# PROCESSES AND PRODUCTS, STRUCTURES AND MEANING IN MATHEMATICS CLASSROOM: SOME SNAPSHOTS FROM THE LAST CENTURY

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*In my contribution I will consider some moments of the evolution of mathematics teaching in the last century, putting into evidence the influences on it, which derived from epistemology and psychology. Some snapshots will illustrate how important changes of priorities in mathematics teaching in school have been more or less directly inspired by dominating positions in those disciplines. In particular, I will discuss the relative importance of processes and products, structures and meanings in mathematics classroom, inspired by different epistemological and cognitive references. The aim of the contribution will be to show the necessity of developing mathematics education as a relatively autonomous scientific discipline, i.e. a research space where to tackle teaching and learning problems with its own tools, as well as tools coming from other disciplines, critically considering their potential and limits, and their consequences on the solution of those problems.*

## SOME INTERESTING CASES

We consider some cases in which the influence of epistemological and psychological theories on the teaching of school mathematics has been strong in the last century. Here, like in the other Sections of this text, essential data for references will be provided.

### The case of formalism and logicism

According to the epistemological work of some mathematicians (see D. Hilbert' "Grundlagen der Geometrie") and philosophers (like G. Frege and B. Russel: see B. Russel's "Philosophy of Mathematics"), one century ago the image of mathematics as a pure formal construction, shaped according to logical rules, spread across the western culture; reference to concrete objects and physical relationships was reduced to a matter of heuristics or applications.

The influence on secondary school mathematics was relevant in continental Europe, though not entirely coherent with those epistemological premises: indeed it encouraged teachers to behave as if abstraction, formal calculations and rigorous deductions were the core of the image of mathematics to be conveyed to students, and the crucial aspects of mathematics to be evaluated in students' performances. The previous Platonist view of mathematics, so widely spread in culture and the school system, mixed up (in spite of fundamental epistemological differences) with the new epistemological orientations. Mathematical structures and logical features of products (e.g. proofs) became much more important than meanings and production processes at the teachers' eyes. Applications of mathematics lost their importance in curricula (with the exception of technical schools).

### The case of modern mathematics

During the sixtieths and the seventieths the wave of "modern mathematics" was highly influential in primary school mathematics teaching in several countries of the world. This happened both in countries that adopted new national programs inspired by the ideas of "modern mathematics" (like France), and in countries where no change of that kind happened (like Italy).

Piaget's theory and the Bourbaki mathematics offered psychological and mathematical ideas for the development of a universal, structural, context-free teaching of basic mathematics (specially for arithmetic in the first grades: sets and relations as key ideas for the approach to natural numbers and arithmetic operations). Mathematics educators and some mathematicians engaged in providing teachers with more or less appropriate and pertinent tools to teach mathematics according to the new ideas (for instance, the use of Venn's diagrams, graphs, etc. was proposed in "pilot textbooks" to illustrate arithmetic concepts and relations, and then spread across all textbooks). In some countries (Belgium, France, Italy, etc.) some mathematicians directly engaged in the production of "pilot textbooks".

In the long run of mathematics education in primary and lower secondary school, modern mathematics tends to privilege structures over meanings, and products over processes. We can say that while the teaching of basic arithmetic had not been greatly influenced by formalism and logicism, modern mathematics allowed to fill the gap between the teaching of basic arithmetic in primary school and the teaching of formal mathematics in secondary school.

The widespread persistence of some aspects of "modern mathematics" in the first grades of primary school (in spite of the evolution of national programs, epistemological orientations and psychological theories) can be explained not only through the inertia of the school system, but also according to the simplification brought by the set approach to natural numbers and arithmetic operations: to count objects and to represent addition as union of two disjoint sets is easier for students and teachers than considering the variety of the uses of natural numbers and the variety and complexity of problem situations needing an addition. In several countries the long term consequences of that simplification on students' mathematical performances are not taken into account.

### The case of constructivism

More recently, in several countries the development of constructivism in mathematics education shaped some prescriptions for primary school curricula, spread across teacher preparation and more or less directly influenced teachers' educational choices.

It is well known that constructivism is not a homogeneous theoretical entity in the domain of psychology: the word "constructivism" covers many theoretical orientations (from Piaget's constructivism to von Glasersfeld's "radical constructivism"; from individual constructive adaptation to challenging situations, to social constructivism).

