ON MINIMAL TRIANGULATIONS OF PRODUCTS OF CONVEX POLYGONS

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Abstract. We give new lower bounds for the minimal number of simplices needed in a triangulation of the product of two convex polygons, improving the lower bounds in [Bo&al05].

1. Introduction

The use of volume of simplices in hyperbolic geometry for combinatorial problems, initiated by Thurston, has proven successful for minimal triangulations of polytopes. For example, embedding an n-gon ideally in the hyperbolic plane, it is elementary to see, since the area of a geodesic triangle is majorized by the (constant) area of an ideal geodesic triangle, that the minimal number of 2-dimensional simplices needed for a triangulation of a convex n-gon is equal to $n-2$. Similarly, the 3-cube can be embedded ideally in the 3-dimensional hyperbolic space in such a way that its triangulation in 5 simplices consists of regular ideal simplices. As regular simplices have maximum volume, this shows that the triangulation is minimal. More generally, Smith gives in [Sm00] the best lower bound so far for the minimum number of n-dimensional simplices in a triangulation of the n-cube as the ratio of the hyperbolic volume of the ideal cube to the ideal regular simplex in hyperbolic n-space. Also, in [SlTaTh88], Sleator, Tarjan and Thurston relate the hyperbolic volume of simplices to the size of minimal triangulations of polytopes and balls, and use this relation to compute the asymptotic combinatorial diameter of the Stasheff polytope (or associahedron).

Here, we will use a cocycle Vol_4 which is cohomologous to the volume form on the product of two copies of the hyperbolic plane to give new lower bounds on the minimal number $T(m,n)$ of top dimensional simplices in a triangulation of the product $P(m) \times P(n)$, where $P(m)$ denotes the convex polygon with m vertices. Note that all our triangulations of polytopes are supposed not to have any more vertices than the original polytope. Previously known are the lower bounds $T(m,n) \geq 2mn - (8/3)(m+n)$ and $T(m,4) \geq 7,5m-3$ obtained in [Bo&al05]. We prove:

Theorem 1. $T(m,n) \geq 3,125 \cdot mn - 5(m+n) + 6$.

Theorem 2. $T(m,4) \geq \frac{23}{3}m - \frac{44}{3}$.

Thus, we are coming closer to the upper bounds $T(m,n) \leq 3,5mn - 6(m+n) + 8$ and $T(m,4) \leq 8m - 16$ established in [Bo&al05] for even m, n. The upper bounds for odd m and n are slightly higher.

Since the arguments used in our proofs are of homological type, our method should give the same lower bounds for the polytopial Gromov norm $\|P(m) \times P(n)\|$ of $P(m) \times P(n)$ (for a definition see [Bo&al05]), for which the upper bounds $\|P(m) \times P(m)\| \leq 3,25m^2 - 8,5m$ is known from [Bo&al05] for even m. Such

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lower bounds would contradict the conjecture in [Bo&al05], that the polytopial Gromov norm of $P(m) \times P(n)$ behaves as $3mn + O(m+n)$.

Our lower bounds are only sharp for incidental low values of m and n. If m or n is odd, then our proof could, with some care, be improved to give slightly better lower bounds. Observe also that for $n = 3$, the value of $T(m,3)$ is known and computed in [Bo&al05].

The combinatorial volume cocycle Vol_4 used for the proofs of Theorems 1 and 2 already appears in [Bu07]: Its sup norm $\|\mathrm{Vol}_4\|_\infty = 2/3$ as a cohomology class in $H^4_c(PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R}))$ relates the simplicial volume of products of surfaces to the simplicial volume of their factors:

$$\|\Sigma_g \times \Sigma_h\| = \frac{3}{2} \|\Sigma_g\| \, \|\Sigma_h\| = 24(g-1)(h-1),$$

where Σ_g and Σ_h are surfaces of genus g and h respectively. This is the first product formula for the simplicial volume, and the first example of an exact value of a nonvanishing simplicial volume for a manifold not admitting a constant curvature metric. We refer to [Bu07] for more details.

This paper is structured as follows: In Section 2, we define several combinatorial volume cocycles for certain low dimensional polytopes P and see that they define cohomology classes in the relative cohomology $H^*(P, \partial P)$. In Section 3, we compute the minimal number of 3-simplices in prisms $P(m) \times [0,1]$, a result which already is known from [DeSaTa01]. In Section 4, we give a (probably sharp) bound on the number of 4-simplices in a triangulation of $P(m) \times P(n)$ having at least one 3-face in the boundary $\partial\left(P(m) \times P(n)\right)$. And finally, in Section 5, we use the combinatorial volume cocycle Vol_4 in order to estimate the number of remaining (interior) 4-simplices in such a triangulation.

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2. Combinatorial volume cocycles

2.1. **Cocycles on products of low dimensional spheres.** Identify the 0-dimensional sphere S^0 with the two endpoints 0 and 1 of the unit interval $[0,1]$ and define a cochain

$$\nu : \left(S^0\right)^2 \longrightarrow \{-1, 0, +1\}$$

as

$$\begin{aligned} \nu(0,0) = \nu(1,1) &= 0, \\ \nu(0,1) &= +1, \\ \nu(1,0) &= -1. \end{aligned}$$

Observe that ν is alternating by definition. Furthermore, ν is a cocycle, i.e. it satisfies the cocycle relation $\delta\nu(x,y,z) = \nu(y,z) - \nu(x,z) + \nu(x,y) = 0$, for every x, y, z in S^0.

Fix now an orientation on the circle S^1 and recall that the *orientation cocycle* Or is defined as

$$\begin{array}{rccl} \mathrm{Or}: & \left(S^1\right)^3 & \longrightarrow & \mathbb{R} \\ & (x_0,x_1,x_2) & \longmapsto & \begin{cases} +1 & \text{if } x_0,x_1,x_2 \text{ are positively oriented,} \\ -1 & \text{if } x_0,x_1,x_2 \text{ are negatively oriented,} \\ 0 & \text{if } x_i = x_j \text{ for } i \neq j. \end{cases} \end{array}$$

Note that the cochain Or is alternating by definition. To check that it is a cocycle, we verify that $\delta\mathrm{Or}(x_0,...,x_3) = 0$ for any 4-tuple of points $x_0,...,x_3$ in S^1. Let us first assume that the points are all distinct. Since $\delta\mathrm{Or}$ is alternating, we can without loss of generality assume that the points $x_0, ..., x_3$ are positively cyclically ordered. In particular $\mathrm{Or}(x_0, ..., \widehat{x}_i, ..., x_3) = +1$ for every i, and

$$\delta\mathrm{Or}(x_0, ..., x_3) = \sum_{i=0}^{3} (-1)^i \mathrm{Or}(x_0, ..., \widehat{x}_i, ..., x_3) = 1 - 1 + 1 - 1 = 0.$$

If the four points $x_0, ..., x_3$ are not all distinct, then we can assume that $x_0 = x_1$ and we have

$$\delta\mathrm{Or}(x_0, ..., x_3) = \mathrm{Or}(x_1, x_2, x_3) - \mathrm{Or}(x_0, x_2, x_3) + 0 - 0 = 0.$$

The cup product of the orientation cocycle with ν, is given by

$$\begin{array}{rccl} \mathrm{Or} \cup \nu : & \left(S^1 \times S^0\right)^4 & \longrightarrow & \mathbb{R} \\ & ((x_0, y_0), ..., (x_3, y_3)) & \longmapsto & \mathrm{Or}(x_0, x_1, x_2) \cdot \nu(y_2, y_3). \end{array}$$

