Simplicial volume of locally symmetric spaces covered by $\mathrm{SL}_3\mathbb{R}/SO(3)$

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ABSTRACT. We give the first complete proof of the strict positivity of the simplicial volume of compact locally symmetric spaces covered by $\mathrm{SL}_3\mathbb{R}/SO(3)$ and show why the proof in [**Sa82**] is incorrect.

1. Introduction

The aim of this paper is twofold: On the one hand, we want to give a simple proof of the following theorem.

THEOREM 1. *Let M be a compact locally symmetric space whose universal cover is $\mathrm{SL}_3\mathbb{R}/SO(3)$. Then the simplicial volume of M is strictly positive.*

On the other hand, we show why the proof in [**Sa82**] of the same result for locally symmetric space whose universal cover is $\mathrm{SL}_n\mathbb{R}/SO(n)$, where $n \geq 2$, is incomplete.

Theorem 1 is complementary to a result which has remarkably just recently been proven by Lafont and Schmidt in almost full generality thus answering affirmatively a conjecture of Gromov.

THEOREM 2 ([**LaSch05**]). *Let M be a compact locally symmetric space whose universal cover is a globally symmetric space of noncompact type and not isomorphic to $\mathrm{SL}_3\mathbb{R}/SO(3)$. Then the simplicial volume of M is strictly positive.*

All proofs of the positivity of the simplicial volume of locally symmetric spaces rely on a uniform bound on the volume of certain top dimensional simplices in their universal cover. In [**InYa82**], Inoue and Yano generalize ideas of Thurston [**Th78**] in order to show that the volume of geodesic simplices in any (fixed) symmetric space of real rank one is uniformly bounded, thus proving Theorem 2 in this case. In [**LaSch05**] the simplices in consideration are constructed with the barycenter method, and the obtained volume bound strongly relies on previous work by Connell and Farb [**CoFa03**]. Note that the proof in [**LaSch05**] does not cover the case of $\mathrm{SL}_3\mathbb{R}$, so that the present paper contains the only volume bound on simplices in $\mathrm{SL}_3\mathbb{R}/SO(3)$ leading to a proof of Theorem 3.

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The simplices we investigate here are those introduced in [**Sa82**], namely convex simplices in the trace 1 model of the symmetric space $\mathrm{SL}_n\mathbb{R}/SO(n)$. The mistake in [**Sa82**] is that it is assumed that the euclidean barycenter of the simplices is invariant under isometries of the symmetric space - which is false. Once observed, this error is easy to point out: Theorem 7.4 in [**Sa82**] - which would imply the claimed result - is not proven as stated. This is explained in the last section of this paper. For more details, see also [**Bu05**]. Note that we do not see how to fix this gap in any straightforward way.

The proof we present here for $n = 3$ is substantially different from Savage's even though we do bound the volume of the same simplices. The various estimates which we use to do so are much sharper, so that the integral we are left with is easier to bound. For $n = 2$, our method shows without using the transitivity of $\mathrm{SL}_2\mathbb{R}$ on nondegenerated ideal geodesic simplices of $\mathrm{SL}_2\mathbb{R}/SO(2)$ that the area of those simplices is up to a sign constant.

This paper is structured as follows: We start by giving some models for the symmetric spaces $\mathrm{SL}_n\mathbb{R}/SO(n)$ and discussing the geometry of their boundaries in Section 2. In Section 3 we indicate how Theorem 1 reduces to proving that the volume of certain simplices of the symmetric space is uniformly bounded. The volume forms for our models of symmetric space are computed in Section 4 and in Section 5 we exhibit a formula for the volume of simplices. We treat the simple example of $n = 2$ in Section 6 and the more complicated case $n = 3$ in Section 7. Finally, Savages's proof is discussed in Section 8.

2. The geometry of the symmetric space

Let Sym_n denote the space of n by n real valued symmetric matrices and let Pos_n be the subset of positive definite matrices,

$$\mathrm{Sym}_n = \{S \in M_n(\mathbb{R}) \mid S = S^t\},$$
$$\mathrm{Pos}_n = \{S \in \mathrm{Sym}_n \mid xSx^t > 0 \text{ for every } 0 \neq x \in \mathbb{R}^n\}.$$

The space of symmetric matrices is a vector space of dimension $n(n+1)/2$. The set $\{E_{ij} | 1 \leq i \leq j \leq n\}$, where E_{ii} is the matrix having the (i, i)-coefficient equal to 1 and all others equal to 0 while E_{ij} is the matrix having the (i, j) and (j, i)-coefficients equal to 1 and all others equal to 0, furnishes a natural basis of Sym_n. Let $\{e_1, ..., e_{n(n+1)/2}\}$ be the canonical basis of $\mathbb{R}^{n(n+1)/2}$. There exists a unique bijection between the sets $\{E_{ij} | 1 \leq i \leq j \leq n\}$ and $\{e_1, ..., e_{n(n+1)/2}\}$ preserving the lexicographic and natural orders respectively. This bijection induces an isomorphism $\mathrm{Sym}_n \cong \mathbb{R}^{n(n+1)/2}$. We will abuse notation and view an element of Sym_n both as an n by n matrix and as a vector in $\mathbb{R}^{n(n+1)/2}$ (via this specific isomorphism). In particular, we shall consider the determinant of $n(n+1)/2$ vectors of $\mathbb{R}^{n(n+1)/2}$ as a function on the product of $n(n+1)/2$ copies of Sym_n, thus as a map

$$\det : (\mathrm{Sym}_n)^{n(n+1)/2} \longrightarrow \mathbb{R}.$$

The groups $\mathrm{GL}_n\mathbb{R}$ and $\mathrm{SL}_n\mathbb{R}$ act on the space $M_n(\mathbb{R})$ of n by n real valued matrices according to the rule

$$\begin{array}{rcl} \rho_g : M_n(\mathbb{R}) & \longrightarrow & M_n(\mathbb{R}) \\ S & \longmapsto & gSg^t, \end{array}$$

for every g in $\mathrm{GL}_n\mathbb{R}$ or $\mathrm{SL}_n\mathbb{R}$. Note that this action is linear. In fact, it is given by the natural inclusion $\mathrm{GL}_n\mathbb{R} \hookrightarrow \mathrm{GL}_{n^2}\mathbb{R}$ defined by $g \longmapsto g \otimes g$. In particular, the character of this representation of $\mathrm{GL}_n\mathbb{R}$ is

$$\begin{array}{rcl} \mathrm{GL}_n\mathbb{R} & \longrightarrow & \mathbb{R} \\ g & \longmapsto & \det(g)^{2n}. \end{array}$$

The action of $\mathrm{GL}_n\mathbb{R}$ obviously restricts to an action on the vector space of symmetric, respectively anti-symmetric, matrices and it can be checked that the corresponding characters are $g \mapsto \det(g)^{n+1}$ and $g \mapsto \det(g)^{n-1}$ respectively. As a consequence, if $S_1, ..., S_d$ are symmetric matrices and g is an element of $\mathrm{GL}_n\mathbb{R}$, then

$$\det(gS_1g^t, ..., gS_dg^t) = \det(g)^{n+1} \det(S_1, ..., S_d). \tag{2.1}$$

The space Pos_n is as an open subset of Sym_n naturally a smooth manifold. Its tangent space at each point is, by translation, identified with Sym_n. The action ρ_g on Sym_n restricts to an action which we still denote by ρ_g on Pos_n. Because it is linear on Sym_n, the induced map on the tangent space is again given as

$$\begin{array}{rcl} (\rho_g)_* : \mathrm{Sym}_n & \longrightarrow & \mathrm{Sym}_n \\ S & \longmapsto & gSg^t. \end{array}$$

It is a standard fact - which we shall not explicitly need here - that the space Pos_n can be endowed with a Riemannian metric for which the transformations ρ_g are isometries for every g in $\mathrm{GL}_n\mathbb{R}$. In fact, the scalar product on $T_S\mathrm{Pos}_n \cong \mathrm{Sym}_n$, for S in Pos_n, can be taken as $\langle X, Y\rangle_S = \mathrm{Tr}(S^{-1}XS^{-1}Y)$, for every X, Y in Sym_n. (Note that a different scaling of this product is also common.)

Let $\mathrm{Pos}_n^{\det}$ denote the hyperspace of Pos_n consisting of those positive definite matrices with determinant equal to 1. It has dimension

$$d = \frac{n(n+1)}{2} - 1.$$

For further use, define a map π as the composition of the natural projection of Pos_n onto $\mathrm{Pos}_n^{\det}$ with the inclusion $\mathrm{Pos}_n^{\det} \subset \mathrm{Pos}_n$:

$$\begin{array}{rcl} \pi : \mathrm{Pos}_n & \longrightarrow & \mathrm{Pos}_n \\ S & \longmapsto & \frac{1}{\det(S)^{1/n}} S. \end{array}$$

The action of $\mathrm{SL}_n\mathbb{R}$ on Pos_n restricts to an action on $\mathrm{Pos}_n^{\det}$. Note that this action is by isometries with respect to the Riemannian metric induced from Pos_n. The stabilizer of the identity is clearly equal to $\mathrm{SO}(n)$, so that the space $\mathrm{Pos}_n^{\det}$ is one possible model for the symmetric space $\mathrm{SL}_n\mathbb{R}/\mathrm{SO}(n)$. But we could choose a different normalization, for example consider the space $\mathrm{Pos}_n^{\mathrm{tr}}$ consisting of all positive definite matrices with trace equal to 1. The action of $\mathrm{SL}_n\mathbb{R}$ on $\mathrm{Pos}_n^{\mathrm{tr}}$ needs then to be normalized also, and is given as

$$\rho_g^{\mathrm{tr}}(S) = \frac{1}{\mathrm{tr}(gSg^t)} gSg^t,$$

for every S in $\mathrm{Pos}_n^{\mathrm{tr}}$ and g in $\mathrm{SL}_n\mathbb{R}$. The model $\mathrm{Pos}_n^{\mathrm{tr}}$ has two major advantages: First, it is a bounded subset of the space of symmetric matrices, and as such has a natural compactification $\overline{\mathrm{Pos}_n^{\mathrm{tr}}} = \mathrm{Pos}_n^{\mathrm{tr}} \cup \partial\mathrm{Pos}_n^{\mathrm{tr}}$. Second, it is a convex subset of the space of symmetric matrices. Thus, given any $i+1$ points $S_0, ..., S_i$ in $\mathrm{Pos}_n^{\mathrm{tr}}$ or possibly in its boundary, we have a canonical choice of i-simplex $\sigma(S_0, ..., S_i)$ given by taking the convex linear combination of the S_j's:

$$\sigma(S_0, ..., S_i)(t_0, ..., t_i) = \sum_{j=0}^{i} t_j S_j,$$

for every $(t_0, ..., t_i)$ in Δ^i. Such a simplex - which we can of course also define for points $S_0, ..., S_i$ in $\overline{\mathrm{Pos}}_n$ - is called a *straight* simplex. If the vertices $S_0, ..., S_i$ are all in the boundary of Pos_n, then we say that the simplex is *ideal*. Note that this in fact provides us with a natural i-filling of the symmetric space $\mathrm{SL}_n\mathbb{R}/\mathrm{SO}(n)$. Observe that lines in the trace 1 model of the symmetric space are the distinguished geodesics with respect to the Hilbert's metric of the convex set $\mathrm{Pos}_n^{\mathrm{tr}}$ (see [**dlH93**]), but they are in general not geodesics for the Riemannian metric indicated above.

