

## **Change of the meaning of mathematical objects due to the passage between their different representations**

### **How other disciplines can be useful to the analysis of this phenomenon**

Bruno D'Amore – Martha Isabel Fandiño Pinilla  
NRD – Department of Mathematics – University of Bologna - Italy

#### **1. Episodes**

In D'Amore (2006) and in D'Amore, Fandiño Pinilla (2007), we have reported and discussed, exclusively from a semiotic structural point of view, episodes taken from classroom situations in which students are mathematics teachers in their initial training, engaged in facing representations problems. The task consisted in this: working in small groups the trainee teachers received a text written in natural language; such texts had to be transformed into algebraic language. Once they had come to the algebraic formulation, this was explained by the group and collectively discussed. Our duty as university teachers was to suggest the further transformation of the obtained algebraic expressions into other algebraic *expressions*, to face collective discussions on their meaning. We present below three examples:

##### *Example 1*

[We omit the original linguistic formulation which, in this case, is not relevant];  
the final algebraic formulation proposed by group 1 is:  $x^2+y^2+2xy-1=0$ , which in natural language is interpreted as follows: «A circumference» [the interpretation error is evident, but we decide to pass over]; we carry out the transformation which leads to:  $x+y=\frac{1}{x+y}$  that after a few attempts is interpreted as «A sum that has the same value of its reciprocal»;

question: «But  $x+y=\frac{1}{x+y}$  is it or not the “circumference” we started with?»;

student A: «Absolutely no, a circumference must have  $x^2+y^2$ »;

student B: «If we simplify, yes».

One can ask whether or not it is the transformation that gives a *sense*: from the episode it seems that if one would perform the inverse passages, then one would return to a “circumference”. But it could also instead be that the meanings are attributed to the specific representations, without links between them, as if the transformation that has a sense for the teacher has not one for who performs it.

##### *Example 2*

The text written in natural language requires the algebraic writing of the sum of three consecutive natural numbers and the proposal of group II is:  $(n-1)+n+(n+1)$  [obviously the doubt remains in the case of  $n=0$ , but we decide to pass over]; we carry out the transformation that leads to the following writing:  $3n$  that is interpreted as: «The triple of a natural number»;

question: «But  $3n$  can be thought as the sum of three consecutive natural numbers?»;

student C: «No, *like this* no, *like this* it is the sum of three equal numbers, that is  $n$ ».

##### *Example 3*

We consider the sum of the first 100 natural positive numbers:  $1+2+\dots+99+100$ ; we perform Gauss classical transformation;  $101\times 50$ ; this representation is recognized as the solution of the problem but not as the representation of the starting object; the presence of the multiplication sign compels all the students to look for a sense in mathematical objects in which the “multiplication” term (or similar terms) appears;

question: «But  $101\times 50$  is it or not the sum of the first 100 positive natural numbers?»;

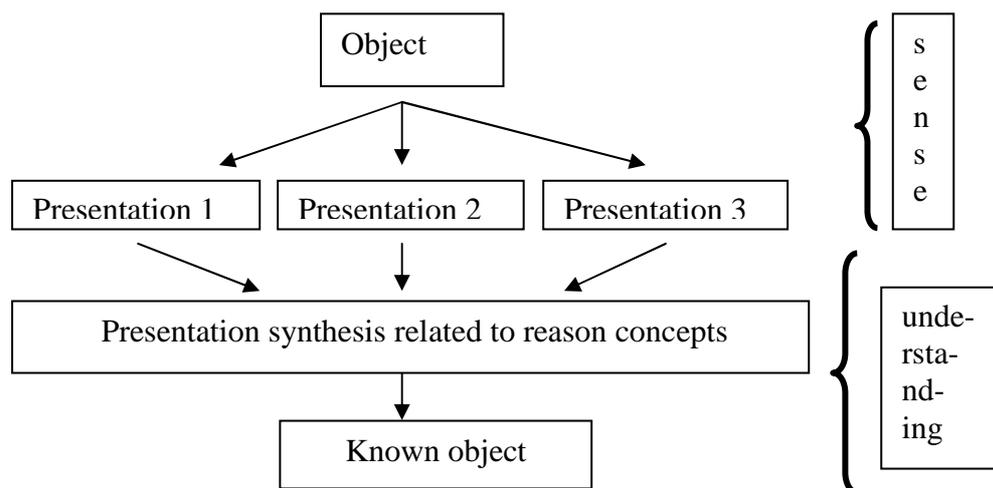
student D: «That one, is not a sum, that is a multiplication; it corresponds to the sum, but it is not the sum».

