Finiteness properties of characteristic classes of flat bundles

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Abstract

We prove that primary characteristic classes of flat G-bundles can be represented by cocycles taking only finitely many values on singular simplices when G is a real algebraic group. Gromov previously showed that a bounded representative exists. In contrast to Gromov's proof, we do not rely on Hironaka's resolution of singularity. Instead our method involves standard techniques of semi-algebraic sets.

1 Introduction

The first boundedness property of characteristic numbers of flat bundles is probably Milnor's characterization of flat bundles over surfaces ([Mi58]), later generalized to the unoriented case by Wood ([Wo71]).

Theorem 1 (Milnor-Wood inequality) *Let ξ be a $\mathrm{SL}_2\mathbb{R}$-bundle over a surface Σ_g of genus $g \geq 1$. The bundle ξ is flat if and only if its Euler class $\varepsilon(\xi) \in H^2(\Sigma_g)$ satisfies*

$$|\varepsilon(\xi)[\Sigma_g]| \leq g - 1.$$

This result, or more precisely one of its implications, can in a natural way be put in the context of singular bounded cohomology. Indeed, the following theorem proven by Ivanov and Turaev in [IvTu82] shows that the Euler class of any flat $\mathrm{SL}_n\mathbb{R}$-bundle can be represented by a bounded cocycle. (A singular cocycle is said to be bounded, if its set of values on singular simplices is bounded, or equivalently, if its norm $\|.\|_\infty$ is finite. See Section 2 for further details.)

Theorem 2 *If ξ is a flat $\mathrm{SL}_n\mathbb{R}$-bundle over a CW-complex B, then its Euler class $\varepsilon(\xi) \in H^n(B)$ satisfies*

$$\|\varepsilon(\xi)\|_\infty \leq \frac{1}{2^n}.$$

This bound on the Euler class, together with the knowledge of the ℓ^1-norm of the fundamental class of a surface Σ_g (also called simplicial volume) implies half of the Milnor-Wood inequality, as pointed out by Ghys in [Ghys87] (see also [Ghys99]). It is a simple consequence of the duality of the two norms.

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In his seminal paper [Gr82], Gromov generalized the boundedness of the Euler class of flat bundles to all characteristic classes:

Theorem 3 *Let G be a real algebraic subgroup of $\mathrm{GL}_n(\mathbb{R})$. Then every primary characteristic class of flat G-bundle can be represented by a bounded cocycle.*

As explained further in Section 3, by a (primary) characteristic class of flat G-bundles is meant a cohomology class in the image of $H^*(BG) \to H^*(BG^\delta)$, where G^δ denotes the group G endowed with the discrete topology.

An immediate corollary of Gromov's theorem (Theorem 3 here) is that a topological space with amenable fundamental group does not possess any nontrivial characteristic class of flat G-bundle, when G is a real algebraic subgroup of $\mathrm{GL}_n(\mathbb{R})$.

The hypothesis in the above theorem of Gromov (Theorem 3) that G be algebraic cannot be removed. Indeed, Goldman gives in [Go81] an example of a flat G-bundle over the 2-torus with nontrivial characteristic class in degree 2. This class cannot be bounded since the bounded cohomology of the torus is trivial. The group G considered is the quotient of the Heisenberg group H of upper triangular unipotent 3 by 3 matrices with the normal subgroup generated by any central element, and the characteristic class in $H^2(BG)$ is the obstruction to the existence of a section of the universal bundle over BG.

We give in the present paper a new proof of Gromov's theorem with the advantage that a representative for every characteristic class of flat bundle can be found whose set of values on singular simplices is not only bounded, but furthermore finite. We thus prove:

Theorem 4 *Let G be a real algebraic subgroup of $\mathrm{GL}_n(\mathbb{R})$. Then every primary characteristic class of flat G-bundle can be represented by a cocycle whose set of value on singular simplices is finite.*

The first step of the proof, which is common to both Gromov's original proof of Theorem 3 and our Theorem 4, is to reduce to a simplicial version of the statement:

Theorem 5 *Let G be a real algebraic subgroup of $\mathrm{GL}_n(\mathbb{R})$ and $\beta \in H^q(BG)$ a characteristic class. There exists a finite subset I of $\mathbb{R}$ such that for every flat G-bundle ξ over a simplicial complex K, the cohomology class $\beta(\xi) \in H^q(|K|)$ can be represented by a cocycle whose set of values on the q-simplices of K is contained in I.*

Again, the case of the Euler class was already well known: Sullivan proved in [Su76] that the Euler class of any flat $\mathrm{SL}_n(\mathbb{R})$-bundle over a simplicial complex can be represented by a simplicial cocycle taking values in $\{-1, 0, 1\}$ and Smillie improved this to $\{-1/2^n, 0, 1/2^n\}$.

Let us point out that both the proofs of the simplicial version of the theorem and the reduction to it are not only completely different from Gromov's but also much more elementary. It is in our case only a technical artifice to show how one can reduce to the simplicial version of the theorem - or to be more precise, a stronger version of it formulated in Theorem 16 where the simplicial cocycle can furthermore be chosen to be the pullback by any given classifying map $F : |K| \to BG^\delta$ of an alternating singular cochain on BG^δ. The main difficulty of Theorem 4 thus really lies in the proof of this simplicial version. While Gromov needs Hironaka's deep resolution of singularities, our main tool is the following bounded version of the existence of a finite triangulation of semi-algebraic sets as developed by Benedetti and Risler in [BeRi90], from which the following theorem can be deduced:

Theorem 6 *Let X be a compact semi-algebraic set, with a semi-algebraic triangulation T. Let K be a simplicial complex and $f : |K| \to X$ a semi-algebraic map, whose complexity is uniformly bounded on every simplex of K. Then there exists a simplicial approximation of f on a uniformly bounded refinement of K.*

The latter theorem appears in this note, in a slightly modified and adapted form as the Technical Lemma 15.

This paper is structured as follows: We start in Section 2 by a quick reminder on singular bounded cohomology. In Section 3 we present the model of classifying space we chose to work with and define characteristic classes of flat bundles. In Section 4 we define semi-algebraic sets and present all the technical results to be needed in the proof of the simplicial version of our main theorem, which we carry through in Section 5. After introducing the necessary tool of inverse limits, we prove the singular version of our main theorem and discuss some alternative proofs in Section 6 .

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2 Bounded cohomology

We review here, mainly in order to fix the notation, the very basics of the theory of singular bounded cohomology, a theory which was introduced by Gromov in [Gr82].

Let X be a topological space. The space $C_q(X)$ of singular q-chains on X is defined to be the (real) vector space over the basis of singular simplices $S_q(X) = \{\sigma : \Delta^q \to X \mid \sigma \text{ is continuous}\}$. Endowed with its natural boundary operator $\partial : C_q(X) \to C_{q-1}(X)$ it becomes a complex whose homology is the singular homology $H_*(X)$ of X. The ℓ^1-norm corresponding to the canonical basis $S_q(X)$ of $C_q(X)$ is defined as

$$\|z\|_1 = \sum_\sigma |z_\sigma|, \quad \text{for } z = \sum_\sigma z_\sigma \sigma \in C_q(X).$$

This norm induces a semi-norm on the homology of X. If X is an oriented, compact manifold of dimension n, the ℓ^1-norm of its fundamental class $[X] \in H_n(X)$ is called the simplicial volume of X.

While the singular cohomology of X is obtained from the cochain complex $C^*(X)$ defined as the algebraic dual of the space of chains, let us instead consider the topological dual of the normed space $C_q(X)$: The space of (singular) bounded cochains on X is defined as

$$C_b^q(X) = \{c \in C^q(X) \mid \|c\|_\infty < \infty\},$$

where

$$\begin{aligned}\|c\|_\infty &= \sup\{|c(z)| \mid z \in C_q(X),\ \|z\|_1 = 1\} \\ &= \sup\{|c(\sigma)| \mid \sigma \in S_q(X)\}.\end{aligned}$$

The dual coboundary operator on $C^*(X)$ restricts to bounded cochains, so that one defines the (singular) bounded cohomology $H_b^*(X)$ of the space X to be the homology of the complex $(C_b^*(X), \delta)$. Note however that this is not a cohomology theory: the excision axiom does not hold.

We will say that a cohomology class $[c] \in H^q(X)$ is bounded if it can be represented by a bounded cocycle, or equivalently, if it is contained in the image of the comparison map

$$H_b^q(X) \longrightarrow H^q(X),$$

induced by the inclusion of complexes $C_b^*(X) \hookrightarrow C^*(X)$.

3 Classifying space and characteristic classes

Classifying space and characteristic classes

Let G be a topological group. A principal G-bundle $\xi_G = PG \to BG$ is said to be *universal* if for every principal G-bundle $\xi = P \to B$ over a CW-complex B there exists a *classifying map* $f : B \to BG$, unique up to homotopy, such that the bundle ξ is isomorphic to the pull back $f^*(\xi_G)$. The base space BG of the universal bundle ξ_G is called the *classifying space*. We shall exhibit a possible model for BG below, or more precisely, a finite dimensional approximation classifying all bundles over simplicial complexes of bounded dimension.

A *characteristic class* c assigns to any principal G bundle ξ over a topological space B a cohomology class $c(\xi) \in H^q(B)$ such that if $f : B' \to B$ is a continuous map then $c(f^*(\xi)) = f^*(c(\xi)) \in H^q(B')$. Characteristic classes are easily seen to be in one to one correspondence with the cohomology of some (and hence any) classifying space BG.

There are many equivalent definitions for the flatness of principal G-bundles. Let us just introduce the ones which we will use in the present note. Denote by G^δ the group G endowed with the discrete topology. The set theoretic inclusion $G^\delta \to G$ induces a map $BG^\delta \to BG$ between the corresponding classifying spaces. A principal G-bundle is said to be *flat* if its classifying map factorizes, up to homotopy, through BG^δ, or equivalently, if there exists a covering of its base space and transition functions relative to this covering which are locally constant. In the differentiable setting, this is the same as to require that the bundle can be endowed with a connection with vanishing curvature.

A *(primary) characteristic class of flat bundles* is now simply a cohomology class in the image of the map

$$H^*(BG) \longrightarrow H^*(BG^\delta),$$

which is induced from the mapping $BG^\delta \to BG$. Observe that Gromov's Theorem (Theorem 3 here) now admits the following reformulation:

Theorem 7 *Let G be a real algebraic group. The image of the map $H^*(BG) \to H^*(BG^\delta)$ is included in the image of the comparison map $H_b^*(BG^\delta) \to H^*(BG^\delta)$.*

The first examples of both nontrivial flat bundles and nontrivial characteristic classes were given by Milnor in [Mi58], where flat bundles over surfaces are characterized in terms of their Euler class. Further examples comprise the Kähler class and the Euler class in higher degree.

While the standard Chern and Pontrjagin classes, and more generally any characteristic classes in the image of the Chern-Weil homomorphism are trivial on flat bundles since the latter homomorphism is given by evaluation on a curvature tensor, one gets more examples of nontrivial characteristic classes of flat bundles whenever the Chern-Weil homomorphism is not surjective.

