Euler Characteristic of nonsingular real tropical hypersurfaces

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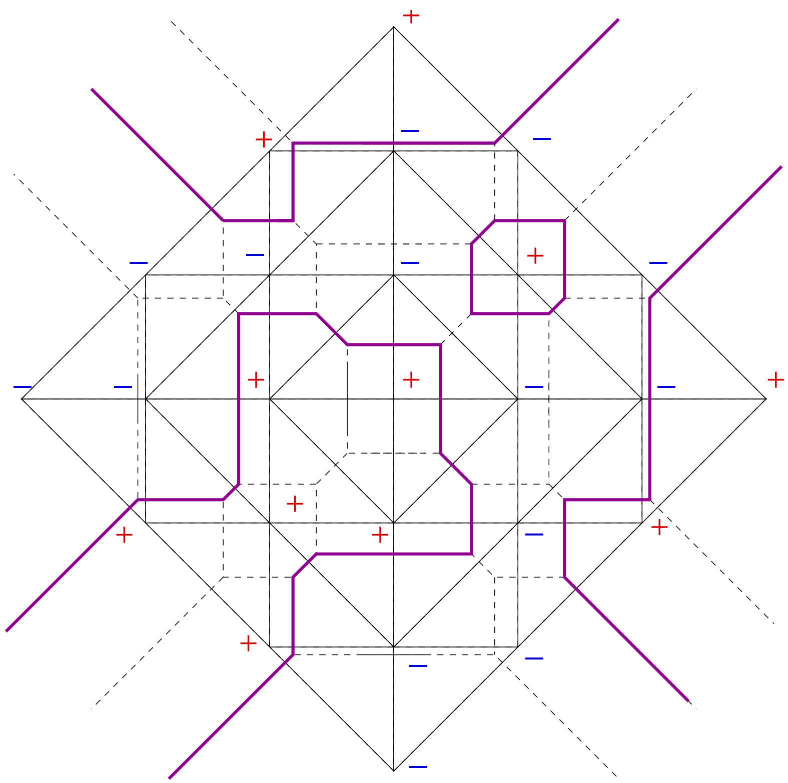
Real tropical curve and its dual subdivision.



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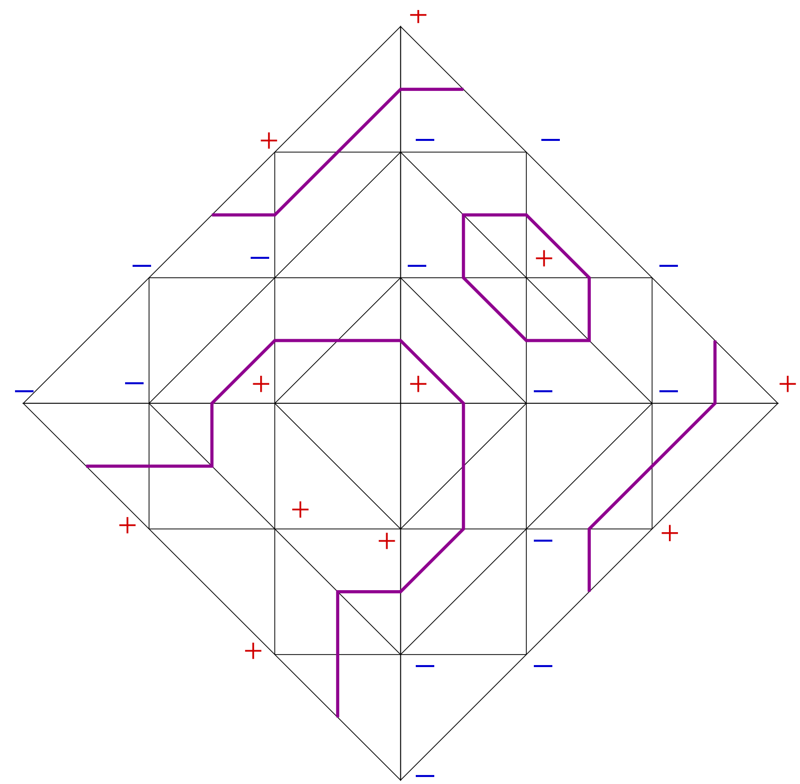


Tropical Varieties

$\mathbb{K}$ field of Puiseux series.

$$g(t) = \Sigma_{r \in R} b_r t^r \in \mathbb{K}.$$

Where $b_r \in \mathbb{C}$, $R \subset \mathbb{Q}$ bounded by below, contained in an arithmetic sequence. Valuation :

$$\mathrm{val}(g(t)) := \min\{r / b_r \neq 0\},$$
$$v(g) := -\mathrm{val}(g).$$

$f(z) = \Sigma_{\omega \in A} \; c_\omega z^\omega$ avec $A \in \mathbb{Z}^n$,

$|A| < +\infty$, $c_\omega \in \mathbb{K}^*$, $z = (z_1, \ldots, z_n)$

$$Z_f := \{z \in (\mathbb{K}^*)^n / f(z) = 0\}$$

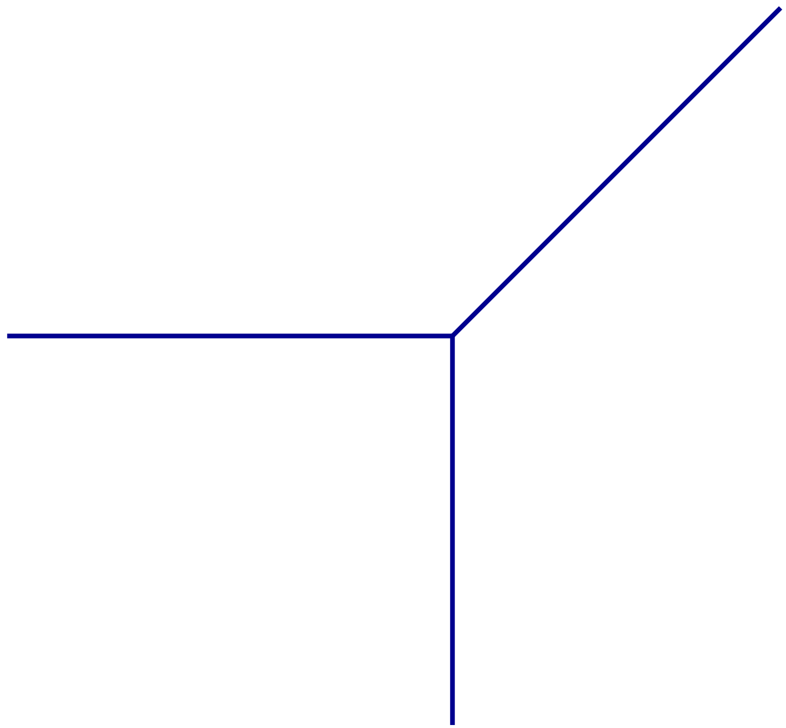
$$\begin{array}{rccc} V : & (\mathbb{K}^*)^n & \longrightarrow & \mathbb{R}^n \\ & z & \longmapsto & (v(z_1), \ldots, v(z_n)) \end{array}$$

Definition 1 *A tropical hypersurface is the closure of the image under* V *of a hypersurface in* $(\mathbb{K}^*)^n$*:*

$$T_f := \overline{V(Z_f)} \subset \mathbb{R}^n$$

Examples

Tropical line



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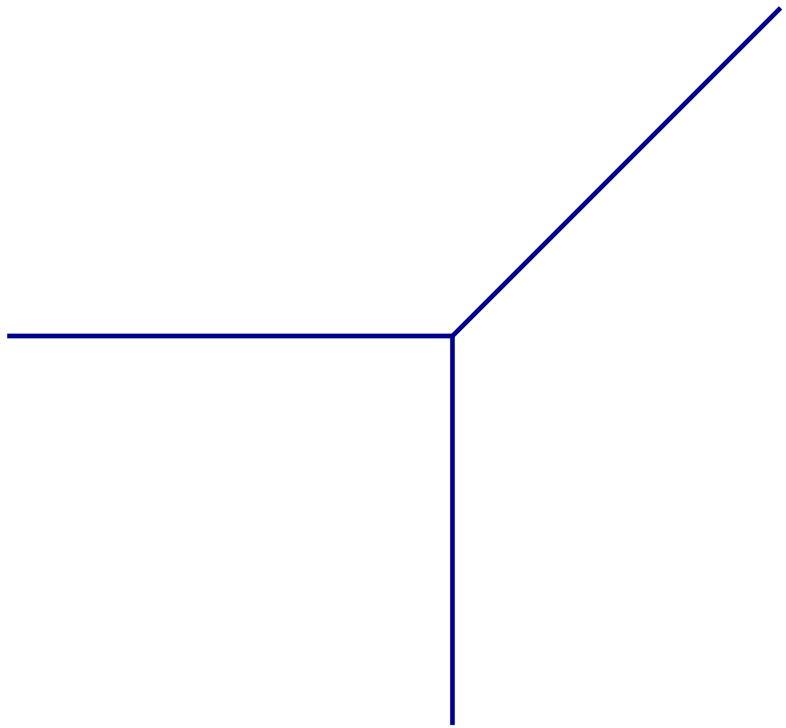
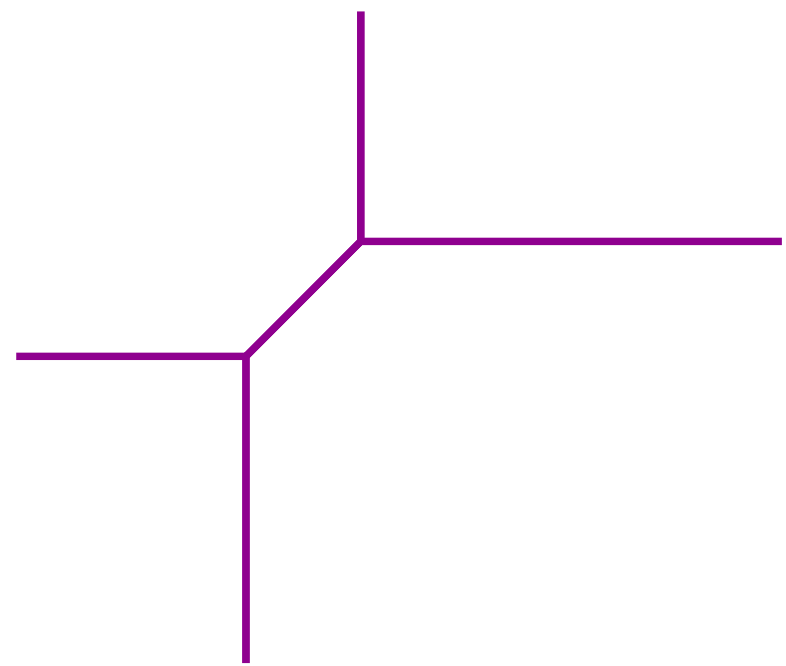


Bidegree (1,1) curve



Kapranov's Theorem

$f(z) = \Sigma_{\omega \in A} c_\omega z^\omega$

$$\text{Put } \nu : \begin{array}{ccc} A & \longrightarrow & \mathbb{R} \\ \omega & \longmapsto & -v(c_\omega) \end{array}$$

$$\mathcal{L}(\nu) : \begin{array}{ccc} (\mathbb{R})^n & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \max(x \cdot \omega - \nu(\omega)) \end{array}$$

The Legendre transform $\mathcal{L}(\nu)$ of ν is a piecewise-linear convex function.

