GRAPH COVERINGS AND TWISTED OPERATORS

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Abstract. Given a graph and a representation of its fundamental group, there is a naturally associated twisted adjacency operator. The main result of this article is the fact that these operators behave in a controlled way under graph covering maps. When such an operator can be used to enumerate objects, or compute a partition function, this has concrete implications on the corresponding enumeration problem, or statistical mechanics model. For example, we show that if $\widetilde{\Gamma}$ is a finite connected covering graph of a graph Γ endowed with edge-weights $\mathsf{x} = \{\mathsf{x}_\mathsf{e}\}_\mathsf{e}$, then the spanning tree partition function of Γ divides the one of $\widetilde{\Gamma}$ in the ring $\mathbb{Z}[\mathsf{x}]$. Several other consequences are obtained, some known, others new.

1. Introduction

The aim of this article is to present a result of algebraic graph theory, probably known to the experts, in a fairly self-contained and elementary manner. This brings into what we believe to be the correct framework several well-known results in combinatorics, statistical mechanics, and L-function theory, but also provides new ones. In order to preserve its non-technical nature, we focus in the present article on relatively direct consequences, leaving the more elaborate implications to subsequent papers [9, 10].

We now explain our main result in an informal way, referring to Section 2 for precise definitions and background, to Theorem 3.6 for the complete formal statement, and to Section 3.2 for its proof. Given a locally finite weighted graph Γ and a representation ρ of its fundamental group, one can define a *twisted adjacency operator* A^ρ_Γ. Consider a covering map $\widetilde{\Gamma} \to \Gamma$ of finite degree between two connected locally finite graphs. Via this map, the fundamental group $\pi_1(\widetilde{\Gamma})$ embeds into $\pi_1(\Gamma)$. As a consequence, any representation ρ of $\pi_1(\widetilde{\Gamma})$ defines an *induced representation* $\rho^\#$ of $\pi_1(\Gamma)$. Our main result is the fact that the operator $\mathsf{A}^\rho_{\widetilde{\Gamma}}$ is conjugate to $\mathsf{A}^{\rho^\#}_\Gamma$.

Let us mention that the existence of an isomorphism between the relevant vector spaces can be understood as a chain-complex version of the so-called *Eckmann-Shapiro lemma*, originally stated in group cohomology (see Remark 3.7). The interesting part of Theorem 3.6, which we have not been able to find in the literature, is the fact that the explicited isomorphism conjugates the aptly defined twisted adjacency operators.

As an immediate consequence of this result, we see that the decomposition of $\rho^\#$ into irreducible representations leads to a direct sum decomposition of $\mathsf{A}^{\rho^\#}_\Gamma$, and therefore of $\mathsf{A}^\rho_{\widetilde{\Gamma}}$. For example, if ρ is taken to be the trivial representation, we readily obtain the fact that A_Γ is a direct summand of $\mathsf{A}_{\widetilde{\Gamma}}$, see Corollary 3.8. (Here, the absence of superscript means that these operators are not twisted, or twisted by the trivial representation.) Furthermore, if

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the covering is normal, then $\mathsf{A}_{\widetilde{\Gamma}}$ factors as a direct sum of the operators A_Γ^ρ twisted by the irreducible representations of the Galois group of the covering, see Corollary 3.9.

Whenever A_Γ^ρ can be used to enumerate combinatorial objects in Γ, or in an associated graph G, these statements have very concrete combinatorial implications. More generally, if these operators can be used to compute some partition functions of the weighted graph (Γ, x), or of an associated weighted graph (G, x), these results have often non-trivial consequences on the corresponding models. Several of these implications are well-known, but others are new. We now state some of them, referring to Section 4 for details.

There is an obvious place to start, namely the *matrix-tree theorem*: the Laplacian Δ_G allows to enumerate spanning trees (STs) and rooted spanning forests (RSFs) in G. More generally, if $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ is endowed with edge-weights $\mathsf{x} = \{\mathsf{x}_\mathsf{e}\}_{\mathsf{e}\in\mathsf{E}}$, then it allows to compute the corresponding partition functions $\mathsf{Z}_{ST}(\mathsf{G}, \mathsf{x})$ and $\mathsf{Z}_{RSF}(\mathsf{G}, \mathsf{x})$, which can be thought of as elements of the polynomial ring $\mathbb{Z}[\mathsf{x}] = \mathbb{Z}[\{\mathsf{x}_\mathsf{e}\}_{\mathsf{e}\in\mathsf{E}}]$. Applying Corollary 3.8 to the Laplacian $\mathsf{A}_\Gamma = \Delta_\mathsf{G}$, we obtain the following result: if $\widetilde{\mathsf{G}}$ is a finite covering graph of a finite connected graph G endowed with edge-weights x, and if $\tilde{\mathsf{x}}$ denotes these weights lifted to the edges of $\widetilde{\mathsf{G}}$, then $\mathsf{Z}_{ST}(\mathsf{G}, \mathsf{x})$ divides $\mathsf{Z}_{ST}(\widetilde{\mathsf{G}}, \tilde{\mathsf{x}})$ and $\mathsf{Z}_{RSF}(\mathsf{G}, \mathsf{x})$ divides $\mathsf{Z}_{RSF}(\widetilde{\mathsf{G}}, \tilde{\mathsf{x}})$ in the ring $\mathbb{Z}[\mathsf{x}]$. As an immediate consequence, the number of spanning trees in G divides the number of spanning trees in $\widetilde{\mathsf{G}}$ (a fact first proved by Berman [4] using a different method), and similarly for rooted spanning forests (to the best of our knowledge, a new result).