3.1 The model of classifying space

Let n and q be positive natural numbers and set $N = (q+1)n$. The *space of n-frames* in $\mathbb{R}^q$, which we denote by $\mathrm{Fr}_n(\mathbb{R}^N)$, consists of ordered n-tuples of linearly independent vectors in $\mathbb{R}^N$. It is naturally identified with the set of N times n matrices with linearly independent columns. There is a natural action of $\mathrm{GL}_n(\mathbb{R})$ from the right (and one of $\mathrm{GL}_N(\mathbb{R})$ from the left) simply given by matrix multiplication.

Let now G be a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. Define

$$PG_q = \mathrm{Fr}_n(\mathbb{R}^N) \quad \text{and} \quad BG_q = PG_q/G,$$

and let $\pi_G : PG_q \to BG_q$ denote the natural projection. It is easy to check that we have thus obtained a principal G-bundle which we denote by ξ_q^G. For a frame A in PG_q, we denote by $[A]_G$ its image by the projection map π_G, that is, its equivalence class in the quotient $BG_q = PG_q/G$.

Observe that for $G = \mathrm{GL}_n(\mathbb{R})$, the space BG_q is precisely the Grassmanian manifold of n-dimensional vector subspaces of $\mathbb{R}^N$, and in general BG_q is a fiber bundle over the Grassmanian, with fiber diffeomorphic to $\mathrm{GL}_n(\mathbb{R})/G$.

3.2 The classifying map for bundles over simplicial complexes

Let G be a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. Let K be a simplicial complex of dimension q, and ξ a principal G-bundle over the geometric realization of K. Let us write π for the bundle map of ξ. We would now like to exhibit a finite covering of $|K|$ on which the bundle ξ can be trivialized. If we were ready to consider coverings with arbitrarily many subsets, we could consider the covering

$$\{\mathrm{star}(v)\}_{v \in K^0}.$$

Indeed, as the stars are contractible, the bundle ξ is trivial over them. However, we would like to bound the number of sets in the covering independently of the simplicial complex (but depending on the dimension q). To do so, we will consider the stars in the first barycentric subdivision of K and take union of stars of barycenters of simplices of K of the same dimension.

More precisely, let K_{bar} denote the first barycentric subdivision of K, and observe that the stars in K_{bar} of two barycenters of simplices of K of same dimension are always disjoint. Defining S_i to be the open subset of $|K_{\text{bar}}|$ consisting of the union of the stars (in K_{bar}) of all barycenters of i-dimensional simplices of K,

$$S_i = \coprod_{\substack{s \in K, \\ \text{Dim} s = i}} \text{star}_{K_{\text{bar}}}(b^s),$$

we conclude that we get a finite covering $\{S_0, ..., S_q\}$ of $|K_{\text{bar}}|$ such that the bundle ξ is trivial when restricted to any of the S_i's. Let

$$\phi_i : \pi^{-1}(S_i) \longrightarrow S_i \times G$$

be some local trivialization of the bundle ξ and

$$g_{ij} : S_i \cap S_j \longrightarrow \text{GL}_n(\mathbb{R})$$

be the corresponding transition functions.

For every i between 0 and q, define a continuous G-equivariant map $\overline{f}_i : \pi^{-1}(S_i) \to PG_q$ as

$$\overline{f}_i(u) = \begin{pmatrix} t_0 g_{0i}(\pi(u))g \\ \vdots \\ t_i g_{ii}(\pi(u))g \\ \vdots \\ t_n g_{ni}(\pi(u))g \end{pmatrix},$$

where u belongs to $\pi^{-1}(S_i)$, the image of u via ϕ_i is $\phi_i(u) = (\pi(u), g)$, and $\pi(u) = \Sigma_{j=0}^{q} t_j b^{s_j}$, with b^{s_j} the vertex in K_{bar} corresponding to the barycenter of the j-th dimensional simplex s_j of K. Of course, the matrix is to be understood as an N times n matrix consisting of $(q+1)$ blocks of square matrices. If $g_{ji}(\pi(u))$ is not defined, then t_j is zero, so that we consider $t_j g_{ji}(\pi(u))$ as the n times n zero matrix. Observe that this N times n matrix really represents a frame, since the block $t_i g_{ii}(\pi(u))g$ has non zero determinant.

It follows from the cocycle relations of the transition functions $\{g_{ij}\}$ that $\overline{f}_i = \overline{f}_j$ on $\pi^{-1}(S_i \cap S_j)$. The maps $\overline{f}_i$, agreeing on their domain's intersection, induce a continuous G-equivariant map

$$\overline{f} : P \longrightarrow PG_q.$$

Let $f : |K| \to BG_q$ be the corresponding map on the base spaces, so that if $\underline{t} = \Sigma_{j=0}^{q} t_i b^{s_i}$, with s_j simplices of K of dimension j we have

$$f(\underline{t}) = \begin{bmatrix} t_0 g_{0i}(\underline{t}) \\ \vdots \\ t_i \text{Id}_n \\ \vdots \\ t_n g_{ni}(\underline{t}) \end{bmatrix}_G,$$

with i chosen so that $t_i \neq 0$. We have just proven that the map $f : |K| \to BG_q$ is a classifying map for the bundle ξ. Consequently, the two bundles ξ and $f^*(\xi_q^G)$ are isomorphic.

Lemma 8 *Let K be a finite q-dimensional simplicial complex and $F : |K| \to BG^\delta$ a continuous map. Then there exists, for the bundle $F^*(PG^\delta)$, a classifying map $f : |K| \to BG$ as above such that for every simplices k_1, k_2 of K and affine isomorphism $\alpha : |k_1| \to |k_2|$, if $F|_{|k_1|} = F|_{|k_2|} \circ \alpha$, then $f|_{|k_1|} = f|_{|k_2|} \circ \alpha$.*

Proof. Let $\{S_i\}_{i=0}^q$ be the covering of $|K|$ as defined above. Since the classifying map f is defined uniquely in terms of transition functions relative to the covering $\{S_i\}_{i=0}^q$, it is clear that to prove the lemma, it is enough to exhibit such transition functions such that for every affine isomorphism $\alpha : |k_1| \to |k_2|$, if $F|_{|k_1|} = F|_{|k_2|} \circ \alpha$, then $g_{ij}(\alpha(x)) = g_{ij}(x)$, for every i, j and x in $|k_1| \cap S_i \cap S_j$.

Any section $\tau_i : S_i \to F^*(PG^\delta)|_{S_i}$ gives rise to a trivialization

$$\begin{array}{rcl} \varphi_i : S_i \times G^\delta & \longrightarrow & F^*(PG^\delta)|_{S_i} \\ (x, g) & \longmapsto & \tau_i(x)g \end{array}$$

(and conversely). Because G^δ is discrete, such a section is completely determined by its value on one point of every connected component of S_i. For every $x \in F(K^0_{\mathrm{bar}}) \subset BG^\delta$, pick $y(x) \in PG^\delta$ in the fiber over x and let $\tau_i : S_i \to F^*(PG^\delta)|_{S_i}$ be the sections determined by

$$\tau_i(b^{s_i}) = (b^{s_i}, y(F(b^{s_i}))) \in F^*(PG^\delta)|_{S_i},$$

for every barycenter $b^{s_i} \in K^0_{\mathrm{bar}}$ of an i-dimensional simplex s_i of K. The transition functions are then given, for $x \in S_i \cap S_j$, by the relation

$$g_{ij}(x)\tau_j(x) = \tau_i(x).$$

By construction, if $\alpha : |k_1| \to |k_2|$ is an affine isomorphism satisfying $F|_{|k_1|} = F|_{|k_2|} \circ \alpha$, then

$$\begin{tikzcd} F^*(PG^\delta)|_{|k_1| \cap S_i} \arrow[r, "(\alpha, \mathrm{Id})"] & F^*(PG^\delta)|_{|k_2| \cap S_i} \\ |k_1| \cap S_i \arrow[u, "\tau_i"] \arrow[r, "f"] & |k_2| \cap S_i \arrow[u, "\tau_i"'] \end{tikzcd}$$

so that the transition functions satisfy, for every i, j and x in $|k_1| \cap S_i \cap S_j$, the relation

$$g_{ij}(\alpha(x)) = g_{ij}(x),$$

which finishes the proof of the lemma. ■

4 Semi-algebraic sets

The aim of this section is to introduce all standard results on semi-algebraic sets which we will need for our proof of Theorem 5. For the sake of conciseness, we omit most proofs, and invite the interested reader to consult Chapter 2 of the book [BeRi90] by Benedetti and Risler.

4.1 Definitions and first properties

A subset X of $\mathbb{R}^n$ is said to be *semi-algebraic* if it admits a representation of the form

$$X = \cap_{i=1}^{s} \cup_{j=1}^{r_i} \{x = (x_1, ..., x_n) \in \mathbb{R}^n \mid P_{i,j}(x) \ \geq \ 0\},$$

where $P_{i,j}(T_1, ..., T_n)$ is a polynomial in n variables belonging to $\mathbb{R}[T_1, ..., T_n]$ for every i and j. Such a representation is by no means unique as will soon be clear.

We can surely measure the complexity of a semi-algebraic set X in terms of the dimension of the affine space X belongs to, and the minimal number and degree of the polynomials involved in a representation of X. More precisely, let R be a representation as above of some semi-algebraic set. Define

$$C(R) = \sum_{i=1}^{s} r_i \quad \text{and } D(R) = \max_{i,j}\{\deg(P_{i,j})\}.$$

Let $n, c, d \in \mathbb{N}$ and set

$$S(n,c,d) := \left\{ X \subset \mathbb{R}^n \;\middle|\; \begin{array}{l} X \text{ is semi-algebraic and admits} \\ \text{a representation } R \text{ with} \\ C(R) \leq c \ \text{ and } \ D(R) \leq d \end{array} \right\}.$$

We say that a semi-algebraic set X *is of complexity* $S(n,c,d)$ if X belongs to $S(n,c,d)$.

For example, algebraic sets are semi-algebraic. In particular, the affine space $\mathbb{R}^n$ is semi-algebraic, and belongs to $S(n,0,0)$. The standard q-simplex

$$\Delta^q = \{(t_1, ..., t_q) \in \mathbb{R}^q \mid t_i \geq 0, \ 1 - \Sigma_{i=1}^{q} t_i \geq 0\}$$

belongs to $S(q, q+1, 1)$, and more generally, any finite simplicial complex K is semi-algebraic of complexity $S(n,c,1)$, where n and c depend on the number of simplices of K. Observe also that the minimal complexity of a semi-algebraic set is not well defined: the semi-algebraic set

$$\{x \in \mathbb{R} \mid x^2 \geq 1\} \ = \ \{x \in \mathbb{R} \mid x \leq -1\} \cup \{x \in \mathbb{R} \mid x \geq 1\}$$

is both of complexity $S(1,1,2)$ and $S(1,2,1)$.

Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ be semi-algebraic. A map $f : X \to Y$ is called *semi-algebraic* if it is continuous and its graph is a semi-algebraic subset of $\mathbb{R}^n \times \mathbb{R}^m$. It is moreover called *semi-algebraic of complexity* $S(n,c,d)$ if its graph is semi-algebraic of complexity $S(n,c,d)$.

Before enumerating some useful properties of semi-algebraic sets and maps which we will need in the proof of our Theorem 5, let us introduce some convenient notation. Let $n_1, ..., n_q$ and n be natural numbers (or more generally functions or various objects). We write $n \lhd (n_1, ..., n_q)$ if the number n is bounded by a number depending only on $n_1, ..., n_q$. As an example, given a polynomial $f \in \mathbb{R}[T]$, denote by $r(f)$ the number of roots of f, and by $\deg(f)$ the degree of f, then $r(f) \lhd \deg(f)$.

Lemma 9 *If $X_1, ..., X_\ell$ are semi-algebraic sets of complexity $S(n,c,d)$, then the intersection $\cap_{i=1}^{d} X_i$ is semi-algebraic of complexity $S(n, \ell c, d)$.*

Lemma 10 *Let X and Y be two algebraic subsets of $\mathbb{R}^n$. If X and Y are of complexity $S(n,c,d)$ then there exists $C, D \lhd n, c, d$ such that their join*

$$X \star Y = \{t(x,0) + (1-t)(y,1) \mid 0 \le t \le 1,\ x \in X,\ y \in Y\} \subset \mathbb{R}^n \times \mathbb{R}$$

is semi-algebraic of complexity $S(n+1, C, D)$.

Theorem 11 (Tarski-Seidenberg) *Let n, m, c, d be natural numbers. Then there exists $C, D \lhd n+m, c, d$ such that for every semi-algebraic sets $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ and for every semi-algebraic map $f : X \to Y$, if $A \subset X$ is a semi-algebraic set of complexity $S(n,c,d)$ and f is of complexity $S(n+m,c,d)$, then $f(A) \subset Y$ is a semi-algebraic subset of $\mathbb{R}^m$ of complexity $S(m, C, D)$.*

Note that this theorem fails to be true for algebraic sets: consider the projection of the sphere in the Euclidean plane onto any one dimensional subvectorspace.

Corollary 12 *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be semi-algebraic sets, $f : X \to Y$ a semi-algebraic map of complexity $S(n+m,c,d)$. Suppose that $A \subset Y$ is a semi-algebraic subset of complexity $S(m,c,d)$, then $f^{-1}(A) \subset X$ is semi-algebraic of complexity $S(n,C,D)$, where $C, D \lhd n, m, c, d$.*

Corollary 13 *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ and $Z \subset \mathbb{R}^p$ be semi-algebraic sets, $f : X \to Y$ and $g : Y \to Z$ semi-algebraic maps. Suppose that f is of complexity $S(n+m,c,d)$ and g of complexity $S(m+p,c,d)$. Then the map $g \circ f : X \to Z$ is semi-algebraic of complexity $S(n+p,C,D)$, where $c, d \lhd n, m, p, c, d$.*

4.2 Triangulations of semi-algebraic sets

Theorem 14 below is the most technical tool which we need for our proof of Theorem 4. It is a bounded version of the existence of semi-algebraic triangulations of semi-algebraic sets. The unbounded version (that is, the existence of a semi-algebraic triangulation with no bound on the number or on the complexity of the simplices) was proven by Hironaka in [Hi74] following the analogous result by Lojasiewicz for semi-analytic sets. It was then observed by Benedetti and Risler, that one straightforwardly obtains the corresponding bounded version, by bounding every step of the constructive proof of Hironaka, as detailed in [BeRi90, Theorem 2.9.4].

Let X be a semi-algebraic set. A triangulation $h : X \to |K|$ of X is said to be a *semi-algebraic triangulation* if the homeomorphism h between X and the geometric realization of the simplicial complex K is semi-algebraic.

Theorem 14 *For every compact semi-algebraic set X and every semi-algebraic subsets $X_1, \ldots, X_\ell \subset X$, if $X_1, ..., X_\ell$ and X are of complexity $S(n,c,d)$ then there exists a semi-algebraic triangulation*

$$h : X \longrightarrow |K|$$

such that

1. *X_i is a finite union of $h^{-1}(s)$ for some simplices s of K, for every i between 1 and ℓ;*

2. *the number of simplices of K is bounded by k, where $k \lhd (n, c, d, \ell)$;*

3. *for every simplex s of K the set $h^{-1}(s)$ is semi-algebraic of complexity $S(n, C, D)$, for some $C, D \lhd (n, c, d, \ell)$.*

It is a straightforward consequence of the existence of semi-algebraic triangulations of semi-algebraic sets that connected subsets of semi-algebraic sets are semi-algebraic. (This is false for algebraic sets.)

4.3 Semi-algebraicity of the classifying space and classifying map

In the sequel we examine the question of semi-algebraicity for the classifying space BG_q and the classifying map f, which we defined in Section 3, in the case where the bundle in consideration is flat.

It is clear that the space of n-frames $\mathrm{Fr}_n(\mathbb{R}^N)$ is semi-algebraic. Indeed, recall that it is naturally identified with the set of all N times n matrices with linearly independent columns. The latter condition being equivalent to the non vanishing of at least one of the maximal minor, the space $\mathrm{Fr}_n(\mathbb{R}^N)$ can be viewed as a semi-algebraic subset of $\mathbb{R}^{Nn}$.

Let G be a real algebraic subgroup of $\mathrm{GL}_n\mathbb{R}$ and let us show that BG_q is semi-algebraic. The main point is that BG_q can in a natural way be viewed as a homogeneous space. Indeed, consider the action of $\mathrm{GL}_N\mathbb{R}$ on BG_q (where, as in Section 3, $N = (q+1)n$) given by left matrix multiplication

$$\begin{aligned} \mathrm{GL}_N\mathbb{R} \times BG_q &\longrightarrow BG_q \\ (A, [X]_G) &\longmapsto [AX]_G. \end{aligned}$$

The stabilizer of the point $\begin{bmatrix} 1_n \\ 0 \end{bmatrix}_G \in BG_q$ is easily checked to be

$$H(\mathbb{R}) = \left\{ \begin{pmatrix} g & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}_N(\mathbb{R}) \middle| \, g \in G \right\}.$$

Our space BG_q is thus diffeomorphic to the homogeneous space

$$\mathrm{GL}_N(\mathbb{R})/H(\mathbb{R}).$$

Since G is algebraic, it is clear that $H(\mathbb{R})$ is a real algebraic subgroup of $\mathrm{GL}_N\mathbb{R}$. It is a consequence of a well known theorem of Chevalley that the homogeneous space

$$Y(\mathbb{C}) = \mathrm{GL}_N(\mathbb{C})/H(\mathbb{C})$$

of the corresponding complex algebraic groups is a complex quasi-projective variety (see [Bo91], §6 or more precisely Theorem 6.8). However, it is in general false that the real points $Y(\mathbb{R})$ of $Y(\mathbb{C})$ form the homogeneous space $\mathrm{GL}_N(\mathbb{R})/H(\mathbb{R})$. To see that let us consider the following examples:

- The quotient of $\mathrm{GL}_1(\mathbb{C})$ by its finite subgroup $\{+1, -1\}$ can naturally be identified with $\mathrm{GL}_1(\mathbb{C})$ in such a way that the quotient mapping is given by

$$\begin{array}{ccc} \mathrm{GL}_1(\mathbb{C}) & \longrightarrow & \mathrm{GL}_1(\mathbb{C}) \\ z & \longmapsto & z^2. \end{array}$$

But in the real case, the quotient $\mathrm{GL}_1(\mathbb{R})/\{+1,-1\}$ is of course not diffeomorphic to $\mathrm{GL}_1(\mathbb{R})$. In fact, it is diffeomorphic to one connected component of $\mathrm{GL}_1(\mathbb{R})$.

- More generally, the quotient of $\mathrm{GL}_n(\mathbb{C})$ by its orthogonal subgroup $\mathrm{O}(n,\mathbb{C})$ is naturally identified with the space of nondegenerated quadratic forms over $\mathbb{C}$, or equivalently, the space of symmetric nondegenerated complex valued $(n \times n)$-matrices. But the nondegenerated quadratic forms over $\mathbb{R}$, contrarily to the complex case, are not all equivalent, so that the action of $\mathrm{GL}_n(\mathbb{R})$ is not transitive: it has precisely $n+1$ orbits corresponding to the signature of the nondegenerated symmetric matrices. The homogeneous space $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n,\mathbb{R})$ actually is diffeomorphic to the orbit of the identity, that is the set of symmetric real valued $(n \times n)$-matrices for which all eigenvalues are strictly positive. It can thus be viewed as a semi-algebraic set.

The problem in the two above examples is that the projection map

$$\mathrm{GL}_N(\mathbb{C}) \longrightarrow \mathrm{GL}_N(\mathbb{C})/H(\mathbb{C}) = Y(\mathbb{C}),$$

which is defined over $\mathbb{R}$, is not surjective anymore when restricted to the underlying real varieties:

$$\mathrm{GL}_N(\mathbb{R}) \longrightarrow Y(\mathbb{R}).$$

Equivalently, the action of $\mathrm{GL}_N\mathbb{R}$ on $Y(\mathbb{R})$ is not transitive.

Let $\overline{1}$ denote the image of the identity via the projection map $\mathrm{GL}_N(\mathbb{C}) \longrightarrow Y(\mathbb{C})$ and let $X(\mathbb{R})$ be its orbit in $Y(\mathbb{R})$ under the action of $\mathrm{GL}_N(\mathbb{R})$. The stabilizer of $\overline{1}$ is then clearly

$$H(\mathbb{C}) \cap GL_N(\mathbb{R}) = H(\mathbb{R}),$$

so that

$$BG_q \cong \mathrm{GL}_N(\mathbb{R})/H(\mathbb{R}) \cong X(\mathbb{R}).$$

Because $X(\mathbb{R})$ is a finite union of connected components of $Y(\mathbb{R})$, it is semi-algebraic. Indeed, it is an easy consequence of Theorem 14 that connected components of semi-algebraic sets are semi-algebraic.

For further use, define $n(BG_q)$ to be equal to the dimension of the affine space that BG_q belongs to. (In particular, BG_q then belongs to $S(n(BG_q), c, d)$ for some c, d.)

Because of the universal property of the quotient (see [Bo91], §6), it is readily seen that the projection map $\pi : PG_q = \mathrm{Fr}_n(\mathbb{R}^N) \rightarrow X(\mathbb{R}) = BG_q$ is a semi-algebraic map.

