The determination of mathematical objects of didactical activities

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Various theoretical constructs coming from epistemology, psychology, semiotics, ethnomathematics, and philosophy, influence maths education in its theoretical orientations and practical implementations. They affect didactical knowledge, for instance concerning the range of activities that favour the emergence of students' knowledge about addition; they afford tools for analysing teaching learning situations like linguistic tools, they fertilise maths education with fields of study as argumentation, instruments, virtual activity, gestures.

Bearing in mind didactical research purposes, I wish to argue about how the determination and the borders of our mathematical objects of teaching may depend on the perspective from which we analyse them For us, researchers, in relation to our theoretical frames, and for the students, because of the opportunity to construct different "meanings" that teacher's didactical choices can favour. I will consider the epistemological and the cognitive perspectives ("cognitive" intended in its broad sense: individual cognition as well as social cognition). I will question the boundaries of some mathematical concepts and, *dually, give some hints about reasoning in mathematical activity*. I shall use the vague expression "mathematical object" to include *notion, concept*, *reasoning* and other *mathematical activities* we wish student to develop.

Starting with epistemological analyses

When aiming at didactical transposition, epistemological analysis of objects of knowledge is needed to guide us towards more dense meanings than the mathematical ones that our own mathematical experience has formed in us. But an epistemological analysis is not unique, nor is it shared by all of us in the same way. And didactical transposition depends on the epistemological perspective. Let us consider natural numbers, to illustrate our argument.

1) <u>A first epistemological analysis</u>

Some epistemological analyses refer essentially to mathematical structures and consider only cardinal and ordinal foundations of integer numbers. Thus the didactical transposition of cardinality and order will be central in didactical settings concerning numerical situations. Generally, measure is introduced as a search of "universal" unity system (use a rubber to evaluate a table's length, find a common unit to compare various tables' lengths), gradually conducting to a third aspect of number, that will allow the construction of decimal or rational numbers. With these three components the object "number" is covered in primary school, which is quite sufficient from the point of view of its mathematical structure. The two first components will also be used to construct the decimal system of integer number writing (through grouping of sets of ten units, sets of ten tens, etc.).

In this example, mathematical objects are analysed as structured constructs. Written mathematics is the main reference (compared to mathematician’s activity) and mathematical constructions in school context tend to approach the elaboration of a mathematical text.

2) Socio-cultural extension of the epistemological analysis

Enlightened by ethnomathematical studies, we may consider a socio-cultural epistemological perspective. We can interpret the “meaning” of mathematical objects through their cultural embedding (or "roots"). Then cardinal, order and measure do not suffice to give “all” the meanings numbers do bear in our culture. And therefore they may not be sufficient to help students catching mathematical objects with their own proper cultural means that they developed within everyday life, nor with the concept’s whole operational potential.

As a matter of fact, a number can also be understood as a value. This “meaning” is not formalised in a mathematical structure, but it is echoed in the money system and is profoundly rooted in many cultural contexts. In this second perspective, grouping and exchanging procedures, and the corresponding symbolic system, can be rooted in the “value meaning”. The priority is no more given to a system of signs organised by cardinality and ordinality, but to the continuity with common social practices. In the first perspective the idea of value is only potential in the exchange and grouping procedure, but is not intentionally exploited as a meaningful basis. Constructing numbers’ writing by transposing the cardinal and ordinal explanation of the organisation of signs is a very different didactical perspective from explaining this organisation in relation to socio cultural practices and conventions (that will also need a didactical transposition, by the way!). The meaning of “1” in “15” can be presented as the symbolic representation of a group of 10 units. It can also be based on an idea of value: the value of “1” can be read with reference to the value of “1” in the price “15 Euro”, conventionally determined (in social practices) as ten “one Euro” coins. This approach corresponds to the child's everyday socio-cultural experience. Exploring this potential familiarity may give the student an opportunity of basing (at least partly) the number concept on easily attainable meanings. A coherent didactical consequence would be to realise and exploit the transposition of these meanings. And this implies that the contour of what is an integer has been modified.

We could develop similar analyses of reasoning and mathematical reasoning, and their developments in school context. The reasoning the student will be able to attain (because of meaningfulness) and to develop will vary… if we allow students to elaborate reasoning containing arguments driven from experiences, which the mathematical objects (like value, above) are based upon. Thus, both referring to grouping and designating grouping (cardinality and order), and referring to value and socio-cultural uses of money, might be acceptable arguments at the beginning of primary school. Students' allowed mathematical reasonings might depend on researchers' epistemological choices.

Epistemological analysis can also be opened to body references. For instance, teaching the concept of angle may be related to the bodily experience of verticality and inclination (M. Serres' history of early elaboration of the concept enhance the long lasting interdependence of the definitions of angle and of inclination, and their implicit relation to body's verticality).