In these episodes we witness a constant change of meaning during the transformations: each new representation has a specific meaning of its own not referable to the one of the starting representations, even if the passage from the first to the second ones has been performed in an evident and shared manner.

## 2. The causes of the changes of meaning

What are the causes of the changes of meaning, what origin do they have?

We can start from this diagram that we appreciate a lot because of its attempt to put in the right place the ideas of *sense* and *understanding* (Radford, 2004a).



The process of meanings endowment moves at the same time within various semiotic systems, simultaneously activated; we are not dealing with a pure classical dichotomy: treatment/conversion, that leaves the meaning prisoner of the internal semiotic structure, but with something much more complex. Ideally, from a structural point of view, the meaning should come from within the semiotic system we are immersed in. Therefore, in *Example 2*, the pure passage from  $(n-1)+n+(n+1)$  to  $3n$  should enter the category: treatment semiotic transformation. But what happens in the classroom practice, and not only with novices in algebra, is different. There is a whole path to cover, that starts from single specific meanings culturally endowed to the signs of the algebraic language ( $3n$  is the triple of something;  $101\times 50$  is a product, not a sum). Thus, there are sources of meanings relative to the algebraic language that anchor to meanings culturally constructed, previously in time; such meanings often have to do with the arithmetic language. From a, so to speak, “external” point of view, we can trace back to seeing the different algebraic writings as equisignificant, since they are obtainable through semiotic treatment, but from inside this vision is almost impossible, bound as it is to the culture constructed by the individual in time. In other words we can say that students (not only novices) turn out bridled to sources of meaning that cannot be simply governed by the syntax of the algebraic language. Each passage gives rise to forms or

symbols to which a specific meaning is recognised because of the cultural processes THROUGH which it has been introduced.

In Luis Radford's semiotic anthropological approach (ASA) mathematical knowledge is seen as the product of a reflexive cognitive mediated praxis. «Knowledge as cognitive praxis (*praxis cogitans*) underlines the fact that what we know and the way we come to know it are underlaid by ontological positions and by cultural processes of meaning production that give form to a certain way of rationality within which certain types of questions and problems are posed. The *reflexive* nature of knowledge must be understood in Ilyenkov's sense, that is, as a distinctive component that makes cognition an intellectual reflexion of the external world in accordance with the forms of individuals' activity (Ilyenkov, 1977, page 252). The mediated nature of knowledge refers to the role played by tools and signs as means of knowledge objectification and as instruments that allow to bring to a conclusion the cognitive *praxis*» (Radford, 2004b, page 17).

On the other hand, «the object of knowledge is not filtered only by our senses, as it appears in Kant, but overall by the cultural modes of signification (...). (...) the object of knowledge is filtered by the technology of the semiotic activity. (...) knowledge is culturally mediated» (Radford, 2004b, page 20). «(...) These terms are the semiotic means of objectification. Thanks to these means, the general object that always remains directly inaccessible starts to take form: it starts to become an "object of consciousness" for the pupils. Although general, these objects however remain *contextual*» (Radford, 2004b, page 23).

The approach to the object and its appropriation on the part of the individual who learns, are the result of personal intentions with which individuals express themselves through experiences that see the objects used in suitable contexts: «Intentions occur in contextual experiences that Husserl called *noesis*. The conceptual content of such experiences he termed *noema*. Thus, noema corresponds to the way objects are grasped and become known by the individuals while noesis relates to the modes of cultural categorial experiences accounting for the way objects become attended and disclosed (Husserl, 1931)» (Radford, 2002, page. 82).

In the cases we presented above, and in general in mathematics, it is specific that the objects are attended from the first moment in their formal expression, in our case in the algebraic language; the individual learns to formally handle these signs, but what happens to the starting mathematical object? What happens to the starting meanings? We suppose that these meanings are tightly bound to the arithmetic experience of the pupil and overall to the way in which such an experience becomes objective through its objective transposition into ordinary language. Deep understanding of algebraic or, more in general, formal manipulation holds a prominent position.