As for the classifying map f described in the previous section, in the case where the bundle ξ is flat, the transition functions relative to the open covering described in Section 3 can be chosen to be locally constant. This means that on every q-dimensional open simplex $\mathrm{Int}(k)$ of K, the classifying map f takes the form

$$\underline{t} = (t_0, ..., t_q) \longmapsto f(\underline{t}) = \begin{bmatrix} t_0 \mathrm{Id}_n \\ t_1 g_{10} \\ \vdots \\ t_q g_{q0} \end{bmatrix}_G ,$$

where the g_{i0}'s are constant elements of G defined by $g_{i0} = g_{i0}(\underline{t})$, for any $\underline{t}$ in $\mathrm{Int}(k)$. By continuity, the map f actually has the above form on the whole (closed) simplex k. We claim that it is semi-algebraic of uniformly bounded complexity when restricted to any simplex of K. To see that, consider its lift to $PG_q = \mathrm{Fr}_n(\mathbb{R}^N)$

$$\underline{t} = (t_0, ..., t_q) \longmapsto \begin{pmatrix} t_0 \mathrm{Id}_n \\ t_1 g_{10} \\ \vdots \\ t_q g_{q0} \end{pmatrix},$$

which clearly is an affine map, now that the g_{ij}'s are constant. It is thus semi-algebraic of uniformly bounded complexity , where the bound only depends on the dimension, thus on n and q. As mentioned above, the projection $PG_q \to BG_q$ is a semi-algebraic map, so that the claim follows.

5 Proof of the simplicial version

Theorem 5 *Let G be a real algebraic subgroup of $\mathrm{GL}_n(\mathbb{R})$ and $\beta \in H^q(BG)$ be a characteristic class. There exists a finite subset $I \subset \mathbb{R}$ such that for every flat principal G-bundle ξ over a finite simplicial complex K the cohomology class $\beta(\xi) \in H^q(K)$ can be represented by a cocycle whose set of value on the q-simplices of K is contained in I.*

Proof. First observe that it is enough to prove the theorem for simplicial complexes of dimension smaller or equal to q. Indeed, a simplicial q-cocycle is defined on the q-dimensional simplices and two q-cocycles represent the same cohomology class if they differ by a coboundary, which also only depends on the q-skeleton.

Now, any principal G-bundle over a q-dimensional simplicial complex, can be obtained as the pull back of the approximation to the universal bundle BG_q, where BG_q is as in Section 3.1.

The space BG_q was shown to be semi-algebraic in Section 4.3, so that in particular its closure $\overline{BG}_q$ admits, by Theorem 14, a finite semi-algebraic triangulation T which we can chose in such a way that it restricts to a triangulation of the boundary of BG_q. To simplify the notation, we identify $\overline{BG}_q$ with the geometric realization $|T|$ of its triangulation T. Upon replacing T by its first barycentric subdivision we can assume that any open simplex contained in BG_q has at least one of its vertex in BG_q. Also recall that the classifying map $f : |K| \to BG_q$ exhibited in Section 3.2 was proven in Section 4.3 to be semi-algebraic, and furthermore of complexity bounded independently of the bundle ξ or even the simplicial complex K, when restricted to any simplex of K.

Our next aim is to find a simplicial approximation of the classifying map $f : |K| \to BG_q \hookrightarrow |T|$ (or to be precise, actually an approximation to a map $\overline{f} : |K| \to BG_q \hookrightarrow |T|$ homotopic to f) such that the homotopy between f and its simplicial approximation has image in BG_q. Of course it is a well known fact that upon passing to an arbitrarily fine subdivision of K this is always possible. Our main point is now precisely that we only need to refine K in a uniformly bounded way. This will follow at once from the following Technical Lemma.

Lemma 15 (Technical Lemma) *There exists a triangulation L of K and a continuous map $\overline{f} : |K| \to |T| = BG_q$ homotopic to f such that*

- *each simplex of K is triangulated by at most d simplices of L, and the bound d is independent of ξ or K,*
- *the interior of every simplex ℓ of L is mapped by $\overline{f}$ inside the interior of some simplex t of T whose interior is contained in BG_q,*

$$\overline{f}(\mathrm{Int}(\ell)) \subset \mathrm{Int}(t).$$

We postpone for the time being the proof of the Technical Lemma and show how the theorem can now be proven. Let $T_{\max}$ be the biggest subcomplex of T contained in BG_q. The simplicial approximation can be defined as follows: For every vertex v of L, define $\varphi(v) \in T^0_{\max}$ to be any vertex of the only open simplex of T containing $\overline{f}(v)$. This indeed defines a simplicial map

$$\varphi : L \longrightarrow T_{\max}$$

since if $v_1, ..., v_q$ generate a q-simplex of L, then $\overline{f}(v_1), ..., \overline{f}(v_q)$ belong to some (closed) simplex t of T. As the only open simplex containing $\overline{f}(v_i)$, for $i \in \{1, ..., q\}$, is necessarily contained in t, it follows that $\varphi(v_i)$ must be one of the vertices of t, so that $\varphi(v_1), ..., \varphi(v_q)$ indeed generate a simplex of T, namely a face of t. Furthermore, as all $\varphi(v_i)$'s belong to $T_{\max}$, the simplex generated by them also lies in $T_{\max}$.

The simplicial map φ is easily verified to be a simplicial approximation to the continuous map $\overline{f} : |K| = |L| \to BG_q \hookrightarrow |T|$: One can check that for every vertex v in L^0, it holds that

$$\overline{f}(\mathrm{star}(v)) \subset \mathrm{star}(\varphi(v)).$$

For our purposes it is however enough to know that the maps $|\varphi|$ and $\overline{f}$ are homotopic, which is obvious since for every point x in $|L|$ its image $\overline{f}(x)$ belongs to the same simplex of T than $|\varphi|(x)$, so that the (positive) convex linear combination of the two maps $|\varphi|$ and $\overline{f}$ is well defined, thus providing the desired homotopy. Note that by construction, it is clear that the whole homotopy lies in BG_q.

Let $b \in Z^q(BG_q)$ be an alternating cocycle representing the cohomology class corresponding to the characteristic class β. We have

$$\beta(\xi) = [f^*(b)] = [|\varphi|^*(b)] \in H^q(K),$$

since $|\varphi|$ is homotopic to $\overline{f}$, and the latter map is itself, by the Technical Lemma, homotopic to f. Let I_T be the set of values taken by the cocycle b when evaluated on q-dimensional simplices of $T_{\max}$. Of course I_T is a finite subset of $\mathbb{R}$ since the simplicial complex $T_{\max}$ is finite. Set

$$I = \left\{ \sum_{i=1}^{d'} (-1)^{\varepsilon_i} r_i \,\middle|\, d' \leq d, \varepsilon_i \in \{0,1\}, r_i \in I_T \right\},$$

where d is as in the Technical Lemma. It is clear that the set I is finite. Observe that the cocycle $\varphi^*(b)$, considered as a simplicial cocycle on L clearly enjoys the property that its evaluation on q-dimensional simplices of L is contained in I_T. A simplicial cocycle on K, representing $\beta(\xi)$, is now obtained as follows: the value of a q-dimensional simplex k of K is the sum of the values of $\varphi^*(b)$ on the the simplices of L appearing in the triangulation of k, and is hence contained in I, which finishes the proof of the theorem. ■

Of course, the so obtained bound is by no means sharp. Observe that it is composed of two parts: the possible values of a cocycle on $BG_q = |T|$ representing the characteristic class β evaluated on the fixed triangulation T, and the amount of simplices (the d from the Technical Lemma) needed to refine the simplicial complex K so as to have a simplicial approximation of the classifying map. The latter bound can actually be computed effectively.

Proof of the Technical Lemma. We will prove the Lemma inductively by showing that for every $0 \leq i \leq q$ there exists constants c_i, d_i and m_i depending only on i, the group G and the dimension q of the simplicial complex, a triangulation L_i of the i-skeleton K^i of K and a continuous map $f_i : |K| \to |T| = BG_q$ homotopic to f such that

1. each simplex of K^i is triangulated by at most m_i simplices of L_i,
2. the image by f_i of the interior of every simplex of L_i is contained in the interior of some simplex t of T,
3. every simplex ℓ of L_i is semi-algebraic of complexity $S(i, c_i, d_i)$,
4. the map f_i restricted to any simplex of K^i is semi-algebraic of complexity $S(i + n(BG_q), c_i, d_i)$.

The two first properties are exactly the conclusion of the Technical Lemma for $i = q$, and the two last ones are added for inductive purposes. For $i = 0$, there is nothing to prove: Take $f_0 = f$ and $L_0 = K^0$ (so that $c_0 = n(BG_q), d_0 = 1$ and $m_0 = 1$). Let us thus assume that a triangulation L_{i-1} of the $(i-1)$-skeleton of K and a continuous map $f_{i-1} : |K| \to |T|$ satisfying the above properties are given.

The strategy of the proof is the following: We are going to triangulate each i-dimensional simplex k of K in such a way that the triangulation on the boundary ∂k of k is precisely the first barycentric subdivision of L_{i-1}. We will thus automatically obtain a triangulation of the i-skeleton of K. To do so, we subdivide every i-dimensional simplex k in two subsets k_{int} and k_{ext}. After defining the map f_i and checking that it satisfies the above property 4 we prove that there exists triangulations of k_{int} and k_{ext} which agree on $k_{\text{int}} \cap k_{\text{ext}}$ and correspond to the first barycentric subdivision of L_{i-1} on ∂k. We show that both the triangulation of k_{int} and k_{ext} satisfy the above properties 1, 2 and 3, thus proving the Technical Lemma.

The subsets k_{int} and k_{ext}. Let k be an i-th dimensional simplex of K and consider the two following subsets of its geometric realization: Choose ε with $0 < \varepsilon < 1$ and define

$$k_{\text{int}} = \left\{ \sum_{j=0}^{i} t_j v_j \;\middle|\; t_i \geq \frac{\varepsilon}{1+i} \;\forall\; j = 0, \ldots, i, \right\}$$

and

$$k_{\text{ext}} = \left\{ \sum_{j=0}^{i} t_j v_j \;\middle|\; \exists\, j \in \{0, ..., i\} \text{ with } t_i \leq \frac{\varepsilon}{1+i} \right\},$$

where of course $v_0, ..., v_i$ are the vertices of k. The subset k_{ext} is the closure of some sufficiently small neighborhood of the boundary of k so that k_{ext} is homotopically equivalent to ∂k. The subset k_{int} is the closure of $k \backslash k_{\text{ext}}$, that is, a homothetic copy of k centered at the barycenter of k and contraction factor strictly smaller than 1.

