SEMIOTIC THEORETICAL FRAMEWORKS: CREATIVITY AND IMAGINATIVE RATIONALITY IN MATHEMATICS EDUCATION

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The evolution of theoretical frameworks in mathematics education research is the broad umbrella under which this paper is situated. It is argued that because the field of mathematics education concerns people and their learning, every discipline relevant to humanity is germane to this research. There has been a movement to broaden the initial focus on psychology—as mathematics education comes of age—to include sociocultural theory. However, this paper focuses on the usefulness of theories borrowed from literary domains that address metaphor, metonymy, and imaginative rationality in semiotics. The recent use of semiotics as a theoretical framework for research includes the strong focus on logical thinking in mathematics education, but also allows for the recognition of creative elements in the learning and doing of mathematics, including the abduction that is an essential component of creative work, preceding, reflexively relating to, and complementing deduction and induction. Illustrative examples are provided, drawn from recent research on the learning of trigonometry and geometry.

Already in 1978, in a paper characterizing and contrasting "Dionysians and Apollonians", Gerald Holton questioned and commented as follows.

> How do scientists go about obtaining knowledge? How *should* they? Few modern research scientists tend to be introspective about these questions. During apprenticeship, most scientists somehow absorb the necessary pragmatic attitude and then go about their business quite successfully, content to leave it to a small handful to become interested in epistemology when some obstinate difficulty blocks scientific advance. (p. 84, his emphasis)

On opposite ends of the epistemological scale, Dionysians were characterized as emphasizing the human elements of the scientific process, which include imagination, affect, insights, and creativity, whereas Apollonians were seen to concentrate on the final, rational product of the endeavor, downplaying the humanism of the journey there. Holton considered the goal of the latter group to be description of "a reality in space and time which is independent of ourselves" (p. 91). Albert Einstein—on the basis of his own averred "basic axiom," namely, the postulation of a real world in which physics is an attempt to grasp reality as it is thought independently of its being observed— was placed in the category of Apollonians. However, Einstein's epistemology is more balanced than Holton's categorization suggests. In an essay on "Science and religion" written in 1941, Einstein (1979) claimed that "Science is the century-old endeavor to bring together by means of systematic thought the perceptible phenomena of this world into as thoroughgoing an association as possible" (p. 24). Mitigating Holton's classification, human conceptualization is indispensable in Einstein's epistemology. He wrote, "To put it boldly, it [physics] is the attempt at the posterior reconstruction of existence by the process of conceptualization" (ibid.). Thus there are elements of humanism in this reconstruction. Albert Einstein was arguably one of the most creative theoretical physicists of the 20th century. His well-known claim that the most beautiful thing we can experience is the mysterious—which is the source of both art and science— places the origin of his creative imagination squarely in the concerns of the Dionysians rather than the Apollonians, contrary to Holton's attribution. There is evidence in the writings of other scholars (Schilpp, 1959), and even in Holton's (1973, 1978) own analyses, that visualization and the juxtaposition of images in Einstein's thought were a source of the imaginative rationality for which his thought experiments are famous. More than that, it was the Janusian polarity of opposites (Rothenberg, 1979), in his personality and thus also in his scientific thinking (Holton, 1978, especially pp. 275-281), that made his break with traditional physics possible.

The extreme views of Dionysians and Apollonians are still rife, and still highly relevant in the epistemology of mathematics (Davis & Hersh, 1981). Although it is still the norm to present the results of *mathematical* research in a published form that hides the creative processes of their genesis, it is increasingly accepted that *mathematics education*, which involves all the complexity of human learning (Presmeg, 1998), partakes of both the arts and the sciences, and thus that theories from all relevant disciplines are germane to research on the teaching and learning of mathematics at all levels.

In this position paper I argue that just as in the case of Einstein's creativity, in mathematics education and therefore in its research, it is essential to take into account the sometimes hidden human qualities of imagination, aesthetic sense and feeling, and creativity, in addition to the logic that is the *sine qua non* of mathematical endeavor. Semiotics is the study of activity with signs (Colapietro, 1993). Several of the semiotic triads posited by Charles Sanders Peirce (1992, 1998) are germane to this balanced view of the teaching and learning of mathematics. Two of these triads are presented in the following sections.