Let $\partial\mathrm{Pos}_n$ denote the boundary of Pos_n with respect to the induced topology from Sym_n and set $\overline{\mathrm{Pos}}_n = \mathrm{Pos}_n \cup \partial\mathrm{Pos}_n$. The boundary $\partial\mathrm{Pos}_n$ of Pos_n decomposes in $n-2$ subsets according to the rank of its matrices. As we shall eventually want to bound the volume of straight simplices in $\mathrm{Pos}_n^{\mathrm{tr}}$ with rank 1 matrices as vertices, we will restrict our attention to this subset of the boundary. Any rank 1 matrix with positive nonzero eigenvalue belongs to $\partial\mathrm{Pos}_n$ and has the form

$$g \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix} g^t = g e_1 e_1^t g_t,$$

for some g in $\mathrm{GL}_n\mathbb{R}$. Thus, taking $x = ge_1 \in \mathbb{R}^n$, we see that such a matrix can be written as xx^t, which shows that the map

$$\begin{array}{rccc} R: & \mathbb{R}^n \setminus \{0\} & \longrightarrow & \partial\mathrm{Pos}_n \\ & x & \longmapsto & xx^t \end{array}$$

surjects onto the rank 1 matrices of $\partial\mathrm{Pos}_n$. The preimage of $R(x)$ is clearly $\{x, -x\}$. Furthermore, the map R is $\mathrm{GL}_n\mathbb{R}$-equivariant:

$$R(gx) = gR(x)g^t,$$

for every x in $\mathbb{R}^n$ and g in $\mathrm{GL}_n\mathbb{R}$.

Since $\|x\|_2^2 = \mathrm{Tr}(R(x))$, the restriction of R to the unit sphere S^{n-1} has image contained in $\mathrm{Pos}_n^{\mathrm{tr}}$. Abusing notation, we also denote this map by R:

$$\begin{array}{rccc} R: & S^n & \longrightarrow & \partial\mathrm{Pos}_n^{\mathrm{tr}} \\ & x & \longmapsto & xx^t. \end{array}$$

Note that this map is also equivariant with respect to the natural action of $\mathrm{GL}_n(\mathbb{R})$ on S^{n-1} (given by $x \mapsto gx/\left\|gx\right\|_2$) and the action ρ_g^{tr} on $\mathrm{Pos}_n^{\mathrm{tr}}$.

Observe that the subspace of Pos_n (respectively $\mathrm{Pos}_n^{\mathrm{tr}}$) consisting of rank 1 boundary points is homeomorphic to $(\mathbb{R}^n \setminus \{0\}) / \pm 1$ (respectively $P^{n-1}\mathbb{R}$).

3. Bounded cohomology and simplicial volume

It is well known that the positivity of the simplicial volume of a compact locally symmetric manifold covered by $\mathrm{SL}_n\mathbb{R}/SO(n)$ is equivalent to the surjectivity of the comparison map

$$H_{c,b}^*(\mathrm{SL}_n\mathbb{R}) \longrightarrow H_c^*(\mathrm{SL}_n\mathbb{R})$$

between the bounded continuous and the continuous real valued cohomology of $\mathrm{SL}_n\mathbb{R}$ in top degree, that is, in degree $d = n(n+1)/2 - 1$. For a proof of this equivalence, see for example [**Bu04**, Ch. 3.2.6].

It is a standard fact that the real valued continuous cohomology of $\mathrm{SL}_n\mathbb{R}$ in degree i is isomorphic to the $\mathrm{SL}_n\mathbb{R}$-invariant differential i-forms on the symmetric space $\mathrm{SL}_n\mathbb{R}/\mathrm{SO}(n)$, which we denote by $A^i(SL_n\mathbb{R}/SO(n))^{\mathrm{SL}_n\mathbb{R}}$ and in top dimension d, it is easy to see that $A^d(SL_n\mathbb{R}/SO(n))^G$, and hence $H^d_c(\mathrm{SL}_n\mathbb{R})$, is 1-dimensional. Indeed, that it is at most 1-dimensional follows from the transitivity of the action of $\mathrm{SL}_n\mathbb{R}$ on its symmetric space and from the fact that the latter space has dimension d. Furthermore, a nontrivial $\mathrm{SL}_n\mathbb{R}$-invariant d-form on $\mathrm{Pos}_n^{\mathrm{tr}}$ (or $\mathrm{Pos}_n^{\det}$) will be exhibited in the next section. This form

$$\omega^{\mathrm{tr}} \in A^d(\mathrm{Pos}_n^{\mathrm{tr}})^G$$

shall be called the volume form. This is a slight abuse of terminology since it is only the volume form in the proper Riemannian sense of the word after an appropriate rescaling of the metric. But in any case, it is well defined up to a nonzero constant, which is all we need here.

The isomorphism between invariant differential forms on the symmetric space and the continuous cohomology of $\mathrm{SL}_n\mathbb{R}$ can be described explicitly as shown by Dupont in [**Du76**]:

$$\begin{aligned} A^i(\mathrm{SL}_n\mathbb{R}/SO(n))^G &\longrightarrow H^*_c(\mathrm{SL}_n\mathbb{R}) \\ \alpha &\longmapsto [c_x(\alpha)], \end{aligned}$$

where $c_x(\alpha)$ is an $\mathrm{SL}_n\mathbb{R}$-invariant i-cocycle given, for $g_0, ..., g_i$ in $\mathrm{SL}_n\mathbb{R}$, as

$$c_x(\alpha)(g_0, ..., g_i) = \int_{\sigma(g_0\cdot x, ..., g_i\cdot x)} \alpha,$$

with x a fixed base point in $\mathrm{SL}_n\mathbb{R}/\mathrm{SO}(n)$ and σ an i-filling. In the trace model $\mathrm{Pos}_n^{\mathrm{tr}}$ of the symmetric space $\mathrm{SL}_n\mathbb{R}/\mathrm{SO}(n)$, the i-filling can be chosen as the straight simplices

$$\begin{array}{cccc} \sigma(S_0, ..., S_i): & \Delta^i & \longrightarrow & \mathrm{Pos}_n^{\mathrm{tr}} \\ & (t_0, ..., t_i) & \longmapsto & \sum_{j=0}^i t_j S_j, \end{array}$$

for every $S_0, ..., S_i$ in $\mathrm{Pos}_n^{\mathrm{tr}}$, which we already defined in the previous section. Moreover, if the integration of a simplex with vertices in the boundary $\partial\mathrm{Pos}_n^{\mathrm{tr}}$ is well defined, then one can similarly define an i-cocycle on $\mathrm{SL}_n\mathbb{R}$ as

$$c_\xi(\alpha)(g_0, ..., g_i) = \int_{\sigma(g_0\cdot \xi, ..., g_i\cdot \xi)} \alpha,$$

where now ξ lies on the boundary $\partial\mathrm{Pos}_n^{\mathrm{tr}}$, which is a measurable cocycle and represents the same cohomology class as $c_x(\alpha)$, for any x in $\mathrm{Pos}_n^{\mathrm{tr}}$. This is the case in top dimension. Let thus R be a boundary point in $\partial\mathrm{Pos}_n^{\mathrm{tr}}$. It follows from the previous discussion that $c_R(\omega^{\mathrm{tr}})$ is a cocycle representing a generator of $H^d_c(\mathrm{SL}_n\mathbb{R})$, so that Theorem 1 will follow at once from the following theorem:

THEOREM 3. *There exists C in $\mathbb{R}$ such that, for every rank 1 vertices $R_0, ..., R_5$ in ∂Pos_3^{tr}, the inequality*

$$\left| \int_{\sigma(R_0,...,R_5)} \omega^{tr} \right| \leq C$$

holds, where $\sigma(R_0,...,R_5) : \Delta^5 \longrightarrow Pos_3^{tr}$ *is the straight simplex given, for every* $(t_0,...,t_5)$ *in* Δ^5*, by*

$$\sigma(R_0,...,R_5)(t_0,...,t_5) = \sum_{j=0}^{5} t_j R_j.$$

An alternative argument to see how Theorem 3 implies the positivity of the simplicial volume is given by Savage ([**Sa82**, Section 2 and Theorem 5.3]).

4. The volume form

We start by exhibiting in Proposition 1 a differential d-form ω on Pos_n which is invariant under the action of $\mathrm{SL}_n\mathbb{R}$. This form ω restricted to $\mathrm{Pos}_n^{\det}$ will be our volume form on the determinant model of the symmetric space $\mathrm{SL}_n\mathbb{R}/SO(n)$. Note that this proposition is Theorem 4.1 in [**Sa82**] but the proof we give here is much simpler.

Recall that symmetric matrices $S = (s_{ij})_{1\leq i,j\leq n}$ in $M_n(\mathbb{R})$ are identified to their images in $\mathbb{R}^{d+1}$, via the isomorphism given in Section 2. The matrix S thus becomes a vector with entries $s_0,...,s_d$, where the s_i's just correspond to a relabelling of the s_{ij}'s, for $i \leq j$, the relabelling being in fact given by the lexicographic order on $\{(i,j) \mid 1 \leq i \leq j \leq n\}$.

Proposition 1. *The differential d-form*

$$\omega = \sum_{i=0}^{d} (-1)^i s_i ds_0 \wedge ... \wedge \widehat{ds_i} \wedge ... \wedge ds_d$$

on Pos_n *is invariant under the action of* $\mathrm{SL}_n\mathbb{R}$ *given by* $S \longmapsto gSg^t$.

Proof. One only need to observe that, for S in Pos_n and $X_1,...,X_d$ vectors in $T_S\mathrm{Pos}_n \cong \mathrm{Sym}_n$, we have

$$\omega_S(X_1,...,X_d) = \det(S, X_1,...,X_d).$$

The $\mathrm{SL}_n\mathbb{R}$-invariance is then a simple consequence of (2.1). Indeed, one has

$$\begin{aligned}\rho_g^*(\omega)_S(X_1,...,X_d) &= \omega_{\rho_g(S)}(\rho_{g*}(X_1),...,\rho_{g*}(X_d)) \\ &= \det(gSg^t, gX_1g^t,...,gX_dg^t) \\ &= \det(g)^{n+1}\det(S, X_1,...,X_d) \\ &= \omega_S(X_1,...,X_d),\end{aligned}$$

for every g in $\mathrm{SL}_n\mathbb{R}$. □

Because the volume form ω^{tr} on the trace model $\mathrm{Pos}_n^{\mathrm{tr}}$ of the symmetric space is clearly equal to the restriction to $\mathrm{Pos}_n^{\mathrm{tr}}$ of the pullback of ω by π, we now compute $\pi^*(\omega)$. Again, those computations are already present in [**Sa82**] and can be found there in the proof of Proposition 4.3.