The map f_i. Define a continuous map $\alpha_k : |k| \to |k|$ to be, on k_{int} the natural affine homothety between k_{int} and k, and on k_{ext} the projection from the barycenter of k onto the boundary ∂k. More precisely, we have

$$\alpha_k \left(\sum_{j=0}^{i} t_j v_j \right) = \begin{cases} \sum_{j=0}^{i} \frac{1}{1-\varepsilon} \left(t_j - \frac{\varepsilon}{i+1} \right) v_j & \text{if } \sum_{j=0}^{i} t_j v_j \in k_{\text{int}}, \\ \sum_{j=0}^{i} \frac{t_j - \min\{ t_j \mid 0 \leq j \leq m\}}{1-(i+1)\min\{ t_j \mid 0 \leq j \leq m\}} v_j & \text{if } \sum_{j=0}^{i} t_j v_j \in k_{\text{ext}}. \end{cases}$$

Clearly α_k is well-defined, continuous and semi-algebraic. Also, since for every i-dimensional simplex k, the map α_k is the identity on ∂k, it defines a continuous map $\alpha : |K^i| \to |K^i|$. Furthermore, it is obvious that it extends to a continuous map $|K| \to |K|$, still denoted by α, which we can moreover assume to map every simplex of K to itself and to be semi-algebraic of complexity $S(2q, c_\alpha, d_\alpha)$, when restricted to any simplex of K, where the constants c_α and d_α do not depend on anything else than i and q. Such a map α is clearly homotopic to the identity.

Define

$$f_i = f_{i-1} \circ \alpha : |K| \longrightarrow |T| = BG_q.$$

Since f_{i-1} is homotopic to f, the same is true for f_i and by Corollary 13, the map f_i is, when restricted to any simplex of K, semi-algebraic of complexity $S(q + n(BG_q), c_i, d_i)$, where $c_i, d_i \lhd q, n(BG_q), c_\alpha, d_\alpha, c_{i-1}, d_{i-1}$, and thus $c_i, d_i \lhd q, G, i$.

The triangulation of k_{int}. The map α is, when restricted to k_{int} a homothety from k_{int} to k. Thus the first barycentric subdivision $(L_{i-1})_{\text{bar}}$ of the triangulation L_{i-1} restricted to the boundary of k naturally induces, via α, a triangulation by semi-algebraic simplices of complexity $S(n_{i-1}, c_{i-1}, d_{i-1})$ of the boundary of k_{int}. We would now like to have a semi-algebraic triangulation of k_{int} agreeing with the following two families of semi-algebraic subsets:

- The simplices of the triangulation of ∂k_{int} induced by $(L_{i-1})_{\text{bar}}$.
- The pull back by f_i of the simplices of T.

We are of course going to apply Theorem 14 to k_{int} and those two families of semi-algebraic subsets, so let us first check that the above sets all are of uniformly bounded complexity, and in uniformly bounded quantity. Note that k_{int} is of complexity $S(i, i+1, 1)$.

- Since each simplex of L_{i-1} is, by induction, of complexity $S(i-1, c_{i-1}, d_{i-1})$, it follows that each simplex of $(L_{i-1})_{\text{bar}}$ is of complexity $S(i-1, c_{i-1}, d_{i-1})$, and the same is true for the corresponding simplices in ∂k_{int}.

  There are at most $(i+1) \cdot m_{i-1} \cdot i!$ such simplices.

- Since the semi-algebraic triangulation T of BG_q is finite, any simplex t of T is of complexity $S(n(BG_q), c_T, d_T)$, for some c_T, d_T depending only on G and q. By Corollary 12 it follows that $f_i^{-1}(t)$ is semi-algebraic of complexity $S(q, C, D)$, where $C, D \lhd q, n(BG_q), c_i, d_i, c_T, d_T$, thus $C, D \lhd q, G, i$. By Lemma 9 we now obtain that $f_i^{-1}(t) \cap k_{\text{int}}$ is semi-algebraic of complexity $S(i, 2\max\{i+1, C\}, \max\{1, D\})$ for every simplex t of T.

  Of course, the number of such sets is majorized by the number of simplices of T, which only depends on q and G.

Let us now apply Theorem 14 to k_{int} and its two above given families of semi-algebraic subsets. We thus obtain a semi-algebraic triangulation L_{int} of k_{int} fulfilling the following properties:

1. - The triangulation L_{int} restricted to the boundary of k_{int} is a refinement of the triangulation corresponding to the first barycentric subdivision of the triangulation L_{i-1} restricted to ∂k.
   - For every simplex t of T, the semi-algebraic set $f_i^{-1}(t) \cap k_{\text{int}}$ is a finite union of simplices of L_{int}, so that the image by f_i of the interior of any simplex of L_{int} is contained in the interior of some simplex of T.
2. The number of simplices of L_{int} is bounded by m_{int}, where m_{int} is a constant depending only on q, G and i.
3. each simplex of L_{int} is semi-algebraic of complexity $S(n, c_{\text{int}}, d_{\text{int}})$, where $c_{\text{int}}, d_{\text{int}}$ are constants depending only on q, G and i.

The triangulation of k_{ext}. It now remains to triangulate k_{ext} in such a way that the triangulation agrees with the first barycentric subdivision of L_{i-1} on ∂k and with the triangulation L_{int} on $k_{\text{ext}} \cap k_{\text{int}} = \partial k_{\text{int}}$. This triangulation should of course also enjoy the desired properties. To do so, we consider the homeomorphism between k_{ext} and $\partial k \times [0,1]$ given by

$$\begin{array}{rccc} \beta : & k_{\text{ext}} & \longrightarrow & \partial k \times [0,1] \\ & x = \sum_{j=0}^{i} t_j v_j & \longmapsto & (\alpha(x), \frac{i+1}{\varepsilon}\min\{t_0, ..., t_i\}). \end{array}$$

The boundary $\partial k \subset k_{\text{ext}}$ is thus mapped by β to $\partial k \times \{0\}$, and $k_{\text{ext}} \cap k_{\text{int}} = \partial k_{\text{int}}$ to $\partial k \times \{1\}$. Denote by L_0 the first barycentric subdivision of the triangulation L_{i-1}, and by L_1 the triangulation of ∂k corresponding to the triangulation L_{int} of $k_{\text{ext}} \cap k_{\text{int}} = \partial k_{\text{int}}$. By construction, L_1 is a refinement of L_0. The triangulation L_{int} of k_{int} is now easily defined as the inverse image, via β^{-1} of the following triangulation of $\partial k \times [0,1]$: For every simplex $\{v_0 < ... < v_{i-1}\}$ of L_0 (where the ordering of the vertices is given from L_0 being the first barycentric of the simplicial complex L_{i-1}), for every simplex $\{w_0, ..., w_n\}$ of L_1 such that the geometric realization of the simplex $\{w_0, ..., w_n\}$ is contained in the geometric realization of the simplex $\{v_0, ..., v_{i-1}\}$ and for every $m \in \{0, ..., i-1\}$ define the simplex

$$\{(v_0, 0), (v_1, 0), ..., (v_m, 0), (w_0, 1), ..., (w_n, 1)\} \text{ of } L_{\text{ext}}$$

to be the join of the simplex $\{(v_0, 0), ..., (v_{i-1}, 0)\}$ of $L_0 \times \{0\}$ and the simplex $\{(w_0, 1), ..., (w_n, 1)\}$ of $L_1 \times \{1\}$. It is straightforward to check that we have thus obtained a triangulation of $\partial k \times [0,1]$ and hence of k_{int}. Moreover we have:

1. A rough bound for the number of simplices of L_{ext} is the number of simplices of L_0 plus 1 multiplied by the number of simplices in L_1 plus 1. But now, by induction, the number of simplices of L_0 is at most m_{i-1} times i, since ∂k has i faces of dimension $i-1$, and the number of simplices of L_1 is surely strictly smaller than the number of simplices of L_{int}, which is bounded by m_{int}. We thus obtain that the amount of simplices of L_{ext} is bounded by
$$m_{\text{ext}} = (i \cdot m_{i-1} + 1)\,(m_{\text{int}} + 1)\,.$$

2. Observe that the diagram
$$\begin{array}{ccc} k_{\text{ext}} & \xrightarrow{\beta} & \partial k \times [0,1] \\ \big\downarrow f_i & & \big\downarrow \text{proj}_1 \\ |T| & \xleftarrow{f_{i-1}} & \partial k, \end{array}$$
where of course proj_1 stands for the projection on the first factor, is commutative. The interior of any simplex of L_{ext} is by construction mapped inside the interior of some simplex of the first barycentric subdivision of L_{i-1} and as by induction the image by f_{i-1} of the interior of any simplex of L_{i-1} and hence also of $(L_{i-1})_{\text{bar}}$ is contained in the interior of some simplex of T the conclusion follows.

3. As L_0 is the first barycentric subdivision of L_{i-1} its simplices are all semi-algebraic of complexity $S(i-1, c_{i-1}, d_{i-1})$. Also, L_1 being the triangulation L_{int} on ∂k_{int} it is semi-algebraic of complexity $S(i-1, c_{\text{int}}, d_{\text{int}})$. By Lemma 10, the join of any simplex of L_0 and L_1, and thus any simplex of L_{ext} is semi-algebraic of complexity $S(i, c_{\text{ext}}, d_{\text{ext}})$, where c_{ext},$d_{\text{ext}} \lhd c_{i-1}, d_{i-1}, c_{\text{int}}, d_{\text{int}}$ and thus c_{ext},$d_{\text{ext}} \lhd i, G, q$.

■

We prove now a slightly stronger version of Theorem 5, where we show that if the bundle ξ is induced by a classifying map $F : |K| \to BG^\delta$, then the simplicial cocycle can furthermore be chosen in the image of the induced map $F^* : C^q_{\text{sing}}(BG^\delta) \to C^q_{\text{simpl}}(K)$. We assume that our space of singular cochains consists of alternating cochains, so that the map F^* admits the following natural description: For c in $C^q_{\text{sing}}(BG^\delta)$ and k an oriented q-simplex of K, that is, a q-simplex of K together with an ordering of its vertices, we have
$$F^*(c)(k) = \text{sign}(\tau)c(F \circ |\tau|),$$
where $\tau : \Delta^q \to k$ is an isomorphism between the two oriented q-simplices Δ^q and k.

Theorem 16 *Let G be a real algebraic subgroup of $GL_n\mathbb{R}$ and $\beta \in H^q(BG^\delta)$ a primary characteristic class. Then there exists a finite subset $I \subset \mathbb{R}$ such that for every finite simplicial complex K and every continuous map $F : |K| \to BG^\delta$, there exists a cochain $b \in C^q_{sing}(BG^\delta)$ such that the simplicial cochain $F^*(b) \in C^q_{simpl}(K)$ is a cocycle representing $F^*(\beta)$ and taking values in I when evaluated on q-simplices.*

Proof. We start the proof with the following claim, which, as we will show below, is a consequence of the easy Lemma 8 and the proof of Theorem 5.

Claim 17 *There exists a cocycle $b \in C^q_{simpl}(K)$ representing $F^*(\beta)$ taking values in I when evaluated on q-simplices of K such that for every affine isomorphism $\alpha : |k_1| \to |k_2|$ between the oriented simplices k_1, k_2 of K, if $F|_{|k_1|} = F|_{|k_2|} \circ |\alpha|$, then $b(k_1) = \mathrm{sign}(\alpha) b(k_2)$.*

Proof of Claim. First note that by Lemma 8, it is enough to prove the claim with F replaced by a classifying map $f : |K| \to BG$ as in Section 3.2.

