Euler Characteristic of nonsingular real tropical hypersurfaces

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Real tropical curve and its dual subdivision.

Viro method: combinatorial patchworking of a cubic.
**Tropical Varieties**

\[ g(t) = \sum_{r \in R} b_r t^r \in \mathbb{K}. \]

Where \( b_r \in \mathbb{C}, \) \( R \subset \mathbb{Q} \) bounded by below, contained in an arithmetic sequence. Valuation:

\[ \text{val}(g(t)) := \min \{ r/b_r \neq 0 \}, \quad v(g) := -\text{val}(g). \]

**Definition 1** A tropical hypersurface is the closure of the image under \( V \) of a hypersurface in \((\mathbb{K}^*)^n:\)

\[ f(z) = \sum_{\omega \in A} c_\omega z^\omega \quad \text{avec} \; A \in \mathbb{Z}^n, \]

\[ |A| < +\infty, \; c_\omega \in \mathbb{K}^*, \; z = (z_1, \ldots, z_n) \]

\[ Z_f := \{ z \in (\mathbb{K}^*)^n / f(z) = 0 \} \]

\[ V : (\mathbb{K}^*)^n \longrightarrow \mathbb{R}^n \]

\[ z \; \longmapsto \; (v(z_1), \ldots, v(z_n)) \]

\[ T_f := \overline{V(Z_f)} \subset \mathbb{R}^n \]
Examples

Tropical line
Examples

Tropical line

Bidegree (1,1) curve
Kapranov’s Theorem

\[ f(z) = \sum_{\omega \in A} c_\omega z^\omega \]

Put \( \nu : A \rightarrow \mathbb{R} \)
\[ \omega \mapsto -\nu(c_\omega) \]

\[ \mathcal{L}(\nu) : (\mathbb{R})^n \rightarrow \mathbb{R} \]
\[ x \mapsto \max(x \cdot \omega - \nu(\omega)) \]

The Legendre transform \( \mathcal{L}(\nu) \) of \( \nu \) is a piecewise-linear convex function.

**Theorem 2 (Kapranov)** \( T_f \) is the nonlinearity domain of \( \mathcal{L}(\nu) \).

\[ V(Z_f) = \text{corner locus}(x \mapsto \max(x \cdot \omega + \nu(c_\omega))) \]
Example in dim. one

\[ f(x) = t.x^0 + 1.x + t^{-2}.x^2 \]

Tropical roots: corner locus of \( x \mapsto \max\{0.x + 1, x + 0, 2x - 2\} \)
Kapranov’s Theorem in dim. 2
Example

Tropical conic
Example

Tropical conic

Another conic
Example

Tropical cubic
Duality

\[ f(z) = \sum_{\omega \in A} c_\omega z^\omega, \quad \Delta = \text{ConvHull}(A) \] Newton polytope of \( f \).

\[ \Gamma := \text{ConvHull}(\omega, \nu(c_\omega)), \omega \in A \]

\[ \nu : \Delta \rightarrow \mathbb{R} \]
\[ x \mapsto \min \{ x/(\omega, x) \in \Gamma \} \]

The linearity domains of \( \nu \) are the \( n \)-cells of a convex polyhedral subdivision \( \tau \) of \( \Delta \).
$T_f$ induces a subdivision $\Xi$ of $\mathbb{R}^n$. Subdivisions $\tau$ and $\Xi$ are dual:

There is a one-to-one inclusion reversing correspondence $L$ between cells of $\Xi$ and cells of $\tau$ such that for any $\xi \in \Xi$,

1. $\dim L(\xi) = \text{codim} \xi$,
2. $L(\xi) \perp \xi$.

