachers' mathematical knowledge and mathematical communication in the classroom: The teachers in our study often attempted to create certain norms that would possibly allow a rich mathematical communication. They encouraged students to think about different solutions or how to prove certain formulas for calculating derivatives in case that they had forgotten. However, most of them conceived these norms at a practical level - the students had to find the shortest solution in order to save time in the exams – and emphasized in their teaching the development of skills. On the other hand, few teachers considered these norms as a way to develop students' understanding and extended the mathematical communication at a metacognitive level by asking the students to compare and evaluate their solutions. For example, teacher C when he revised students' homework in the classroom, he encouraged his students to describe different solution methods they had used and to discuss in the classroom their relations, possibilities and limitations. This teacher has the mathematical knowledge which allows him to reflect on different solution methods and to seek the important mathematical characteristics that differentiate these methods. This knowledge indicates teacher's awareness in the discipline of mathematics and allows him to transform classroom communication to a real mathematical communication.

Mathematics knowledge for teaching is content specific: This claim is illustrated by two examples. The pedagogical knowledge that one teacher (teacher D) developed especially during his postgraduate studies helped him to think and try to implement alternative teaching approaches in his teaching. This had an effect on the way that the teacher interacted with the students. However, this knowledge by being independent from teaching and learning mathematics was not enough to develop teacher's mathematical and pedagogical awareness in the specific area of calculus teaching. On the other hand, teacher C had attended at a postgraduate level a course on didactics of calculus and he considered the knowledge that he gained was crucial in his professional development. However, he could not "transfer" automatically this knowledge to other mathematical areas. Talking about relations between geometry and calculus, he considered that "geometry is only formulas and if the students know these, they can be successful … analysis has difficult concepts and understanding them is important for the students". In the case of geometry, this teacher's knowledge is the formal mathematical knowledge that he developed at school and university while in the case of calculus this knowledge has become specialised for teaching.

POINTS FOR FURTHER DISCUSSION

The mathematics knowledge that is required for mathematics teaching in secondary schools is beyond the knowledge that someone with a mathematics degree can have. This knowledge seems that it can be distinguished, as in the case of primary school teachers, in common and specialised mathematical knowledge (Ball et al, 2005). However, in the case of secondary education the common content knowledge can be defined as the knowledge that an adult with a mathematics degree can have. The specialised mathematical knowledge is the knowledge that allows the teacher to integrate effectively mathematical and pedagogical knowledge and create a rich mathematical environment in the classroom. It is characterised by teacher's ability to make connections between different mathematical areas; to be aware of the role of these connections in the discipline of mathematics; to extend these connections to other disciplines; to blend mathematical and pedagogical dimensions. From our data, it also appears that the specialised mathematical knowledge seems to be content specific and it requires to be developed parallel to the pedagogical content knowledge, something which has been addressed by other researchers in the area of mathematics teacher education (Cooney and Wiegel, 2003). More research is needed on teacher knowledge in different mathematical areas concerning secondary education that would allow us to examine in what ways existed frameworks on teacher knowledge can be used or extended. Another important issue that needs to be further investigated in secondary mathematics teacher education is the ways that the specialised mathematics knowledge for teaching can be developed.

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