In mathematics education different theoretical contributions from psychology "constructivisms" were re-elaborated according to the specificity of the content field and developed into a pervasive, multifaceted educational orientation where it is not easy to recognize the original references. In spite of the variety of underlying theoretical positions, some common features characterise teaching practices inspired by constructivism: students must play an active role in the individual (or social) construction (or re-construction) of mathematical knowledge, the main role of the teacher being to build and manage the scenario where students play their parts; the "adaptive" construction of meanings rooted in problem situations is the main aim (while structural analogies are discovered and investigated afterwards); processes are intended as carriers of meanings, while their products are seen only as their final outcomes to be "institutionalised". No dialectic tension is usually considered between products and processes, or between structures and meanings.

Some of these features of constructivism in mathematics education on one side reveal the lack of robust epistemological elaboration concerning mathematics (which could depend on the psychological sources of constructivism in mathematics education), and on the other could explain the difficulty to achieve several standard goals in students' mathematical preparation.

## SOME REFLECTIONS ON THE ABOVE CASES

Psychological and epistemological investigations do not work (as their main aim) for a better learning of mathematics. In spite of this fact we have seen how the development of epistemological and psychological theories has a more or less direct and coherent influence on the teaching of mathematics. I would like to discuss why and how it happens.

Epistemological theories are aimed at describing and framing some aspects of mathematics, while psychological theories are intended to describe, interpret and, possibly, predict learners' laboratory behaviour on a given area of paradigmatic tasks. Validity is not a universal and a-temporal character of epistemological and psychological theories: we can find different theories according to the concerned field of mathematics, the objects of investigation, the historical period. It is also necessary to consider the fact that epistemological and psychological theories (and we could add: mathematical theories as well!) do not develop in the vacuum. Since the Greeks, their birth and development is influenced by contemporary scientific and philosophical culture, which reciprocally can receive contributions from them.

These considerations partly explain why the teaching of mathematics is so strongly influenced by epistemological and psychological theories: the cultural environment acts not only as an inspiring source for ideas in epistemology and psychology, but also as a multiplier of specific hints coming from those disciplines when they are "received" by mathematics educators and teachers. However, in the reality of the school teaching of mathematics, what comes from mathematics, epistemology and psychology is mediated by the complex school culture (textbooks, materials, tradition, programs…). If we adopt the Chevallard's term of "noosphere" (see Chevallard's "La transposition didactique") to designate the system of institutions and people who manages the relationships between mathematicians' mathematics and taught mathematics, in general we can see how processes in the noosphere are sensitive to external influences (coming from politics, culture, etc) but they develop with a relative autonomy and inertia.

In spite of autonomy and inertia, those members of the noosphere that have special responsibilities in teachers' preparation and curriculum development (in particular, researchers in mathematics education) frequently act as if some epistemological and psychological theories would carry the truth about what mathematics is, and how students learn it. Frequently they assume an important role in "transposing" those theories in the school system, mainly through teachers' professional preparation. The consequences of the substantial lack of autonomy of mathematics education from epistemological and psychological theories can provoke some phenomena that are evident in the history of the teaching and learning of mathematics in the last three decades in the USA: if we consider the importance of proof and proving in the NCTM standards and compare standards issued in 1990 with those issued in the year 2000, we see a dramatic change from a substantial marginalization of proof to a restored centrality of it. Specific epistemological influences advocating the "death of proof" contributed to determine the orientation of 1990 Standards, while the mathematicians' pressures were influential on 2000 Standards. And recent "math wars" can be read in terms of conflicts between one part of the mathematicians' community, on one side, and mathematics educators sensitive to the influence of constructivism, on the other, in a cultural and political situation where ideological pressures on education and traditional educational values frequently assume political relevance.