Cognitive analyses

Cognitive and developmental psychological theories also influence the determination of the objects of teaching, thus the objects of transposition and the didactical choices.

1) <u>Vergnaud's "conceptual fields", and related cognitive phenomena.</u>

Vergnaud defines a "concept" through its three components. The set of its reference situations (opening the way to cultural roots of concepts, and to the impact of experience); the set of its operational invariants (allowing to consider purposeful activity and the existence of non explicit elaborations of a concept); and the set of its linguistic representations. And, he invites us to consider conceptual fields, rather than specific mathematical concepts. Mathematical conceptual field's notions and concept components are linked through mathematical activity in a wide sense. From this cognitive point of view, mastering mathematical objects cannot be reduced to the mastery of its representation means (symbols, linguistic expression, schemata) and the rules of reasoning that ensure coherency to the representational system. The mathematical object we determine, in a didactical perspective, expands to references and (dually) *to purposeful activity that supports it and carries its "meanings"*. Activity reflects operational invariants when they bear, implicitly or not, mathematical properties.

Thus, in a didactical perspective, the object of knowledge can be seen in the use of a property, for instance, to solve a problem that may not be posed in a mathematical context, or not using a mathematical register.

Example: When reasoning on a virtual situation (concerning the task to measure a segment with a ruler broken at 5), a student says that he imagines each number of the graduation sliding 5 grades to the right and 0 coming at the place of 5. He uses an implicit theorem ("theorem in action" concerning the additivity of measure): *the number I read is the sum of* *the broken space' length (5cm) plus the length of the segment I measure*. A didactical perspective can allow this contextualised and implicit "theorem" to be part of the concept of addition, since the student develops an operational invariant in the conceptual field of additive structures, which gradually extends from integer numbers to measure. Another example: Producing schemata to represent physical phenomena or mathematical relations and make them thinkable, as well as building reasoning upon them, are highly mathematical activities. As for virtual situations, schemata carry the double aspect of reference situations and operational invariants. They favour the development of these concept components of mathematical objects.

2) <u>Vygotsky's everyday/scientific concepts dialectics</u>

Vygotsky's seminal work will help us inject a dynamical and cultural determination of our mathematical objects. Exploring his scientific concepts/every day concepts dialectics to analyse student's evolution of conceptualisation, we can consider that the mathematical objects can move between two main qualities of uses: everyday and scientific ways of using it. Everyday concepts are rather intuitive. Scientific ones are used voluntarily with explicit reference to systemic links. The first ones do not need to belong only to common culture, but can be knowledge familiar enough to be part of intuition and of common culture in a specific community; the second ones do not need to belong to scientific domains, but concern any structured knowledge (as in the case of grammar).

Completing previous perspectives with this vygotskyan one allows us to deepen the analysis of how an operational invariant underlying activity is part of a mathematical object's construction, and how reference situations and activity are relevant for such constructions.

According to Vygotsky, school aims at developing scientific treatment of concepts. In this perspective, an adequate didactical setting would aim at bringing the students to use consciously a mathematical concept and make use of its systemic links. But how to assert that the mathematical object is effectively in construction if the activity is not consciously related to the given mathematical object? Operational invariants, when not related to consciously mobilised explicit knowledge, cannot be considered yet as parts of that mathematical knowledge that school must transfer to new generations.

Thus it is necessary to consider the evolutionary character of a mathematical object, instead of a series of pieces of knowledge.

Epistemological Vs cognitive analyses

Following Vygostky's theoretical elaboration, scientific concepts are developed at school through the steps offered by every day knowledge (which dialectically evolve to become more explicitly mastered). If we recall the epistemological analyses, we can now underline the very special role culturally embedded roots and body anchored references play in mathematical objects' construction. For instance, in our number construction example, we see why extended epistemological perspective offers a deep change in the didactical frames and justifies the change of boundaries of the mathematical object.

This discussion leads us to include, in the determination of a concept, its modelling potential related to other activities: because of the relation of its representation component to the reference situations and the operational invariants, on one hand, and because of the dialectical relation between scientific concepts and everyday concepts, on the other.

In conclusion:

Determining a mathematical object is a question that several researchers pose, often because some accepted activity at one stage will not be acceptable later. But researchers' epistemological and cognitive positions strongly affect the way mathematical objects are determined, as can be witnessed by various theoretical works. M.Mariotti deals with "theorem" as a system of theory, statement and proof. L.Radford's extends algebraic thinking to include generalisation steps of calculation patterns detectable in gesture and voice rhythm. G.Arsac's discussion of the socio-cultural and historical scientific circumstances of the origin of proof suggests different perspectives of analysis and evokes different determination of what is proof.