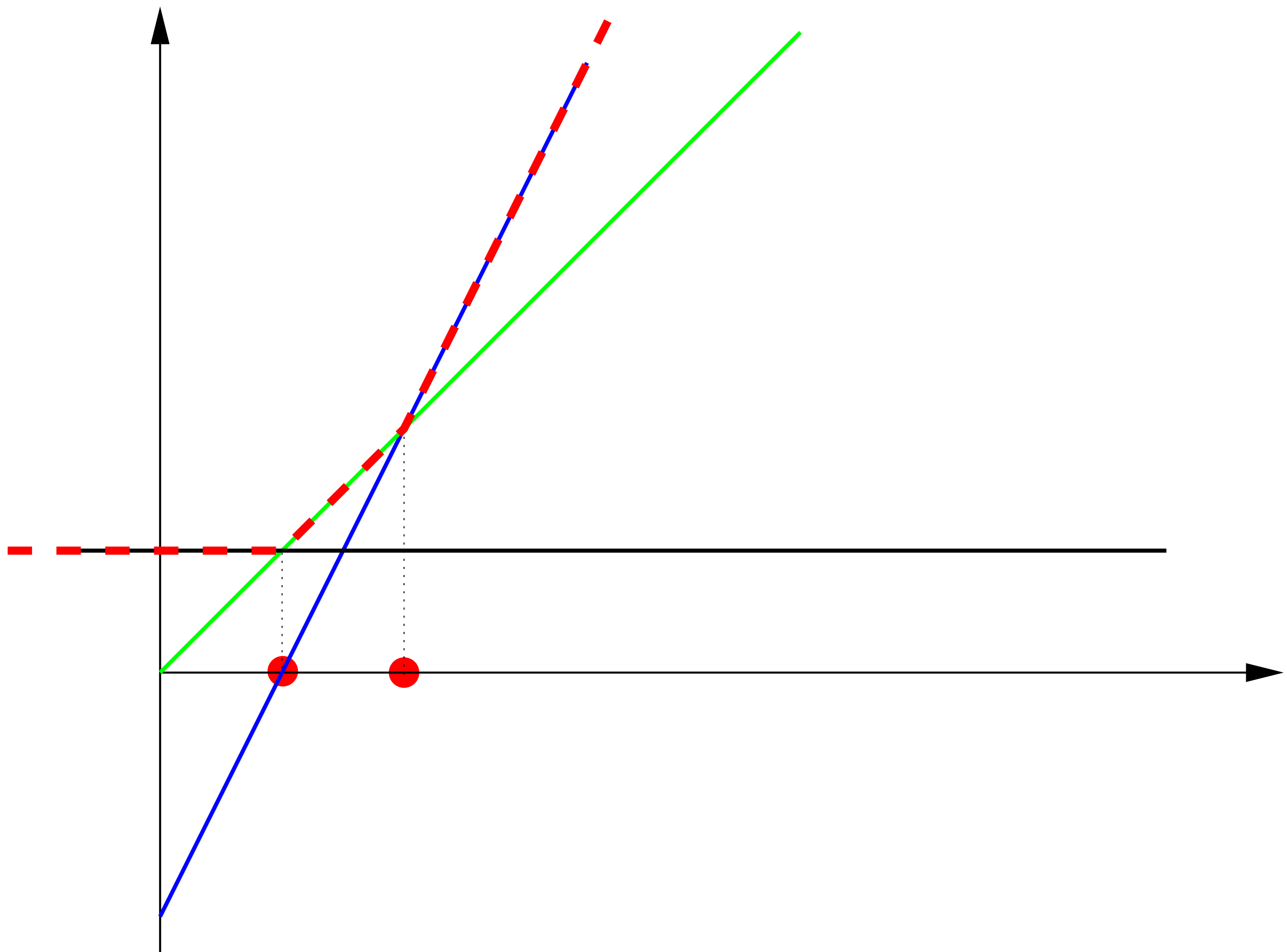
Theorem 2 (Kapranov) T_f *is the nonlinearity domain of* $\mathcal{L}(\nu)$.

$$V(Z_f) = \textit{corner locus}(x \mapsto \max(x \cdot \omega + v(c_\omega))).$$

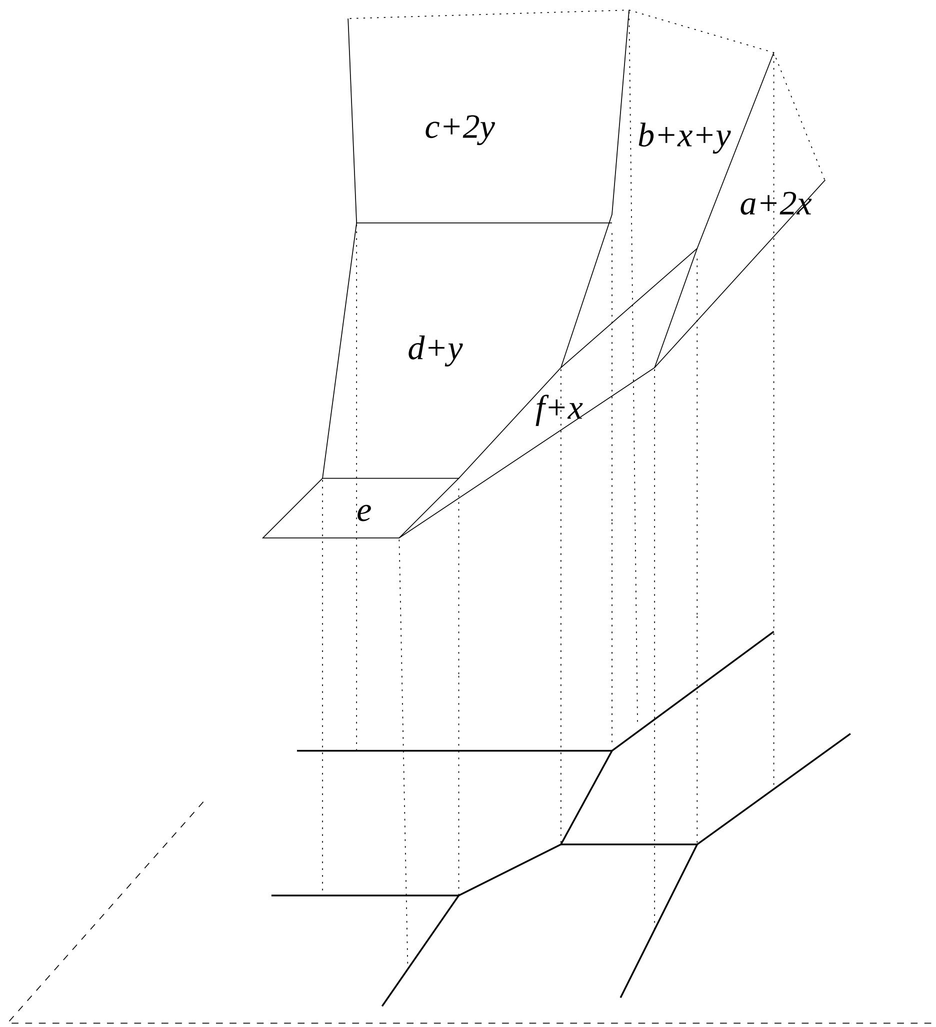
Example in dim. one

$f(x) = t.x^0 + 1.x + t^{-2}.x^2$

Tropical roots: corner locus of $x \mapsto \max\{0.x + 1, x + 0, 2x - 2\}$

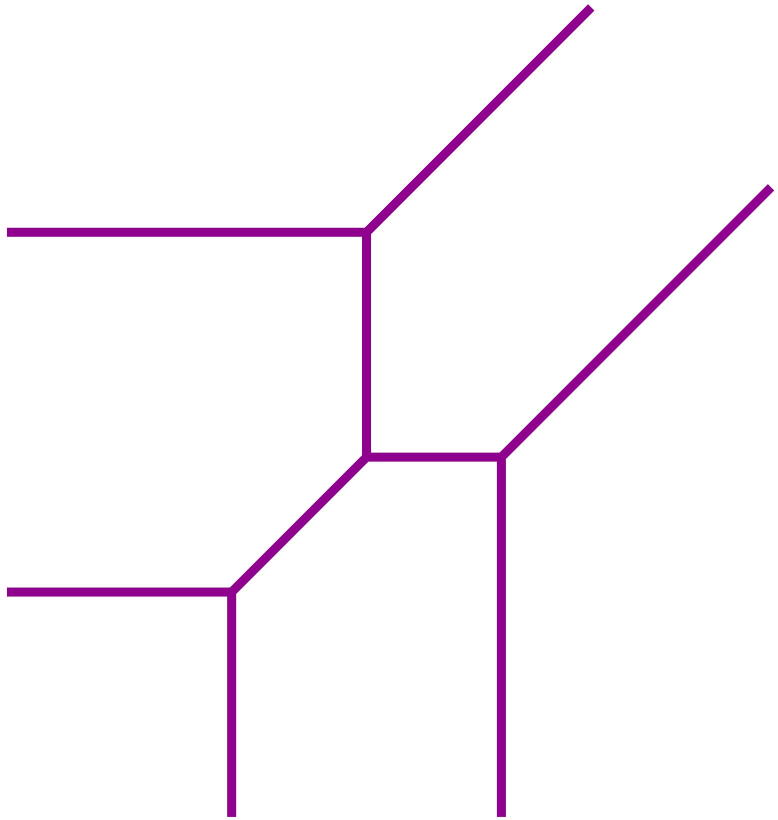


Kapranov's Theorem in dim. 2



Example

Tropical conic



Example

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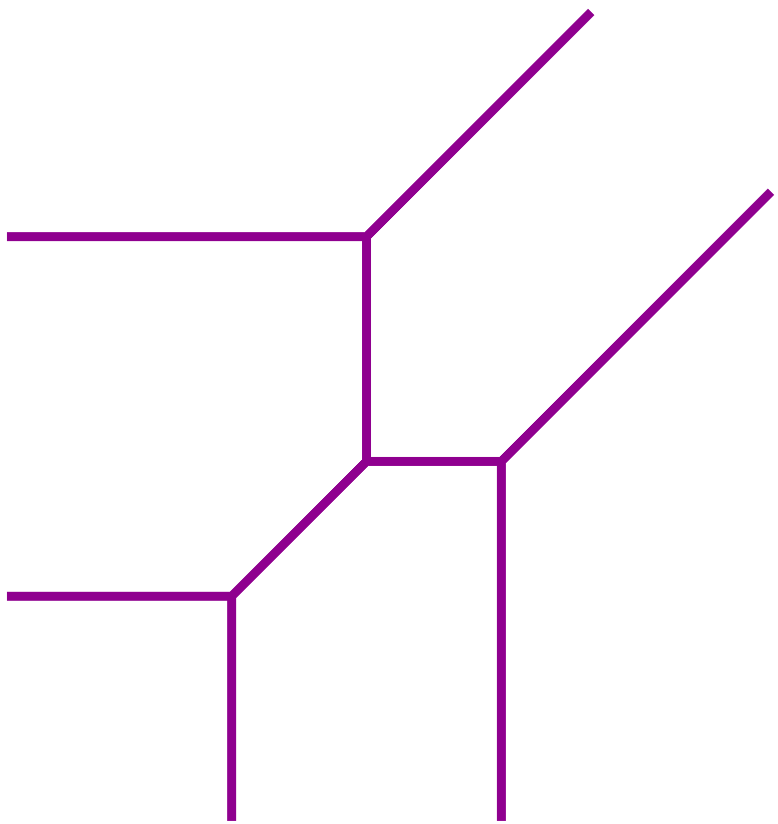
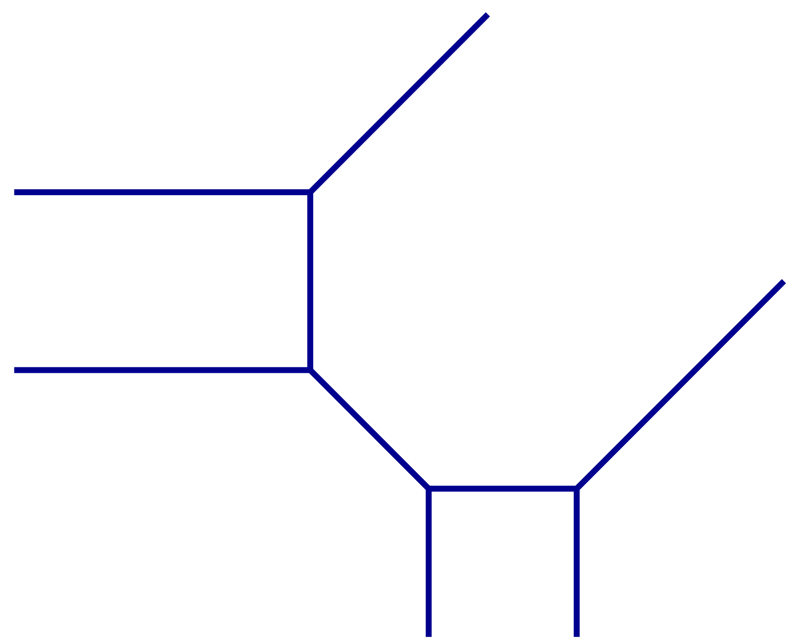
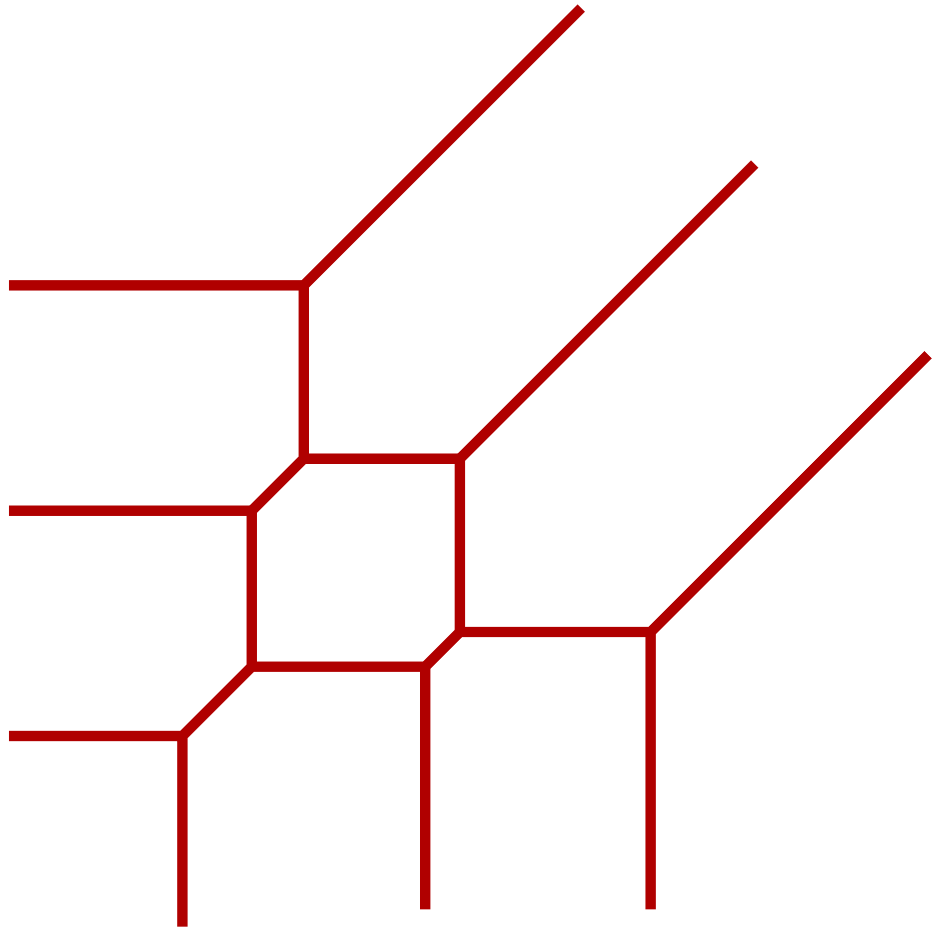