## TOWARDS AUTONOMY AND IDENTITY OF MATHEMATICS EDUCATORS

Given the above analysis, what should be the task of mathematics educators (researchers in mathematics education, teachers' educators, curriculum developers, etc.)? I do not think that mathematics educators can develop a completely autonomous and autarchic science of the teaching of mathematics in school. This is an illusion for two reasons: on one side, teachers come from a given school or university mathematics culture and are embedded in a given cultural environment, and mathematics educators are prepared in given cultural institutions; thus it is not possible to ignore what teachers and mathematics educators know and think about the teaching and learning of mathematics. On the other, if mathematics educators want to go beyond mere descriptions of what happens in the mathematics classroom they need to consider what mathematics is, and how mathematics is appropriated by student; thus they need to deal with scientific results coming from epistemology and psychology.

The unavoidable reference to epistemology and psychology can be denied or underestimated, but in that case what usually happens is that implicit assumptions are made, or explicit assumptions are assumed as unquestionable truth. Some didactical theories are intended to play an autonomous role, but let us consider the example of Brousseau's theory of didactical situations (T. D. S., originating from the aim to develop an autonomous field of research concerning what happens in real mathematics classes: cf Brousseau's articles in Recherches en Didactique des Mathematiques, 1980; 1986; and his book on the "Theory of didactical situations"): we can recognize that learning is assumed to happen according to a mechanism of Piagetian "adaptation", and some other aspects of the T.D.S. reflect specific Piagetian hypotheses. In particular the distinction between "action" situations and "formulation" situations reflects the role attributed by Piaget to language; the direct cultural intervention of the teacher only in the phase of "institutionalisation" of knowledge built by students reflects what Piaget writes when he attributes to the teacher the only cultural role to establish links between children's constructions and cultural traditions. And the Bourbaki mathematics is an implicit underlying epistemological reference for many aspects of the T.D.S. (including the lack of specific elaborations for mathematical modelling).

The problem is what choices to make and how to perform them, keeping into account the variety of results and perspectives provided by epistemology and psychology. In my opinion the task of mathematics educators is not to choose an epistemological position or a psychological theory as an "all purpose" and universal reference (each outstanding epistemological position being culturally situated, each psychological theory having a limited domain of validity). In my opinion, what mathematics educators can do is to identify important teaching and learning problems, consider different existing theories and try to understand their potential and limitations in order to tackle the identified problems. However his statement is still vague for two reasons. First, to identify important teaching and learning problems requires some preliminary theoretical assumptions concerning the importance of the competence at stake and the way to ascertain related learning difficulties. Second, it is necessary to point out some keys (suggested by epistemological and psychological analyses) to avoid a dispersive view of the whole panorama of the teaching and learning of mathematics. A dialectic process should be developed: our epistemological and psychological culture together with our knowledge of what happens in school suggest to consider specific educational problems; in order to tackle those problems we need to identify appropriate tools from epistemology and psychology (and, in some cases, history of mathematic, sociology, etc.). It may happen that such tools allow us to re-formulate the original educational problems, or to identify further related problems.

In the title I have considered two couples of terms (processes and products, structures and meanings) that I consider interesting if we want to deal with some important teaching and learning problems. Drawing teachers' attention to the relationships between products and processes in the mathematics classroom means to allow them to consider their relative importance in mathematics and in the teaching and learning of mathematics, and tackle the problem of the tension that must be nurtured between them. For instance, in the case of proof students need to learn to produce conjectures and build their validation, with an eye to the cultural characters of the products to be achieved (statements and proofs). Considering structures and meanings should help teachers to become aware of the importance of structural facts arising from the comparison of different mathematical domains or constructions together with the relevance of meanings rooted in the specific, different contexts of use of mathematical notions.

Once we consider processes and products, structures and meanings, we can realize why we need epistemological and psychological theories to frame investigations concerning the relationships between them: what tools existing epistemological and psychological theories can provide us with, in order to describe, interpret and manage the tension between constructive processes and the cultural requirements of their products, or the tension between the discovery and formalization of structural analogies, and the rooting of meanings in specific situations?

When dealing with specific mathematics teaching and learning problems, we must recognize that in many cases existing tools elaborated by epistemology, psychology, sociology, etc. need to be adapted and re-elaborated. This is a first ground where the (relative) autonomy of mathematics educators can be exercised. A second ground concerns the need of new specific tools related to specific features of mathematics and mathematical activity. For instance, the need for the construct of the "didactic contract" (proposed by G. Brousseau) or the construct of "socio-mathematical norms" (proposed by the group leaded by P. Cobb) is not so evident and relevant in other disciplines.