Second, observe that the triangulation L of K and the simplicial map $\varphi : L \to T_{\max}$ constructed in the proof of Theorem 5 being defined inductively on the skeleton of K in such a way that they depend only on the classifying map f, can be chosen so that α maps the restriction to k_1 of the triangulation L to the restriction to k_2 of the triangulation L and furthermore $\varphi = \varphi \circ \alpha : L|_{k_1} \to T_{\max}$, whenever α is as in the hypothesis of the claim. It is immediate that the resulting simplicial cochain on K, constructed as in the proof of Theorem 5, will satisfy the claimed assertion. ■

To finish the proof of the theorem, define a cochain $b' \in C^q_{\mathrm{sing}}(BG^\delta)$ as follows: If $\sigma : \Delta^q \to BG^\delta$ is a singular simplex such that there exists an isomorphism $\tau : \Delta^q \to k$, where k is an oriented q-simplex of K, such that $\sigma = F \circ \tau$, then set

$$b'(\sigma) = \mathrm{sign}(\tau) b(\tau(\Delta^q)).$$

Otherwise, define $b'(\sigma)$ to be arbitrary. By the claim, the cochain b' is well defined. Indeed, suppose that $F \circ \tau_1 = F \circ \tau_2$, for isomorphisms $\tau_i : \Delta^q \longrightarrow k_i$, where $i = 1, 2$. Then $\alpha = \tau_2^{-1}\tau_1 : k_1 \to k_2$ furnishes an isomorphism between k_1 and k_2 with sign(α) =sign(τ_1)sign(τ_2), so that by the claim,

$$\mathrm{sign}(\tau_1) b(\tau_1(\Delta^q)) = \mathrm{sign}(\tau_1) b(k_1) = \mathrm{sign}(\tau_1)\mathrm{sign}(\alpha) b(k_2) = \mathrm{sign}(\tau_2) b(\tau_2(\Delta^q)).$$

By definition, we have $F^*(b') = b$. Thus, the simplicial cocycle b satisfies all the requirements of the theorem. ■

6 Proof of the singular version

We are now almost ready to give a proof of Theorem 4, which will be a simple consequence of its simplicial version (Theorem 16) by an argument of inverse limit. We start by recalling the elementary definitions of inverse systems and limits.

A *directed set* is a non-empty, partially ordered set $(\Lambda, \geq)$ such that

$$\forall\ \lambda, \mu \in \Lambda,\ \exists\ \nu \in \Lambda \ \text{ with } \nu \geq \lambda,\ \nu \geq \mu.$$

An *inverse system* $(X_\lambda, \pi_{\mu\lambda})$ of sets over a directed set Λ is a family of sets $(X_\lambda)_{\lambda\in\Lambda}$ together with maps $\pi_{\mu\lambda} : X_\lambda \to X_\mu$ whenever $\lambda \geq \mu$ satisfying the two conditions $\pi_{\lambda\lambda} = \mathrm{Id}_{X_\lambda}$, for every λ in Λ, and $\pi_{\nu\mu}\pi_{\mu\lambda} = \pi_{\nu\lambda}$, for $\lambda \geq \mu \geq \nu$. The *inverse limit* of the inverse system $(X_\lambda, \pi_{\mu\lambda})$ is defined as

$$\varprojlim X_\lambda = \{(g_\lambda) \in \prod_{\lambda\in\Lambda} X_\lambda \mid \pi_{\mu\lambda}(g_\lambda) = g_\mu\ \forall\ \lambda \geq \mu\}.$$

The following proposition is a well known and straightforward consequence of Tychonov's theorem. It gives a simple criterion for inverse limits to be nonempty.

Proposition 18 *If $(X_\lambda, \pi_{\mu\lambda})$ is an inverse system of nonempty compact spaces over a directed set Λ, then*

$$\lim_{\leftarrow} X_\lambda \neq \varnothing.$$

Our main example of directed set and inverse system. Let X be a nonempty topological space. Set

$$\Lambda = \left\{ (K,\sigma) \;\middle|\; \begin{array}{l} K \text{ finite simplicial complex,} \\ \sigma : |K| \to X \text{ continuous} \end{array} \right\}.$$

It is nonempty since X is nonempty. Put the following partial order on Λ: Let $(K_1,\sigma_1),(K_2,\sigma_2) \in \Lambda$, then

$$(K_2,\sigma_2) \geq (K_1,\sigma_1) \text{ if } \quad \begin{array}{l} \exists \text{ a simplicial injection } i : K_1 \to K_2 \\ \text{such that } \sigma_2 \circ |i| = \sigma_1. \end{array}$$

It is readily seen that Λ is a directed set. Indeed, for $(K_1,\sigma_1),(K_2,\sigma_2)$ in Λ, define $\sigma : |K_1 \sqcup K_2| \to X$ as $\sigma(x) = \sigma_i(x)$ if x belongs to $|K_i|$. It is clear that $(K,\sigma) \geq (K_i,\sigma_i)$, for $i = 1,2$.

Let X be a topological space, $\beta \in H^q_{\mathrm{sing}}(X)$ a singular cohomology class on X and I a compact subset of $\mathbb{R}$. For every (K,σ) in Λ, define

$$Y_{(K,\sigma)} = \left\{ b \in Z^q_{\mathrm{simpl}}(K) \;\middle|\; \begin{array}{c} [b] = \sigma^*(\beta),\ b \in \sigma^*(C^q_{\mathrm{sing}}(X)), \\ b(k) \in I \ \forall\ q\text{-simplex } k \in K \end{array} \right\}.$$

If $(K_1,\sigma_1) \leq (K_2,\sigma_2)$, the simplicial inclusion $i : K_1 \to K_2$ induces a map

$$i^* : Y_{(K_2,\sigma_2)} \longrightarrow Y_{(K_1,\sigma_1)}.$$

Note that from the requirement that any cocycle of $Y_{(K,\sigma)}$ belongs to the image of σ^* it follows that the map i^* does not depend on the choice of simplicial injection i. Indeed, suppose $j : K_1 \to K_2$ is another simplicial injection with $\sigma_2 \circ |j| = \sigma_1 = \sigma_2 \circ |i|$, then

$$i^* \circ \sigma_2^* = (\sigma_2 \circ |i|)^* = \sigma_1^* = (\sigma_2 \circ |j|)^* = j^* \circ \sigma_2^*,$$

so that i^* and j^* agree on the image of σ_2^* in which $Y_{(K_2,\sigma_2)}$ is contained. Observe moreover that

- for every (K,σ) in Λ, the map $Y_{(K,\sigma)} \to Y_{(K,\sigma)}$ is the identity since it is induced by the identity on K,
- if $(K_1,\sigma_1) \leq (K_2,\sigma_2) \leq (K_3,\sigma_3)$ with simplicial injections $i : K_1 \to K_2$, $j : K_2 \to K_3$, then $(j \circ i)^*_{|Y_{(K_3,\sigma_3)}} = (i^* j^*)_{|Y_{(K_3,\sigma_3)}} = i^*_{|Y_{(K_2,\sigma_2)}} \circ j^*_{|Y_{(K_3,\sigma_3)}}$.

We have thus proven that $\{Y_{(K,\sigma)}\}$ forms an inverse system over Λ.

Proof of Theorem 4. Let G be a real algebraic subgroup of $\mathrm{GL}_n\mathbb{R}$ and $\beta \in H^q(BG^\delta)$ a primary characteristic class. Let Λ be the directed set constructed above for $X = BG^\delta$, and $\{Y_{(K,\sigma)}\}$ the inverse system obtained from $X = BG^\delta$, $\beta \in H^q(BG^\delta)$ and the finite subset I of $\mathbb{R}$ from Theorem 16. The conclusion of Theorem 16 is exactly equivalent to $Y_{(K,\sigma)}$ being nonempty for every (K,σ) in Λ. Moreover, the $Y_{(K,\sigma)}$'s are compact: Indeed, for every (K,σ) in Λ, the space $Y_{(K,\sigma)}$ is the subspace of the finite dimensional vector space $Z^q_{\mathrm{simpl}}(K)$ formed of the intersection of an affine subspace (the image of the coboundary δ), a linear subspace (the image of σ^*), and a compact subset (from that b takes its values in the finite set I). It now follows from Proposition 18, that the inverse limit of the inverse system $\{Y_{(K,\sigma)}\}$ is nonempty:

$$\varnothing \neq \varprojlim Y_{(K,\sigma)}.$$

Let thus $(b_{(K,\sigma)})$ be an element in the inverse limit, and define a singular cochain $b \in C^q_{\mathrm{sing}}(X)$ by $b(\sigma) = b_{(\Delta^q,\sigma)}(\Delta^q)$, for every singular simplex $\sigma : \Delta^q \to X$. It is clear from the definition of b, that the cochain b takes its values in I on singular simplices.

It remains to show that the cochain b is a cocycle representing β. Let thus c be an arbitrary cocycle representing β. By the Universal Coefficient Theorem, it is enough to show that b and c agree on integral singular cycles. Let $z = \Sigma_{i=1}^r a_i\sigma_i \in Z_q(X)$ be such a singular cycle on X, where we can without loss of generality assume that the coefficients a_i lie in $\{-1,+1\}$. We want now to represent the homology class $[z]$ by a continuous map from the realization of a finite simplicial complex K into X:

Claim 19 *Let $z = \Sigma_{i=1}^r a_i\sigma_i$ be a singular cycle in X with $a_i = \pm 1$. Then there exists a closed q-dimensional simplicial complex K and a continuous map $\zeta : |K| \to X$ such that the induced map*

$$\zeta^* : C^*_{sing,alt}(X) \longrightarrow C^*_{simpl}(K)$$

satisfies the relation

$$\zeta^*(c)\,([K]) = c\,(z)\,,$$

for every singular, alternating cochain c in $C^q_{sing,alt}(X)$.

Intuitively, one would like to build up the simplicial complex K from r copies of the standard simplex Δ^q, define ζ on each of those as σ_i, for $i = 1,..,r$, and glue up the $(q-1)$-faces of the standard simplices according to the cycle relation of z. However, for the glued up object to be a simplicial complex with the desired property, we need to add sufficiently many simplices between the copies of the standard simplices. This will be made more precise in the proof of the claim, at the end of this section.

Assuming the claim, we are now left with checking that the cochain b and the cocycle c agree on z: We first show that

$$b_{(K,\zeta)} = \zeta^*(b). \tag{1}$$

To see that, let k be an oriented q-dimensional simplex of K and recall that by definition of the induced map ζ^*, we have

$$\zeta^*(b)(k) = b(\zeta \circ |\tau|),$$

where τ is an orientation preserving isomorphism $\tau : \Delta^q \cong k$, so that $\zeta \circ |\tau| : \Delta^q \to X$ is a singular simplex. By definition of b, we further get

$$b(\zeta \circ |\tau|) = b_{(\Delta^q, \zeta \circ |\tau|)}(\Delta^q).$$

Let now $i : \Delta^q \cong k \hookrightarrow K$ be the simplicial inclusion given by the composition of τ with the canonical inclusion $k \subset K$. Clearly, the simplicial inclusion i gives us

$$(K, \zeta) > (\Delta^q, \zeta \circ |\tau|),$$

so that, since $(b_{(K,\sigma)})$ belongs to the inverse limit of the $Y_{(K,\sigma)}$'s,

$$b_{(\Delta^q, \zeta \circ |\tau|)} = i^*(b_{(K,\zeta)}).$$

As $i_*(\Delta^q) = k$, we finally conclude

$$b_{(\Delta^q, \zeta \circ |\tau|)}(\Delta^q) = b_{(K,\zeta)}(k),$$

which proves Equality (1).