Deduction, induction, and abduction

According to Peirce (1992), *trichotomic* is the art of making three-fold divisions. By his own admission, he showed a proclivity for the number three in his philosophical thinking. "But it will be asked, why stop at three?" he wrote (Peirce, 1992, p.251), and his reply to the question is as follows:

> [W]hile it is impossible to form a genuine three by any modification of the pair, without introducing something of a different nature from the unit and the pair, four, five, and every higher number can be formed by mere complications of threes (ibid.).

Accordingly, he used triads not only in his semiotic model including object, sign (sometimes called the representamen), and interpretant, but also in the types of each of these components. Of the ten triads that Peirce introduced in his writings over several decades, two are of particular relevance to the purpose of this paper. The first of these is Peirce's recognition of the importance of a third kind of logic, namely *abduction*, in addition to the well known forms of deduction and induction that underlie all mathematical cognition, and indeed, scientific thinking in general.

Intuitions or hunches have been demonstrated as being central features in the creative thinking that resulted in many scientific innovations (e.g., Hadamard, 1945). This kind of thinking is an example of what Peirce (1998) called *abduction*. In his own words, "[The] step of adopting a hypothesis as being suggested by the facts, is what I call *abduction*. I reckon it as a form of inference, however problematical the hypothesis may be held" (p. 95). There is no guarantee that an abduction will turn out to be useful, or even correct, as Peirce (ibid.) suggested:

> An *Abduction* is a method of forming a general prediction without any positive assurance that it will succeed either in the special case or usually, its justification being that it is the only possible hope of regulating our future conduct rationally, and that Induction from past experience gives us strong encouragement to hope that it will be successful in the future. (p. 299)

Most of Sherlock Holmes's so-called deductions are, strictly speaking, abductions, because they are guesses—based on evidence—about what took place (Colapietro, 1993). As Colapietro pointed out, "They are guesses carefully framed and then carefully tested" (p. 1). An abduction may result from only one instance, unlike induction, which requires a number of particular instances to suggest its principle. Colapietro defines the ways in which abduction differs from deduction and induction as a means of inference, as follows:

> Deduction is the logical operation by which we derive the necessary consequences from some purely hypothetical situation (for example, if it is true that A is greater than B and B is greater than C, then it is necessarily true that A is greater than C). Induction is the operation by which we test hypotheses in terms of consequences derived by deduction; abduction is that by which hypotheses are framed in the first place. In other words, deduction proves that something *must* be the case; induction shows that something *actually is* the case; and abduction suggests that something *might be* the case. (p. 2)

In this account, Colapietro does not distinguish a view of *induction* as a form of formal proof—as in mathematical induction—from the less formal use of the word to indicate a process of inferring a general law from particular instances (Concise Oxford Dictionary). The essential difference between induction and abduction is that abduction may be a guess based on the evidence of only one instance, unlike induction in which several particular instances are necessary for the general inference to be made.

A short example will illustrate the significance of abduction (in addition to induction and deduction) in mathematical problem solving. In spring of 2006, I and two colleagues (Jeff Barrett and Sharon McCrone) investigated the ways that prospective elementary school teachers progressively construct generalizations in a course called Geometric Reasoning: Geometry as Earth Measures. Early in the course, the teacher (Jeff) gave the class, working in groups, the very open task of finding the best position to situate a bridge over a river between two towns in such a way that the road distance between the towns was as short as possible. The class had negotiated in open discussion that the towns could be represented by points, A and B, and that the bridge should be perpendicular to the banks of the river. After working in groups, the students were required to propose a solution for homework, which was collected in the following session, before further group and whole-class discussion. Many students in the class adjusted the task to make the line segment joining the two towns perpendicular to the river: but this trivialized the situation and was not a general solution. Several students knew that a straight line segment would give the shortest distance between A and B, but the problem remained that the bridge had to be perpendicular to the banks of the river. Some students wanted to place the bridge at the midpoint of intersection of this line with the banks of the river. In observing the class, and thinking about the task on my own, I had the abduction that the point of placement had to be not in the middle, but proportionate to the distances of the towns from the river. This abduction led to my working with similar triangles to solve the problem. However, one student, Sam (pseudonym) had a more dynamic abduction that led him, and later his group, to a much simpler and more elegant solution. He realized that the river could be “shifted” to be adjacent to either of the towns A or B, without changing the structure of the task. AC and DB represent the width of the river (see figure 1). Then the long sides of the parallelogram ACBD could be taken to intersect with the river, wherever it may be, supplying the position for placement, EF, of the bridge. The path AEFB would then always be the shortest possible, and it would be equal in length to sides AD or BC of the parallelogram plus the width of the river.

