WHAT KNOWLEDGE IS INVOLVED IN CHOOSING AND GENERATING USEFUL INSTRUCTIONAL EXAMPLES?[1]

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EXAMPLES AND TEACHER KNOWLEDGE

The knowledge teachers need in order to select and construct useful examples in their mathematics classroom involves sound mathematics and deep pedagogy. In this paper I will illustrate the complex considerations and sophisticated knowledge that is often required for making educated choices and generating appropriate examples for specific learning goals and contexts.

Examples are an integral part of mathematics and a significant element of expert knowledge (Rissland-Michener, 1978). They are essential for generalization, abstraction and analogical reasoning. Examples are inherently connected to explanations and mathematical discourse (Leinhardt, 2001). The construction of an explanation for teaching heavily relies on the specific choice of examples, as expressed by Leinhardt, Zaslavsky, and Stein (1990):

> Explanations consist of the orchestrations of demonstrations, analogical representations, and examples. […]. A primary feature of explanations is the use of well-constructed examples, examples that make the point but limit the generalization, examples that are balanced by non- or counter-cases. (p. 6)

Bills, Dreyfus, Mason, Tsamir, Watson and Zaslavsky (2006) indicate two main attributes that make an example pedagogically useful. Accordingly, an example should be transparent, that is, make it relatively easy to direct attention to the features that make it exemplary. It should also foster generalization, that is, it should highlight the necessary specific features that constitute an example of the target case, and at the same time point to the arbitrary and changeable features. Clearly, the extent to which an example is transparent or useful is subjective. The specific elements and representation of examples, and the respective focus of attention facilitated by the teacher, have bearing on what learners notice, and consequently, on their mathematical understanding. Thus, the role of the teacher is to offer learning opportunities that involve a large enough variety of 'useful examples' to address the diverse needs and characteristics of the learners.

In spite of the critical roles examples play in learning and teaching mathematics, we know rather little about teachers' choice and treatment of examples. Rowland, Thwaites and Huckstep (2003) identify three types of elementary teachers' poor choice of examples:

- choices of instances that obscure the role of variables (for example, in learning to tell time, using the example of half past six or in a coordinate system using examples of points with the same values for both coordinates);

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- choices of numbers to illustrate a certain arithmetic procedure when another procedure would be more sensible to perform for the selected numbers (for example, using 49x4 to illustrate conventional multiplication);
- randomly generated examples (for example, by a dice) when careful choices should be made.

Rowland et al's (2003) findings concur with the concerns raised by Ball, Bass, Sleep, and Thames (2005) regarding the knowledge base teachers need in order to carefully select appropriate examples. Not surprisingly, the choice of examples in secondary mathematics is far more complex and involves a wide range of considerations (Zaslavsky & Lavie, 2005; Zodik & Zaslavsky, 2007; Zaslavsky & Zodik, 2007).

The choice of examples presents the teacher with a challenging responsibility, requiring considerations of many competing features, especially since the specific choice and treatment of examples may facilitate or impede learning. Exemplification can be seen as core knowledge needed for teaching as well as a driving force for enhancing teachers' knowledge. It builds on and enhances teachers' knowledge of pedagogy, mathematics, and student epistemology.

From a mathematical perspective, an example must satisfy certain mathematical conditions depending on the concept or principle it is meant to illustrate; from a pedagogical perspective, an example needs to be presented in a way that conveys its 'message'; and from an epistemological perspective, it is necessary to be aware of what the students actually 'see' in an example (Mason and Pimm, 1984) and of the danger of over-generalizing or under-generalizing from examples.

In spite of the above, most mathematics teacher education programs do not explicitly address this issue and do not systematically prepare prospective teachers to deal with exemplification and particularly with the choice of instructional examples in an educated way. Thus, it seems that the skills required for effective treatment of examples is obtained mostly through one's own teaching experience and thus constitutes craft knowledge. Teachers' craft knowledge is acquired mostly over time through their experiences; to a large extent it is a-theoretical and idiosyncratic (Kennedy, 2002). By making sense of mathematics teachers' craft knowledge regarding treatment of examples we may gain insight into specific aspects of their knowledge that could be used as a basis for designing professional development activities that may facilitate teachers' construction of systematic knowledge. Zaslavsky and Zodik (2007) provide an analysis of experienced teachers' treatment of examples in terms of strengths and difficulties associated with exemplifications in the mathematics classroom.

AN ILLUSTRATION OF THE COMPLEXITY INVOLVED IN CONSTRUCTING USEFUL INSTRUCTIONAL EXAMPLES

Peled and Zaslavsky (1997) analyzed a collection of counter-examples that secondary preservice and inservice mathematics teachers generated for their (supposed) students, in order to convince them that the following statement was false:

<u>A False Statement</u>:

Two rectangles having congruent diagonals are congruent

This task represents a potential classroom situation in which a teacher deliberately presents a false statement or a student makes a false claim spontaneously. In either case, the need to provide a convincing counter-example arises. Peled and Zaslavsky's finding (ibid) indicate the possible difficulty teachers face in this situation: 25% (mostly the pre-service teachers) were not able to generate an adequate counter-example and instead gave an inadequate example that does not refute the statement.