Another conic



Example

Tropical cubic

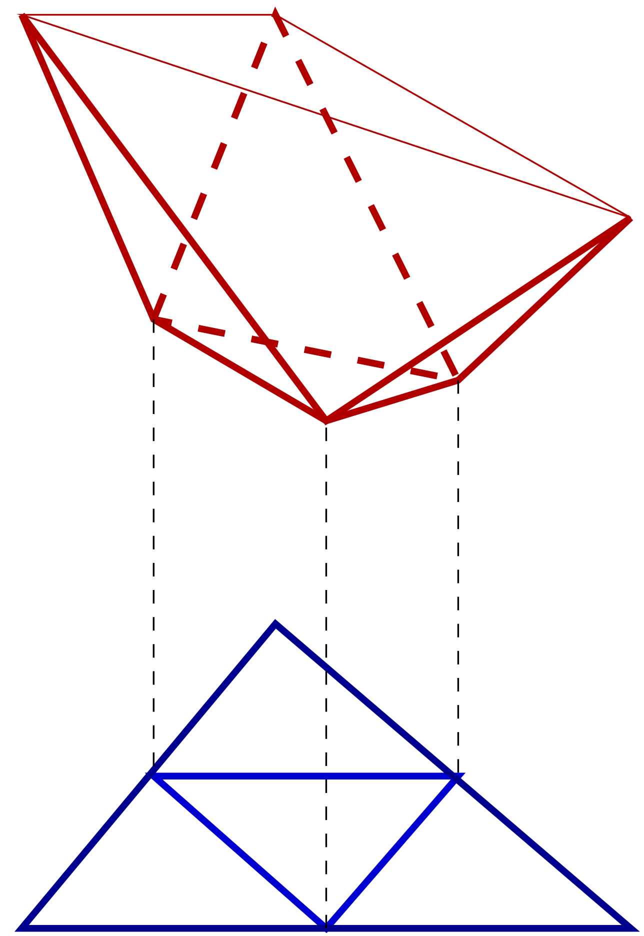


Duality

$f(z) = \Sigma_{\omega \in A}\ c_\omega z^\omega$, $\Delta = \mathrm{ConvHull}(A)$ Newton polytope of f.

$\Gamma := \mathrm{ConvHull}\,(\omega, v(c_\omega)), \omega \in A$

$$\begin{array}{rccc} \nu : & \Delta & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \min\{x/(\omega, x) \in \Gamma\} \end{array}$$

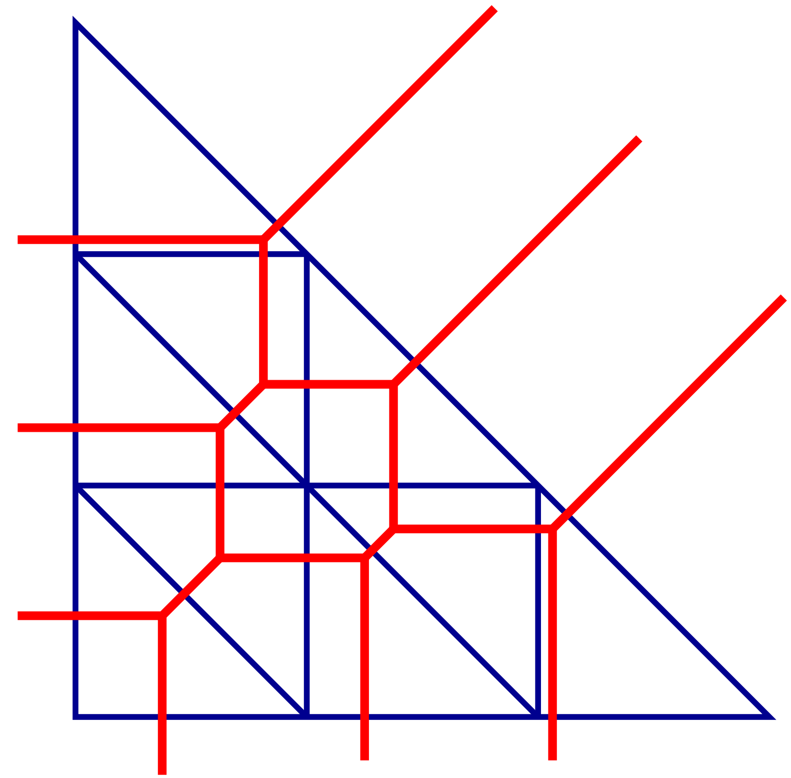


The linearity domains of ν are the n-cells of a convex polyhedral subdivision τ of Δ.

Duality

T_f induces a subdivision Ξ of $\mathbb{R}^n$. Subdivisions τ and Ξ are dual:
There is a one-to-one inclusion reversing correspondance L between cells of Ξ and cells of τ such that for any $\xi \in \Xi$,

1. $\dim L(\xi) = \operatorname{codim} \xi$,
2. $L(\xi) \perp \xi$.

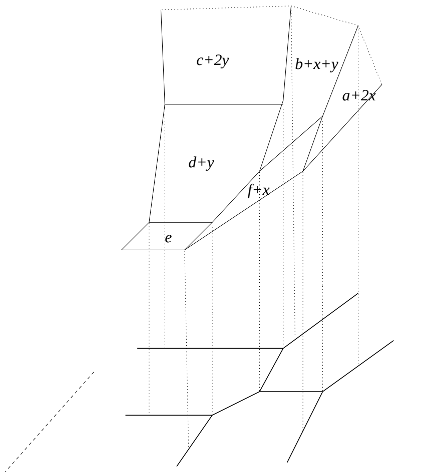


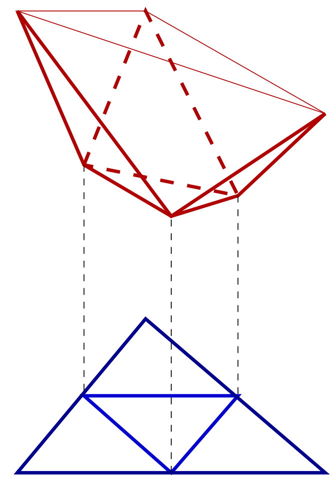
T_f is said nonsingular if τ is primitive (n-simplices have volume $\frac{1}{n!}$).

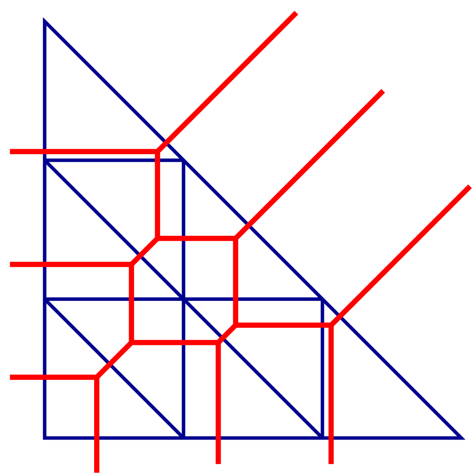
Duality

$$\begin{array}{rccc} f_{\mathrm{trop}}: & (\mathbb{R})^n & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \max(x \cdot \omega - \nu(\omega)) \end{array}$$

$$\begin{array}{rccc} \nu: & A & \longrightarrow & \mathbb{R} \\ & \omega & \longmapsto & -v(c_\omega) \end{array}$$







Complex tropical hypersurfaces

$\mathbb{K}$ Field of Puiseux series. $g(t) = \Sigma_{r\in R} b_r t^r \in \mathbb{K}$.
valuation : $\mathrm{val}(g(t)) = min\{r / b_r \neq 0\}$, $v(g) := -\,\mathrm{val}(g)$, $f(z) = \Sigma_{\omega\in A}\, c_\omega z^\omega$.

$$Z_f := \{z \in (\mathbb{K}^*)^n / f(z) = 0\},\ \ \arg(g(t)) := \arg(b_{\mathrm{val}(g(t))}).$$

$$\begin{array}{rccl} W := V \times \mathrm{Arg} : & (\mathbb{K}^*)^n & \longrightarrow & \mathbb{R}^n \times (S^1)^n \simeq (\mathbb{C}^*)^n \\ & z & \longmapsto & ((v(z_1), \ldots, v(z_n)), (\arg(z_1), \ldots, \arg(z_n))) \end{array}$$

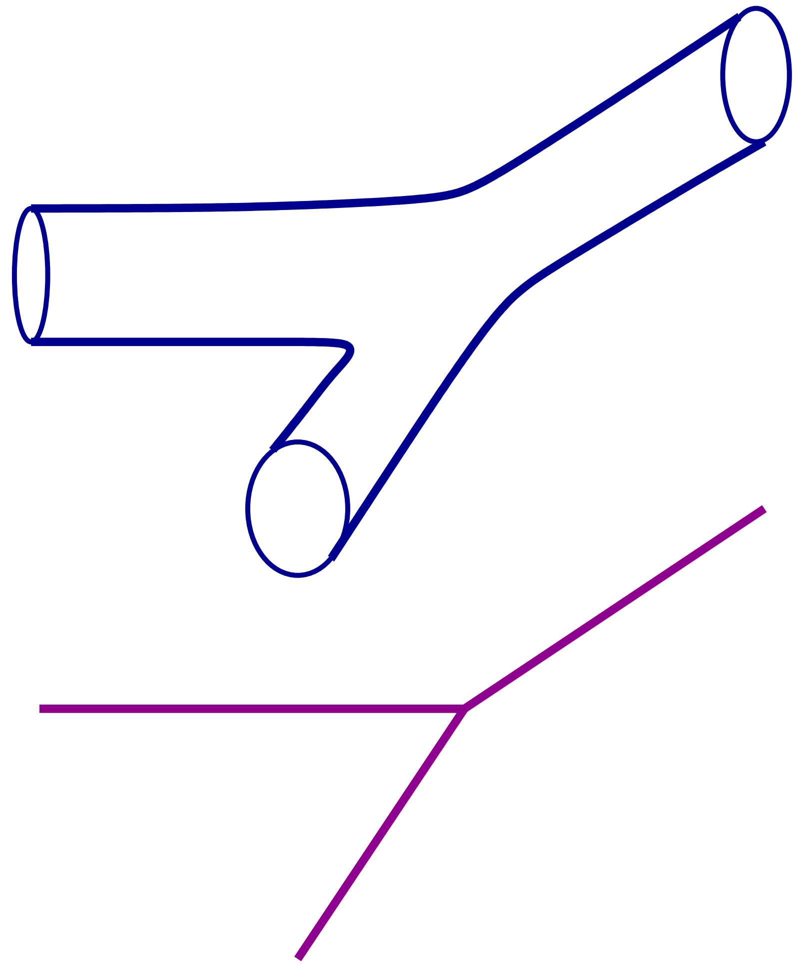
$$\mathcal{W}(z) := (e^{v(z_1)+i\arg(z_1)}, \ldots, e^{v(z_n)+i\arg(z_n)}).$$

Definition 3 *A complex tropical hypersurface is the closure of the image under $\mathcal{W}$ of a hypersurface in $(\mathbb{K}^*)^n$:*

$$\mathbb{C}T_f := \overline{\mathcal{W}(Z_f)} \subset \mathbb{C}^n$$

Example

Complex tropical line



Real tropical hypersurfaces

$f(z) = \Sigma_{\omega \in A} c_\omega z^\omega$ with $c_\omega = \Sigma \alpha_r t^r$ and $\alpha_r \in \mathbb{R}$.