Proposition 2. *The pullback of* ω *by* π *is given by*

$$\pi^*(\omega) = \frac{1}{\det(S)^{(n+1)/2}}\omega.$$

PROOF. The induced map $\pi_* : T\mathrm{Pos}_n \to T\mathrm{Pos}_n$ on the tangent bundle of Pos_n furnishes for each point S in Pos_n a map $T_S\mathrm{Pos}_n \cong \mathrm{Sym}_n \to T_{\pi(S)}\mathrm{Pos}_n \cong \mathrm{Sym}_n$ given, when viewed as a map on the space Sym_n of symmetric matrices, by the $(d+1)$-square matrix $(\partial\pi_i/\partial s_j(S))_{0\le i,j\le d}$. Let us compute its coefficients:

$$\frac{\partial \pi_i}{\partial s_j}(S) = \begin{cases} \frac{-1}{(\det S)^{2/n}} s_i \frac{\partial}{\partial s_j}(\det S)^{1/n} & \text{if } i \neq j, \\ \frac{1}{(\det S)^{2/n}} \left(\det(S)^{1/n} - S_i \frac{\partial}{\partial s_j}(\det S)^{1/n}\right) & \text{if } i = j. \end{cases}$$

The matrix $(\partial\pi_i/\partial s_j(S))_{0\le i,j\le d}$ thus takes the form

$$\left(\frac{\partial \pi_i}{\partial s_j}(S)\right)_{0\le i,j\le d} =$$
$$= \frac{1}{(\det S)^{1/n}}\left(\mathrm{Id}_d - \frac{1}{(\det S)^{1/n}}\begin{pmatrix} S_0 \\ \vdots \\ S_d\end{pmatrix}\left(\frac{\partial}{\partial s_0}(\det S)^{1/n} ... \frac{\partial}{\partial s_d}(\det S)^{1/n}\right)\right).$$

For any symmetric matrix X in Sym_n, there exists a real number $\lambda_X \in \mathbb{R}$ (depending also on S) such that

$$\frac{\partial \pi_{ij}}{\partial s_{k\ell}}(S)(X) = \frac{1}{(\det S)^{1/n}} X + \lambda_X S.$$

If now, $X_1, ..., X_d$ are arbitrary vectors in $T_S\mathrm{Pos}_n \cong \mathrm{Sym}_n$, we have

$$\pi^*(\omega)_S(X_1, ..., X_d) = \omega_{\pi(S)}(\pi_*(X_1), ..., \pi_*(X_d))$$
$$= \det\left(\tfrac{1}{\det(S)^{1/n}} S, \tfrac{1}{(\det S)^{1/n}} X_1 + \lambda_{X_1} S, ..., \tfrac{1}{(\det S)^{1/n}} X_n + \lambda_{X_n} S\right)$$
$$= \tfrac{1}{\det(S)^{(d+1)/n}} \det(S, X_1, ..., X_n)$$
$$= \tfrac{1}{\det(S)^{(n+1)}} \omega_S(X_1, ..., X_n),$$

since $d = n(n+1)/2 - 1$. □

LEMMA 1. *Let $S_0, ..., S_d$ be matrices in $\overline{\mathrm{Pos}}_n$ and $\mu_0, ..., \mu_d$ be nonvanishing real numbers. Then*

$$\int_{\sigma(S_0,...,S_d)} \pi^*(\omega) = \int_{\sigma(\mu_0 S_0,...,\mu_d S_d)} \pi^*(\omega).$$

PROOF. This is a simple consequence of Stoke's theorem. After all, the singular simplices $\pi \circ \sigma(S_0, ..., S_d)$ and $\pi \circ \sigma(\mu_0 S_0, ..., \mu_d S_d)$ have the same image. They are only parametrized differently, but surely have the same volume as we shall now prove: Define a homotopy $H : \Delta^d \times [0,1] \to \mathrm{Pos}_n^{\det}$ between $\pi_*\sigma$ and $\pi_*\overline{\sigma}$ as

$$H(x,t) = \pi((1-t)\sigma(x) + t\overline{\sigma}(x)),$$

for x in Δ^d and t in $[0,1]$. The map H enjoys the property that for every inclusion of face $\Delta^{d-1} \hookrightarrow \Delta^d$, the image $H(\Delta^{d-1} \times [0,1])$ is at most $(d-1)$-dimensional so that its volume must vanish. Now since the volume form is a closed form (any $\mathrm{SL}_n\mathbb{R}$-invariant form on $\mathrm{SL}_n\mathbb{R}/SO(n)$ is automatically closed), the volume of $H(\partial(\Delta^d \times$

$[0,1])$) is zero, so that

$$\int_{\pi_*\sigma(S_0,\ldots,S_d)} \omega = \int_{\pi_*\sigma(\mu_0 S_0,\ldots,\mu_d S_d)} \omega,$$

as desired. □

In view of Lemma 1 we are now interested in the restriction of $\pi^*(\omega)$ not only to $\mathrm{Pos}_n^{\mathrm{tr}}$ but to different affine subspaces of Pos_n.

Proposition 3. *If $A = \{S \in Pos_n \mid \sum_{i=0}^d \lambda_i s_i = 1\}$ is a d-dimensional affine subspace of Pos_n for some $\lambda_i \in \mathbb{R}$, then $\pi^*(\omega)$ restricted to A takes the form*

$$\frac{\lambda_0^{-1}}{\det(S)^{(n+1)/2}} ds_1 \wedge ds_2 \wedge \ldots \wedge ds_d.$$

Note that it is automatic that λ_0 is nonzero. Indeed, it follows from A being a d-dimensional subspace of Pos_n, that it must contain a positive multiple of E_1, say μE_1, where $\mu > 0$. Thus the condition $\Sigma \lambda_i s_i = 1$ becomes $\lambda_0 \mu = 1$ for $S = \mu E_1$.

Proof. For the second assertion of the proposition, let $A = \{S \in \mathrm{Pos}_n \mid \sum_{i=0}^d \lambda_i s_i = 1\}$ be a d-dimensional subspace of Pos_n and recall that $\lambda_0 \neq 0$. Derivating the relation $\sum_{i=0}^d \lambda_i s_i = 1$ gives

$$ds_0 = -\lambda_0^{-1} \sum_{i=1}^d \lambda_i ds_i.$$

Substituting ds_0 by the right hand side of the above equation leads to

$$ds_0 \wedge \ldots \wedge \widehat{ds_i} \wedge \ldots \wedge ds_n = (-1)^i \lambda_0^{-1} \lambda_i ds_1 \wedge \ldots \wedge ds_n,$$

for every i between 0 and d. Indeed, if $i = 0$, there is nothing to prove. Suppose $i > 0$, then

$$\begin{aligned} ds_0 \wedge \ldots \wedge \widehat{ds_i} \wedge \ldots \wedge ds_n &= \left(-\lambda_0^{-1} \sum_{j=1}^d \lambda_j ds_j \right) \wedge ds_1 \wedge \ldots \wedge \widehat{ds_i} \wedge \ldots \wedge ds_n \\ &= -\lambda_0^{-1} \lambda_j ds_i \wedge ds_1 \wedge \ldots \wedge \widehat{ds_i} \wedge \ldots \wedge ds_n \\ &= (-1)^i \lambda_0^{-1} \lambda_i ds_1 \wedge \ldots \wedge ds_n. \end{aligned}$$

The form $\pi^*(\omega)$ restricted to A thus takes the form

$$\begin{aligned} \pi^*(\omega) &= \frac{1}{\det(S)^{(n+1)/2}} \sum_{i=0}^d (-1)^i s_i ds_0 \wedge \ldots \wedge \widehat{ds_i} \wedge \ldots \wedge ds_n \\ &= \frac{1}{\det(S)^{(n+1)/2}} \left(\sum_{i=0}^d \lambda_0^{-1} \lambda_i s_i \right) ds_1 \wedge \ldots \wedge ds_n \\ &= \frac{\lambda_0^{-1}}{\det(S)^{(n+1)/2}} ds_1 \wedge \ldots \wedge ds_n, \end{aligned}$$

as claimed. □

Observe that in particular, our volume form ω^{tr} on the trace model $\mathrm{Pos}_n^{\mathrm{tr}}$ of the symmetric space, which is equal to the pullback of ω by π restricted to $\mathrm{Pos}_n^{\mathrm{tr}}$ is given by

$$\omega^{\mathrm{tr}} = \pi^*(\omega)|_{\mathrm{Pos}_n^{\mathrm{tr}}} = \frac{1}{\det(S)^{(n+1)/2}} ds_1 \wedge ... \wedge ds_d.$$

This is Proposition 4.3 in [**Sa82**].

5. A volume formula

We shall now exhibit a simple formula for the computation of the volume of a simplex of the form $\pi \circ \sigma$, where σ is a straight simplex. This expression is implicit in [**Sa82**] (at least in the case where the R_i's are rank 1 boundary points in $\partial \mathrm{Pos}_n^{\mathrm{tr}}$) and is used there in the beginning of the proof of Theorem 7.4.

THEOREM 4. *Let $R_0, ..., R_d$ be positive definite matrices in $\overline{Pos_n}$. Then*

$$\int_{\sigma(R_0,...,R_d)} \pi^*(\omega) = \det(R_0, ..., R_d) \int_{\Delta^d} \frac{dt_1...dt_d}{\det(\Sigma_{i=0}^d t_i R_i)^{(n+1)/2}}.$$

Of course, the variable t_0, in the above integral, is to be understood as being equal to $1 - t_1 - ... - t_d$.