Through an interesting comparison, Radford expresses himself on this point as follows: «While Russell (1976, page 218) considered the formal manipulations of signs as empty descriptions of reality, Husserl stressed the fact that such a manipulation of signs requires a shift of intention, a noematic change: the focus becomes the signs themselves, but not as signs *per se*. And he insisted that the abstract manipulation of signs is supported by new meanings arising from rules resembling the rules of a game (Husserl 1961, page 79), which led him to talk about signs having a *game signification* (...)» (Radford, 2002, page 88).

After having shown the broad and complex significance of the phenomenon, we must refer to other disciplines in order to understand better and better the issue of the different meanings of algebraic expressions, that is, in order to give a significant contribution to this aspect of mathematics education.

### **3. Analysis of the phenomenon thanks to theories "external" of mathematics education**

We believe that some theories "external" of mathematics education can have, and that in fact they already have, a strong influence on the analyses of various phenomena, like the ones here described,

therefore giving a contribution to changing the theoretical frame of our discipline, in its future research developments.

**Philosophy.** We have seen in section 2 how philosophy (Husserl's phenomenology) is able to give a remarkable contribution and we will not repeat ourselves.

Learning is taking consciousness of a general object in accordance with the modes of rationality of the culture one belongs to.

**Sociology.** In D'Amore (2005), D'Amore and Godino (2007), we show how the results of the analyses relative to the behaviours of individuals engaged in an activity of conceptual learning of mathematical objects, their transformations of the descriptions of such objects from ordinary language to formal language, the manipulations of such formalizations can be framed within a sociological interpretation key: the learning environment is framed within a sociological interpretation key and the individuals' behaviours are interpreted through the notion of "practice" and its "meta-practice" evolution. Essentially the individuals shift from a shared practice, recognized as characteristic of the social group they belong to, to a meta-practice that modifies such characteristic; the interpretative behaviour therefore ceases to be global and social and becomes local and personal; the notions that come into play in such interpretations are specific of the occasion and not of the situation in its entirety.

We pass over this point, referring back to the quoted texts.

**Anthropology.** In D'Amore and Godino (2006, 2007) we go into strongly anthropological details in order to explain the nature of the choices of the individual who learns mathematics. In such articles we highlight how «Having obliged the researcher to point all his attention to the activities of human beings who have to do with mathematics (not only solving problems, but also communicating mathematics) is one of the merits of the anthropological point of view, inspiring other points of view, amongst which the one that today we call "anthropological" in the proper sense: the ATD, anthropological theory of didactics (of mathematics) (Chevallard, 1999; page 221). Why this adjective "anthropological"? It is not an exclusiveness of the approach created by Chevallard in 80s, as he himself declares (Chevallard, 1999), but an "effect of the language" (page 222); it distinguishes the theory, identifies it, but it is not peculiar to such theory in a univocal way » (D'Amore, Godino, 2006, page 15). The ATD is almost exclusively centred on the institutional dimension of mathematical knowledge, as a development of the research program started with fundamental didactics. The crucial point is that «ATD places the *mathematical* activity, and therefore *the study* in mathematics activity, *in the set of human activities and of social institutions*» (Chevallard, 1999).

This kind of analyses, although subjected to criticisms in D'Amore, Godino (2006, 2007), has opened the way to the use of anthropology as a critical instrument, as a new theoretical frame at research into mathematics education, in accordance with what has been already highlighted in the above quoted articles. It is the human being, strong of the acquired culture, strong of the specific expressive, communicative luggage, who handles formal writings and gives them a meaning that it cannot be anything else but coherent with his social history; every meaning of each formal expression is the result of an anthropological comparison between a lived history and a here-and-now that must be coherent with that history.

We pass over this point, referring back to the quoted texts.