$T_f$ is said nonsingular if $\tau$ is primitive ($n$-simplices have volume $\frac{1}{n!}$).
Duality

\[ f_{\text{trop}} : \quad (\mathbb{R})^n \rightarrow \mathbb{R} \quad x \rightarrow \max(x \cdot \omega - \nu(\omega)) \]

\[ \nu : \quad A \rightarrow \mathbb{R} \quad \omega \rightarrow -\nu(c_\omega) \]
Complex tropical hypersurfaces

\( \mathbb{K} \) Field of Puiseux series. \( g(t) = \sum_{r \in \mathbb{R}} b_r t^r \in \mathbb{K}. \)
valuation: \( \text{val}(g(t)) = \min \{ r/b_r \neq 0 \}, \) \( v(g) := -\text{val}(g), \)
\( f(z) = \sum_{\omega \in A} c_\omega z^\omega. \)

\[ Z_f := \{ z \in (\mathbb{K}^*)^n / f(z) = 0 \}, \quad \text{arg}(g(t)) := \text{arg}(b_{\text{val}(g(t))}). \]

\[ W := V \times \text{Arg} : (\mathbb{K}^*)^n \longrightarrow \mathbb{R}^n \times (S^1)^n \cong (\mathbb{C}^*)^n \]
\[ z \longmapsto ((v(z_1), \ldots, v(z_n)), (\text{arg}(z_1), \ldots, \text{arg}(z_n))) \]

\[ \mathcal{W}(z) := (e^{v(z_1)+i\text{arg}(z_1)}, \ldots, e^{v(z_n)+i\text{arg}(z_n)}). \]

Definition 3 A complex tropical hypersurface is the closure of the image under \( \mathcal{W} \) of a hypersurface in \((\mathbb{K}^*)^n:\)

\[ \mathbb{C}T_f := \overline{\mathcal{W}(Z_f)} \subset \mathbb{C}^n \]
Example

Complex tropical line
Real tropical hypersurfaces

\[ f(z) = \sum_{\omega \in A} c_\omega z^\omega \text{ with } c_\omega = \sum \alpha_r t^r \text{ and } \alpha_r \in \mathbb{R}. \]

Definition 4 \( \mathbb{R}T_f := \mathbb{C}T_f \cap (\mathbb{R}^n \times \{0, \pi\}^n). \)
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For $\mathbb{R}T \cap (\mathbb{R}^n \times \{p\})$, $p \in \{0, \pi\}^n$, $\text{sign } \omega := e^{i<p, \omega>} \cdot \text{sign } c_\omega$. 
T-construction

Let $\Delta$ be a polytope with integer vertices.
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$D$ a sign distribution at the vertices of $\tau$. 
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- Extend the sign distribution to $\tau^*$.
- Separate $+$ and $-$ in each simplex by hyperplane pieces.
- Identify facets of $\Delta^*$ according to the parity of their primitive integer normal vectors $\rightarrow \overrightarrow{\Delta}, H$. 
Viro’s Theorem

Theorem 5 (Viro) There exists a real algebraic hypersurface $Z$ in $X_\Delta$ with Newton polytope $\Delta$ and a homeomorphism $h : \mathbb{R}X_\Delta \to \overline{\Delta}$ such that $h(\mathbb{R}Z) = H$. 
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The above construction is equivalent to:

Draw the symmetric copies of the tropical hypersurface (and of its dual triangulation) in each orthant.
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Viro method: combinatorial patchworking of a cubic.
Theorem

Assume $X_\Delta$ is nonsingular and $\tau$ is primitive (simplices have volume $\frac{1}{n!}$).

Let $Z$ be the hypersurface from Viro’s Theorem. (It is an algebraic hypersurface with Newton polytope $\Delta$.)

$$\sigma(Z) := \sum_{p+q=0} (-1)^p h^{p,q}(Z) = \begin{cases} 
\text{signature of } Z & \text{if } \dim_{\mathbb{C}} Z = 0[2], \\
0 & \text{otherwise.}
\end{cases}$$

Theorem 6 \(\chi(H) = \sigma(Z).\)
Proof

The triangulation $\tau$ of $\Delta$ induces a cellular decomposition of $H$: each $k$-simplex of $\tau^*$ contains at most one $(k-1)$-cell.

Remark 7  The number $n_k$ of $(k - 1)$-cells in the symmetric copies of a $k$-simplex $s$ depends neither on the sign distribution nor on $s$.