Definition 4 $\mathbb{R}T_f := \mathbb{C}T_f \cap (\mathbb{R}^n \times \{0, \pi\}^n)$.

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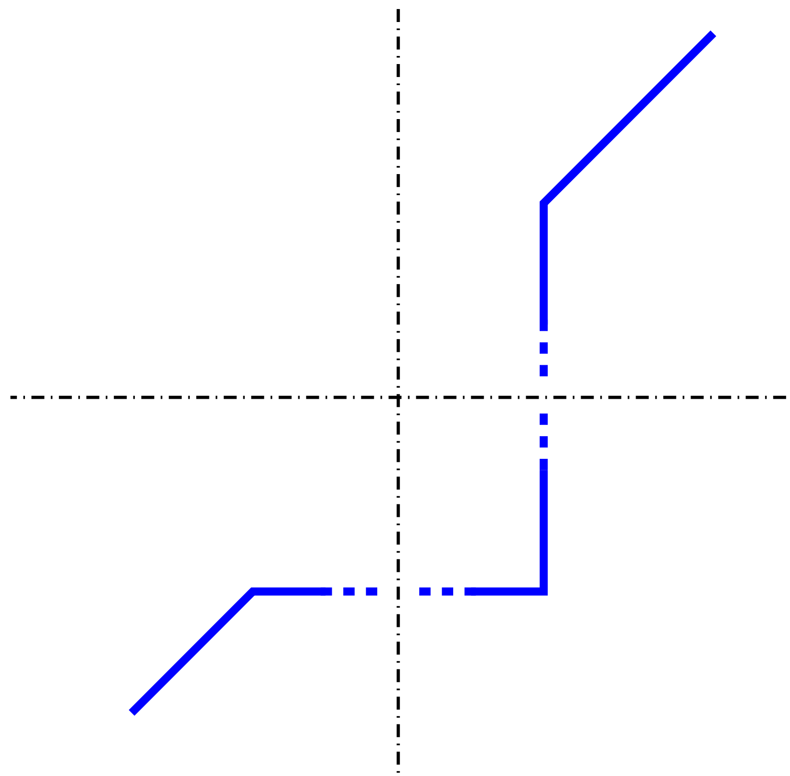

Real tropical hypersurfaces

- Assume $\mathbb{R}T$ is nonsingular i.e. τ is primitive.

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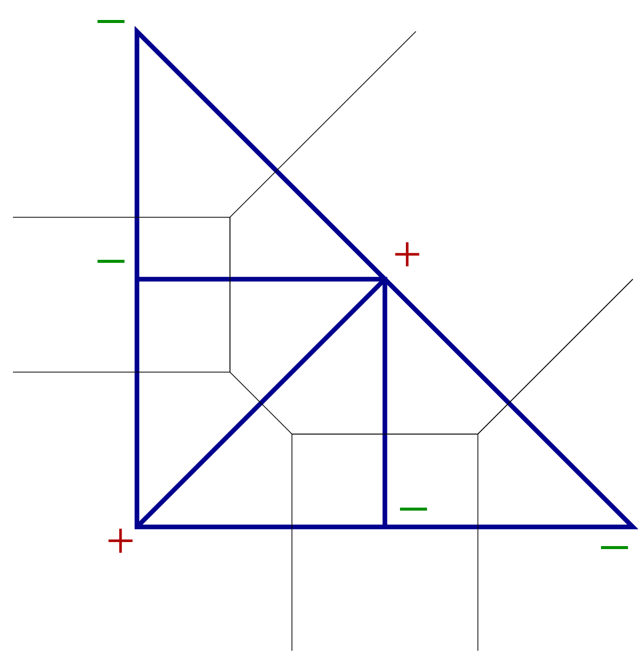
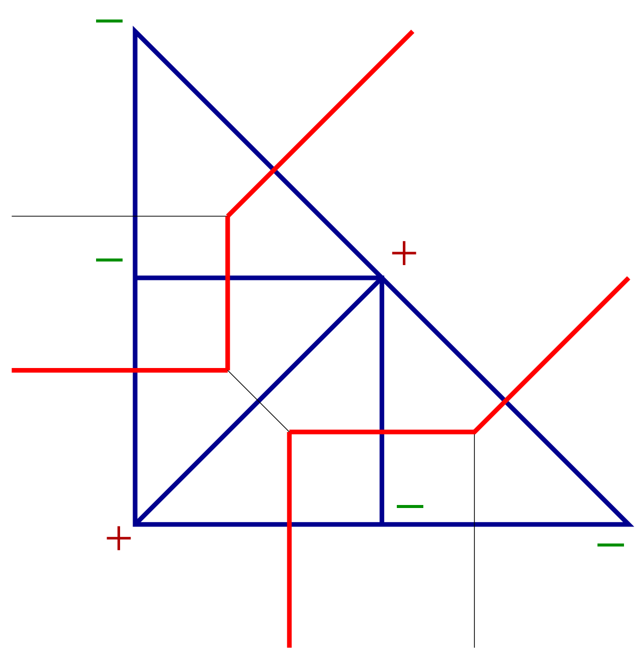
- sign c_ω $:= \operatorname{sign} \alpha_{\mathrm{val}(c_\omega)}$.

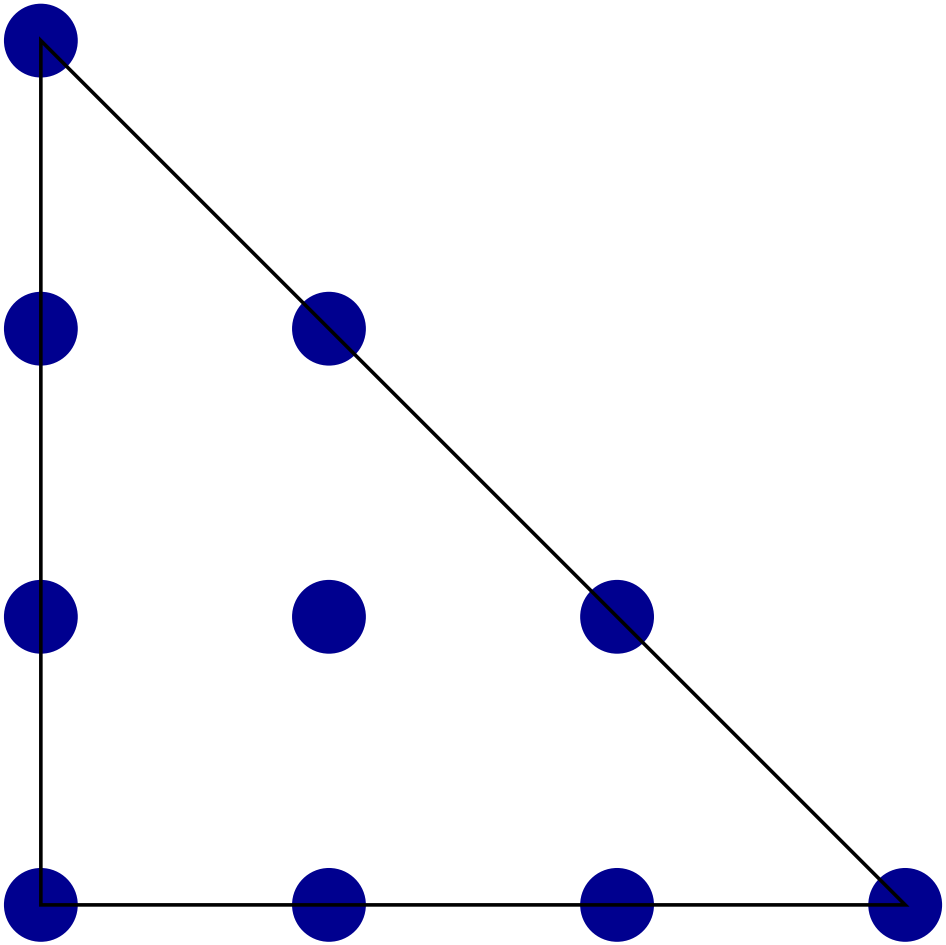


Real tropical hypersurfaces

- Assume $\mathbb{R}T$ is nonsingular i.e. τ is primitive.
- $\text{sign}\, c_\omega := \text{sign}\, \alpha_{\text{val}(c_\omega)}$.
- For $\mathbb{R}T \cap (\mathbb{R}^n \times \{p\})), p \in \{0, \pi\}^n$, $\text{sign}\, \omega := e^{i<p,\omega>}.\, \text{sign}\, c_\omega$.



T-construction



- Let Δ be a polytope with integer vertices.

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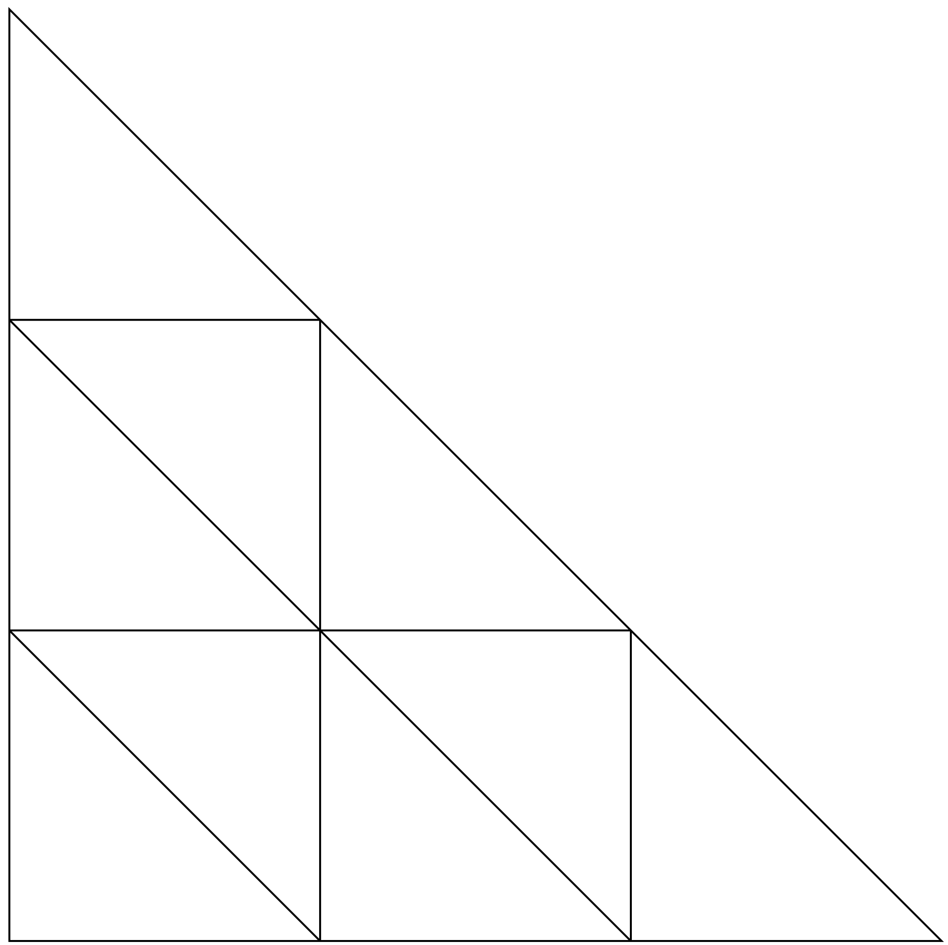

- Let Δ be a polytope with integer vertices.
- τ a convex triangulation of Δ.