PROOF. Let A be the affine linear combination of the R_i's, that is,

$$A = \{S \in \mathrm{Pos}_n \mid S = \Sigma_{i=0}^d t_i R_i,\ t_i \in \mathbb{R}\}.$$

Clearly, there exists real numbers $\lambda_0, ..., \lambda_d$ such that

$$A = \{S \in \mathrm{Pos}_n \mid \Sigma_{i=0}^d \lambda_i s_i = 1\}.$$

If A has dimension strictly smaller than d, then both the right and the left hand side of the equality of the theorem are 0. Otherwise, by Proposition 3, the form $\pi^*(\omega)$ restricted to A takes the form

$$\frac{\lambda_0^{-1}}{\det(S)^{(n+1)/2}} ds_1 \wedge ... \wedge ds_d.$$

Let $\sigma : \Delta^d \to \overline{\mathrm{Pos}_n}$ be the straight singular simplex $\sigma(R_0, ..., R_d)$, so that

$$\sigma(t_0, ..., t_d) = \Sigma_{i=0}^d t_i R_i = R_0 + \Sigma_{i=1}^d t_i \left(R_i - R_0\right),$$

where the last equality comes from the relation $t_0 + ... + t_d = 1$. We then have that

$$\begin{aligned} \mathrm{Vol}(x_0, ..., x_d) &= \int_{\pi_*(\sigma)} \omega = \int_\sigma \pi^*(\omega) \\ &= \int_\sigma \frac{\lambda_0^{-1}}{\det(S)^{(n+1)/2}} ds_1 \wedge ... \wedge ds_d \\ &= \int_{\Delta^d} |\det(\sigma')| \frac{\lambda_0^{-1}}{\det(\sigma(t_0, ..., t_d))^{(n+1)/2}} dt_1 \cdot ... \cdot dt_d, \end{aligned}$$

from the chain rule. Since σ is a linear map, its Jacobian $|\det(\sigma')|$ is easy to compute. Indeed,

$$\frac{\partial \sigma_j}{\partial t_i}(t_0, ..., t_d) = \left(R_i\right)_j - \left(R_0\right)_j,$$

so that $|\det(\sigma')|$ is the determinant of the d vectors $R_1 - R_0, ..., R_d - R_0$ with the 0-coordinate removed. Since every R_i by definition belongs to A, we have for the 0-coordinate of $R_i - R_0$:

$$(R_i)_0 - (R_0)_0 = \lambda_0^{-1} \sum_{j=1}^{d} \lambda_j \left((R_i)_j - (R_0)_j\right).$$

Let us now compute the determinant of the $d+1$ vectors $R_0, R_1, ..., R_d$:

$$\det(R_0, R_1, ..., R_d) = \det(R_0, R_1 - R_0, ..., R_d - R_0).$$

The first row of the matrix $(R_0, R_1 - R_0, ..., R_d - R_0)$ (corresponding to the 0-coordinate of each column vector) is

$$\begin{aligned}(\lambda_0^{-1} - \Sigma_{j=1}^{d} \lambda_0^{-1} \lambda_j \, (R_0)_j \, , \Sigma_{j=1}^{d} \lambda_0^{-1} \lambda_j \, ((R_1)_j - (R_0)_j \, , ..., \lambda_j \, ((R_d)_j - (R_0)_j) = \\ = (\lambda_0^{-1}, 0, ..., 0) - \sum_{j=1}^{d} \lambda_0^{-1} \lambda_j \left((R_0)_j \, , (R_1)_j - (R_0)_j \, , ..., (R_d)_j - (R_0)_j \right),\end{aligned}$$

and the latter sum is clearly a linear combination of the rows 1 up to d of the matrix $(R_0, R_1 - R_0, ..., R_d - R_0)$. Thus

$$\det(R_0, R_1 - R_0, ..., R_d - R_0) = \det \begin{pmatrix} \lambda_0^{-1} & 0 \\ (R_0)_1 & \\ \vdots & ((R_i)_j - (R_0)_j)_{1 \leq j, i \leq n} \\ (R_0)_d & \end{pmatrix},$$

which shows that

$$\det(R_0, R_1, ..., R_d) = \lambda_0^{-1} \left|\det(\sigma')\right|,$$

and finishes the proof of the theorem. □

The following lemma, which is Theorem 5.1 in [**Sa82**] provides a better understanding of the denominator of the integrand appearing in Theorem 4, when the vertices R_i all have rank 1 so that they take the form $R_i = x_i x_i^t$ for some vectors x_i in $\mathbb{R}^n$. It is proven by means of elementary linear algebra.

Lemma 2. *Let $x_1, ..., x_N$ be vectors in $\mathbb{R}^n$, then*

$$\det\left(\sum_{i=1}^{N} t_i x_i x_i^t\right) = \sum_{j_1 < ... < j_n} \left(\prod_{i=1}^{n} t_{j_i}\right) \det(x_{j_1}, ..., x_{j_n})^2,$$

for any real positive numbers $t_1, ..., t_N$.

Proof of Lemma 2. First note that upon replacing every vector x_i by $t_i^{1/2} x_i$ we can without loss of generality assume that $t_0 = ... = t_N = 1$. Let $x_1, ..., x_N$ be vectors in $\mathbb{R}^n$. Let x_i^j denote the j-th coordinate of the vector x_i, so that

$$x_i = \begin{pmatrix} x_i^1 \\ \vdots \\ x_i^n \end{pmatrix},$$

for $i = 1,..,N$. Observe that the j-th column of the matrix $x_i x_i^t$ is equal to $x_i^j x_i$. By the multilinearity of the determinant, we have

$$\det\left(\sum_{i=1}^N x_i x_i^t\right) = \det\left(\sum_{i=1}^N x_i^1 x_i, ..., \sum_{i=1}^N x_i^n x_i\right)$$
$$= \sum_{i_1,...,i_n=1}^N x_{i_1}^1 \cdot ... \cdot x_{i_n}^n \det(x_{i_1}, ..., x_{i_n}).$$

Obviously, if $i_k = i_\ell$ for some $1 \le k \neq \ell \le n$, then $\det(x_{i_1}, ..., x_{i_n})$ vanishes, so that it is enough to sum over indices $i_1, ..., i_n$ which are all distinct. Those can be written in a unique way as $i_{\tau(1)}, ..., i_{\tau(n)}$, where $i_1 < ... < i_n$ and τ is a permutation of the set $\{1, ..., n\}$. The above expression can thus be rewritten as

$$\sum_{1\le i_1<...<i_n\le N} \sum_{\tau\in S_n} x_{i_{\tau(1)}}^1 \cdot ... \cdot x_{i_{\tau(n)}}^n \det(x_{i_{\tau(1)}}, ..., x_{i_{\tau(n)}}) =$$
$$= \textstyle\sum_{1\le i_1<...<i_n\le N} \left(\sum_{\tau\in S_n} \operatorname{sign}(\tau) x_{i_{\tau(1)}}^1 \cdot ... \cdot x_{i_{\tau(n)}}^n\right) \det(x_{i_1}, ..., x_{i_n})$$
$$= \textstyle\sum_{1\le i_1<...<i_n\le N} \det(x_{i_1}, ..., x_{i_n})^2,$$

as desired. □

6. The case $n = 2$

For $n = 2$ the symmetric space $\mathrm{SL}_2\mathbb{R}/SO(2)$ is, upon rescaling the metric appropriately, isometric to the 2-dimensional hyperbolic space. Of course, using the transitivity of the action of $\mathrm{SL}_2\mathbb{R}$ on oriented triples of distinct points on $\partial\mathrm{Pos}_2^{\mathrm{tr}}$, it is readily seen that the area of nondegenerated ideal geodesic triangles is up to a sign constant. We shall however reprove this elementary fact, mainly in order to exemplify in this simple case the method we will use in the next section for $\mathrm{SL}_3\mathbb{R}$. Note that we will do so without using the transitivity of $\mathrm{SL}_2\mathbb{R}$ on the boundary of the symmetric space $\mathrm{Pos}_2^{\mathrm{tr}}$.

Lemma 3. *Let x, y, z be vectors in $\mathbb{R}^2$. Then*

$$\det(R(x), R(y), R(z)) = \det(x, y)\det(x, z)\det(y, z).$$

Proof. This is a straightforward consequence of the Vandermonde determinant formula: Let x_i, y_i, z_i, for $i = 1, 2$ denote the coordinates of x, y, z respectively. By definition, the left hand side of the desired equality is equal to

$$\det\begin{pmatrix} x_1x_1 & y_1y_1 & z_1z_1 \\ x_1x_2 & y_1y_2 & z_1z_2 \\ x_2x_2 & y_2y_2 & z_2z_2 \end{pmatrix} = x_1^2y_1^2z_1^2 \det\begin{pmatrix} 1 & 1 & 1 \\ x_2/x_1 & y_2/y_1 & z_2/z_1 \\ (x_2/x_1)^2 & (y_2/y_1)^2 & (z_2/z_1)^2 \end{pmatrix}.$$

The latter matrix being a Vandermonde matrix, its determinant is equal to

$$\left(\frac{x_2}{x_1} - \frac{y_2}{y_1}\right)\left(\frac{y_2}{y_1} - \frac{z_2}{z_1}\right)\left(\frac{z_2}{z_1} - \frac{x_2}{x_1}\right).$$

Multiplying this expression by $x_1^2y_1^2z_1^2$, we clearly obtain the right hand side of the Lemmas's equation. □

Let x_0, x_1, x_2 be arbitrary points on S^1 and let $\sigma : \Delta^2 \to \mathrm{Pos}_2^{\mathrm{tr}}$ be the straight singular simplex

$$\sigma = \sigma(R(x_0), R(x_1), R(x_2)).$$

Observe that if the points $R(x_0), R(x_1), R(x_2)$ are not all distinct (which happens precisely when $x_i = \pm x_j$ for some $i \neq j$), the simplex σ is degenerated and hence has zero area. Let us thus assume that this is not the case. Set

$$D_0 = \det(y, z), \quad D_1 = \det(z, x), \quad D_2 = \det(x, y),$$

and note that the D_i's are all nonzero. Define another straight singular simplex $\overline{\sigma} : \Delta^2 \to \mathrm{Pos}_2$ as

$$\overline{\sigma} = \sigma(R(D_0 x_0), R(D_1 x_1), R(D_2 x_2)).$$

Since $R(D_i x_i) = D_i^2 R(x_i)$, for $i = 0, 1, 2$, we are in the situation of Lemma 1, so that

$$\int_\sigma \omega^{\mathrm{tr}} = \int_\sigma \pi^*(\omega) = \int_{\overline{\sigma}} \pi^*(\omega).$$

The latter integral is, by Theorem 4 equal to

$$\det(R(D_0x_0), R(D_1x_1), R(D_2x_2)) \int_{\Delta^2} \frac{dt_1 dt_2}{\det(\overline{\sigma}(t_0, t_1, t_2))^{3/2}}.$$

On the one hand, we now get from Lemma 3 that

$$\begin{aligned}
&\det(R(D_0x_0), R(D_1x_1), R(D_2x_2)) = \\
&\qquad = \det(D_0x_0, D_1x_1)\det(D_1x_1, D_2x_2)\det(D_2x_2, D_0x_0) \\
&\qquad = (D_0D_1D_2)^3.
\end{aligned}$$

On the other hand, we have, with the help of Lemma 2, the following expression for the denominator of the integrand:

$$\begin{aligned}
&\det(\overline{\sigma}(t_0, t_1, t_2))^{3/2} = \det(t_0R(D_0x_0) + t_1R(D_1x_1) + t_2R(D_2x_2))^{3/2} \\
&= (t_0t_1 \det(D_0x_0, D_1x_1)^2 + t_0t_2 \det(D_0x_0, D_2x_2,)^2 + t_1t_2 \det(D_1x_1, D_2x_2)^2)^{3/2} \\
&= |D_0D_1D_2|^3 (t_0t_1 + t_0t_2 + t_1t_2)^{3/2}.
\end{aligned}$$

Thus, we are now reduced to the simple expression

$$\begin{aligned}
\int_\sigma \pi^*(\omega) &= \frac{(D_0D_1D_2)^3}{|D_0D_1D_2|^3} \int_{\Delta^2} \frac{dt_1 dt_2}{(t_0t_1 + t_0t_2 + t_1t_2)^{3/2}} \\
&= \pm \int_{\Delta^2} \frac{dt_1 dt_2}{(t_0t_1 + t_0t_2 + t_1t_2)^{3/2}}.
\end{aligned}$$

Observe that the latter integral can be computed and is in fact equal to 2π. (The form ω is only up to a constant the Riemannian volume form corresponding to the hyperbolic metric.)