Proposition 8 (Itenberg)

$$n_k = 2^n - 2^{n-k}.$$
Proof

If \( s \in \partial \Delta \), one has to consider identifications:
if \( s \) is contained in \( j \) facets then \( s \) contributes for

\[
\frac{2^n - 2^{n-k}}{2j} \ (k - 1) \ - \ \text{cells}.
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Theorem 10 (Ehrhart’s polynomial)  The number of integer points in a multiple \( \lambda \Delta \) of the polytope \( \Delta \) is given by a polynomial in \( \lambda \) of degree \( n = \dim \Delta \).

\[
Ehr_{\Delta}(\lambda) = \sum_{i=0}^{n} a_i^{\Delta} \lambda^i
\]

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Proof

The number of simplices of dimension $k$ of a primitive triangulation $\tau$ depends only on $\Delta$.

**Proposition 11 (Dais)** *The number of $k$-dimensional simplices in the interior of $\Delta$ is:*

$$
nbs_k^n = \sum_{l=k+1}^{n+1} k! S_2(l, k+1)(-1)^{n-l+1}.a_{l-1}^\Delta,
$$

where $S_2(i, j) = 1/(j)! \sum_{m=0}^{j}(-1)^{j-m}C_j^m m^i$ is the second Stirling number.
Proof

Then

\[ \chi(H) = \sum_{i=1}^{n} \sum_{F \in \mathcal{F}_i(\Delta)} \sum_{l=2}^{i+1} \chi_{l,i+1} a_{i-1}^F \]

with \( \chi_{l,i+1} := (-1)^{i-l+1} \sum_{j=0}^{i-1} \frac{(2^i - 2^j)}{i-j+1} \sum_{k=0}^{i-j+1} (-1)^k \binom{k}{i-j+1} k^l \)
We have: \( \sigma(Z) = \sum_{p+q=0} (-1)^p h^{p,q}(Z) \).

**Theorem 12 (Danilov and Khovanskii)**

\[
\begin{align*}
    h^{p,p}(Z) &= (-1)^{p+1} \sum_{i=p+1}^{n} (-1)^i C_{i}^{p+1} f_i(\Delta) \\
    h^{\frac{n-1}{2}, \frac{n-1}{2}}(Z) &= (-1)^{\frac{n+1}{2}} \sum_{i=\frac{n+1}{2}}^{n} (-1)^i C_{i}^{\frac{n+1}{2}} f_i(\Delta) - \sum_{i=\frac{n+1}{2}}^{n} \sum_{F \in \mathcal{F}_i(\Delta)} (-1)^i \Psi_{\frac{n+1}{2}}(F) \\
    h^{p,n-1-p}(Z) &= (-1)^n \sum_{i=p+1}^{n} \sum_{F \in \mathcal{F}_i(\Delta)} (-1)^i \Psi_{p+1}(F) \\
    h^{p,q}(Z) &= 0 \text{ otherwise.}
\end{align*}
\]

With \( \Psi_{p+1}(F) = \sum_{\alpha=1}^{i+1} \sum_{a=0}^{p+1} (-1)^a C_{i+1}^{a} (p + 1 - a)^{\alpha-1} a^F_{\alpha-1} \).
Proof

\[ \sigma(Z) = \sum_{i=1}^{n} \sum_{F \in \mathcal{F}_i(\Delta)} \sum_{l=2}^{i+1} \sigma_{l,i+1} a_{l-1}^F, \]

with \( \sigma_{l,i+1} := \sum_{p=0}^{n-1} (-1)^i (-1)^{p+1} \sum_{q=0}^{p+1} (-1)^q C_{i+1}^q (p + 1 - q)^{l-1}. \)
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with \[ \chi_{l,i+1} = (-1)^{i-l+1} \sum_{j=0}^{i-1} \frac{(2^i - 2j)}{i-j+1} \sum_{k=0}^{i-j+1} (-1)^k \binom{k}{i-j+1} k^l. \]