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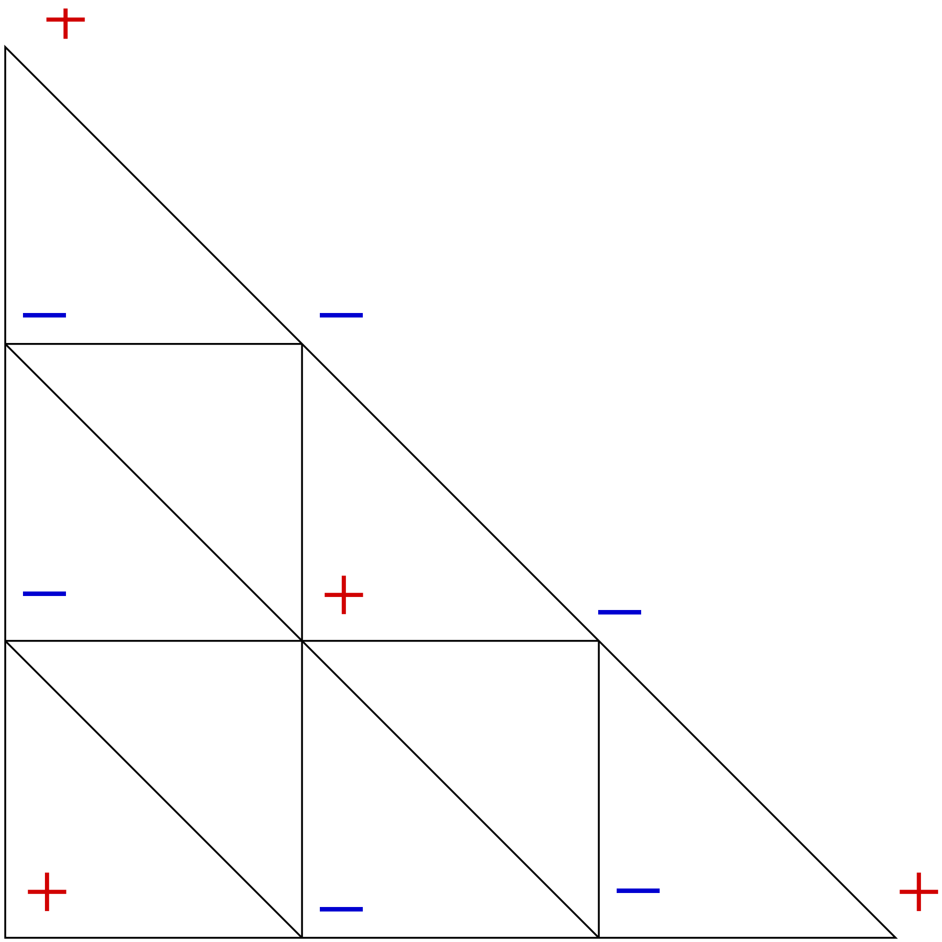
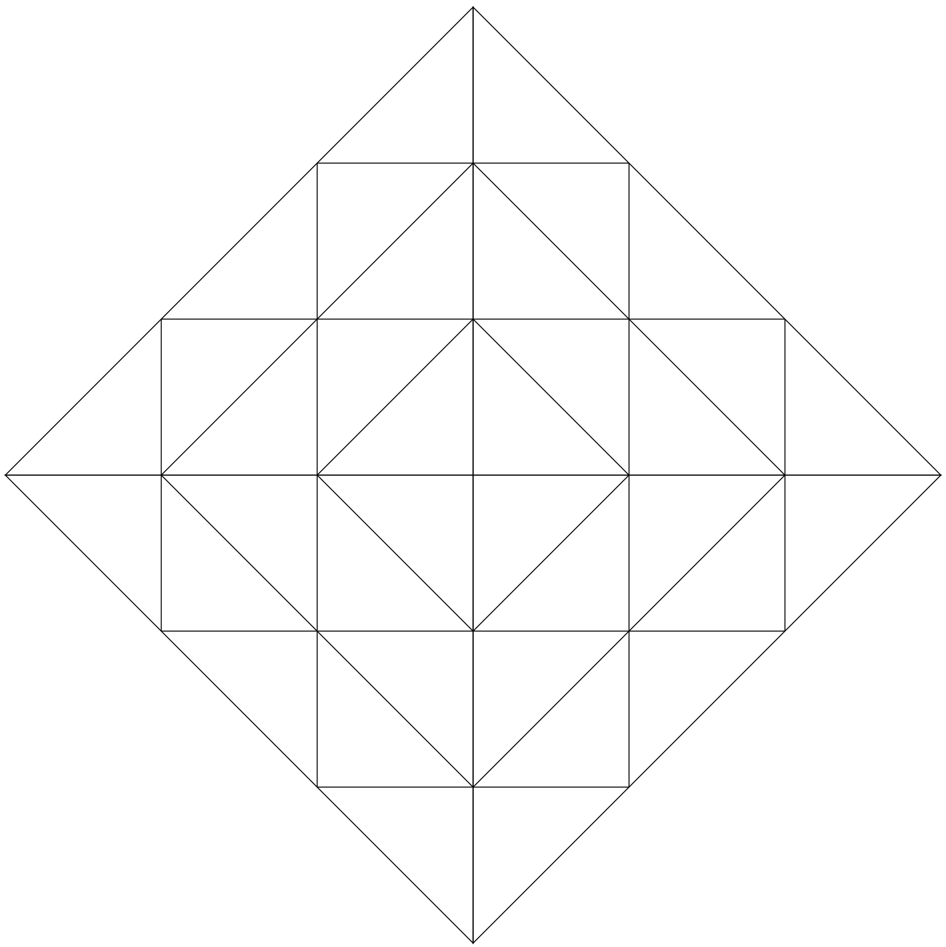


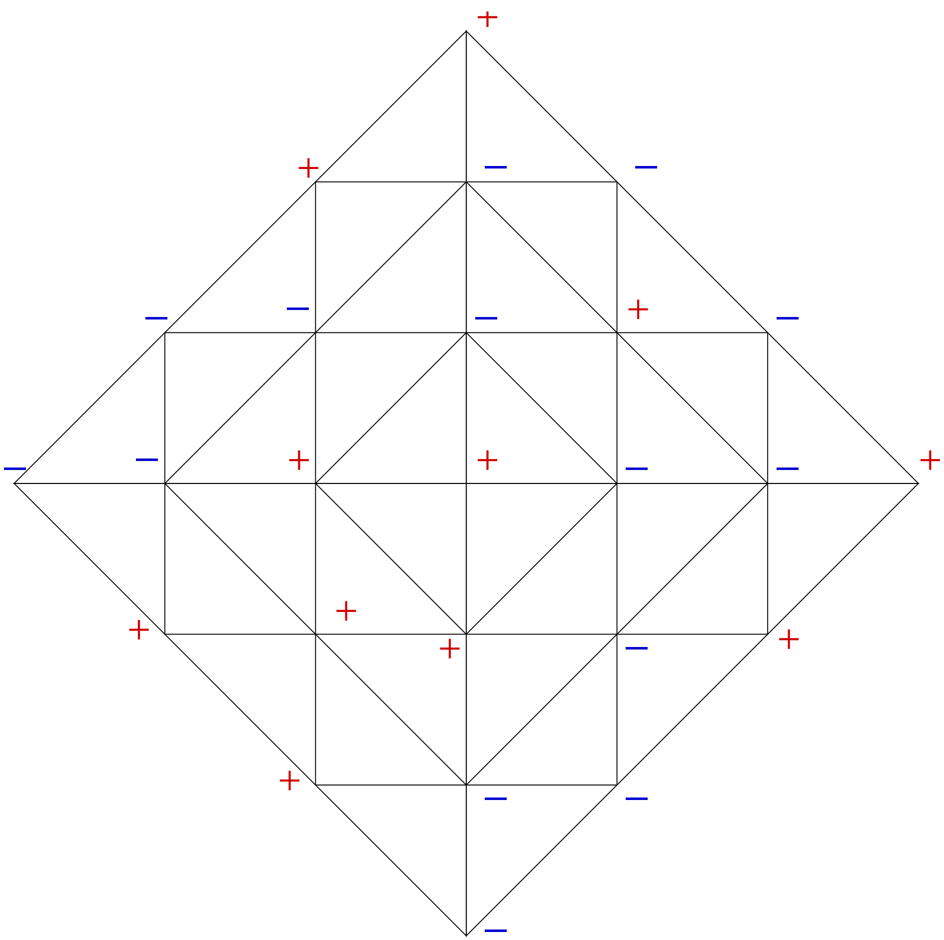
- Let Δ be a polytope with integer vertices.
- τ a convex triangulation of Δ.
- D a sign distribution at the vertices of τ.

T-construction



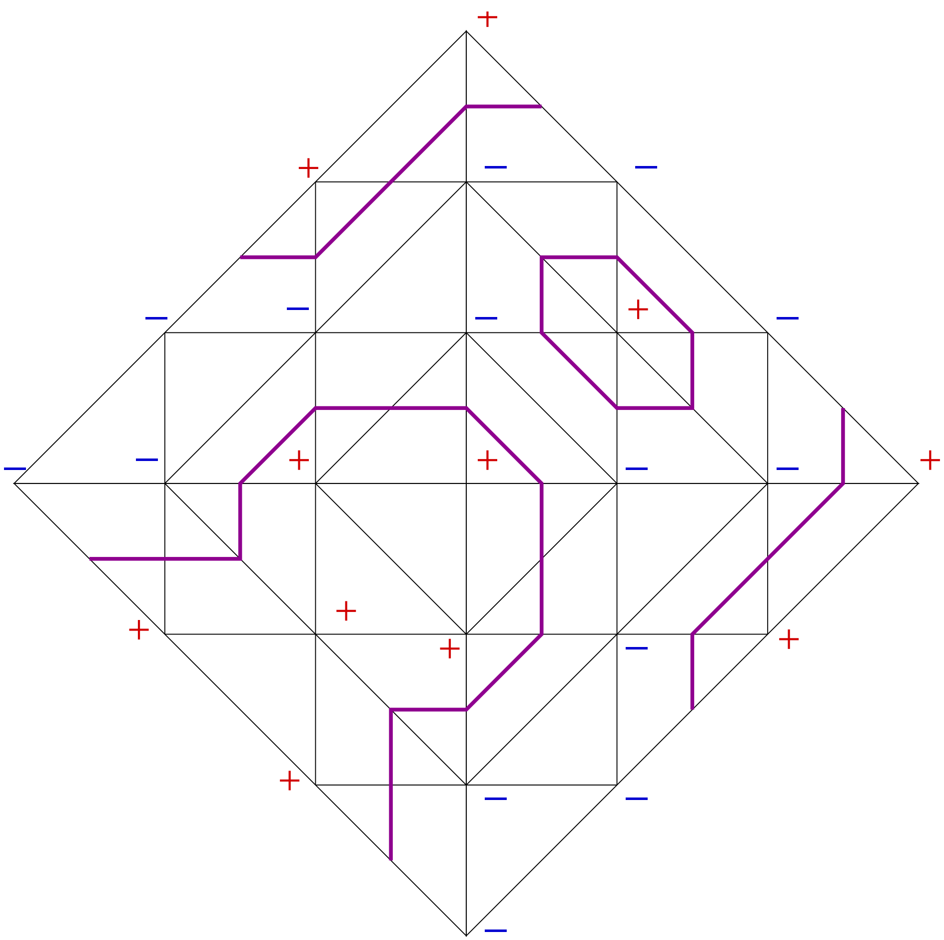
- Take symmetric copies of Δ and τ to obtain Δ^* and τ^*.

T-construction



- Take symmetric copies of Δ and τ to obtain Δ^* and τ^*.
- Extend the sign distribution to τ^*.