7. The case $n = 3$

We are now ready to prove Theorem 3, that is, that the form $\omega^{\mathrm{tr}} = \pi^*(\omega)|_{\mathrm{Pos}_3^{\mathrm{tr}}}$ is uniformly bounded when integrated on straight ideal simplices of $\mathrm{Pos}_3^{\mathrm{tr}}$ with rank 1 vertices. The proof consists of a succession of reductions.

First reduction. We show that it is enough to bound those straight simplices with rank 1 vertices $\sigma(R(x_1),...,R(x_6))$ for which three among the six vectors $x_1,...,x_6$ of $\mathbb{R}^3$ span a 2-dimensional subspace. Thus, we will prove that Theorem 3 is a consequence of the following proposition:

PROPOSITION 4. *There exists a positive constant K such that for every nonzero vectors x_1,x_2,x_3,y_1,y_2,y_3 in $\mathbb{R}^3$ such that x_1,x_2,x_3 span a 2-dimensional vector space the inequality*

$$\int_{\sigma(R(x_1),R(x_2),R(x_3),R(y_1),R(y_2),R(y_3))} \pi^*(\omega) \le K$$

holds.

To see how Proposition 4 implies Theorem 3, let $x_1,...,x_6$ be arbitrary vectors of $\mathbb{R}^3$. If there exists $1 \le i \ne j \le 6$ such that $x_i = \pm x_j$, then $R(x_i) = R(x_j)$ and the simplex $\sigma(R(x_1),...,R(x_6))$ is degenerated so that its volume is zero. Let us thus assume that this is not the case, so that the vector spaces $\langle x_1, x_2\rangle$ and $\langle x_3, x_4\rangle$ are 2-dimensional. Being subspaces of $\mathbb{R}^3$, their intersection is at least 1-dimensional. Let x_0 be a point of norm 1 on the intersection of the spaces $\langle x_1, x_2\rangle$ and $\langle x_3, x_4\rangle$. By the cocycle relation, we have

$$\mathrm{Vol}(\sigma(R(x_1),...,R(x_6)) = \sum_{i=1}^{6} (-1)^{i+1}\mathrm{Vol}(\sigma(R(x_0),R(x_1),...,\widehat{R(x_i)},...,R(x_6)).$$

We claim that each of the simplex appearing in the right hand side of the equality is a as in Proposition 4. Indeed, if i is equal to 1 or 2, then we have that x_0,x_3,x_4 span a 2-dimensional subspace, and if i is greater or equal to 3, then x_0,x_1,x_2 do. Thus, the volume of an arbitrary simplex with rank 1 boundary points is bounded by 6 times the maximal volume of a special simplex, so that the constant C of Theorem 3 can be taken to be equal to $6K$, where K is the constant of Proposition 4.

The advantage of considering this type of simplices is in the simple expression which we have to express the determinant of their vertices, as shown in the next proposition.

PROPOSITION 5. *Let $x^1,x^2,x^3 \in \mathbb{R}^3$ be spanning a 2-dimensional vector space and $y^1,y^2,y^3 \in \mathbb{R}^3$ be arbitrary vectors. Then*

$$\begin{aligned}\det(R(x^1),R(x^2),R(x^3),R(y^1),R(y^2),R(y^3)) =\\ = \det(x^1,x^2,y^3)\det(x^1,y^2,x^3)\det(y^1,x^2,x^3)\det(y^1,y^2,y^3).\end{aligned}$$

PROOF. We start by proving the proposition in the particular case when

$$x^1 = e_1,\ x^2 = e_2 \text{ and } y^3 = e_3.$$

Since x^3 belongs to the plane generated by x^1 and x^2, its third coordinate must vanish, so that it takes the form

$$x^3 = \begin{pmatrix} x^3_1 \\ x^3_2 \\ 0 \end{pmatrix}.$$

Also, we denote by y^i_j the j-th coordinate of y^i, so that

$$y^i = \begin{pmatrix} y^i_1 \\ y^i_2 \\ y^i_3 \end{pmatrix},$$

for $i=1,2,3$. The matrix $(R(x^1), R(x^2), R(x^3), R(y^1), R(y^2),\, R(y^3))$ now takes the explicit form

$$\left(\begin{array}{ccc|ccc} 1 & 0 & x^3_1x^3_1 & & & \\ 0 & 0 & x^3_1x^3_2 & & * & \\ 0 & 1 & x^3_2x^3_2 & & & \\ & & & y^1_1y^1_3 & y^2_1y^2_3 & y^3_1y^3_3 \\ & 0 & & y^1_2y^1_3 & y^2_2y^2_3 & y^3_2y^3_3 \\ & & & y^1_3y^1_3 & y^2_3y^2_3 & y^3_3y^3_3 \end{array}\right).$$

Its determinant is clearly equal to the product of the determinants of the two 3 by 3 matrices on the diagonal, that is,

$$\det\begin{pmatrix} 1 & 0 & x^3_1x^3_1 \\ 0 & 0 & x^3_1x^3_2 \\ 0 & 1 & x^3_2x^3_2 \end{pmatrix} \det\begin{pmatrix} y^1_1y^1_3 & y^2_1y^2_3 & y^3_1y^3_3 \\ y^1_2y^1_3 & y^2_2y^2_3 & y^3_2y^3_3 \\ y^1_3y^1_3 & y^2_3y^2_3 & y^3_3y^3_3 \end{pmatrix} = x^3_2x^3_1y^1_3y^2_3y^3_3 \det(y^1,y^2,y^3).$$

Since $\det(x^1,x^2,y^3)=1$ and $y^3_3=1$, it thus remain to prove that

$$\det(x^1,y^2,x^2)\det(y^1,x^2,x^3) = x^3_2x^3_1y^1_3y^2_3.$$

But this is readily computed: We have

$$\det(x^1,y^2,x^3) = \det\begin{pmatrix} 1 & y^2_1 & x^3_1 \\ 0 & y^2_2 & x^3_2 \\ 0 & y^2_3 & 0 \end{pmatrix} = -y^2_3x^3_2$$

and

$$\det(y^1,x^2,x^3) = \det\begin{pmatrix} y^1_1 & 0 & x^3_1 \\ y^1_2 & 1 & x^3_2 \\ y^1_3 & 0 & 0 \end{pmatrix} = -y^1_3x^3_1.$$

Let now the x^i's and the y^i's be arbitrary vectors of $\mathbb{R}^3$ as in the hypothesis of the proposition. If the vectors x^1,x^2,y^3 were not linearly independent, the face generated by $R(x^1),R(x^2),R(x^3)$ and $R(y^3)$ of the simplex spanned by the $R(x^i)$'s and $R(y^i)$'s would be contained in a 2-dimensional subspace (isomorphic to the trace model of the symmetric space $\mathrm{SL}_2\mathbb{R}/SO(2)$) of the boundary of the symmetric space. But a face generated by 4 points is degenerated if its dimension is strictly smaller than 3. Thus, the determinant of the $R(x^i)$'s and $R(y^i)$'s has to vanish. As for the right hand side of the equality, we have $\det(x^1,x^2,y^3)=0$.

Let us now assume that the vectors x^1,x^2,y^3 are linearly independent. There exists a unique element g in $\mathrm{GL}_3\mathbb{R}$ such that $x^1=ge_1$, $x^2=ge_2$ and $y^3=ge_3$. Define $\widetilde{x}^i = g^{-1}x^i$ and $\widetilde{y}^i = g^{-1}y^i$, for $i=1,2,3$. In view of equality (2.1) of Section 2, we have

$$\begin{aligned}\det(R(x^1),R(x^2),R(x^3),R(y^1),R(y^2),R(y^3)) = \\ = \det(g)^4\det(R(\widetilde{x}^1),R(\widetilde{x}^2),R(\widetilde{x}^3),R(\widetilde{y}^1),R(\widetilde{y}^2),R(\widetilde{y}^3)).\end{aligned}$$

By our above computations, the latter expression is equal to

$$\begin{aligned}\det(g)^4 \det(\widetilde{x}^1,\widetilde{x}^2,\widetilde{y}^3)\det(\widetilde{x}^1,\widetilde{y}^2,\widetilde{x}^3)\det(\widetilde{y}^1,\widetilde{x}^2,\widetilde{x}^3)\det(\widetilde{y}^1,\widetilde{y}^2,\widetilde{y}^3) = \\ = \det(x^1,x^2,y^3)\det(x^1,y^2,x^3)\det(y^1,x^2,x^3)\det(y^1,y^2,y^3),\end{aligned}$$

and the proposition is proven. □

Remark 1. *For arbitrary vectors x^1,x^2,x^3y^1,y^2,y^3 in $\mathbb{R}^3$, we can more generally prove that*

$$\begin{aligned}\det(R(x^1),R(x^2),R(x^3),R(y^1),R(y^2),R(y^3)) = \\ = \det(x^1,x^2,y^3)\det(x^1,y^2,x^3)\det(y^1,x^2,x^3)\det(y^1,y^2,y^3) \\ - \det(y^1,y^2,x^3)\det(y^1,x^2,y^3)\det(x^1,y^2,y^3)\det(x^1,x^2,x^3),\end{aligned}$$

which clearly implies Proposition 5 since in this case $\det(x^1,x^2,x^3)=0$.

Second reduction. Let x_1,x_2,x_3 in S^2 be spanning a 2-dimensional subspace of $\mathbb{R}^3$ and y_1,y_2,y_3 be arbitrary points in S^2. Let ε be a positive number, typically small. We claim that, if the straight simplex with vertices $R(x_i)$ and $R(y_i)$ is nondegenerated, then, upon interchanging x_2 with x_3 and replacing y_i by $-y_i$, there exists g in $\mathrm{SL}_3\mathbb{R}$ such that

$$\frac{1}{\|gx_1\|_2}gx_1 = e_1, \quad \frac{1}{\|gx_2\|_2}gx_2 = \begin{pmatrix}\cos\frac{2\pi}{3}\\ \sin\frac{2\pi}{3}\\ 0\end{pmatrix}, \quad \frac{1}{\|gx_3\|_2}gx_3 = \begin{pmatrix}-\sin\frac{2\pi}{3}\\ \cos\frac{2\pi}{3}\\ 0\end{pmatrix},$$

and furthermore

$$\left\|\frac{1}{\|gy_i\|_2}gy_i - e_3\right\|_2 \leq \varepsilon,$$

for $i=1,2,3$. To see that, start by sending the plane generated by x_1,x_2,x_3 onto the plane $\langle e_1,e_2\rangle$ and use the transitivity of $\mathrm{SL}_2\mathbb{R}$ on triple of distinct points of $P^1\mathbb{R}$ to achieve the first condition. Then act with a diagonal matrix with diagonal entries $\lambda^{-1},\lambda^{-1},\lambda^2$, where λ is big enough for the second condition to be achieved.