Observe that, in view of the well-known and straightforward formula

$$\delta\left(\mathrm{Or} \cup \nu\right) = \delta\mathrm{Or} \cup \nu - \mathrm{Or} \cup \delta\nu = 0,$$

the cochain $\mathrm{Or} \cup \nu$ is also a cocycle. However, it is clearly not alternating, so that we define a cochain

$$\mathrm{Vol}_3 : \left(S^1 \times S^0\right)^4 \longrightarrow \mathbb{R}$$

as the alternation of $\mathrm{Or} \cup \nu$:

$$\mathrm{Vol}_3(z_0, ..., z_3) = \frac{1}{4!} \sum_{\sigma \in Sym(4)} \mathrm{sign}(\sigma) \left(\mathrm{Or} \cup \nu\right) (z_{\sigma(0)}, ..., z_{\sigma(3)}),$$

for every 4-tuple $(z_0, ..., z_3)$ in $\left(S^1 \times S^0\right)^4$. Observe that

$$\mathrm{Or} \cup \nu - \mathrm{Vol}_3 = \delta b_3,$$

where $b_3 : \left(S^1 \times S^0\right)^3 \longrightarrow \mathbb{R}$ is the cochain defined as

$$b_3((x_0, y_0), (x_1, y_1), (x_2, y_2)) = \frac{1}{3} \mathrm{Or}(x_0, x_1, x_2) \cdot \left(\nu(y_1, y_2) + \nu(y_0, y_2)\right).$$

In particular, the cochain Vol_3 is also a cocycle.

The cup product of the orientation cocycle with itself, is given by

$$\begin{array}{rccl} \mathrm{Or} \cup \mathrm{Or} : & \left(S^1 \times S^1\right)^5 & \longrightarrow & \mathbb{R} \\ & ((x_0, y_0), ..., (x_4, y_4)) & \longmapsto & \mathrm{Or}(x_0, x_1, x_2) \cdot \mathrm{Or}(y_2, y_3, y_4), \end{array}$$

and it is a cocycle in view of the formula

$$\delta\left(\mathrm{Or} \cup \mathrm{Or}\right) = \delta\mathrm{Or} \cup \mathrm{Or} + \mathrm{Or} \cup \delta\mathrm{Or} = 0.$$

We define a cochain

$$\mathrm{Vol}_4 : \left(S^1 \times S^1\right)^5 \longrightarrow \mathbb{R}$$

as the alternation of $\mathrm{Or} \cup \mathrm{Or}$:

$$\mathrm{Vol}_4(z_0, ..., z_4) = \frac{1}{5!} \sum_{\sigma \in Sym(5)} \mathrm{sign}(\sigma) \left(\mathrm{Or} \cup \mathrm{Or}\right) (z_{\sigma(0)}, ..., z_{\sigma(4)}),$$

for every 5-tuple $(z_0, ..., z_4)$ in $\left(S^1 \times S^1\right)^5$. Here also, $\mathrm{Or} \cup \mathrm{Or}$ and Vol_4 differ by a coboundary. Indeed, letting $b_4 : \left(S^1 \times S^1\right)^4 \to \mathbb{R}$ be the cochain

$$\begin{aligned} b_4((x_0, y_0), ..., (x_3, y_3)) = & \frac{1}{3} \mathrm{Or}(x_1, x_2, x_3) \left(\mathrm{Or}(x_1, x_3, x_4) + \mathrm{Or}(x_2, x_3, x_4)\right) \\ & + \frac{1}{12} \left(\mathrm{Or}(x_1, x_2, x_4) \left(\mathrm{Or}(x_1, x_3, x_4) + \mathrm{Or}(x_2, x_3, x_4)\right)\right. \\ & + \mathrm{Or}(x_1, x_3, x_4) \left(-\mathrm{Or}(x_1, x_2, x_4) + \mathrm{Or}(x_2, x_3, x_4)\right) \\ & \left. + \mathrm{Or}(x_2, x_3, x_4) \left(-\mathrm{Or}(x_1, x_2, x_4) - \mathrm{Or}(x_1, x_3, x_4)\right)\right), \end{aligned}$$

one can check that $\mathrm{Or} \cup \mathrm{Or} - \mathrm{Vol}_4 = \delta b_4$. Note that in particular, Vol_4 is a cocycle.

2.2. **Some homological algebra.** Let P be any convex polytope and let

$$C_q(P) = \left\{ \sum_{\text{finite}} a_\sigma \sigma \;\middle|\; \begin{array}{l} a_\sigma \in \mathbb{R},\ \sigma : \Delta^q \to P \text{ affine,} \\ \text{vertices of } \sigma \subset P^0 \end{array} \right\}$$

be the real vector space over the basis of affine simplices $\sigma : \Delta^q \to P$, where Δ^q denotes the standard q-simplex, mapping the vertices of Δ^q to the vertices P^0 of P. The *relative chain complex* of $(P, \partial P)$ is given, in degree q, by

$$C_q(P, \partial P) = C_q(P)/C_q(\partial P).$$

The boundary operator $\partial : C_q(P) \to C_{q-1}(P)$ induces a boundary operator on the quotient $C_*(P, \partial P)$ and the resulting homology group $H_*(P, \partial P)$ is the *relative homology* of $(P, \partial P)$.

Any triangulation T of the n-dimensional polytope P determines in a unique (after the choice of an orientation on P) and natural way an affine cycle z_T in $C_n(P, \partial P)$, which in turn gives a relative homology class

$$[z_T] \in H_n(P, \partial P).$$

(All our triangulations are assumed not to have more vertices than P.) If z_1, z_2 are two affine cycles arising from two triangulations of P, then $[z_1] = [z_2]$ in $H_n(P, \partial P)$. Indeed, it is easy to see that there exists chains c in $C_{n+1}(P)$ and b in $C_n(\partial P)$ such that

$$z_2 - z_1 = \partial c + b.$$

Let $C^q(P)$ denote the algebraic dual of the space of chains $C_q(P)$. The *relative cochain complex* of $(P, \partial P)$ is given, in degree q, by

$$C^q(P, \partial P) = \{ f \in C^q(P) \mid f(c) = 0 \text{ for every } c \in C_q(\partial P) \}.$$

The coboundary operator $\delta : C^q(P) \to C^{q+1}(P)$ clearly restricts to the relative cochain complex $C^*(P, \partial P)$ and the resulting cohomology group $H^*(P, \partial P)$ is the *relative cohomology* of $(P, \partial P)$.

Observe that the evaluation of chains on cochains induces a well-defined pairing

$$\langle .,. \rangle : H^q(P, \partial P) \otimes H_q(P, \partial P) \longrightarrow \mathbb{R}.$$

2.3. **Combinatorial volume.** We start with a trivial case: The cocycle $\nu : (S^0)^2 \to \mathbb{R}$ determines a cocycle

$$\nu \in C^1([0,1], \{0,1\}),$$

which we still denote by ν, defined as

$$\nu(\sigma) = \nu(\sigma_0, \sigma_1),$$

where σ_0, σ_1 are the (ordered) vertices of the affine simplex σ. Note that it is obvious that ν is well defined, since it vanishes on 1-chains contained in the boundary. If T is the only triangulation of the interval with all its vertices in the boundary, then the relative cycle z_T, which consists of one affine chain, satisfies $\nu(z_T) = 1$. Obviously, ν is nothing else than the Euclidean volume.