**Psychology.** In D'Amore and Godino (2006) we show how the shift from the anthropological vision to the onto-semiotic one is made necessary (amongst other things) by the need of not trivializing the presence of psychology in the study of learning and, in general, classroom situations. In D'Amore (1999) we show, for example, how ideas on representation drawn from psychology, regarding the explanation of the passage from image (weak) to model (stable) of concepts (Paivio,

1971; Kosslyn, 1980; Johnson Laird, 1983; Vecchio, 1992), can be placed as a unitary basis of the explanation of several didactic phenomena, as intuitive models, the shift from internal to external models, the figural concepts, up to misconceptions, studied mainly in the 80s. Also the ideas of frame and script (Bateson, 1972; Schank, Abelson, 1977) have been used for the same purpose.

## References

- Bateson G. (1972). *Steps to an Ecology of Mind*. Novato Ca.: Chandler Publishing Company.
- Chevallard Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques* (Grenoble, France). 19, 2, 221-266.
- D'Amore B. (2005). Pratiche e metapratiche nell'attività matematica della classe intesa come società. Alcuni elementi rilevanti della didattica della matematica interpretati in chiave sociologica. *La matematica e la sua didattica* (Bologna, Italy). 3, 325-336.
- D'Amore B. (2006). Objetos, significados, representaciones semióticas y sentido. In: Radford L., D'Amore B. (eds.) (2006). *Semiotics, Culture and Mathematical Thinking*. Special issue of the journal *Relime* (Cinvestav, México DF., México). 177-196.
- D'Amore B., Fandiño Pinilla M.I. (2007). How the sense of mathematical objects changes when their semiotic representations undergo treatment and conversion. *La matematica e la sua didattica* (Bologna, Italy). Vol. 21, n. 1, pagg. 87-92. Proceedings of : Joint Meeting of UMI-SIMAI/SMAI-SMF: *Mathematics and its Applications*. Panel on Didactics of Mathematics. Department of Mathematics, University of Turin. July 6, 2006.
- D'Amore B., Godino D.J. (2006). Punti di vista antropologico ed ontosemiotico in Didattica della Matematica. *La matematica e la sua didattica* (Bologna, Italy). 1, 9-38.
- D'Amore B, Godino D.J. (2007). El enfoque ontosemiótico como un desarrollo de la teoría antropológico en didáctica de la matemática. *Relime* (Cinvestav, México DF., México). Accepted, forthcoming.
- Husserl E. (1931). *Ideas*. London (U. K.): The Macmillan Company.
- Husserl E. (1961). *Recherches logiques*. (Recherches I et II). Paris (France): PUF.
- Ilyenkov E.V. (1977). *Dialectical Logic*. Moscow (URSS): Progress Publishers.
- Johnson-Laird P.N. (1983). *Mental Models*. Cambridge Ma.: Cambridge Univ. Press.
- Kosslyn S.M. (1980). *Image and Mind*. Cambridge Ma.: Harvard Univ. Press.
- Paivio A. (1971). *Imagery and Verbal Processes*. New York: Holt, Rinehart and Winston.
- Radford L. (2002). On heroes and the collapse of narratives: a contribution on the study of symbolic thinking. In: Cockburn A.D., Nardi E. (eds.) (2002). *Proceedings of the 16<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education, PME 26*. Vol. 4, 81-88.
- Radford L. (2003). Gestures, speech and the sprouting of signs. *Mathematics Thinking and Learning* (Florence, Kentucky, USA). 5(1), 37-70.
- Radford L. (2004a). Cose sensibili, essenze, oggetti matematici ed altre ambiguità. *La matematica e la sua didattica* (Bologna, Italy). 1, 4-23.
- Radford L. (2004b). La généralisation mathématique comme processus sémiotique. In: Arrigo G. (ed.) (2004). Proceedings of the Ticino Mathematics Education Conference. Quaderni Alta Scuola Pedagogica. Bellinzona (Switzerland): Centro didattico cantonale. 11-27.
- Russell B. (1976). *An Inquiry into Meaning and Truth*. London (U.K.): G. Allen and Unwin.
- Schank R.C., Abelson R.P. (1977). *Scripts plans goals and understanding*. Hillsdale N.J.: Lea.
- Vecchio L. (1992). Breve storia delle ricerche sull'immagine mentale. In: Vecchio L. (ed.) (1992). *Le immagini mentali*. Proceedings of the homonymous Meeting of Pavia (October, 1990). Firenze, Italy: La Nuova Italia. 15-48.