We have now shown that Proposition 4 follows from Proposition 6 below. Note that the constants K appearing in both propositions can be taken to be equal.

Proposition 6. *There exists a positive constant K such that if*

$$x_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad x_2 = \begin{pmatrix}\cos\frac{2\pi}{3}\\ \sin\frac{2\pi}{3}\\ 0\end{pmatrix}, \quad x_3 = \begin{pmatrix}-\sin\frac{2\pi}{3}\\ \cos\frac{2\pi}{3}\\ 0\end{pmatrix},$$

and y_1,y_2,y_3 are vectors of $\mathbb{R}^3$ satisfying

$$\left\|\frac{1}{\|gy_i\|_2}gy_i - e_3\right\|_2 \leq \varepsilon,$$

for $i=1,2,3$, then

$$\left|\int_{\sigma(R(x_1),R(x_2),R(x_3),R(y_1),R(y_2),R(y_3))}\pi^*(\omega)\right| \leq K.$$

Proof of Proposition 6. Let x_1, x_2, x_3 and y_1, y_2, y_3 be as in the proposition. To simplify the notation, set

$$\begin{aligned} D_0 &= \det(y_1, y_2, y_3), \\ D_1 &= \det(y_1, x_2, x_3), \\ D_2 &= \det(x_1, y_2, x_3), \\ D_3 &= \det(x_1, x_2, y_3). \end{aligned}$$

Observe that if the simplex is nondegenerated, then the D_i's are nonzero. Let us assume that this is the case and let $\overline{x}_1, \overline{x}_2, \overline{x}_3$ and $\overline{y}_1, \overline{y}_2, \overline{y}_3$ be the following nonzero vectors of $\mathbb{R}^3$:

$$\begin{aligned} \overline{x}_1 &= |D_0 D_1|^{1/2}\, x_1, \quad \overline{x}_2 = |D_0 D_2|^{1/2}\, x_2, \quad \overline{x}_3 = |D_0 D_3|^{1/2}\, x_3, \\ \overline{y}_1 &= |D_2 D_3|^{1/2}\, y_1, \quad \overline{y}_2 = |D_1 D_3|^{1/2}\, y_2, \quad \overline{y}_3 = |D_1 D_2|^{1/2}\, y_3. \end{aligned}$$

Let σ and $\overline{\sigma}$ be the two straight simplices, $\sigma, \overline{\sigma} : \Delta^5 \to \overline{\mathrm{Pos}_3}$, defined respectively as

$$\sigma = \sigma(R(x_1), R(x_2), R(x_3), R(y_1), R(y_2), R(y_3))$$

and

$$\overline{\sigma} = \sigma(R(\overline{x}_1), R(\overline{x}_2), R(\overline{x}_3), R(\overline{y}_1), R(\overline{y}_2), R(\overline{y}_3)).$$

By Lemma 1 we have

$$\int_\sigma \omega^{\mathrm{tr}} = \int_\sigma \pi^*(\omega) = \int_{\overline{\sigma}} \pi^*(\omega),$$

and the latter integral can be, by Theorem 4, rewritten as

$$\det(R(\overline{x}_1), R(\overline{x}_2), R(\overline{x}_3), R(\overline{y}_1), R(\overline{y}_2), R(\overline{y}_3)) \int_{\Delta^5} \frac{\tau}{\det(\overline{\sigma}(t_0, ..., t_5))^2},$$

where $\tau = dt_1 \wedge ... \wedge dt_5$.

Clearly, it is now enough to bound the above expression when the integral is taken over an arbitrary simplex of the first barycentric subdivision of Δ^5. Let thus $\prec$ be an arbitrary order on the set $\{0, ..., 5\}$ and let

$$\Delta^5_\prec = \{(t_0, ..., t_5) \in \Delta^5 \mid t_i \le t_j \text{ whenever } i \prec j\}$$

be the corresponding subsimplex of Δ^5. For obvious symmetry reasons, it will be easier to write the coordinates of a point in Δ^5 as $(r_1, r_2, r_3, s_1, s_2, s_3)$. On $\Delta^5_\prec$, we have

$$r_{i_1} \le r_{i_2} \le r_{i_3} \text{ and } s_{j_1} \ge s_{j_2} \ge s_{j_3},$$

for some i_k, j_k such that $\{i_1, i_2, i_3\} = \{j_1, j_2, j_3\} = \{1, 2, 3\}$. Let λ be the permutation of $\{1, 2, 3\}$ sending i_k to j_k, for $k = 1, 2, 3$.

Before going any further, we need some preliminary easy estimates: It is clear that because, for any $1 \le i \ne j \le 3$, the absolute value of the determinant $\det(x_i, x_j, e_3)$ is equal to $\sqrt{3}/2$, it follows from the hypothesis of the proposition, that

$$\sqrt{3}/2 - \varepsilon \le |\det(x_i, x_j, y_k)| \le \sqrt{3}/2 + \varepsilon,$$

for every $1 \le i, j, k \le k$ with $i \ne j$. In particular,

$$\sqrt{3}/2 - \varepsilon \le |D_1|, |D_2|, |D_3| \le \sqrt{3}/2 + \varepsilon. \tag{7.1}$$

Using this estimates, we can further compute the inequalities:

$$\begin{aligned}\det(\overline{y}_1,\overline{y}_2,\overline{y}_3)^2 &= D_1^2D_2^2D_3^2\det(y_1,y_2,y_3)^2 = (D_0D_1D_2D_3)^2\\ &\geq D_0^2\left(\sqrt{3}/2-\varepsilon\right)^6,\end{aligned} \tag{7.2}$$

and for $1 \leq i,j,k \leq 3$ with $i \neq j$,

$$\det(\overline{x}_i,\overline{x}_j,\overline{y}_k)^2 \geq D_0^2D_iD_j\left(\sqrt{3}/2-\varepsilon\right)^2\det(x_i,x_j,y_k)^2 \geq D_0^2\left(\sqrt{3}/2-\varepsilon\right)^6. \tag{7.3}$$

Recall that we are left with finding a bound for

$$\det(R(\overline{x}_1),R(\overline{x}_2),R(\overline{x}_3),R(\overline{y}_1),R(\overline{y}_2),R(\overline{y}_3))\int_{\Delta^5_{\prec}}\frac{\tau}{\det(\overline{\sigma}(r_1,r_2,r_3,s_1,s_2,s_3))^2}, \tag{7.4}$$

where τ is now the differential 5-form on $\Delta^5_{\prec}$ consisting of the wedge of the differentials of all but one coordinate (which is well defined since the sum of the coordinates is equal to a constant). On the one hand, we now see from Proposition 5 and Equation (7.1) that

$$\begin{aligned}&\det(R(\overline{x}_1),R(\overline{x}_2),R(\overline{x}_3),R(\overline{y}_1),R(\overline{y}_2),R(\overline{y}_3)) =\\ &\qquad= \det(\overline{y}_1,\overline{y}_2,\overline{y}_3)\det(\overline{x}_1,\overline{x}_2,\overline{y}_3)\det(\overline{x}_1,\overline{y}_2,\overline{x}_3)\det(\overline{y}_1,\overline{x}_2,\overline{x}_3)\\ &\qquad= (D_0D_1D_2D_3)^4 \leq D_0^4\left(\sqrt{3}/2+\varepsilon\right)^{12}.\end{aligned}$$

On the other hand, Lemma 2 allows us to express $\det(\overline{\sigma}(r_1,r_2,r_3,s_1,s_2,s_3))$ as a sum of expressions of the form $t_1t_2t_3\det(z_1,z_2,z_3)^2$, where $\{t_1,t_2,t_3\}\subset\{r_1,r_2,r_3,s_1,s_2,s_3\}$ and the z_i's are the corresponding vectors among the $\overline{x}_j$'s and $\overline{y}_j$'s. As all the summands are positive, restricting to a subsum we obtain the majoration

$$\begin{aligned}&\det(\overline{\sigma}(r_1,r_2,r_3,s_1,s_2,s_3)) \geq\\ &\qquad\geq s_1s_2s_3\det(\overline{y}_1,\overline{y}_2,\overline{y}_3)^2 + s_{\lambda(1)}r_2r_3\det(\overline{y}_{\lambda(1)},\overline{x}_2,\overline{x}_3)^2\\ &\qquad\quad + r_1s_{\lambda(2)}r_3\det(\overline{x}_1,\overline{y}_{\lambda(2)},\overline{x}_3)^2 + r_1r_2s_{\lambda(3)}\det(\overline{x}_1,\overline{x}_2,\overline{y}_{\lambda(3)})^2\\ &\qquad\geq D_0^2\left(\sqrt{3}/2-\varepsilon\right)^6\left(s_1s_2s_3+s_{\lambda(1)}r_2r_3+r_1s_{\lambda(2)}r_3+r_1r_2s_{\lambda(3)}\right),\end{aligned}$$

where the last inequalities follows from (7.2) and (7.3). It remains to plug into (7.4) those two last inequalities so as to obtain the bound

$$\begin{aligned}&\left|\int_{\overline{\sigma}|_{\Delta^5_{\prec}}}\pi^*(\omega)\right| \leq\\ &\qquad\left(\frac{\sqrt{3}/2+\varepsilon}{\sqrt{3}/2-\varepsilon}\right)^{12}\int_{\Delta^5_{\prec}}\frac{\tau}{\left(s_1s_2s_3+s_{\lambda(1)}r_2r_3+r_1s_{\lambda(2)}r_3+r_1r_2s_{\lambda(3)}\right)^2}.\end{aligned}$$

The theorem will now follow from the next lemma, where we show that the latter integral (which clearly is independent of the starting points x_i and y_i) converges. Note that this integral only converges for specific orders on the vertices (we could not have r_1 and $s_{\lambda(1)}$ as the two smallest coordinates for example): This is why we first chose an order and then used the appropriate majoration on the denominator of our integral.

Lemma 4. *Let $\Delta^5_{\prec}$ be such that for $(r_1, r_2, r_3, s_1, s_2, s_3)$ in $\Delta^5_{\prec}$, the inequalities $r_1 \geq r_2 \geq r_3$ and $s_1 \leq s_2 \leq s_3$ hold. Then the integral*

$$\int_{\Delta^5_{\prec}} \frac{\tau}{(s_1 s_2 s_2 + s_1 r_2 r_3 + r_1 s_2 r_3 + r_1 r_2 s_3)^2}$$

converges.