Choose an embedding of the convex polygon $P(m)$ with m vertices in the closed unit disk $\mathbb{D}^2$ in such a way that all the vertices $P(m)^0$ of $P(m)$ lie on the boundary S^1 of $\mathbb{D}^2$. The orientation cocycle determines a cocycle

$$\mathrm{Or} \in C^2(P(m), \partial P(m)),$$

which we still denote by Or, defined as

$$\mathrm{Or}(\sigma) = \mathrm{Or}(\sigma_0, \sigma_1, \sigma_2),$$

where $\sigma_0, \sigma_1, \sigma_2 \in P(m)^0 \subset S^1$ are the (ordered) vertices of the affine simplex σ. Note that Or is well defined since if σ is an affine simplex in $\partial P(m)$, then two of its vertices must be equal, so that $\mathrm{Or}(\sigma) = 0$. If T is a triangulation of $P(m)$ in $m-2$ simplices of dimension 2 with all its vertices in $P(m)^0$, then the relative cycle z_T satisfies $\mathrm{Or}(z_T) = m-2$. Moreover, since the evaluation of Or on two cycles arising from two triangulations of $P(m)$ is constant and $\mathrm{Or}(\sigma) = 0$ if and only if $\sigma \subset \partial P(m)$, it is immediate, that any triangulation of $P(m)$ with all its vertices in the boundary has to have precisely $m-2$ simplices of dimension 2. Of course, taking $\mathbb{D}^2$ to be Klein's projective model for the hyperbolic 2-space, $\pi \cdot \mathrm{Or}(\sigma)$ is nothing else than the hyperbolic volume (or area) of σ.

The cocycles Vol_3 and $\mathrm{Or} \cup \nu : (S^1 \times S^0)^3 \to \mathbb{R}$ determine cocycles

$$\mathrm{Vol}_3 \text{ and } \mathrm{Or} \cup \nu \in C^3(P(m) \times [0,1], \partial\,(P(m) \times [0,1])),$$

by

$$\mathrm{Vol}_3(\sigma) = \mathrm{Vol}_3(\sigma_0, \sigma_1, \sigma_2, \sigma_3) \text{ and } \mathrm{Or} \cup \nu(\sigma) = \mathrm{Or} \cup \nu(\sigma_0, \sigma_1, \sigma_2, \sigma_3),$$

where $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are the (ordered) vertices of the affine simplex σ. To check that those two cocycles are well defined we need to verify that they vanish on affine chains contained in $\partial\,(P(m) \times [0,1])$. Such chains are linear combinations of two types of affine simplices: those contained in $P(m) \times \{*\}$ and those contained in $\tau \times [0,1]$, where τ is an exterior edge of $P(m)$. In the former case, all the ν-factors appearing in the definitions of Vol_3 and $\mathrm{Or} \cup \nu$ vanish and in the latter case, all the Or-factors do, so in either cases, the two cocycles vanish. For the same reason, the cochain b_3 (whose coboundary is the difference between $\mathrm{Or} \cup \nu$ and Vol_3) also is a cochain in $C^2(P(m) \times [0,1], \partial\,(P(m) \times [0,1]))$, so that

$$[\mathrm{Vol}_3] = [\mathrm{Or} \cup \nu] \in H^3(P(m) \times [0,1], \partial\,(P(m) \times [0,1])).$$

Proposition 3. *Let T be any triangulation of the prism $P(m) \times [0,1]$ with all its vertices in $(P(m) \times [0,1])^0$. Then*

$$\mathrm{Vol}_3(z_T) = m-2.$$

Proof. If z and z' are two affine chains on $P(m) \times [0,1]$ coming from two triangulations T and T' of the prism, then they determine the same homology class $[z]$ in $H_3(P(m) \times [0,1], \partial\,(P(m) \times [0,1]))$ and

$$\mathrm{Vol}_3(z) = \langle [\mathrm{Vol}_3], [z] \rangle = \mathrm{Vol}_3(z').$$

Let now T_0 be a triangulation of $P(m)$ in $m-2$ simplices of dimension 2. After choosing a numbering of the vertices of $P(m)$ and those of $[0,1]$, we get a canonical triangulation T of $P(m) \times [0,1]$ (in $3(m-2)$ simplices of dimension 3). Denoting by z_0 and z_T the affine cycles arising from the triangulations of T_0 and T respectively, we have

$$\mathrm{Vol}_3(z_T) = \langle [\mathrm{Vol}_3]\,, [z_T] \rangle = (\mathrm{Or} \cup \nu)\,(z_T) = \mathrm{Or}(z_0) \cdot \nu([0,1]) = m-2,$$

which finishes the proof of the proposition. □

The cocycles Vol_4 and $\mathrm{Or} \cup \mathrm{Or} : (S^1 \times S^1)^5 \to \mathbb{R}$ determine cocycles

$$\mathrm{Vol}_4 \text{ and } \mathrm{Or} \cup \mathrm{Or} \in C^5(P(m) \times P(n), \partial\,(P(m) \times P(n))),$$

by

$$\mathrm{Vol}_4(\sigma) = \mathrm{Vol}_4(\sigma_0, ..., \sigma_4) \text{ and } \mathrm{Or} \cup \mathrm{Or}(\sigma) = \mathrm{Or} \cup \mathrm{Or}(\sigma_0, ..., \sigma_4),$$

where $\sigma_0, ..., \sigma_4$ are the (ordered) vertices of the affine simplex σ. To check that those two cocycles are well defined we need to verify that they vanish on affine chains contained in $\partial\,(P(m) \times P(n))$. Such chains are linear combinations of two types of affine simplices: those contained in $P(m) \times \tau$ and those contained in $\tau \times P(n)$, where τ is an exterior edge of $P(n)$ or $P(m)$ respectively. In the former case, all the Or-factors in the second factor appearing in the definitions of Vol_4 and $\mathrm{Or} \cup \mathrm{Or}$ vanish and in the latter case, all the Or-factors in the first factor do, so in either cases, the two cocycles vanish. For the same reason, the cochain b_4 (whose coboundary is the difference between $\mathrm{Or} \cup \mathrm{Or}$ and Vol_4) also is a cochain in $C^3(P(m) \times P(n), \partial(P(m) \times P(n)))$, so that

$$[\mathrm{Vol}_4] = [\mathrm{Or} \cup \mathrm{Or}] \in H^4(P(m) \times P(n), \partial(P(m) \times P(n))).$$

Proposition 4. *Let T be any triangulation of the product $P(m) \times P(n)$. Then*

$$\mathrm{Vol}_4(z_T) = (m-2)(n-2).$$

Proof. The proof that $\mathrm{Vol}_4(z_T)$ is independent of the triangulation T is identical to the proof of the analogous statement in Proposition 3.

Let now T_m and T_n be triangulations of $P(m)$, respectively $P(n)$, in $m-2$, resp. $n-2$, simplices of dimension 2. After choosing a numbering of the vertices of $P(m)$ and $P(n)$, we get a canonical triangulation T of $P(m) \times P(n)$ (in $6(m-2)(n-2)$ simplices of dimension 4). Denoting by z_m, z_n and z_T the affine cycles arising from the triangulations of T_m, T_n and T respectively, we have

$$\mathrm{Vol}_4(z_T) = \langle [\mathrm{Vol}_4], [z_T] \rangle = (\mathrm{Or} \cup \mathrm{Or})(z_T) = \mathrm{Or}(z_m) \cdot \mathrm{Or}(z_n) = (m-2)(n-2),$$

which finishes the proof of the proposition. □

We will call the cocycles $\nu, \mathrm{Or}, \mathrm{Vol}_3$ and Vol_4 combinatorial volume cocycles. In fact, for products of intervals, triangles and squares, we obtain the Euclidean volume cocycle (up to a constant coming from that our triangles have area equal to 1 instead of 1/2). In particular, our proof of the minimality of triangulations of the 3-cube and 4-cube with 5, respectively 16, top dimensional simplices is nothing else than a classical (Euclidean) volume argument.