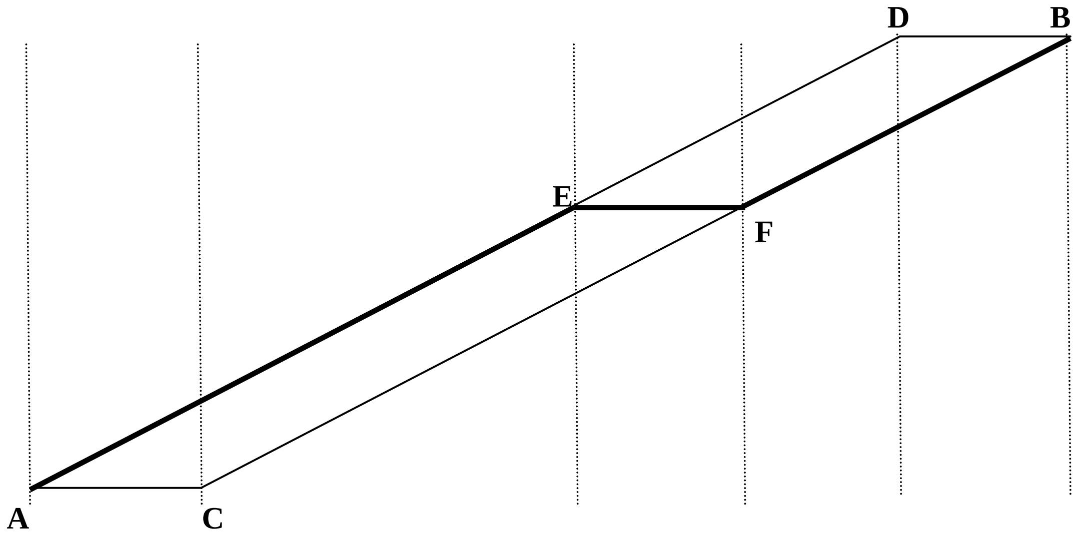

Figure 1. A dynamic solution of the bridge placement problem.

In this case, Sam’s abduction—which appeared rather tentatively in the discussions at first—led to an accurate general solution of the problem. However, it is always necessary to check that the results of an abduction are indeed valid.

Abductive logic has a role to play along with induction and deduction in the teaching and learning of mathematics, as part of the recognition of the human qualities inherent in this endeavor. A second Peircean triad that is relevant concerns the signs that are used in representing mathematical objects and processes.

Iconic, indexical, and symbolic signs

According to Peirce (1998), signs may be *iconic*, *indexical*, or *symbolic*. These types are not inherent in the signs themselves, but depend on the interpretation of the constituent relationship between a sign vehicle (the representamen) and its object. To illustrate by using some of Peirce’s examples, in an *iconic* sign, the sign vehicle and the object share a physical resemblance, e.g., a photograph of a person representing the actual person. Signs are *indexical* if there is some physical connection between sign vehicle and object, e.g., smoke invoking the interpretation that there is fire, or a sign-post pointing to a road. The nature of *symbolic* signs is that there is an element of convention in relating a particular sign vehicle to its object (e.g., algebraic symbolism).

The literary figures of metaphor and metonymy may be related quite closely with iconic and indexical signs respectively (Presmeg, 2005). Briefly, on the one hand, the inherent unidirectional structure of metaphor has an iconic propensity to illuminate comparable structures in two disparate domains. On the other hand, because metonymy depends on context for its efficacy it is indexical in nature. Both these figures have been shown to be relevant as lenses for understanding aspects of teaching and learning mathematics (Presmeg, 1998b, in press). In the learning of trigonometry, both idiosyncratic metaphors (such as Alison's "water level") and canonical metaphors (such as Laura's "bow tie" metaphor adopted from class discussion) were powerful in helping students to construct meaning for trigonometric definitions in the four quadrants of the coordinate plane. Along with the ubiquitous symbolism of mathematics, iconic and indexical signs are essential elements of mathematical meaning.

Theoretical developments that emphasize the humanism of mathematics education as a field are an essential aspect of research in this field.

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