Proof. We begin with an easy assertion.

Claim 1. *For any positive real numbers α_i, β_i satisfying $\alpha_i, \beta_i \geq 2/3$ and $\alpha_i + \beta_i = 2$, for $i = 1, 2, 3$, we have*

$$(s_1 s_2 s_3 + s_1 r_2 r_3 + r_1 s_2 r_3 + r_1 r_2 s_3)^2 \geq r_1^{\alpha_1} r_2^{\alpha_2} r_3^{\alpha_3} s_3^{\beta_3} s_2^{\beta_2} s_1^{\beta_1}.$$

Proof of Claim. We start by showing that whenever $\{i, j, k\} = \{1, 2, 3\}$, we have

$$s_1 s_2 s_3 + s_1 r_2 r_3 + r_1 s_2 r_3 + r_1 r_2 s_3 \geq \max\{s_i (s_j s_k r_j r_k)^{1/2}, r_i (s_j s_k r_j r_k)^{1/2}\}.$$

To see that, observe that by symmetry, we can without loss of generality assume that $i = 1$, $j = 2$ and $k = 3$. Since all the summands of the right hand side of the inequality are positive, we clearly have

$$s_1 s_2 s_3 + s_1 r_2 r_3 + r_1 s_2 r_3 + r_1 r_2 s_3 \geq \max\{s_1 s_2 s_3 + s_1 r_2 r_3, r_1 s_2 r_3 + r_1 r_2 s_3\}.$$

From the inequality between arithmetic and geometric means we further have

$$s_1 (s_2 s_3 + r_2 r_3) \geq s_1 (s_2 s_3 r_2 r_3)^{1/2} \text{ and } r_1 (s_2 r_3 + r_2 s_3) \geq r_1 (s_2 s_3 r_2 r_3)^{1/2},$$

as desired.

Let now $\overline{\alpha_i}$ and $\overline{\beta_i}$ be arbitrary positive real numbers, for $i = 1, 2, 3$. From the above inequalities, we compute

$$\begin{aligned}(s_1 s_2 s_3 + s_1 r_2 r_3 + r_1 s_2 r_3 + r_1 r_2 s_3)^{\overline{\alpha_1} + \overline{\beta_1}} &\geq r_1^{\overline{\alpha_1}} (r_2 r_3 s_2 s_3)^{\overline{\alpha_1}/2} s_1^{\overline{\beta_1}} (r_2 r_3 s_2 s_3)^{\overline{\beta_1}/2} \\ &= r_1^{\overline{\alpha_1}} s_1^{\overline{\beta_1}} (r_2 r_3 s_2 s_3)^{(\overline{\alpha_1} + \overline{\beta_1})/2},\end{aligned}$$

Similarly, we obtain the two inequalities

$$\begin{aligned}(s_1 s_2 s_3 + s_1 r_2 r_3 + r_1 s_2 r_3 + r_1 r_2 s_3)^{\overline{\alpha_2} + \overline{\beta_2}} &\geq r_2^{\overline{\alpha_2}} s_2^{\overline{\beta_2}} (r_1 r_3 s_1 s_3)^{(\overline{\alpha_2} + \overline{\beta_2})/2} \\ (s_1 s_2 s_3 + s_1 r_2 r_3 + r_1 s_2 r_3 + r_1 r_2 s_3)^{\overline{\alpha_3} + \overline{\beta_3}} &\geq r_3^{\overline{\alpha_3}} s_3^{\overline{\beta_3}} (r_1 r_2 s_1 s_2)^{(\overline{\alpha_3} + \overline{\beta_3})/2}.\end{aligned}$$

For α_i, β_i's as in the Claim, we set $\overline{\alpha_i} = \alpha_i - 2/3 \geq 0$ and $\overline{\beta_i} = \beta_i - 2/3 \geq 0$. Note that for each i, we have $\overline{\alpha_i} + \overline{\beta_i} = 2/3$. We can now apply each of the three above inequalities and we obtain

$$\begin{aligned}&(s_1 s_2 s_3 + s_1 r_2 r_3 + r_1 s_2 r_3 + r_1 r_2 s_3)^2 \geq \\ &\qquad \geq \left(r_1^{\overline{\alpha_1}} s_1^{\overline{\beta_1}} (r_2 r_3 s_2 s_3)^{1/3}\right) \left(r_2^{\overline{\alpha_2}} s_2^{\overline{\beta_2}} (r_1 r_3 s_1 s_3)^{1/3}\right) \left(r_3^{\overline{\alpha_3}} s_3^{\overline{\beta_3}} (r_1 r_2 s_1 s_2)^{1/3}\right) \\ &\qquad = r_1^{\alpha_1} r_2^{\alpha_2} r_3^{\alpha_3} s_3^{\beta_3} s_2^{\beta_2} s_1^{\beta_1},\end{aligned}$$

as claimed. □

We start the proof of the lemma by a preliminary case, to illustrate our strategy. Suppose the defining order $\prec$ of $\Delta^5_\prec$ would give

$$r_1 \geq r_2 \geq r_3 \geq s_3 \geq s_2 \geq s_1.$$

Then we apply Claim 1 to $\alpha_1 = 5/4, \beta_1 = 3/4$ and $\alpha_2 = \beta_2 = \alpha_3 = \beta_3 = 1$, so that

$$(s_1s_2s_3 + s_1r_2r_3 + r_1s_2r_3 + r_1r_2s_3)^2 \geq r_1^{5/4} r_2r_3s_3s_2s_1^{3/4}.$$

Define $\varphi : \Delta^5_\prec \to \Delta^5_<$ to be the natural bijection mapping the order $\prec$ to the (anti-)natural order $<$ on the indices of $(t_0, ..., t_5)$, so that

$$\varphi(r_1, r_2, r_3, s_3, s_2, s_1) = (t_0, t_1, t_2, t_3, t_4, t_5).$$

(In particular, $t_i \geq t_j$ whenever $i < j$.) The integral of the lemma can now be rewritten as

$$\int_{\Delta^5_<} \frac{dt_1 \cdot ... \cdot dt_5}{t_0^{5/4} t_1t_2t_3t_4t_5^{3/4}},$$

and is easily estimated. Observe that the integral consists of integrating the variables t_5 to t_1 with for each t_i, for $i = 2, ..., 5$, the integration bounds 0 to t_{i-1} and 0 to 1/2 for $i = 1$. Note also that $t_0 \geq 1/6$. Let us now compute a bound for this integral:

$$\begin{aligned}\int_{\Delta^5_<} \frac{dt_1 \cdot ... \cdot dt_5}{t_0^{5/4} t_1t_2t_3t_4t_5^{3/4}} &\leq 6^{5/4} \int_{\Delta^5_<} \frac{dt_1 \cdot ... \cdot dt_5}{t_1t_2t_3t_4t_5^{3/4}} = 6^{5/4} \int_{\Delta^4_<} t_4^{1/4} \frac{dt_1 \cdot ... \cdot dt_4}{t_1t_2t_3t_4} \\ &= 6^{5/4} \int_{\Delta^3_<} t_3^{1/4} \frac{dt_1dt_2dt_3}{t_1t_2t_3} = 6^{5/4} \int_{\Delta^2_<} t_2^{1/4} \frac{dt_1dt_2}{t_1t_2} \\ &= 6^{5/4} \int_0^{1/2} t_1^{1/4} \frac{dt_1}{t_1} = 6^{5/4} \cdot 4 \cdot \left(\frac{1}{2}\right)^{1/4}.\end{aligned}$$

Let now $\prec$ be an arbitrary order defining $\Delta^5_\prec$ and suppose that for $(r_1, r_2, r_3, s_1, s_2, s_3)$ in $\Delta^5_\prec$, the inequalities $r_1 \geq r_2 \geq r_3$ and $s_1 \leq s_2 \leq s_3$ hold. It is clear that either r_1 or s_3 is maximal and either r_3 or s_1 is minimal. We distinguish four cases:

(1) If r_1 is maximal, we set $\alpha_1 = 5/4$ and $\beta_1 = 3/4$.

(a) If s_1 is minimal, we further define $\alpha_2 = \beta_2 = \alpha_3 = \beta_3 = 1$. From Claim 1, we get

$$(s_1s_2s_3 + s_1r_2r_3 + r_1s_2r_3 + r_1r_2s_3)^2 \geq r_1^{5/4} r_2r_3s_3s_2s_1^{3/4}.$$

If $\varphi : \Delta^5_\prec \to \Delta^5_<$ is the natural map preserving the respective orders, we see that the integral of the lemma becomes precisely the same integral as in the preliminary case.

(b) If r_3 is minimal, we instead put $\alpha_3 = 3/4$, $\beta_3 = 5/4$ and $\alpha_2 = \beta_2 = 1$. From Claim 1, we now get

$$(s_1s_2s_3 + s_1r_2r_3 + r_1s_2r_3 + r_1r_2s_3)^2 \geq r_1^{5/4} r_2r_3^{3/4} s_3^{5/4} s_2s_1^{3/4}.$$

Again, let $\varphi : \Delta^5_\prec \to \Delta^5_<$ be the natural map preserving the respective orders. The denominator of the resulting integral is now

$$t_0^{5/4} t_1^{\gamma_1} t_2^{\gamma_2} t_3^{\gamma_3} t_4^{\gamma_4} t_5^{3/4},$$

where two of the γ_i's are equal to 1, one is equal to 5/4 and another is equal to 3/4. Furthermore, if $\gamma_i = 5/4$ and $\gamma_j = 3/4$ then $i < j$

(which comes from that $s_3 \geq s_1$). Explicitly, this means that the denominator is one of

$$t_0^{5/4}t_1^{5/4}t_2t_3t_4^{3/4}t_5^{3/4}, \quad t_0^{5/4}t_1^{5/4}t_2t_3^{3/4}t_4t_5^{3/4}, \quad t_0^{5/4}t_1^{5/4}t_2^{3/4}t_3t_4t_5^{3/4},$$
$$t_0^{5/4}t_1t_2^{5/4}t_3t_4^{3/4}t_5^{3/4}, \quad t_0^{5/4}t_1t_2^{5/4}t_3^{3/4}t_4t_5^{3/4}, \quad t_0^{5/4}t_1t_2t_3^{5/4}t_4^{3/4}t_5^{3/4}.$$

But as in the preliminary case, one can compute a bound (which is the same) for the corresponding integrals.