2.4. **Values of** Vol_3**.** In order to examine the various possible triangulations of the prism $P(m) \times [0,1]$, we need to study the possible values of the combinatorial volume cocycle Vol_3. Expanding the sum in the defining expression for Vol_3, we get

$$\begin{aligned}
\mathrm{Vol}_3((x_0,y_0),...,(x_3,y_3)) = \frac{1}{12}\,[&\mathrm{Or}(x_0,x_1,x_2)\,(\nu(y_2,y_3)+\nu(y_1,y_3)+\nu(y_0,y_3)) \\
&+ \mathrm{Or}(x_0,x_1,x_3)\,(\nu(y_2,y_3)-\nu(y_1,y_2)-\nu(y_0,y_2)) \\
&+ \mathrm{Or}(x_0,x_2,x_3)\,(-\nu(y_1,y_3)-\nu(y_1,y_2)+\nu(y_0,y_1)) \\
&+ \mathrm{Or}(x_1,x_2,x_3)\,(\nu(y_0,y_3)+\nu(y_0,y_2)+\nu(y_0,y_1))],
\end{aligned}$$

for every 4-tuple $((x_0,y_0),...,(x_3,y_3))$ in $\left(S^1 \times S^0\right)^4$. If $y_0 = y_1 = y_2 = y_3$, then $\nu(y_i, y_j) = 0$ for every i, j, so that the volume cocycle Vol_3 clearly vanishes. If the y_i's are not all equal, then the cardinality of the set $\{y_0, ..., y_3\}$ has to be equal to 2, in which case, upon permuting the variables, and the points 0 and 1 of S^0, we distinguish two cases:

- $y_0 = y_1 = y_2 = 0$ and $y_3 = 1$: Using the cocycle relation $\delta\mathrm{Or}(x_0,x_1,x_2,x_3) = 0$, we compute that $\mathrm{Vol}_3((x_0,y_0),...,(x_3,y_3))$ is equal to

$$\begin{aligned}
&\frac{1}{12}\,[3\mathrm{Or}(x_0,x_1,x_2)+\mathrm{Or}(x_0,x_1,x_3)-\mathrm{Or}(x_0,x_2,x_3)+\mathrm{Or}(x_1,x_2,x_3)] \\
&\qquad = \frac{1}{3}\mathrm{Or}(x_0,x_1,x_2) \in \left\{-\frac{1}{3}, 0, \frac{1}{3}\right\}.
\end{aligned}$$

- $y_0 = y_1 = 0$ and $y_2 = y_3 = +1$: Let first the points $x_0, ..., x_3 \in S^1$ be arbitrary. We have

$$\mathrm{Vol}_3((x_0, y_0), ..., (x_3, y_3)) =$$
$$\frac{1}{6}\left[\mathrm{Or}(x_0, x_1, x_2) - \mathrm{Or}(x_0, x_1, x_3) - \mathrm{Or}(x_0, x_2, x_3) + \mathrm{Or}(x_1, x_2, x_3)\right].$$

Let D_{01} and D_{23} denote the straight lines between x_0, x_1 and x_2, x_3 respectively. There are now four possibilities for the intersection of those lines:

  – $D_{01} = D_{23}$: Up to permuting (x_2, y_2) and (x_3, y_3), this means that $x_0 = x_2$ and $x_1 = x_3$, and hence Vol_3 vanishes.
  – $D_{01} \cap D_{23} = \varnothing$: Either x_0, x_1, x_2, x_3 or x_0, x_1, x_3, x_2 are cyclically ordered (either positively or negatively). In either cases, Vol_3 vanishes.
  – $D_{01} \cap D_{23}$ intersect in one boundary point in S^1: In view of the obvious symmetries, we can suppose that $x_1 = x_3$, and then
  $$\mathrm{Vol}_3((x_0, y_0), ..., (x_3, y_3)) = \frac{1}{3}\mathrm{Or}(x_0, x_1, x_2).$$
  – $D_{01} \cap D_{23}$ intersect in one interior point: x_0, x_3, x_1, x_2 are cyclically ordered, either positively or negatively, and
  $$\mathrm{Vol}_3((x_0, y_0), ..., (x_3, y_3)) = \frac{2}{3}\mathrm{Or}(x_0, x_1, x_2).$$

2.5. **Values of** Vol_4**.** As we want to estimate the value of Vol_4 on different configurations of points, it would be convenient to have a simpler expression for it. We will now show that in the defining expression for Vol_4, it is enough to average over those permutations mapping 2 to 0, so that we will obtain the expression $(\diamondsuit)$ below. Fix $(z_0, ..., z_4)$ in $(S^1 \times S^1)^5$. We start by showing that

$$\Phi(i) := \frac{1}{4} \sum_{\substack{\sigma \in Sym(5) \\ \sigma(2) = i}} \mathrm{sign}(\sigma)\,(\mathrm{Or} \cup \mathrm{Or})\,(z_{\sigma(0)}, z_{\sigma(1)}, z_i, z_{\sigma(3)}, z_{\sigma(4)})$$

is independent of i. By definition, $\Phi(0)$ is equal to

$$\begin{aligned}
&\mathrm{Or}(x_0, x_1, x_2)\mathrm{Or}(y_0, y_3, y_4) + \mathrm{Or}(x_0, x_3, x_4)\mathrm{Or}(y_0, y_1, y_2)\\
&- \mathrm{Or}(x_0, x_1, x_3)\mathrm{Or}(y_0, y_2, y_4) - \mathrm{Or}(x_0, x_2, x_4)\mathrm{Or}(y_0, y_1, y_3)\\
&+ \mathrm{Or}(x_0, x_1, x_4)\mathrm{Or}(y_0, y_2, y_3) + \mathrm{Or}(x_0, x_2, x_3)\mathrm{Or}(y_0, y_1, y_4),
\end{aligned}$$

where we have used, on the first factors, the fact that Or is alternating. From the cocycle relation for Or, we have, for i, j in $\{2, 3, 4\}$,

$$\mathrm{Or}(x_0, x_i, x_j) = \mathrm{Or}(x_1, x_i, x_j) + \mathrm{Or}(x_0, x_1, x_j) - \mathrm{Or}(x_0, x_1, x_i)$$

and the analogous formula for the y_i's. The previous expression thus becomes

$$\begin{aligned}
&\mathrm{Or}(x_0, x_1, x_2)\left(\mathrm{Or}(y_1, y_3, y_4) + \mathrm{Or}(y_0, y_1, y_4) - \mathrm{Or}(y_0, y_1, y_3)\right)\\
&+ \left(\mathrm{Or}(x_1, x_3, x_4) + \mathrm{Or}(x_0, x_1, x_4) - \mathrm{Or}(x_0, x_1, x_3)\right)\mathrm{Or}(y_0, y_1, y_2)\\
&- \mathrm{Or}(x_0, x_1, x_3)\left(\mathrm{Or}(y_0, y_2, y_4) + \mathrm{Or}(y_0, y_1, y_4) - \mathrm{Or}(y_0, y_1, y_2)\right)\\
&- \left(\mathrm{Or}(x_1, x_2, x_4) + \mathrm{Or}(x_0, x_1, x_4) - \mathrm{Or}(x_0, x_1, x_2)\right)\mathrm{Or}(y_0, y_1, y_3)\\
&+ \mathrm{Or}(x_0, x_1, x_4)\left(\mathrm{Or}(y_0, y_2, y_3) + \mathrm{Or}(y_0, y_1, y_4) - \mathrm{Or}(y_0, y_1, y_2)\right)\\
&+ \left(\mathrm{Or}(x_1, x_2, x_3) + \mathrm{Or}(x_0, x_1, x_3) - \mathrm{Or}(x_0, x_1, x_2)\right)\mathrm{Or}(y_0, y_1, y_4).
\end{aligned}$$