In their analysis of the adequate counter-examples that teachers offered, Peled and Zaslavsky (ibid) differentiate three types of counter-examples, according to their explanatory power: *specific*, *semi-general* and *general* examples. The authors maintain that general and semi-general counter-examples offer some explanation and provide insight to why the statement is not true as well as ideas about how to generate more counter-examples to the same statement. Thus, generating such examples involves both mathematical and pedagogical considerations. For example, a general counter-example to the above statement could be the one shown in Figure 1 (taken from Peled and Zaslavsky, 1997). This counter-example communicates a general argument why the claim is false and how to generate an infinite number of different rectangles with the same diagonal.

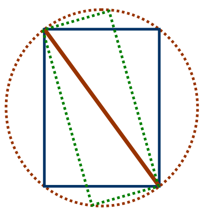

Figure 1: A *general* example of two non-congruent rectangles with the same diagonal

In spite of the seemingly advantages of a more general example for explaining and convincing, as discussed by Peled and Zaslavsky (ibid), if we enter student epistemology into the picture, there could be cases in which a specific example would be more convincing. Consider a student that needs hands-on experiences to better understand and be convinced. This hypothetical student may not be able to see through the general case and relate it to the statement in question. This student would be better convinced if he could actually draw on a grid paper two non-congruent rectangles and compare the lengths of their diagonals. For him to be able to draw a rectangle he would need to know its width and length, or in other words, the two length measurements of the sides of the rectangle. Apparently, the specific examples that teachers generated in Peled and Zaslavsky's study would not suffice for this purpose. Figure 2 depicts a typical specific counter-example that was suggested in their study:

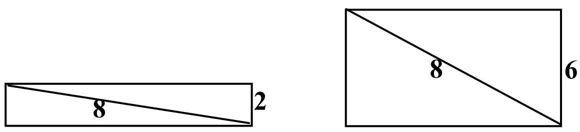

Figure 2: A teacher-generated *specific* example of two non-congruent rectangles with the same diagonal

Clearly, the teachers drew on a lot of their mathematical knowledge. They knew the Pythagoras Theorem, and could easily tell that these two rectangles are not congruent, since they differ in both sides: 2 & $\sqrt{60}$, and 6 & $\sqrt{28}$; They also knew that a rectangle is uniquely determined by its diagonal and one of its sides, and that its diagonal is longer than each of its sides. They could have given the lengths of the other two sides of the rectangles, but chose not to do so, because it involved irrational numbers.

Let us examine this example (Figure 2) in the eyes of our hypothetical student. This counter example is not helpful for him. He still cannot sketch these rectangles precisely on a grid paper. It may not seem as a plausible example because not all the measurements of the sides of the rectangles are given (there is evidence from Zaslavsky & Shir, 2005, that there are students that do not consider the diagonal of a square as part of the square).

In order to address the need of this particular student, it would be good to offer him a specific counter-example in which all lengths of the rectangles are "manageable" integers that can fit in a grid paper. This requirement would take into account student epistemology, and convey sensitivity to special students' needs. However, meeting this requirement presents the teacher with a much more challenging mathematical problem:

If a & b are *integer* lengths of one rectangle and c & d are *integer* lengths of another rectangle, find values for a, b, c, d that satisfy:

$$a^2 + b^2 = c^2 + d^2$$

A small investigation amongst expert mathematics teacher educators, including self reflection on my own solution process, points to the difficulty of finding appropriate integer solutions. One solution that would probably be helpful and manageable for our hypothetical student appears in Figure 3):

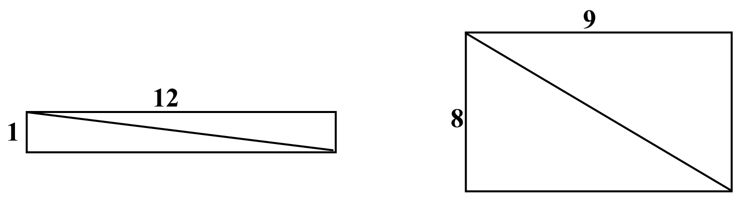

Figure 3: An expert-generated *specific* example of two non-congruent rectangles with the same diagonal

Now our student can easily and precisely draw these two rectangles on a grid paper, measure their diagonals, and within the limitations of measurement tools, realize that these are, in fact, two non-congruent rectangles with equal diagonals. Students, who do not need the hands-on experiences, can calculate the lengths of the diagonals and become convinced that they are equal.

CONCLUDING REMARKS

The above illustration conveys the complexity involved in choosing and generating useful counter-examples for disproving a false claim. Similar considerations may apply to other kinds of examples.

This illustration sheds light on the interplay between mathematical and pedagogical knowledge and adds to the picture the need for attending to special students' needs, in the context of examples.

One of the main questions this illustration raises is how to prepare secondary mathematics teachers to be "example literate" and to be able to deal with the demands that instructional examples present. It also raises the need to further study experienced teachers in order to identify the strengths and weaknesses of the craft knowledge they develop with respect to treatment of examples.

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