(2) If s_3 is maximal, we set $\alpha_3 = 3/4$ and $\beta_3 = 5/4$.
   (a) If s_1 is minimal, we further define $\alpha_1 = 5/4$, $\beta_1 = 3/4$ and $\alpha_2 = \beta_2 = 1$. We have, from Claim 1, that

$$(s_1s_2s_3 + s_1r_2r_3 + r_1s_2r_3 + r_1r_2s_3)^2 \geq r_1^{5/4}r_2r_3^{3/4}s_3^{5/4}s_2s_1^{3/4}.$$

   This gives exactly the same integrals as in case 1(b).
   (b) If r_3 is minimal, we instead put $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$. The inequality obtained from Claim 1 now takes the form

$$(s_1s_2s_3 + s_1r_2r_3 + r_1s_2r_3 + r_1r_2s_3)^2 \geq r_1r_2r_3^{3/4}s_3^{5/4}s_2s_1.$$

   Once again, we are back to computing the integral of the preliminary case.

□

Remark 2. *The same method shows that certain ideal simplices in the higher dimensional symmetric spaces* Pos_n^{tr}, $n \geq 4$, *have a uniformly bounded volume. Those simplices are those for which up to a renumbering of their vertices, for every* i *between* 2 *and* $n-1$, *the* $i(i+1)/2$ *first vertices lie in a copy of* Pos_i^{tr} *in* ∂Pos_n^{tr}. *Thus, all but the first step of our proof for* $n = 3$ *generalize to higher dimensions.*

8. On Savage's proof

In this last section, we will briefly explain the proof presented in [**Sa82**] of the positivity of the simplicial volume of compact manifolds covered by $\mathrm{SL}_n\mathbb{R}/SO(n)$ is false. For more details, we refer the reader to [**Bu05**], where we go through Savage's computation step by step. As mentioned in the introduction, the mistake in [**Sa82**] is that it is not realized that the considered barycentric subdivisions are not invariant under isometries of the symmetric space.

Savage starts with arbitrary rank 1 matrices $P_0, ..., P_d$ in $\partial\mathrm{Pos}_n^{\mathrm{tr}}$ and aims at bounding the volume of the straight singular simplex

$$\begin{array}{cccc} \sigma: & \Delta^d & \longrightarrow & \overline{\mathrm{Pos}_n^{\mathrm{tr}}} \\ & (t_0, ..., t_d) & \longmapsto & \sum_{i=0}^d t_i P_i. \end{array}$$

By symmetry, it is enough to bound the volume of σ restricted to the simplex Δ_0^d of the first barycentric subdivision Δ^d:

$$\Delta_0^d = \{(t_0, ..., t_d) \mid t_0 \geq ... \geq t_d\}.$$

Such a bound would imply the positivity of the simplicial volume.

Using the high transitivity of $\mathrm{SL}_n\mathbb{R}$ on rank 1 boundary points, Savage puts his simplex in a special position, as we extract in the next Theorem. It is simple to prove and we refer the reader to either [**Sa82**], beginning of Section 7, or [**Bu05**, Theorem 2].

Theorem 5. *Let $P_0, ..., P_d$ be rank 1 matrices in ∂Pos_n^{tr} generating a nondegenerated simplex. Then there exists g in $SL_n\mathbb{R}$ and integers $0 = \beta_1 < ... < \beta_n \leq d$ satisfying*

$$\beta_i \leq \frac{(i-1)i}{2}$$

such that

$$\rho_g^{tr}(P_{\beta_i}) = E_i = e_i e_i^t,$$

and furthermore $P_1, ..., P_{\beta_i - 1}$ lie in a copy of Pos_{i-1}^{tr} in ∂Pos_n^{tr}, for every i in $\{1, ..., n\}$.

The group element g in $\mathrm{SL}_n\mathbb{R}$ appearing in Theorem 5 induces an isometry ρ_g^{tr} of the symmetric space, which we denote by h, in accordance with the notation in [**Sa82**]. The simplex that Savage now wants to bound, is the restriction to Δ_0^d of the composition of σ with h - and this would of course imply the desired theorem - but what he actually bounds is the restriction to Δ_0^d of the straight simplex $f = \sigma(h(P_0), ..., h(P_d))$. And in general, not only

$$f|_{\Delta_0^d} \neq h \circ \sigma|_{\Delta_0^d},$$

but more problematically, $f(\Delta_0^d) \neq h \circ \sigma(\Delta_0^d)$ and hence

$$\int_{f(\Delta_0^d)} \omega^{\mathrm{tr}} \neq \int_{h\circ\sigma(\Delta_0^d)} \omega^{\mathrm{tr}}.$$

(Note that if we had not restricted to the first barycentric subdivision, but instead considered f and σ on the whole simplex Δ^d, then of course we would still have that $f \neq h \circ \sigma$ but the integral would agree, since the image of f and $h \circ \sigma$ would in this case be equal.)

This mistake, once observed, is easy to point out. Indeed, a volume bound for $h \circ \sigma|_{\Delta_0^d}$ is claimed in Theorem 7.4 of [**Sa82**], but the proven bound is a volume bound for $f|_{\Delta_0^d}$. In passing, Savage seems to have assumed that $f|_{\Delta_0^d} = h \circ \sigma|_{\Delta_0^d}$. Before we can state the unproven Theorem 7.4, and its true proven version, we need some more notation. Choose w_i on the unit sphere of $\mathbb{R}^n$ such that $h(P_i) = w_i w_i^t$, for $i = 0, ..., d$. Let $\langle ., . \rangle$ be the standard scalar product on $\mathbb{R}^n$. Choose $\alpha_1, ..., \alpha_n$ between 0 and d such that $\langle w_{\alpha_i}, e_n \rangle$ has maximal absolute value. Note that by construction, $\beta_1 < ... < \beta_{n-1} \leq \alpha_i$, for $i = 1, .., n$.

Unproven Theorem 7.4 of [**Sa82**]. *Notation as above. Let T be a subset of the image of $h \circ \sigma|_{\Delta_0^d}$ and let $\Delta_T \subset \Delta_0$ be its preimage $\Delta_T = (h \circ \sigma)^{-1}(T)$. Then there exists a constant $C(n)$ such that*

$$\mathrm{Vol}(T) \leq C(n) \left| \prod_{i=1}^n \langle w_{\alpha_i}, e_n \rangle \right| \int_{\Delta_T} \frac{dt_1 ... dt_d}{\left(\left(\Pi_{k=1}^{n-1} t_{\beta_k} \right) \left(\Sigma_{i=1}^n t_{\alpha_i} \langle w_{\alpha_i}, e_n \rangle^2 \right) \right)^{(n+1)/2}}.$$

The wrong proof. The first equation of the proof - which is correct - just relies on the fact that the volume form is, up to a constant denoted by $C_0(n)$, the form ω^{tr} computed in either Theorem 4.3 in [**Sa82**] or Proposition 3 here. Thus one has

$$\mathrm{Vol}(T) = \int_T \frac{C_0(n)}{(\det(S))^{(n+1)/2}} dx_1 \wedge ... \wedge dx_d.$$

The mistake is now that Savage applies the change of variable formula to the map $f : \Delta_0^d \to \mathrm{Pos}_n^{\mathrm{tr}}$, while he replaces the integrand, not by $f^{-1}(T)$ as he should, but by $\Delta_T = (h \circ \sigma)^{-1}(T)$. In this way, he concludes, using his Theorem 5.14 (Lemma 2 here) that

$$\begin{aligned}\mathrm{Vol}(T) &= \int_{\Delta_T} \frac{C_0(n) dt_1 \cdot \ldots \cdot dt_d}{\left(\det(f(t_1, \ldots, t_d))\right)^{(n+1)/2}} \\ &= \int_{\Delta_T} \frac{C_0(n) dt_1 \cdot \ldots \cdot dt_d}{\left(\Sigma_{j_1 < \ldots < j_n} \left(\Pi_{i=1}^n t_{j_i}\right) \det(w_{j_1}, \ldots, w_{j_n})\right)^{(n+1)/2}},\end{aligned}$$

while he should have concluded that

$$\mathrm{Vol}(T) = \int_{f^{-1}(T)} \frac{C_0(n) dt_1 \cdot \ldots \cdot dt_d}{\left(\Sigma_{j_1 < \ldots < j_n} \left(\Pi_{i=1}^n t_{j_i}\right) \det(w_{j_1}, \ldots, w_{j_n})\right)^{(n+1)/2}}.$$

The rest of the computations are correct, so that the true statement is contained in the next theorem. □

True Theorem 7.4. *Notation as above. Let T be a subset of the image of $h \circ \sigma$ and let $\overline{\Delta_T} \subset \Delta_0$ be its preimage $\overline{\Delta_T} = (f)^{-1}(T)$. Then there exists a constant $C(n)$ such that*

$$\mathrm{Vol}(T) \leq C(n) \left| \prod_{i=1}^{n} \langle w_{\alpha_i}, e_n \rangle \right| \int_{\overline{\Delta_T}} \frac{dt_1 \ldots dt_d}{\left(\left(\Pi_{k=1}^{n-1} t_{\beta_k}\right)\left(\Sigma_{i=1}^n t_{\alpha_i} \langle w_{\alpha_i}, e_n \rangle^2\right)\right)^{(n+1)/2}}.$$

After pages of unnecessarily complicated computations, Savage concludes that the integrand appearing in (both versions of) Theorem 7.4 is uniformly bounded when integrated on the simplex Δ_0^d. And of course, this now only implies that the simplex $f(\Delta_0^d)$ has uniformly bounded volume, but not $(h \circ \sigma)(\Delta_0^d)$ as is claimed.

We do not see any way to save the proof in [**Sa82**]: Theorem 7.4 is the starting point for the only volume bound given in [**Sa82**] and it can not be used to prove that the volume of $h \circ \sigma$ (and hence σ) is bounded, since in fact it diverges when integrated on the whole simplex Δ^d.

References

[Bu04] M. (Bucher-)Karlsson, *Characteristic classes and bounded cohomology*, Ph.D. Thesis, ETHZ (2004).

[Bu05] M. Bucher-Karlsson, *A detailed study of Savage's paper "The space of positive Definite matrices and Gromov's invariant"*, preprint (2005).

[CoFa03] C. Connel, B. Farb, *The degree theorem in higher rank,* J. Diff. Geom. 65 (2003), no. 1, 19-59.

[Du76] J. Dupont, *Simplicial de Rham cohomology and characteristic classes of flat bundles*, Topology Vol. 15 (1976), 233-245.

[Gr82] M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. No. 56 (1982), 5–99.

[dlH93] P. de la Harpe, *On Hilbert's metric for simplices*, Geometric group theory, Vol. 1 (Sussex, 1991), London Math. Soc. Lecture Note Ser. 181 (1993), 97–119.

[InYa82] H. Inoue, K. Yano, *The Gromov invariant of negatively curved manifolds,* Topology Vol. 21 (1982), 83-89.

[LaSch05] J.-F. Lafont, B. Schmidt, *Simplicial volume of closed locally symmetric spaces of non-compact type*, preprint (2005).

[Sa82] R.P. Savage Jr. *The space of positive definite matrices and Gromov's invariant*. Trans. Amer. Math. Soc. 274 (1982), no. 1, 239–263.

[Th78] W. Thurston, *Geometry and topology of* 3*-manifolds*, Lecture Notes, Princeton (1978).

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