Now note that all the terms of the form $\mathrm{Or}(x_0, x_1, x_i)\mathrm{Or}(y_0, y_1, y_j)$, for i, j in $\{2, 3, 4\}$ cancel out two by two, and what we are left with is precisely $\Phi(1)$. We have thus proven that $\Phi(0) = \Phi(1)$. Applying the cyclic permutation $(0, 1, ..., 4)$ and its powers to the indices in the proof of the latter equality, it is immediate that $\Phi(1) = \Phi(2) = \Phi(3) = \Phi(4)$. We thus have $\mathrm{Vol}_4((x_0,y_0),...,(x_4,y_4)) = (1/30)(\Phi(0)+\Phi(1)+\Phi(2)+\Phi(3)+\Phi(4)) = (1/6)\Phi(0)$, and we have hence proven the following equation:

$$\begin{aligned}
&\mathrm{Vol}_4((x_0,y_0),...,(x_4,y_4)) \qquad\qquad (\Diamond)\\
&= \frac{1}{6}\left(\mathrm{Or}(x_0,x_1,x_2)\mathrm{Or}(y_0,y_3,y_4) + \mathrm{Or}(x_0,x_3,x_4)\mathrm{Or}(y_0,y_1,y_2)\right.\\
&\quad - \mathrm{Or}(x_0,x_1,x_3)\mathrm{Or}(y_0,y_2,y_4) - \mathrm{Or}(x_0,x_2,x_4)\mathrm{Or}(y_0,y_1,y_3)\\
&\quad \left. + \mathrm{Or}(x_0,x_1,x_4)\mathrm{Or}(y_0,y_2,y_3) + \mathrm{Or}(x_0,x_2,x_3)\mathrm{Or}(y_0,y_1,y_4)\right)
\end{aligned}$$

Lemma 5. *If three among the x_i's are equal or if three among the y_i's are equal then*

$$|\mathrm{Vol}_4((x_0,y_0),...,(x_4,y_4))| \leq \frac{1}{6}.$$

Proof. As Vol_4 is alternating and symmetric in the first and second factor, we can without loss of generality assume that $x_0 = x_1 = x_2$. But then, the expression $(\Diamond)$ for Vol_4 reduces to

$$\frac{1}{6}\mathrm{Or}(x_0,x_3,x_4)\mathrm{Or}(y_0,y_1,y_2) \in \left\{-\frac{1}{6}, 0, \frac{1}{6}\right\}.$$

□

Lemma 6. *If there exists i_1, i_2 and j_1, j_2 all distinct such that either $x_{i_1} = x_{i_2}$ and $x_{j_1} = x_{j_2}$, or $y_{i_1} = y_{i_2}$ and $y_{j_1} = y_{j_2}$, then*

$$|\mathrm{Vol}_4((x_0,y_0),...,(x_4,y_4))| \leq \frac{1}{3}.$$

Proof. As Vol_4 is alternating and symmetric in the first and second factor, we can without loss of generality assume that $x_0 = x_1$ and $x_2 = x_3$. The expression $(\Diamond)$ for Vol_4 now reduces to

$$\frac{1}{6}\left(\mathrm{Or}(x_0,x_3,x_4)\mathrm{Or}(y_0,y_1,y_2) - \mathrm{Or}(x_0,x_2,x_4)\mathrm{Or}(y_0,y_1,y_3)\right).$$

□

Lemma 7. *If there exists $i \neq j$ such that $x_i = x_j$ or $y_i = y_j$, then*

$$|\mathrm{Vol}_4((x_0,y_0),...,(x_4,y_4))| \leq \frac{1}{2}.$$

Proof. As Vol_4 is alternating, we can without loss of generality assume that $x_0 = x_1$, so that the expression $(\Diamond)$ for Vol_4 reduces to

$$\begin{aligned}
\tfrac{1}{6}\left[\mathrm{Or}(x_0,x_3,x_4)\mathrm{Or}(y_0,y_1,y_2) - \mathrm{Or}(x_0,x_2,x_4)\mathrm{Or}(y_0,y_1,y_3)\right.\\
\left.+\mathrm{Or}(x_0,x_2,x_3)\mathrm{Or}(y_0,y_1,y_4)\right].
\end{aligned}$$

□

Lemma 8. *Let $((x_0,y_0),...,(x_4,y_4))$ be an arbitrary 5-tuple in $(S^1 \times S^1)^5$, then*

$$|\mathrm{Vol}_4((x_0,y_0),...,(x_4,y_4))| \leq \frac{2}{3}.$$

Proof. First note that if either all the x_i's are not distinct or all the y_i's are not distinct, then by Lemma 7, the evaluation of Vol_4 is at most $1/2$. Thus we can assume that this is not the case. Since Vol_4 is alternating, we can furthermore assume that the x_i's are positively cyclically ordered. In particular, $\mathrm{Or}(x_i,x_j,x_k) = +1$, whenever $0 \leq i < j < k \leq 4$. The expression $(\Diamond)$ for Vol_4 now becomes

$$\begin{aligned}
\frac{1}{6}&\left[\mathrm{Or}(y_0,y_1,y_2) - \mathrm{Or}(y_0,y_1,y_3) + \mathrm{Or}(y_0,y_1,y_4)\right.\\
&\left.+\mathrm{Or}(y_0,y_3,y_4) - \mathrm{Or}(y_0,y_2,y_4) + \mathrm{Or}(y_0,y_2,y_3)\right].
\end{aligned}$$

Using the cocycle relation

$$\delta\mathrm{Or}(y_0, y_2, y_3, y_4) = 0,$$

we see that the above expression can be rewritten as

$$\frac{1}{6}\left(\mathrm{Or}(y_0, y_1, y_2) - \mathrm{Or}(y_0, y_1, y_3) + \mathrm{Or}(y_0, y_1, y_4) + \mathrm{Or}(y_2, y_3, y_4)\right) \leq 2/3,$$

which proves the lemma. □

3. Minimal triangulations of the prism $P(m) \times [0,1]$

The following result was first proven in [DeSaTa01].

Theorem 9. *The minimal number of* 3*-simplices in a triangulation of the prism* $P(m) \times [0,1]$ *is equal to*

$$\begin{cases} \frac{5}{2}m - 5 & \text{if } m \text{ is even,} \\ \\ \frac{5}{2}m - \frac{9}{2} & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Let T be a fixed triangulation of $P(m) \times [0,1]$. By restriction, the triangulation T gives triangulations of each of the m-gons $P(m) \times \{0\}$ and $P(m) \times \{1\}$ in $m-2$ simplices of dimension 2. Each of those 2-simplices is a 2-face of one and only one 3-simplex of T. We call such a 3-simplex a simplex of type Tr (for triangle in an m-gon). There are precisely

$$2(m-2)$$

simplices of type Tr. In view of the computations of Section 2.4, a simplex σ of type Tr has combinatorial volume

$$|\mathrm{Vol}_3(\sigma)| = \frac{1}{3}.$$

All the remaining simplices of T have two vertices in $P(m) \times \{0\}$ and two in $P(m) \times \{1\}$. We will say that such an ordered simplex σ is of type A if $\mathrm{Vol}_3(\sigma) = +2/3$ and of type B if $\mathrm{Vol}_3(\sigma) = +1/3$. Denote by a, respectively b, the number of simplices of type A, resp. B, in T.

Let z_T be the affine cycle arising from the triangulation T. It follows from Proposition 3, that $\mathrm{Vol}_3(z_T) = m - 2$ and hence

$$m - 2 \leq \frac{1}{3} \cdot 2(m-2) + \frac{2}{3}a + \frac{1}{3}b,$$

where the last inequality comes from that we have only counted the contribution to the combinatorial volume coming from simplices of type Tr, A and B. There may be more simplices, but as computed in Section 2.4, they have $\mathrm{Vol}_3(\sigma) \leq 0$.

Since b is positive, we get from the previous inequality that

$$a + b \geq a + \frac{1}{2}b \geq \frac{1}{2}(m-2).$$

The number of 3-dimensional simplices of T is hence greater or equal to

$$2(m-2) + a + b \geq \frac{5}{2}(m-2).$$

Rounding up when m is an odd integer gives the claimed lower bound.