T-construction



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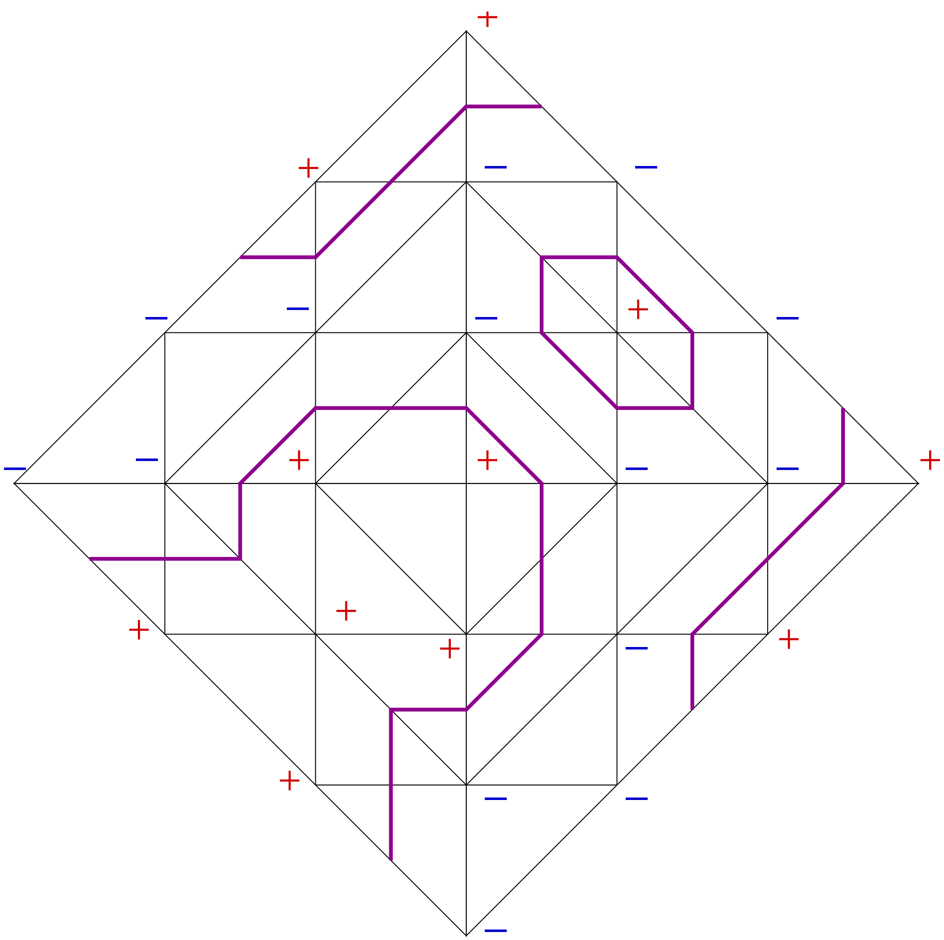
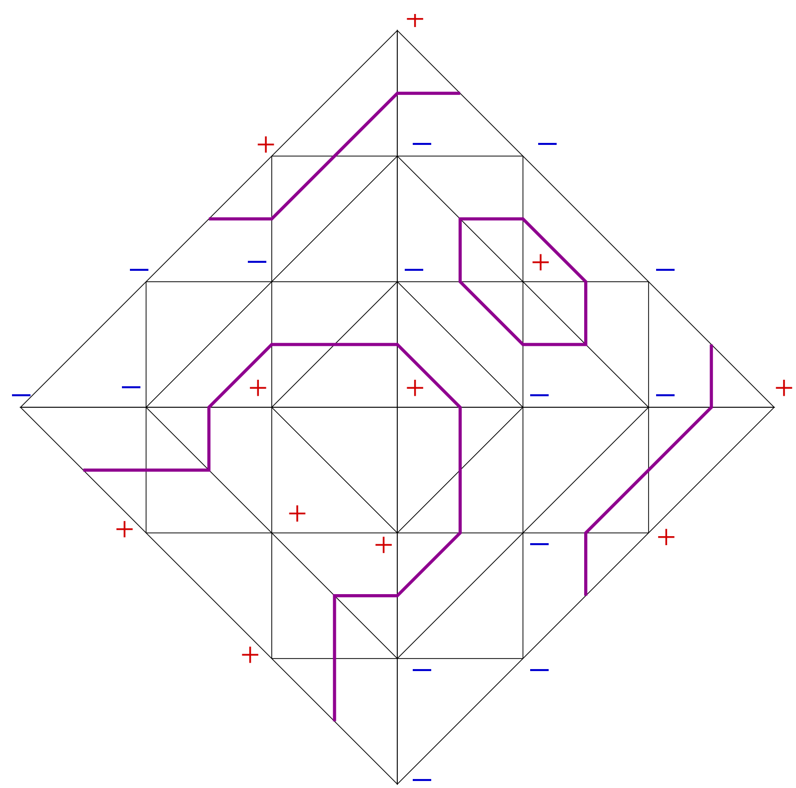


- Take symmetric copies of Δ and τ to obtain Δ^* and τ^*.
- Extend the sign distribution to τ^*.
- Separate $+$ and $-$ in each simplex by hyperplane pieces.
- Identify facets of Δ^* according to the parity of their primitive integer normal vectors $\rightarrow \overline{\Delta},\ H$.

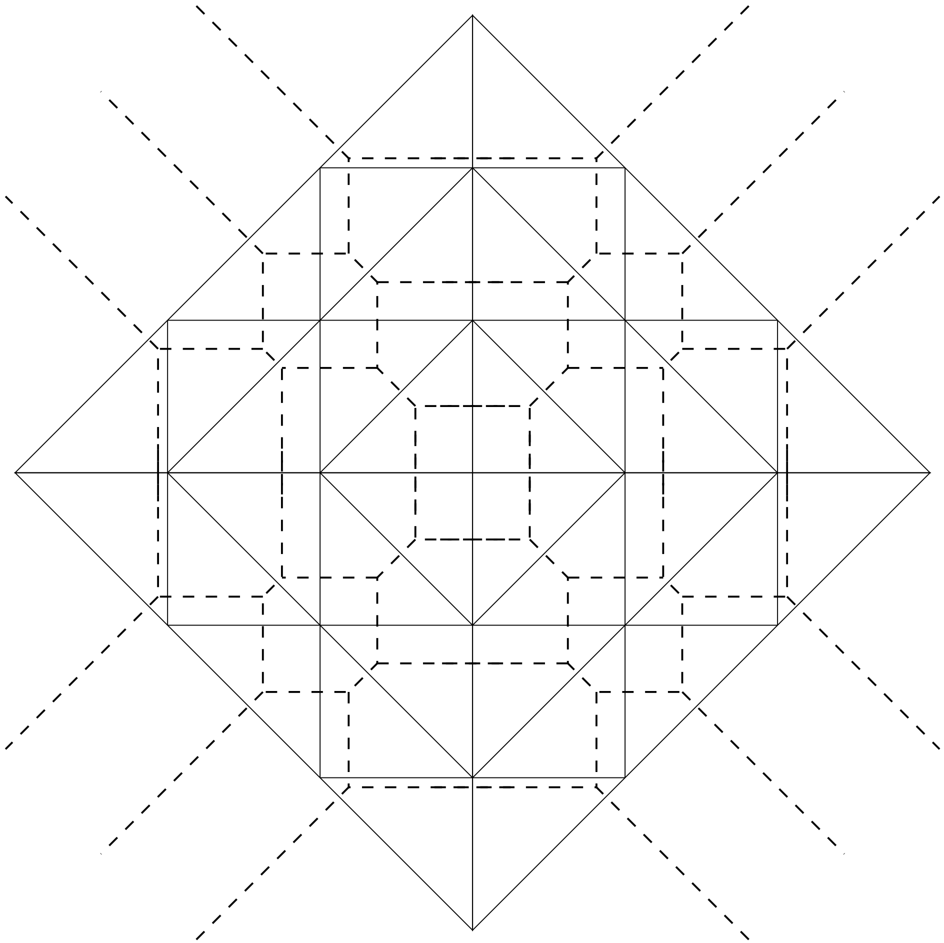
Viro's Theorem

Theorem 5 (Viro) *There exists a real algebraic hypersurface Z in X_Δ with Newton polytope Δ and a homeomorphism $h : \mathbb{R}X_\Delta \to \overline{\Delta}$ such that $h(\mathbb{R}Z) = H$.*



"Tropical" Patchworking

The above construction is equivalent to:



- Draw the symmetric copies of the tropical hypersurface (and of its dual triangulation) in each orthant.

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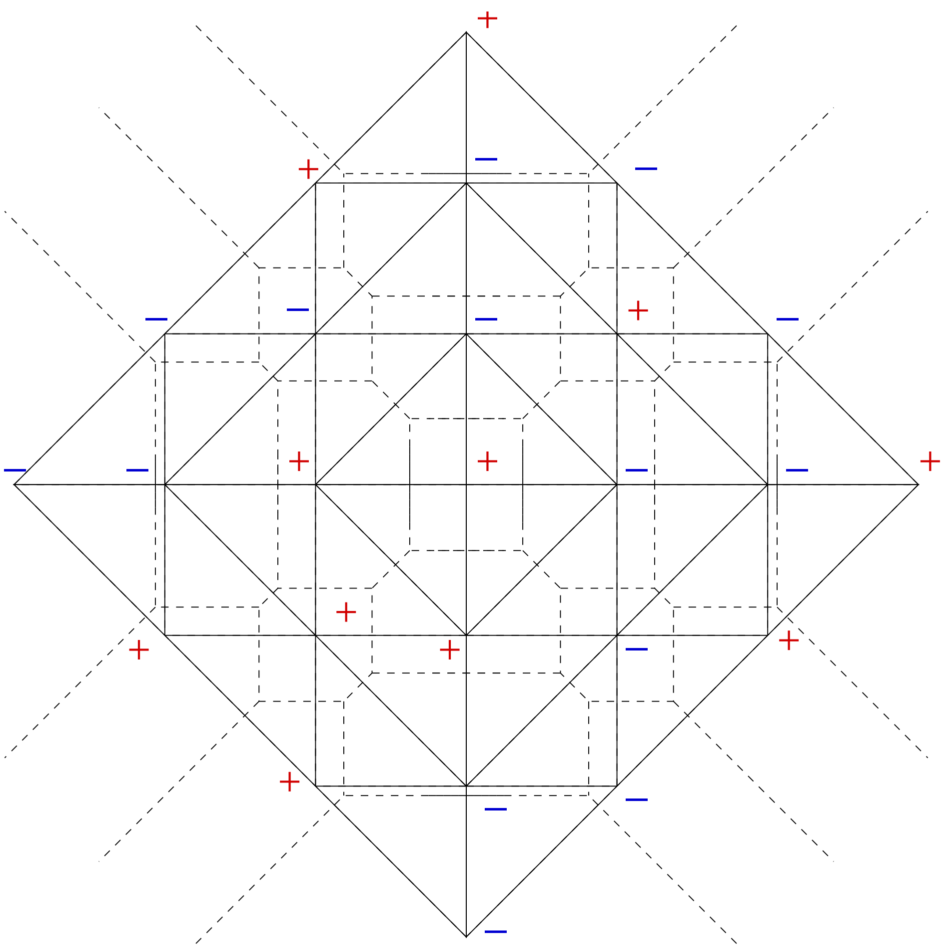

- Draw the symmetric copies of the tropical hypersurface (and of its dual triangulation) in each orthant.
- Take the sign distribution as above.