Recall that by definition, $b_{(K,\zeta)} = \zeta^*(b)$ is a simplicial cocycle representing $\zeta^*(\beta) = \zeta^*([c])$. In particular, the evaluation of $\zeta^*(c)$ and $\zeta^*(b)$ on the simplicial cycle $[K]$ must agree:

$$\zeta^*(b)\,([K]) = \zeta^*(c)\,([K])\,. \tag{2}$$

Applying now Claim 19 to both the cochain b and the cocycle c, we see that Equation (2) becomes

$$b\,(z) = c\,(z)\,.$$

Thus, modulo Claim 19, the Theorem is proven. ■

Proof of Claim 19. We inductively construct triangulations T^q, for $q \geq 0$, of the standard q-simplex Δ^q together with simplicial projections $p_q : T^q \to t_q$ onto a favorite simplex t_q of T^q such that the triangulations T^q restrict to triangulations T^q_i, for $i = 0, ..., q$, of the i-th $(q-1)$-dimensional face of Δ^q which are isomorphic to T^{q-1} under the i-th face inclusion $\eta_i : \Delta^{q-1} \cong |T^{q-1}| \hookrightarrow \Delta^q$. The subtriangulations of T^q arising from j-dimensional faces of Δ^q (in short, j-faces of Δ^q) will be called *j-faces of T^q*. Furthermore, T^q will be shown to satisfy the following two properties:

1. For every simplex t in T^q and for any $0 \leq j \leq q-1$, the boundary ∂t of t intersects at most one j-face of T^q.

2. For every q-simplices t, s in T^q, if there exists $i \neq j$ such that $t \cap T^q_i \neq \varnothing$ and $s \cap T^q_j \neq \varnothing$ then t possesses a vertex in the interior of $\Delta^q \cong T^q$ which does not belong to s.

For $q = 0$, take $T^0 = \{*\}$. Assume now that $p_i : T^i \to t_i$ are constructed for $i < q$. View the q-simplex Δ^q as

$$\Delta^q \simeq \Delta^q \cup_{\partial\Delta^q} \partial\Delta^q \times [0,1],$$

where $\partial\Delta^q$ is identified with $\partial\Delta^q \times \{0\}$.

Inductively choose an order of the vertices T^{q-1} which restricts to the order on its j-faces given by the j-th face inclusion of the inductively chosen order on T^{q-2}. Let S^{q-1} be the canonical product triangulation of $\Delta^{q-1} \times [0,1]$

given by the triangulation T^{q-1} of Δ^{q-1} and the triangulation of $[0,1]$ in one 1-dimensional simplex together with the chosen order on T^{q-1} and the order $0<1$ on $[0,1]$.

Consider now the equivalence relation on S^{q-1} defined by

$$(v,0)\sim(w,0) \iff p_{q-1}(v)=p_{q-1}(w),$$

and let R^{q-1} be the refinement of the quotient simplicial complex $S^{q-1}/\sim$ obtained by subdividing once more all q-simplices of $S^{q-1}/\sim$ by adding one vertex on their barycenter. Thus, a q-simplex of $S^{q-1}/\sim$ is in R^{q-1} triangulated into $q+1$ simplices of dimension q, while its boundary remains unchanged. The triangulation R^{q-1} of $\Delta^{q-1}\times[0,1]$ restricts to the triangulations

$$\begin{array}{lll} \Delta^{q-1} & \text{on} & \Delta^{q-1}\times\{0\}, \\ T^{q-1} & \text{on} & \Delta^{q-1}\times\{1\}, \quad \text{and} \\ R^{q-2} & \text{on} & ((q-2)\text{-face of } \Delta^{q-1})\times[0,1]. \end{array}$$

In particular, it glues up to a triangulation of $\partial\Delta^q\times[0,1]$. Denoting by t_q the middle simplex Δ^q in $\Delta^q\cup_{\partial\Delta^q}\partial\Delta^q\times[0,1]$, we have thus obtained a triangulation T^q of $\Delta^q\simeq\Delta^q\cup_{\partial\Delta^q}\partial\Delta^q\times[0,1]$. Note that the favorite simplex t_q comes with a canonical isomorphism $t_q\cong\Delta^q$.

The product triangulation S^{q-1} admits a canonical projection $S^{q-1}\to T^{q-1}$ and by definition of the quotient $S^{q-1}/\sim$, the composition of this projection with t_{q-1} factors through $S^{q-1}/\sim$:

$$\begin{tikzcd} S^{q-1} \arrow[r, two heads] \arrow[d] & T^{q-1} \arrow[r, two heads, "p_{q-1}"] & t_{q-1}. \\ S^{q-1}/\sim \arrow[urr, "s_{q-1}"'] & & \end{tikzcd}$$

Let $r_{q-1}:R^{q-1}\to S^{q-1}/\sim$ denote the simplicial projection obtained by sending those vertices of R^{q-1} which were initially in $S^{q-1}/\sim$ to themselves, and the new vertices to any of the vertex of the q-simplex in $S^{q-1}/\sim$ which they are the barycenter of. Observe that the composition $s_{q-1}\circ r_{q-1}:R^{q-1}\to t_{q-1}$ restricts to $s_{q-2}\circ r_{q-2}:R^{q-2}\to t_{q-2}$ on the subtriangulations R^{q-2} of the $((q-2)$-face of $\Delta^{q-1})\times[0,1]$'s. In particular, it glues up to the desired simplicial projection

$$p_q:T^q\to t_q.$$

Let us further check that the claimed properties are satisfied:

1. Let t be a simplex in T^q. If $t=t_q$, then ∂t is in the interior of T^q and does not meet any proper face of T^q. If $t\neq t_q$, then t belongs to $\partial\Delta^q\times[0,1]$ and hence to one of the triangulations R^{q-1} of $\Delta^{q-1}\times[0,1]\subset\partial\Delta^q\times[0,1]$. But then, $\partial t\cap\partial T^q$ is included in $\Delta^{q-1}\times\{1\}$ which is triangulated by T^{q-1}, so that the assertion follows by induction.

2. Let t,s be q-dimensional simplices in T^q. Suppose that there exists $i\neq j$ such that $t\cap T^q_i\neq\varnothing$ and $s\cap T^q_j\neq\varnothing$. In particular, t and s belong to two different copies of the triangulation R^{q-1} of $\Delta^{q-1}\times[0,1]\subset\partial\Delta^q\times[0,1]$ (corresponding to the i-th and j-th $(q-1)$-faces of Δ^q). By construction

of R^{q-1}, the q-simplex t has a vertex in the interior of R^{q-1} (and hence in the interior of T^q), namely the one which is the barycenter of a q-simplex in $S^{q-1}/\sim$. Since s belongs to a different copy of R^{q-1}, this vertex of t can not belong to s.

Let $z = \Sigma_{i=1}^r a_i \sigma_i$ be a singular cycle in X with $a_i = \pm 1$. Set $L_i = T^q$ and $L_{ij} = T_j^q$, for $i = 1, .., r$ and $j = 0, .., q$. Define

$$L = L_1 \coprod ... \coprod L_r$$

and $\sigma : |L| \to X$ on every $|L_i|$ as the composition

$$\sigma|_{|L_i|} : |L_i| = |T^q| \xrightarrow{p_q} |t_q| \cong \Delta^q \xrightarrow{\sigma_i} X$$

where $|t_q| \cong \Delta^q$ is the canonical isomorphism described above. Because $\partial z = 0$, there exists a finite (in general nonunique) partition $\mathcal{S} = \coprod_{k=1}^{s} \mathcal{S}^k$ of the set $\mathcal{S} = \{(i,j) \mid i = 1, .., r,\ j = 0, .., q\}$ such that for every k in $\{1, .., s\}$, the set $\mathcal{S}^k$ contains two elements $(i,j) \neq (i',j')$ for which we have

$$\begin{aligned} \sigma_i \circ \eta_j = \sigma_{i'} \circ \eta_{j'} &: \Delta^{q-1} \longrightarrow X, \\ (-1)^j a_i = -(-1)^{j'} a_{i'}&, \end{aligned}$$

where $\eta_j : \Delta^{q-1} \hookrightarrow \Delta^q$ is the j-th face inclusion.

For every j-face L_{ij} of L_i, there exists a canonical isomorphism $L_{ij} \cong \Delta^{q-1}$ such that the composition

$$L_{ij} \cong \Delta^{q-1} \xrightarrow{\eta_j} \Delta^q \cong L_i$$

is the canonical inclusion. Define an equivalence relation $\sim$ on L generated by the relations

$$\begin{aligned} v \sim w \iff\ & \exists k \text{ such that } v \in L_{ij},\ w \in L_{i'j'}, \text{ for } (i,j),(i',j') \in \mathcal{S}^k \text{ and} \\ & v \text{ is mapped to } w \text{ via the isomorphism } L_{ij} \cong \Delta^{q-1} \cong L_{i'j'}, \end{aligned}$$

where the isomorphisms $L_{ij} \cong \Delta^{q-1} \cong L_{i'j'}$ are the canonical ones described above.

Lemma 20 *$v \sim w$ implies either $v = w$ or v, w belong to different j-faces of L.*

Proof. Suppose $v \sim w$ and v, w belong to the same j-face $F \subset L_i$ of L. This means that there exists a finite sequence of isomorphisms

$$\tau_r : L_{i_r, j_r} \xrightarrow{\cong} L_{i'_r, j'_r}$$

with $(i_r, j_r), (i'_r, j'_r) \in \mathcal{S}^k$, for some k, and $i'_r = i_{r+1}$, and such that

$$w = \tau_R \circ ... \circ \tau_0(v).$$

Observe now that $\tau_R \circ ... \circ \tau_0$ induces an automorphism of the face F. Let $f_0, ..., f_j \in \{e_0, ..., e_q\}$ be the vertices of $F \subset L_i \cong \Delta^q$. By definition of the equivalence relation $\sim$, each of the isomorphism τ_r preserves the order by numbering of the vertices $\{e_0, ..., \widehat{e_{j_r}}, ..., e_q\} \subset L_{i_r, j_r} \cong \Delta^q$ and $\{e_0, ..., \widehat{e_{j'_r}}, ..., e_q\} \subset L_{i'_r, j'_r} \cong \Delta^q$. In particular, the automorphism of the face F induced by $\tau_R \circ ... \circ \tau_0$ preserves the order of the vertices $\{f_0, ..., f_j\}$ and has hence to be the identity, so that $v = w$. ■

Define $K = L/\sim$ and let $p : L \to K$ denote the canonical projection.