For the equality, note that there exists a triangulation of $P(3) \times [0,1]$ in 3 simplices (2 of type Tr and 1 of type B). There exists a triangulation of $P(4) \times [0,1]$ in 5 simplices (4 of type Tr and 1 of type A). Decompose $P(2n) \times [0,1]$ in $(2n-2)$ cubes $P(4) \times [0,1]$, and triangulate each of those minimally in 5 simplices. Decompose $P(2n+1) \times [0,1]$ in $(2n-2)$ cubes $P(4) \times [0,1]$ and one prism $P(3) \times [0,1]$, and triangulate each of those minimally in 5, respectively 3, simplices. □

4. The boundary simplices of a triangulation of $P(m) \times P(n)$

Theorem 10. *Let T be any triangulation of $P(m) \times P(n)$, The number of 4-simplices of T with a 3-face in $\partial\,(P(m) \times P(n))$ is greater or equal to*

$$\frac{5}{2}mn - 3(m+n).$$

Observe that this already gives a lower bound for the minimal number $T(m,n)$ of 4-simplices in a triangulation of $P(m) \times P(n)$ which improves the lower bound of $2mn - (8/3)(m+n)$ computed in [Bo&al05]. Using our combinatorial volume cocycle Vol_4, we will further improve this lower bound in the next section.

Proof. Let T be a fixed triangulation of $P(m) \times P(n)$. Its restriction to every prism $P(m) \times \tau$ or $\tau \times P(n)$, where τ is a boundary edge of $P(n)$ or $P(m)$, has 3-simplices of (at least) three types, according to the terminology introduced in the proof of Theorem 9, type Tr, type A and type B. We will say that a 4-simplex of the triangulation T is of type A_m or B_m (respectively A_n or B_n) if one of its 3-face is contained in some prism $P(m) \times \tau$ (resp. $\tau \times P(n)$) and of type A, respectively B. As for the 4-simplices with a 3-face of type Tr, we distinguish two cases, according to the number of 3-faces of type Tr in $P(m) \times \tau$ (resp. $\tau \times P(n)$): A 4-simplex is of type $\mathrm{Tr}_{m,1}$ (respectively $\mathrm{Tr}_{n,1}$) if it has precisely one 3-faces of type Tr in a prism of the form $P(m) \times \tau$ (resp. $\tau \times P(n)$) and of type $\mathrm{Tr}_{m,2}$ (respectively $\mathrm{Tr}_{n,2}$) if it has precisely two 3-faces of type Tr in prisms of the form $P(m) \times \tau$ (resp. $\tau \times P(n)$). Note that a 4-simplex can not have three faces of type Tr in prisms of the form $P(m) \times \tau$ (resp. $\tau \times P(n)$). Set

$$\begin{aligned} \mathcal{TYPE} &= \{\mathrm{Tr}_{m,1}, \mathrm{Tr}_{m,2}, A_m, B_m, \mathrm{Tr}_{n,1}, \mathrm{Tr}_{n,2}, A_n, B_n\}, \\ \mathcal{TYPE}_m &= \{\mathrm{Tr}_{m,1}, \mathrm{Tr}_{m,2}, A_m, B_m\} \quad \text{and} \\ \mathcal{TYPE}_n &= \{\mathrm{Tr}_{n,1}, \mathrm{Tr}_{n,2}, A_n, B_n\}. \end{aligned}$$

For two types of 4-simplices X, Y in $\mathcal{TYPE}$, we will say that a simplex is of type $X \cap Y$, obviously, if it is both of type X and of type Y. For $k = m$ or n, define $t_{k,1}, t_{k,2}, a_k, b_k$ to be the number of 4-simplices in the triangulation T of type $\mathrm{Tr}_{k,1}, \mathrm{Tr}_{k,2}, A_k$ or B_k respectively. Also, for $x = t_{m,1}, t_{m,2}, a_m, b_m$ and $y = t_{n,1}, t_{n,2}, a_n, b_n$, we let $x \cap y$ be the number of simplices of the corresponding types.

Claim 11. *If X, Y belong to $\mathcal{TYPE}_m$, then $X \cap Y = \varnothing$.*

Proof. Let σ be a 4-simplex of T with a 3-face in a given prism $P(m) \times \tau$. Thus, four of the vertices, say $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ of σ are vertices of $P(m) \times \tau$. The fifth vertex σ_0 of σ can not also belong to $P(m) \times \tau$ (otherwise the 4-dimensional simplex σ would be contained in the 3-dimensional prism $P(m) \times \tau$). Suppose that another 3-face of σ, say the one generated by $\sigma_0, \sigma_1, \sigma_2, \sigma_3$, belongs to another prism $P(m) \times \tau'$. Necessarily, τ and τ' have a vertex y in common, and

$$\sigma_1, \sigma_2, \sigma_3 \in P(m) \times \{y\}.$$

Thus, if two 3-faces of σ belong to prisms of the form $P(m) \times \tau$, then σ is of type $\mathrm{Tr}_{m,2}$. Observe furthermore that no other 3-face of σ can belong to a prism of the form $P(m) \times \tau$. □

Clearly, the same conclusion holds for X, Y in $\mathcal{TYPE}_n$. As a consequence, we see that the intersection of any three different types has to be empty. Furthermore, it immediately follows from Claim 11 that for every X in $\mathcal{TYPE}_m$ and $Y \neq Z$ in $\mathcal{TYPE}_n$, the set of simplices of type $X \cap Y$ is disjoint from those of type $X \cap Z$.

Thus, as $\mathrm{Tr}_{m,1} \cap \mathrm{Tr}_{n,1}$, $\mathrm{Tr}_{m,1} \cap \mathrm{Tr}_{n,2}$ and $\mathrm{Tr}_{m,1} \cap B_n$ are disjoint subsets of $\mathrm{Tr}_{m,1}$, we immediately obtain the inequality

$$t_{m,1} \geq t_{m,1} \cap t_{n,1} + t_{m,1} \cap t_{n,2} + t_{m,1} \cap b_n.$$

The inequalities

$$\begin{aligned} t_{n,1} &\geq t_{m,1} \cap t_{n,1} + t_{m,2} \cap t_{n,1} + b_m \cap t_{n,1}, \\ b_m &\geq b_m \cap t_{n,1} + b_m \cap t_{n,2} + b_m \cap b_n, \\ b_n &\geq t_{m,1} \cap b_n + t_{m,2} \cap b_n + b_m \cap b_n \end{aligned}$$

are obtained analogously.

Claim 12. *For any X in $\mathcal{TYPE}_m$, one has $X \cap A_n = \varnothing$.*

Proof. Let σ be a 4-simplex in the triangulation T. Let p_1 and p_2 denote the projections on the first, respectively second, factor of $P(m) \times P(n)$. Denote by $n_1(\sigma)$, respectively $n_2(\sigma)$, the cardinality of the image under p_1, resp. p_2, of the vertices of σ. If σ is a simplex of type A_n, then $n_1(\sigma) \leq 3$ and $n_2(\sigma) \geq 4$. But if σ is of type X, for some X in $\mathcal{TYPE}_m$, then $n_2(\sigma) \leq 3$, which is not possible. □

Claim 13. $2a_m + b_m \geq n(m-2)$ *and* $2a_n + b_n \geq m(n-2)$.

Proof. By symmetry, it is enough to prove the first inequality of the claim. As in the proof of Theorem 9, in each prism $P(m) \times \tau$, the number a of simplices of type A, and the number b of simplices of type B satisfy the inequality

$$2a + b \geq m - 2.$$

As there are n such prisms, and each 3-simplex of type A or B in $P(m) \times \tau$ belongs to one and only one 4-simplex of type A_m or B_m, the claim follows. □

Claim 14. $t_{m,1} + 2t_{m,2} = 2n(m-2)$ *and* $t_{n,1} + 2t_{n,2} = 2m(n-2)$.