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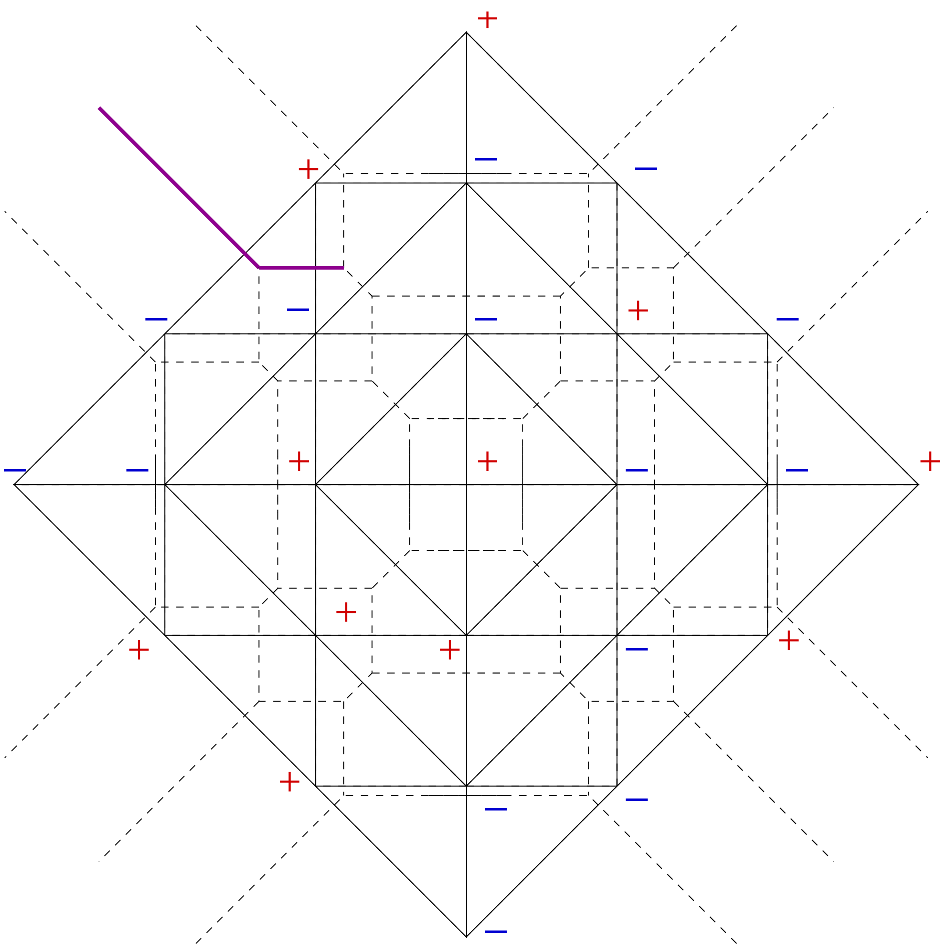

- Draw the symmetric copies of the tropical hypersurface (and of its dual triangulation) in each orthant.
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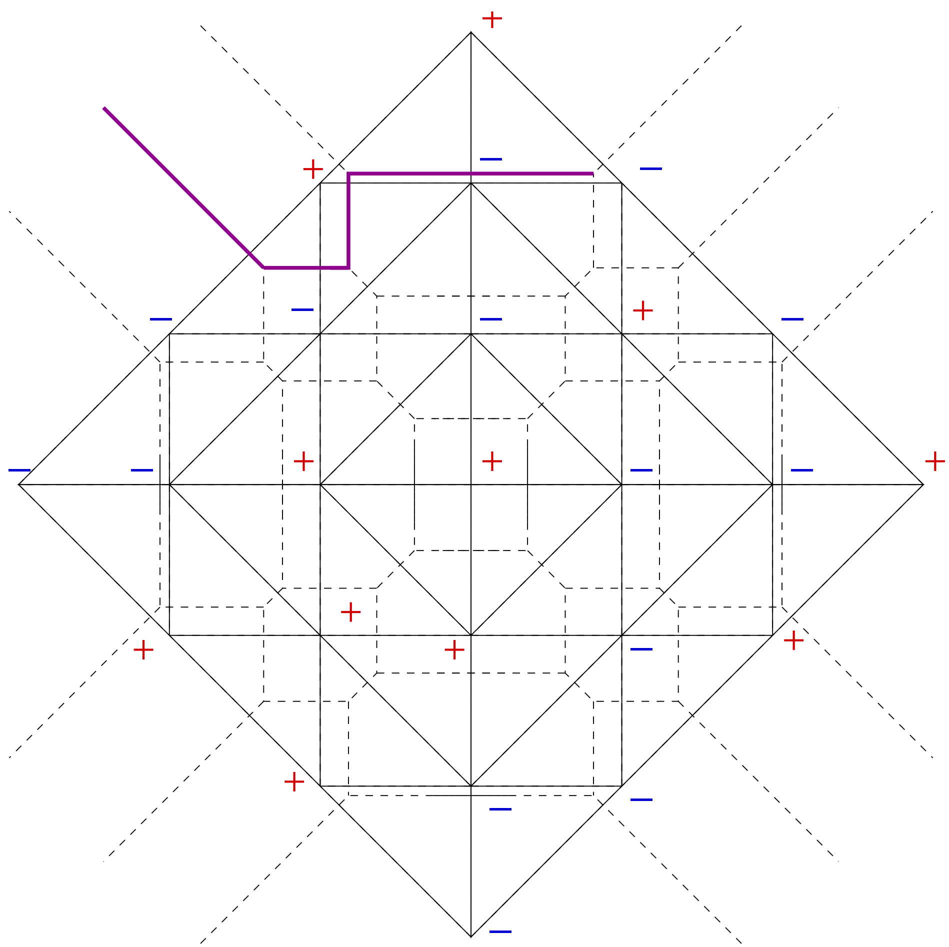

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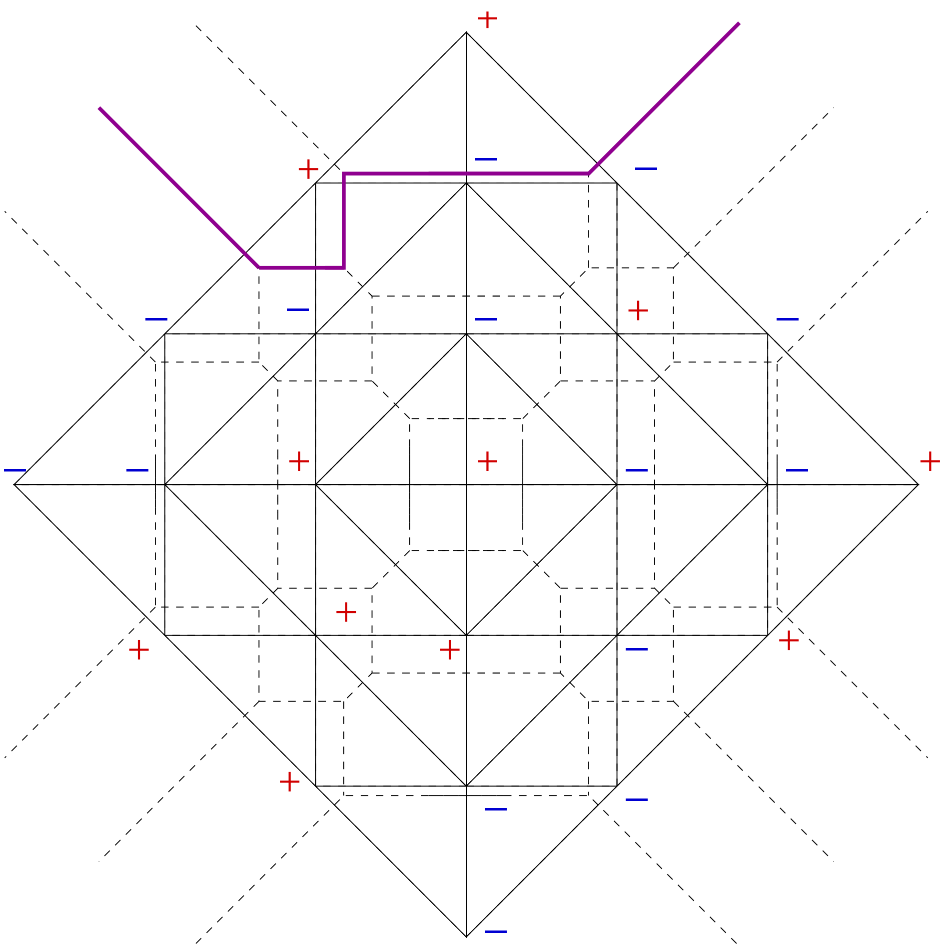

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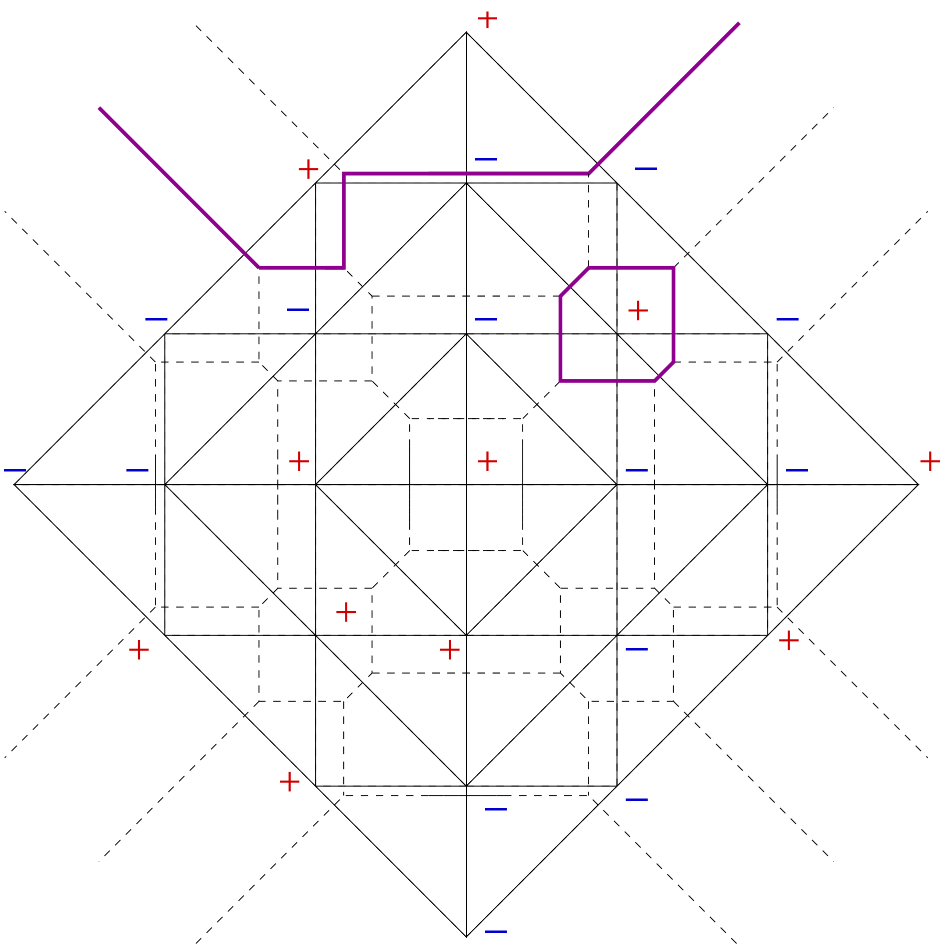

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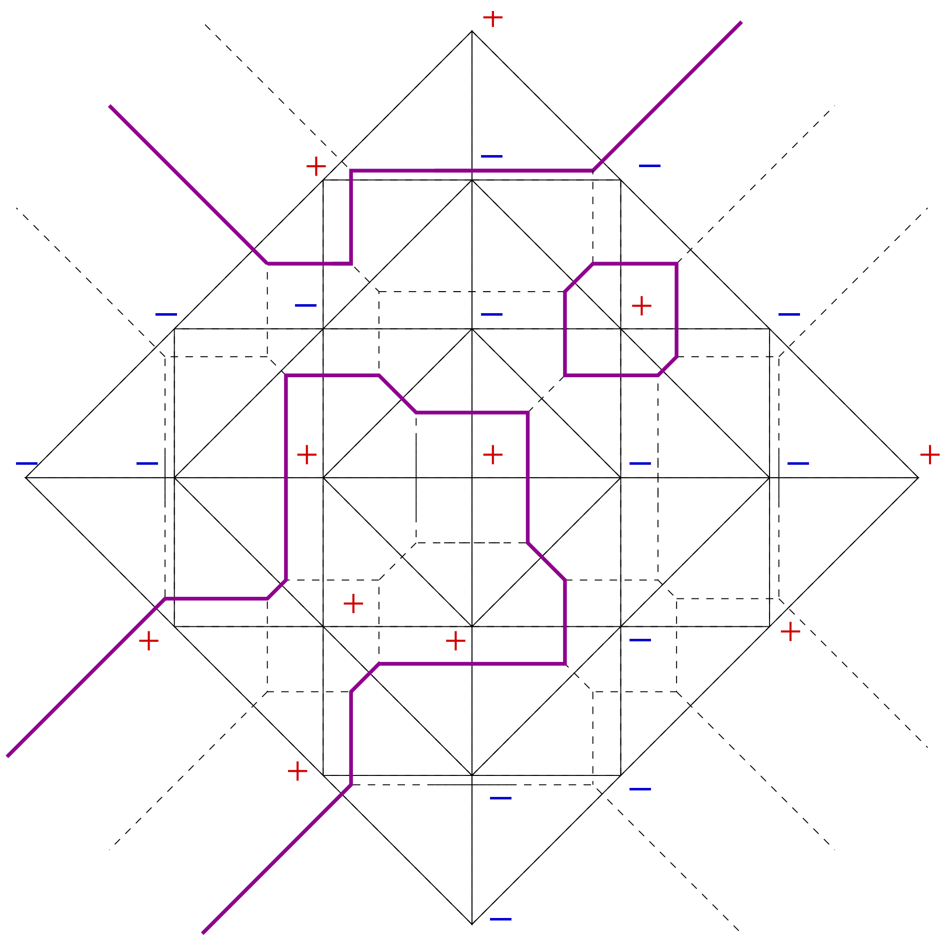