Lemma 21 *The projection $p : L \to K$ induces a one-to-one correspondence between the q-simplices of L and those of K.*

Proof. First, we check that p maps q-simplices to q-simplices. To see this, we show that if $\langle v, w\rangle$ is a 1-simplex of L then $p(v) \neq p(w)$. Suppose $v \sim w$. By Lemma 20 and since $v \neq w$ this implies that v and w belong to different i-faces of L. But this contradicts property 1).

Second, we verify that for any q-simplices k, ℓ of L, if $p(k) = p(\ell)$ then $k = \ell$. Let $v_0, ..., v_q$ be the vertices of k and $w_0, ..., w_q$ be those of ℓ. If $p(k) = p(\ell)$ then up to renumbering the w_i's we have $v_i \sim w_i$ for every $0 \leq i \leq q$. By Lemma 20 this means that either $v_i = w_i$ or v_i and w_i belong to different j-faces of L. Up to permuting the index set $\{0, ..., q\}$, we can suppose that $v_i = w_i$ for $i \leq i_0$ and $v_i \neq w_i$ for $i > i_0$, for some $-1 \leq i_0 \leq q$. If $i_0 = q$, then $k = \ell$. If $i_0 < q$, then v_q and w_q belong to different j-faces and hence also to different (closed) $(q-1)$-faces L_{ij} and $L_{i'j'}$ of L.

If $i \neq i'$, then k belongs to L_i and ℓ belongs to $L_{i'} \neq L_i$, so that none of the vertices of k and ℓ can agree, and in particular, $i_0 = -1$. Thus, all the v_i's (and w_i's) belong to the boundary of L, which contradicts the assumption that k (and ℓ) is q-dimensional: Indeed, if all the vertices of k would lie on the boundary of $L_i \cong T^q$, then they all would have to lie on the same $(q-1)$-face of L_i, since by Property 1) the boundary ∂t of t, which contains $\{v_0, ..., v_q\}$ intersects at most one $(q-1)$-face of L_i. The vertices of k being contained in a $(q-1)$-dimensional simplicial complex, it now follows that the dimension of k is strictly smaller than q.

If $i = i'$, then k and ℓ are both contained in $L_i = T^q$ and we have $v_q \in k \cap L_{ij} \neq \varnothing$ and $w_q \in \ell \cap L_{ij'} \neq \varnothing$, so that by Property 2), k possesses a vertex in the interior of L_i which does not belong to ℓ. But then this vertex can not be equivalent to any of the vertices in ℓ: contradiction. ■

Thus, K is obtained from L by identifying the $(q-1)$-faces L_{ij} with $L_{i'j'}$ for every $(i,j), (i',j')$ in $\mathcal{S}^k$. Note that no other identification can occur. Observe that since the diagram

$$\begin{tikzcd}
L_{ij} \arrow[r, phantom, "\cong"] \arrow[dr, "\sigma|_{L_{ij}}"'] & \Delta^{q-1} \arrow[r, phantom, "\cong"] & L_{i'j'} \arrow[dl, "\sigma|_{L_{i'j'}}"] \\
& X &
\end{tikzcd}$$

commutes, the map $\sigma : |L| \to X$ automatically factors through $|K|$. We denote the resulting map by $\zeta : |K| \to X$.

Let $k_1, ..., k_r$ be the image under the projection $p : L \to K$ of the r favorite simplices $t_q \in T^q \cong L_i \subset L$, for $i = 1, .., r$, and let $k_{r+1}, ..., k_R$ be the remaining q-simplices of K. Denote by $\overline{k}_j$ the simplex k_j endowed with the orientation of its vertices giving the affine oriented simplex $p^{-1}(\overline{k}_j) \subset L_i \cong \Delta^q$ the same orientation as $\Delta^q \cong L_i$. For $j = 1, ..., R$, define b_j in $\{-1, +1\}$ as $b_j = a_i$ if k_j belongs to $p(L_i)$. Observe that $b_i = a_i$ for $i = 1, ..., r$.

Lemma 22 *The chain $\Sigma_{j=1}^R b_j \overline{k}_j$ in $C_q(K)$ is a simplicial cycle representing the fundamental cycle $[K]$.*

Proof. By construction, it is clear that if $\Sigma_{j=1}^R b_j \overline{k}_j$ is a cycle, then it represents the fundamental cycle $[K]$. To check that it is a cycle, note that its boundary is a sum of $(q-1)$-oriented simplices with coefficients in $\{-1,+1\}$, where each of the $(q-1)$-simplices of K appears exactly twice. It thus remains to show that they appear with opposite sign. For $(q-1)$-simplices whose preimage in L belongs to the interior of L, this is clear. If k is not such a $(q-1)$-simplex, then its preimage by p consists of two $(q-1)$-simplices, belonging to L_{i_1,j_1} and L_{i_2,j_2} respectively, for some $(i_1,j_1)\neq(i_2,j_2)\in\mathcal{S}^k$.

Now, k is the m_1-th $(q-1)$-face of an oriented q-simplex $\overline{k}_{n_1}=\langle v_0^1,...,v_q^1\rangle$ of K with $p^{-1}(\overline{k}_{n_1})\in L_{i_1}$ and the m_2-th $(q-1)$-face of an oriented q-simplex $\overline{k}_{n_2}=\langle v_0^2,...,v_q^2\rangle$ of K with $p^{-1}(\overline{k}_{n_2})\in L_{i_2}$. In particular, the vertices of k consists of the sets

$$\left\{v_0^1,...,\widehat{v_{m_1}^1},...,v_q^1\right\}=\left\{v_0^2,...,\widehat{v_{m_2}^2},...,v_q^2\right\}.$$

Let $\tau'_\ell:\Delta^q\to\overline{k}_{n_\ell}$, for $\ell=1,2$, be orientation preserving isomorphisms mapping e_{j_ℓ} to $v_{m_\ell}^\ell$. (Such isomorphisms exist by definition of the order on the $\overline{k}_i$'s.) Let $\tau_\ell:\Delta^{q-1}\to\left\langle v_0^\ell,...,\widehat{v_{m_\ell}^\ell},...,v_q^\ell\right\rangle$, for $\ell=1,2$, be the isomorphisms of oriented $(q-1)$-simplices obtained by composing the j_ℓ-th $(q-1)$-face inclusion $\eta_{j_\ell}:\Delta^{q-1}\hookrightarrow\Delta^q$ with τ'_ℓ. Observe that

$$1=\mathrm{sign}(\tau'_\ell)=(-1)^{j_\ell-m_\ell}\mathrm{sign}(\tau_\ell),\quad\text{for }\ell=1,2.$$

Moreover, we have the equality of oriented $(q-1)$-simplices

$$\left\langle v_0^1,...,\widehat{v_{m_1}^1},...,v_q^1\right\rangle=\mathrm{sign}(\tau_1)\mathrm{sign}(\tau_2)\left\langle v_0^2,...,\widehat{v_{m_2}^2},...,v_q^2\right\rangle.$$

Thus, the simplex k appears in $\partial\left(\Sigma_{j=1}^R b_j\overline{k}_j\right)$ as

$$\begin{aligned}a_{i_1}(-1)^{m_1}\left\langle v_0^1,...,\widehat{v_{m_1}^1},...,v_q^1\right\rangle+a_{i_2}(-1)^{m_2}\left\langle v_0^2,...,\widehat{v_{m_2}^2},...,v_q^2\right\rangle=\\\left(a_{i_1}(-1)^{m_1}+a_{i_2}(-1)^{j_1-m_1}(-1)^{j_2-m_2}\right)\left\langle v_0^1,...,\widehat{v_{m_1}^1},...,v_q^1\right\rangle.\end{aligned}\tag{3}$$

Finally, we use the defining relation $(-1)^{j_1}a_{i_1}=-(-1)^{j_2}a_{i_2}$ to conclude that the coefficient in (3) vanishes, which finishes the proof of the lemma. ■

Let $c\in C^*_{\mathrm{sing,alt}}(X)$ be a singular, alternating cochain on X. Let $\tau:\Delta^q\to k_i$ be the composition of the canonical isomorphism $\Delta^q\cong t_q\subset L_i\subset L$ and of the projection $p:L\to K$. Note that τ is orientation preserving and that $\zeta\circ|\tau|=\sigma_i$. We thus obtain

$$\zeta^*(c)(\overline{k_i})=c(\zeta\circ|\tau|)=c(\sigma_i),\quad\text{for }i=1,...,r.\tag{4}$$

Observe that, by construction, the map $\zeta:|K|\to X$ restricted to any of the k_i's, for $r+1\leq i\leq R$, factors via a simplicial projection through a strictly lower dimensional simplex. In particular, it is immediate that

$$\zeta^*(c)(\overline{k_i})=0,\quad\text{for }i=r+1,...,R.\tag{5}$$

Lemma 22, Equalities (4) and (5) now imply

$$\zeta^*(c)\left([K]\right)=\sum_{i=1}^r\zeta^*(c)(\overline{k_i})+\sum_{i=r+1}^R\zeta^*(c)(\overline{k_i})=\sum_{i=1}^r a_ic(\sigma_i)+0=c\left(z\right),$$

which proves the claim. ■

Alternative proofs

Boundedness. If one is merely interested in the boundedness of characteristic classes of flat bundles, then the following argument furnishes a new proof of Gromov's original result (Theorem 3): For every closed real algebraic subgroup G of $\mathrm{GL}_n\mathbb{R}$, there exists an approximation BG_q to the classifying space which is a *compact* manifold and has the property that for every G-bundle over a q-dimensional simplicial complex K there exists a classifying map $f : |K| \to BG_q$ which is semi-algebraic of uniformly bounded complexity on every simplex of K and piecewise differentiable.

Now any cohomology class $\beta \in H^q(BG_q)$ can be represented by a closed differential q-form ω. Observe that the simplicial cochain

$$k \longmapsto \int_{f_*(k)} \omega,$$

where k is an oriented q-dimensional simplex of K, is a cocycle representing $f^*(\beta) \in H^q_{\mathrm{simpl}}(K)$. Because BG_q is compact and f is semi-algebraic of uniformly bounded complexity on every simplex k of K, it follows that $\int_{f_*(k)} \omega$ is uniformly bounded, so that $f^*(\beta)$ is represented by a uniformly bounded cocycle. The bound is independent of K and f.

Finiteness. From appropriate triangulations of the Cartesian products G^i, for $i = 0, ..., q$, it is not hard (but rather cumbersome) to exhibit a triangulation of the model of the classifying space BG given by the join construction which projects, via the natural projection $BG \to \Delta^q$ onto the first barycentric subdivision of Δ^q. A classifying map $f : |K| \to BG^\delta \to BG$ having the property that composed with the natural projection $BG \to \Delta^q$, it maps simplices of K isomorphically to simplices of Δ^q (such a map can always be found), does admit a simplicial approximation, upon passing to the *first* barycentric subdivision of K. The set I of Theorem 4 is hence potentially much sharper. However, we are not aware of explicit triangulations of the products G^i's.

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