Proof. By symmetry, it is enough to prove the first equality of the claim. Recall from the proof of Theorem 9, that in the restriction of the triangulation T to any prism $P(m) \times \tau$, there are exactly $2(m-2)$ simplices of type Tr. As there are n prisms of the form $P(m) \times \tau$ in $P(m) \times P(n)$, we have a total of

$$2n(m-2)$$

3-simplices of type Tr in such prisms. (We do not count the simplices in prisms of the form $\tau \times P(n)$.)

By definition, a simplex of type $\mathrm{Tr}_{m,1}$, respectively $\mathrm{Tr}_{m,2}$, has precisely one, resp. two, 3-simplices as above. Since a 3-simplex in the boundary $\partial(P(m) \times P(n))$ belongs to one and only one 4-simplex of the triangulation T it follows that

$$t_{m,1} + 2t_{m,2} = 2n(m-2),$$

as claimed. □

Claim 15. $4t_{m,2} \cap t_{n,2} + 2t_{m,1} \cap t_{n,2} + 2t_{m,2} \cap t_{n,1} + 2t_{m,2} \cap b_n + 2b_m \cap t_{n,2} \leq 2mn$

Proof. We count the number of triangles (i.e. 2-simplices) in the restriction of the triangulation T to the union of the squares of the form $\tau_m \times \tau_n$, where τ_m and τ_n are boundary edges of $P(m)$ and $P(n)$ respectively. Clearly, each square is triangulated in two triangles and there are mn squares and hence a total of $2mn$ triangles of the above form.

In a simplex of type $\mathrm{Tr}_{m,2} \cap \mathrm{Tr}_{n,2}$, there exists precisely 4 triangles of the above form: A simplex of type $\mathrm{Tr}_{m,2}$ necessarily has three vertices in $P(m) \times \{y\}$, for some vertex y in $P(n)$, and one in each of $P(m) \times \{y_1\}$ and $P(m) \times \{y_2\}$, where y_1 and y_2 are the opposite vertices of the two boundary edges with vertex y. Symmetrically for a simplex of type $\mathrm{Tr}_{n,2}$. Thus, a simplex of type $\mathrm{Tr}_{m,2} \cap \mathrm{Tr}_{n,2}$ has vertices $(x,y),(x,y_1),(x,y_2),(x_1,y),(x_2,y)$, where $\langle x_1,x\rangle$ and $\langle x,x_2\rangle$ are two boundary edges in $P(m)$ and $\langle y_1,y\rangle$ and $\langle y,y_2\rangle$ are two boundary edges in $P(n)$. The 4-simplex hence contains the 4 triangles $\langle (x,y),(x,y_1),(x_1,y)\rangle$, $\langle (x,y),(x,y_1),(x_2,y)\rangle$, $\langle (x,y),(x,y_2),(x_1,y)\rangle$ and $\langle (x,y),(x,y_2),(x_2,y)\rangle$. There are no other triangle.

In a simplex of type $\mathrm{Tr}_{m,2}\cap\mathrm{Tr}_{n,1}$ or $\mathrm{Tr}_{m,1}\cap\mathrm{Tr}_{n,2}$, there exists precisely 2 triangles of the above form: A simplex of type $\mathrm{Tr}_{m,2}\cap\mathrm{Tr}_{n,1}$ has vertices $(x,y),(x,y_1),(x,y')$, $(x_1,y),(x_2,y)$, where $\langle x_1,x\rangle$ and $\langle x,x_2\rangle$ are the two vertices of two boundary edges in $P(m)$, $\langle y_1,y\rangle$ is a boundary edges in $P(n)$ and y' is arbitrary. There are the 2 triangles $\langle (x,y),(x,y_1),(x_1,y)\rangle$ and $\langle (x,y),(x,y_1),(x_2,y)\rangle$. If there were any more triangle, than the simplex would not be of type $\mathrm{Tr}_{n,1}$, but of type $\mathrm{Tr}_{n,2}$. Symmetrically for a simplex of type $\mathrm{Tr}_{m,1} \cap \mathrm{Tr}_{n,2}$.

In a simplex of type $\mathrm{Tr}_{m,2}\cap B_n$ or $B_m\cap\mathrm{Tr}_{n,2}$, there exists at least 2 triangles of the above form: A simplex of type $\mathrm{Tr}_{m,2}\cap B_n$ has vertices $(x,y),(x_1,y_1),(x_2,y_2),(x_1,y)$, (x_2,y), where $\{x_1,x_2\}$ is a boundary edge in $P(m)$ and $\{y_1,y\}$ and $\{y,y_2\}$ are the two vertices of two boundary edges in $P(n)$. There are now at least 2 triangles, namely $\langle (x_1,y_1),(x_1,y),(x_2,y)\rangle$ and $\langle (x_1,y),(x_2,y),(x_2,y_2)\rangle$. There may be more triangles. (If $\langle x,x_1\rangle$ or $\langle x,x_2\rangle$ forms a boundary edge in $P(m)$.) Symmetrically for a simplex of type $B_m \cap \mathrm{Tr}_{n,2}$.

To prove the claim, it now remains to show that none of the above considered triangles has been counted twice. To see that, observe that a triangle in $\tau_m \times \tau_n$ is the 2-face of a unique 3-simplex in $P(m)\times\tau_n$ and a unique 3-simplex in $\tau_m \times P(n)$. Now, for every 4-simplex σ and each of its triangles t considered above, we have that if the triangle t belongs to $\tau_m \times \tau_n$, then the 3-simplices in $P(m)\times\tau_n$ and $\tau_m \times P(n)$ which t is a 2-face of, are 3-faces of σ. As a 4-simplex is completely determined by two of its 3-faces, the claim follows. $\square$

In view of Claims 11 and 12, the number of 4-simplices with a 3-face of type Tr, type A or type B in $\partial\,(P(m)\times P(n))$ is equal to

$$\begin{aligned} t_{m,1} + t_{m,2} + a_m + b_m + t_{n,1} + t_{n,2} + a_n + b_n - t_{m,1}\cap t_{n,1} - t_{m,1}\cap t_{n,2} - t_{m,1}\cap b_n \\ - t_{m,2}\cap t_{n,1} - t_{m,2}\cap t_{n,2} - t_{m,2}\cap b_n - b_m \cap b_n. \end{aligned}$$

This expression can be rewritten as

$$\begin{aligned}
&\underbrace{\frac{1}{2}t_{m,1} + t_{m,2}}_{=n(m-2)} + \underbrace{\frac{1}{2}t_{n,1} + t_{n,2}}_{=m(n-2)} + \frac{1}{2}\underbrace{\left[t_{m,1} - t_{m,1}\cap t_{n,1} - t_{m,1}\cap t_{n,2} - t_{m,1}\cap b_n\right]}_{\geq 0} \\
&\qquad + \frac{1}{2}\underbrace{\left[t_{n,1} - t_{m,1}\cap t_{n,1} - t_{m,2}\cap t_{n,1} - b_m\cap t_{n,1}\right]}_{\geq 0} \\
&+ \underbrace{a_m + \frac{1}{2}b_m}_{\geq \frac{1}{2}n(m-2)} + \underbrace{a_n + \frac{1}{2}b_n}_{\geq \frac{1}{2}m(n-2)} + \frac{1}{2}\underbrace{\left[b_m - b_m\cap t_{n,1} - b_m\cap t_{n,2} - b_m\cap b_n\right]}_{\geq 0} \\
&\qquad + \frac{1}{2}\underbrace{\left[b_n - t_{m,1}\cap b_n - t_{m,2}\cap b_n - b_m\cap b_n\right]}_{\geq 0} \\
&+ \frac{1}{2}\underbrace{\left[-t_{m,1}\cap t_{n,2} - t_{m,2}\cap t_{n,1} - t_{m,2}\cap b_n - b_m\cap t_{n,2} - 2t_{m,2}\cap t_{n,2}\right]}_{\geq -mn} \\
&\geq \frac{5}{2}mn - 3(m+n),
\end{aligned}$$

where we have used the preliminary computations of Claims 13, 14 and 15 as well as the inequalities preceding Claim 12. □

In fact, the statement of Theorem 10 can be improved as follows, since in its proof, we have really only counted simplices with a face of type Tr, type A or type B. Furthermore, the same partial estimates can be used to give a lower bound on the number of simplices with a face of type Tr.