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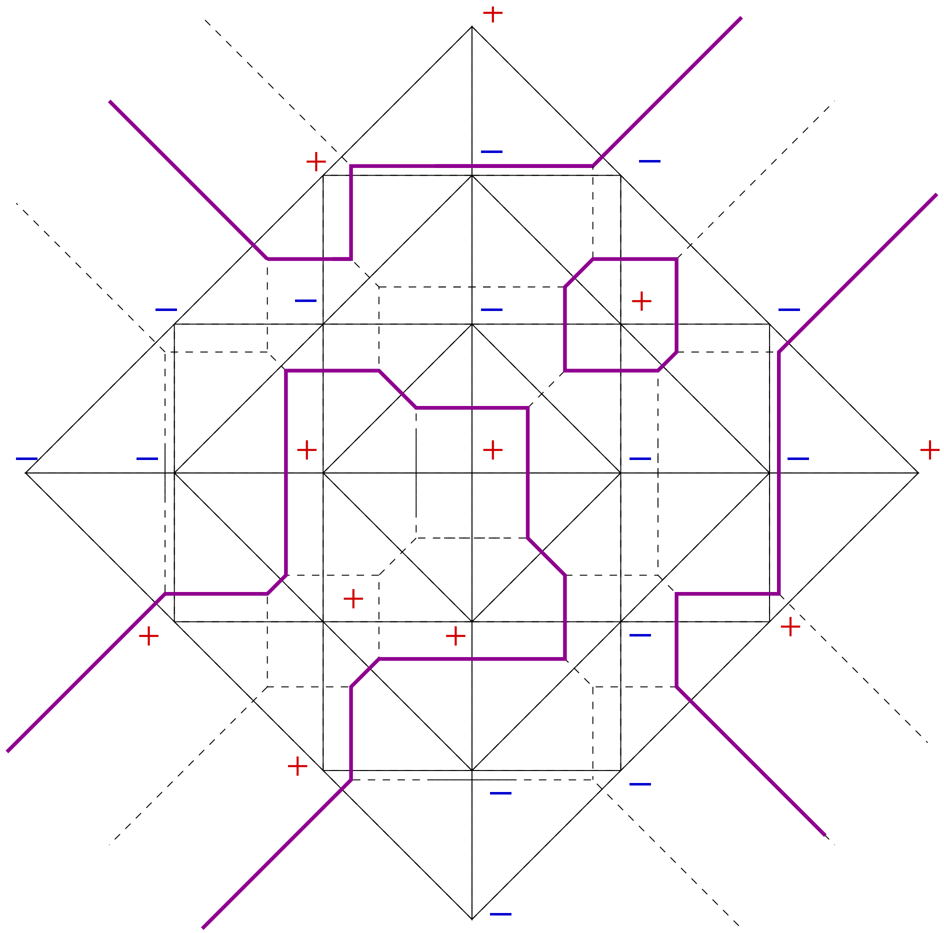

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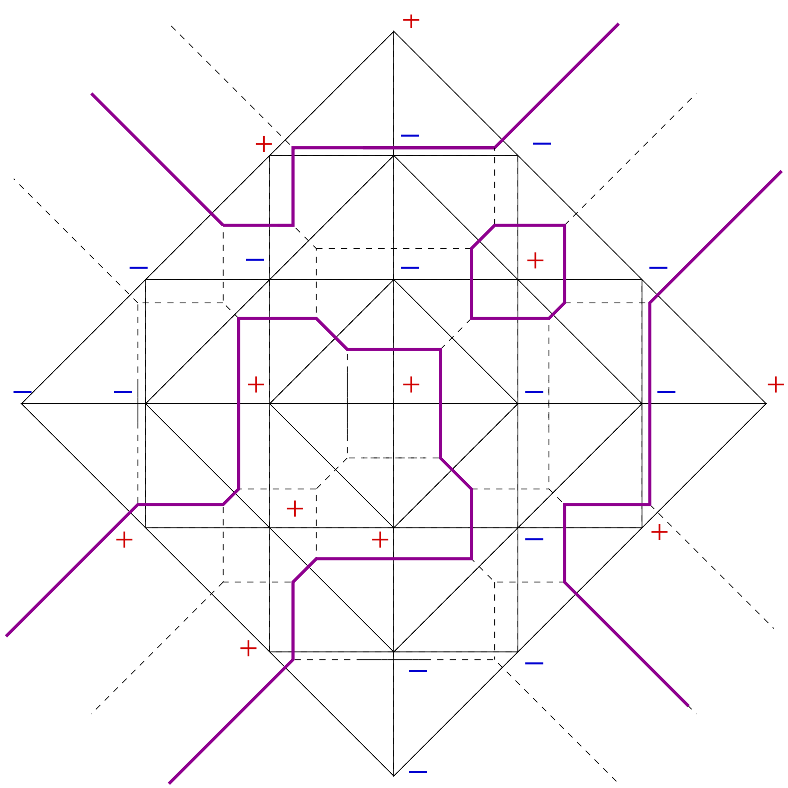
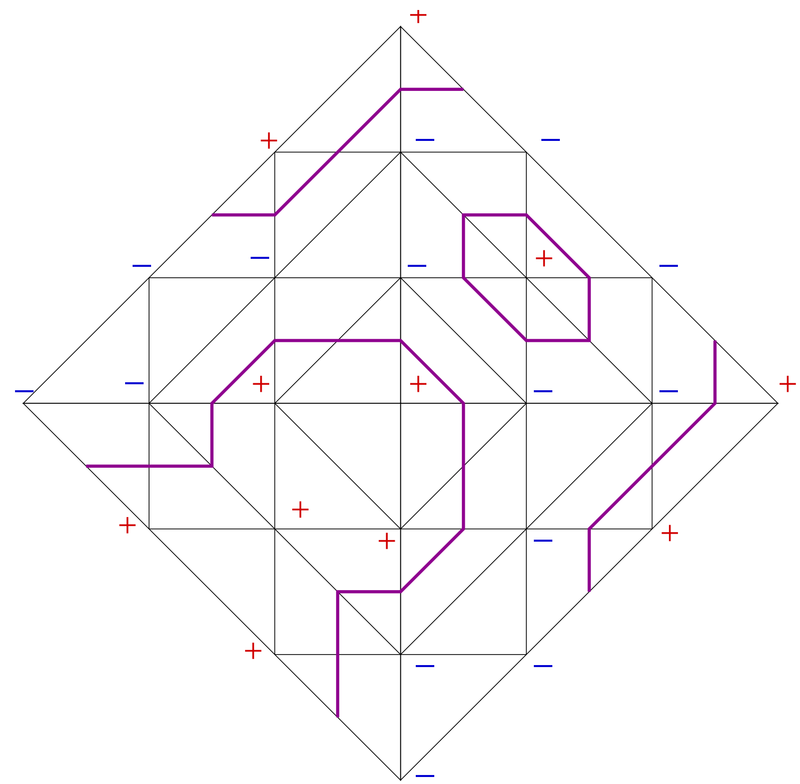
Pachworking

Real tropical curve and its dual subdivision.



Viro method: combinatorial patchworking of a cubic.



Theorem

Assume X_Δ is nonsinguliar and τ is primitive (simplices have volume $\frac{1}{n!}$).

Let Z be the hypersurface from Viro's Theorem. (It is an algebraic hypersurface with Newton polytope Δ.)

$$\sigma(Z) := \sum_{p+q=0\ [2]} (-1)^p h^{p,q}(Z) = \begin{cases} \text{signature of } Z & \text{if } \dim_{\mathbb{C}} Z = 0[2]\,, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6 $\chi(H) = \sigma(Z)$.

Proof

The triangulation τ of Δ induces a cellular decomposition of H: each k-simplex of τ^* contains at most one $(k-1)$-cell.

Remark 7 *The number n_k of $(k-1)$-cells in the symmetric copies of a k-simplex s depends neither on the sign distribution nor on s.*

Proposition 8 (Itenberg)

$$n_k = 2^n - 2^{n-k}.$$

Proof

If $s \in \partial\Delta$, one has to consider identifications:
if s is contained in j facets then s contributes for

$$\frac{2^n - 2^{n-k}}{2^j} \ (k-1)- \text{ cells.}$$

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Theorem 10 (Ehrhart's polynomial) *The number of integer points in a multiple $\lambda\Delta$ of the polytope Δ is given by a polynomial in λ of degree $n = \dim\Delta$.*

$$Ehr_\Delta(\lambda) = \sum_{i=0}^{n} a_i^\Delta \lambda^i$$

Proof

The number of simplices of dimension k of a primitive triangulation τ depends only on Δ.

Proposition 11 (Dais) *The number of k-dimensional simplices in the interior of Δ is:*

$$\mathrm{nbs}_k^\Delta = \sum_{l=k+1}^{n+1} k! S_2(l, k+1)(-1)^{n-l+1} . a_{l-1}^\Delta ,$$

where $S_2(i,j) = 1/(j)! \sum_{m=0}^{j} (-1)^{j-m} \mathrm{C}_j^m m^i$ is the second Stirling number.

Proof

Then

$$\chi(H) = \sum_{i=1}^{n} \sum_{F \in \mathcal{F}_i(\Delta)} \sum_{l=2}^{i+1} \chi_{l,i+1} a_{l-1}^{F}$$

with $\chi_{l,i+1} := (-1)^{i-l+1} \sum_{j=0}^{i-1} \frac{(2^i - 2^j)}{i-j+1} \sum_{k=0}^{i-j+1} (-1)^k \mathrm{C}_{i-j+1}^{k} k^l$

Danilov and Khovanskii Formulae

We have : $\sigma(Z) = \sum_{p+q=0\ [2]} (-1)^p h^{p,q}(Z)$.

Theorem 12 (Danilov and Khovanskii)

$$
\begin{aligned}
h^{p,p}(Z) &= (-1)^{p+1} \sum_{i=p+1}^{n} (-1)^i \mathrm{C}_i^{p+1} f_i(\Delta) \\
h^{\frac{n-1}{2},\frac{n-1}{2}}(Z) &= (-1)^{\frac{n+1}{2}} \sum_{i=\frac{n+1}{2}}^{n} (-1)^i \mathrm{C}_i^{\frac{n+1}{2}} f_i(\Delta) - \sum_{i=\frac{n+1}{2}}^{n} \sum_{F\in\mathcal{F}_i(\Delta)} (-1)^i \Psi_{\frac{n+1}{2}}(F) \\
h^{p,n-1-p}(Z) &= (-1)^n \sum_{i=p+1}^{n} \sum_{F\in\mathcal{F}_i(\Delta)} (-1)^i \Psi_{p+1}(F) \\
h^{p,q}(Z) &= 0 \text{ otherwise.}
\end{aligned}
$$

With $\Psi_{p+1}(F) = \sum_{\alpha=1}^{i+1} \sum_{a=0}^{p+1} (-1)^a \mathrm{C}_{i+1}^a (p+1-a)^{\alpha-1} a_{\alpha-1}^F$.

Proof

$$\sigma(Z) \quad = \quad \sum_{i=1}^{n} \sum_{F \in \mathcal{F}_i(\Delta)} \sum_{l=2}^{i+1} \sigma_{l,i+1} a_{l-1}^{F},$$

with $\sigma_{l,i+1} \quad := \quad \sum_{p=0}^{n-1} (-1)^i (-1)^{p+1} \sum_{q=0}^{p+1} (-1)^q \mathrm{C}_{i+1}^{q} (p+1-q)^{l-1}.$

Proof

$$\sigma(Z) \;=\; \sum_{i=1}^{n} \sum_{F\in\mathcal{F}_i(\Delta)} \sum_{l=2}^{i+1} \sigma_{l,i+1} a_{l-1}^{F},$$

with $\sigma_{l,i+1} \;:=\; \sum_{p=0}^{n-1} (-1)^i (-1)^{p+1} \sum_{q=0}^{p+1} (-1)^q \mathrm{C}_{i+1}^{q} (p+1-q)^{l-1}.$

$$\chi(H) \;=\; \sum_{i=1}^{n} \sum_{F\in\mathcal{F}_i(\Delta)} \sum_{l=2}^{i+1} \chi_{l,i+1} a_{l-1}^{F},$$

with $\chi_{l,i+1} \;=\; (-1)^{i-l+1} \sum_{j=0}^{i-1} \frac{(2^i - 2^j)}{i-j+1} \sum_{k=0}^{i-j+1} (-1)^k \mathrm{C}_{i-j+1}^{k} k^l.$