Theorem 16. *Let T be any triangulation of $P(m) \times P(n)$, The number of 4-simplices of T with a 3-face of type* Tr*, type A or type B in $\partial\left(P(m) \times P(n)\right)$ is greater or equal to*

$$\frac{5}{2}mn - 3(m+n).$$

Theorem 17. *Let T be a triangulation of $P(m) \times P(n)$. The number of 4-simplices of T with a 3-face of type* Tr *in $\partial(P(m) \times P(n))$ is greater or equal to*

$$\frac{3}{2}mn - 2(m+n).$$

Proof. The number of 4-simplices considered in the statement of the theorem is equal to

$$t_{m,1} + t_{m,2} + t_{n,1} + t_{n,2} - t_{m,1} \cap t_{n,1} - t_{m,1} \cap t_{n,2} - t_{m,2} \cap t_{n,1} - t_{m,2} \cap t_{n,2}.$$

This expression can be rewritten as

$$\begin{aligned}
&\underbrace{\frac{1}{2}t_{m,1} + t_{m,2}}_{=n(m-2)} + \underbrace{\frac{1}{2}t_{n,1} + t_{n,2}}_{=m(n-2)} + \frac{1}{2}\underbrace{\left[t_{m,1} - t_{m,1} \cap t_{n,1} - t_{m,1} \cap t_{n,2}\right]}_{\geq 0} \\
&+ \frac{1}{2}\underbrace{\left[t_{n,1} - t_{m,1} \cap t_{n,1} - t_{m,2} \cap t_{n,1}\right]}_{\geq 0} + \frac{1}{2}\underbrace{\left[-t_{m,1} \cap t_{n,2} - t_{m,2} \cap t_{n,1} - 2t_{m,2} \cap t_{n,2}\right]}_{\geq -mn} \\
&\geq \frac{3}{2}mn - 3(m+n).
\end{aligned}$$

□

5. Lower bounds

Proposition 18. *If σ is a 4-simplex which contains a 3-face of type* Tr*, then*

$$|\mathrm{Vol}_4(\sigma)| \leq \frac{1}{6}.$$

Proof. If σ contains a 3-face of type Tr, then by definition, it contains a 2-simplex of $P(m) \times \{*\}$ or $\{*\} \times P(n)$, so that three of its vertices (the vertices of this 2-simplex) have the same second (respectively first) coordinate, an the proposition then follows from Lemma 5. □

Proposition 19. *If σ is a 4-simplex which contains a 3-face of type A or B, then*

$$|\mathrm{Vol}_4(\sigma)| \leq \frac{1}{3}.$$

In fact, if σ is a 4-simplex which contains a 3-face of type B, then $|\mathrm{Vol}_4(\sigma)| \leq 1/6$.

Proof. By symmetry, suppose without loss of generality that σ has a 3-face of type A_m or B_m. Thus, σ has a 3-face α in some prism $P(m) \times \tau$. Denoting by y, y' the vertices of the edge τ, observe furthermore that α has two vertices in $P(m) \times \{y\}$ and two in $P(m) \times \{y'\}$. The proposition now follows from Lemma 6. □

Proof of Theorem 1. Let T be a triangulation of $P(m) \times P(n)$ and let $[z_T]$ denote the corresponding affine cycle. By Proposition 4, we have

$$\mathrm{Vol}_4([z_T]) = (m-2)(n-2).$$

By Theorem 17, there are at least

$$\frac{3}{2}mn - 2(m+n)$$

4-simplex in T which contain a 3-face of type Tr. Pick $(3/2)mn - 2(m+n)$ of them (rounded down, for simplicity, if m and n are both odd). In view of Proposition 18, they contribute by a combinatorial volume of at most

$$\frac{1}{6}\left(\frac{3}{2}mn - 2(m+n)\right).$$

By Theorem 16, there are at least

$$mn - (m+n)$$

further 4-simplex in T which contain a 3-face of type Tr, type A or type B. Pick $mn - (m+n)$ of them. In view of Propositions 18 and 19, they contribute by a combinatorial volume of at most

$$\frac{1}{3}\left(mn - (m+n)\right).$$

Thus, the remaining volume is at least

$$\begin{aligned}&(m-2)(n-2) - \frac{1}{6}\left(\frac{3}{2}mn - 2(m+n)\right) - \frac{1}{3}\left(mn - (m+n)\right)\\ &= \frac{5}{12}mn - \frac{4}{3}(m+n) + 4.\end{aligned}$$

Since the evaluation of the combinatorial volume cocycle on one 4-simplex is at most $2/3$ (Lemma 8), we need at least

$$\frac{3}{2}\left(\frac{5}{12}mn - \frac{4}{3}(m+n) + 4\right) = \frac{5}{8}mn - 2(n+m) + 6$$

more 4-simplices, so that

$$T(m,n) \geq \frac{5}{2}mn - 3(n+m) + \frac{5}{8}mn - 2(n+m) + 6 = \frac{25}{8}mn - 5(m+n) + 6.$$

□

Proof of Theorem 2. If $n = 4$, then exactly as in the proof of Theorem 1, the combinatorial volume of a triangulation of $P(m) \times P(4)$ is equal to $2(m-2)$. We can pick $4m - 8$ simplices which contain a 3-face of type Tr, which contribute by a combinatorial volume of at most $1/6(4m-8)$. We can further pick $3m-4$ simplices which contain a 3-face of type Tr, type A or type B, which contribute by a combinatorial volume of at most $1/3(3m-4)$. Thus the remaining volume is $1/3(m-4)$.

The difference now is that in view of Lemma 7, the combinatorial volume of a 4-simplex with vertices in the vertices of $P(m) \times P(4)$ is at most $1/2$. Thus, we need at least

$$2 \cdot \frac{1}{3}(m-4)$$

more 4-simplices, so that

$$T(m,4) \geq 4m - 8 + 3m - 4 + \frac{2}{3}m - \frac{8}{3} = \frac{23}{3}m - \frac{44}{3}.$$

□

References

[Bo&al05] L. Bowen, J. A. De Loera, M. Develin, F. Santos, *The Gromov norm of the product of two surfaces*, Topology 44 (2005), 321-339.

[Bu07] M. Bucher-Karlsson. *The simplicial volume of closed manifolds covered by* $\mathbb{H}^2 \times \mathbb{H}^2$. Preprint (2007).

[DeSaTa01] J.A. De Loera, F. Santos, F. Takeuchi, *Extremal properties for dissections of convex 3-polytopes*, SIAM J. Discrete Math. 14 (2) (2001) 143-161.

[SlTaTh88] D.D. Sleator, R.E. Tarjan, W. P. Thurston, *Rotation distance triangulations and hyperbolic geometry*, J. Amer. Math. Soc 1 (1988) 647-679.

[Sm00] W. Smith, *A lower bound for the simplexity of the n-cube via hyperbolic volumes*, Combinatorics of polytopes. European J. Combin. 21 (2000), no. 1, 131-137.

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