Another interesting class of operators is given by the weighted skew-adjacency matrices defined by Kasteleyn [20, 21] in his study of the *dimer model* on surface graphs. For this model, Corollary 3.8 can only be applied to cyclic coverings, yielding a known result [15, 25]. Applying Corollary 3.9 to the case of a graph embedded in the torus yields an immediate proof of the classical fact that the dimer characteristic polynomial behaves multiplicatively under so-called *enlargement of the fundamental domain* [24, Theorem 3.3]. However, applying our results to the study of the dimer model on graphs embedded in the Klein bottle leads to new powerful results, that are harnessed in the parallel article [9].

Let us finally mention that our main result can be interpreted as the fact that the operators A_Γ^ρ satisfy the so-called *Artin formalism*, a set of axioms originating from the study of L-series of Galois field extensions [2, 3]. As a consequence, we obtain several results on the associated L-series $L(\Gamma, \mathsf{x}, \rho) = \det(\mathrm{I} - \mathsf{A}_\Gamma^\rho)^{-1}$, providing a wide generalization of the results of Stark and Terras [33, 34], see Section 3.3.

We conclude this introduction with one final remark. There are two ways to consider graphs: either as combinatorial objects, or as topological ones (namely 1-dimensional CW-complexes). Hence, there are two corresponding ways to define and study the associated fundamental groups and covering maps. In our pursuit of simplicity, we have chosen the combinatorial one. As a result, we provide the reader with a brief and dry but self-contained treatment of the required parts of algebraic topology translated from the topological to the combinatorial category, see Sections 2.1–2.3.

This paper is organised as follows. Section 2 deals with the necessary background material and claims no originality: we start from scratch with graphs, their fundamental groups and covering maps, before moving on to connections on graphs, and basics of representation theory of groups. Section 3 contains the definition of the twisted operators, our main result with its proof and corollaries, together with the analogy with algebraic number theory via the Artin formalism. Finally, Section 4 deals with the aforementioned combinatorial applications.

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2. Background on graphs and representations

In this section, we first recall the combinatorial definitions of the fundamental group of a graph and of graph coverings, adapting the standard references [32] and [14] to our purposes, see also [26]. We then proceed with connections on graph vector bundles following [22], and linear representations of groups following [31].

2.1. Graphs and their fundamental groups.

This first paragraph deals with the elementary concepts of graph and directed graph. Since there is no universal agreement on the relevant terminology and notation, we record here these formal definitions following [32].

Definition 2.1. A *directed graph* (or *digraph*) Γ consists of a set V of *vertices*, a set D of *(directed) edges*, together with maps $s,t\colon \mathsf{D}\to\mathsf{V}$ assigning to each edge $e\in\mathsf{D}$ its *source* vertex $s(e)\in\mathsf{V}$ and its *target* vertex $t(e)\in\mathsf{V}$.

A *graph* Γ consists of sets V,D and maps $s,t\colon\mathsf{D}\to\mathsf{V}$ as above, together with an involution of D assigning to each edge $e\in\mathsf{D}$ its *inverse* $\overline{e}\in\mathsf{D}$ such that $\overline{e}\neq e$ and $s(\overline{e})=t(e)$. We let $\mathsf{E}=\mathsf{D}/(e\sim\overline{e})$ denote the set of *unoriented edges*, and write $\mathsf{e}\in\mathsf{E}$ for the unoriented edge corresponding to $e,\overline{e}\in\mathsf{D}$.

A (directed) graph is *locally finite* if for all $v\in\mathsf{V}$, the sets $\mathsf{D}_v=\{e\in\mathsf{D}\,|\,s(e)=v\}$ and $\mathsf{D}^v=\{e\in\mathsf{D}\,|\,t(e)=v\}$ are finite. It is called *finite* if both sets V and D are finite.

Note that these graphs are not simple in general: we allow multiple edges and loops. Note also that in this formalism, graphs are special types of directed graphs. Moreover, given a directed graph Γ, one can build an associated graph (still denoted by Γ) by formally adding an inverse $\overline{e}$ to each edge $e\in\mathsf{D}$.

Let us fix a directed graph Γ. A *path of length* $n\geq 1$ is a sequence $\gamma=(e_1,e_2,\dots,e_n)$ of edges such that $t(e_i)=s(e_{i+1})$ for all $i\in\{1,\dots,n-1\}$. We shall write $s(\gamma)=s(e_1)$ and $t(\gamma)=t(e_n)$ for the source and target of γ, respectively. A *path of length* 0, or *constant path* γ, is given by a vertex, which is both the source and target of γ. A *loop* (based at v) is a path γ with $s(\gamma)=t(\gamma)=v$.

The directed graph Γ is said to be *connected* if for any $v,w\in\mathsf{V}$, there is a path γ with $s(\gamma)=v$ and $t(\gamma)=w$.

2.2. The fundamental group of a graph.

Let us now assume that Γ is a graph, and fix a vertex $v\in\mathsf{V}$.

Note that the set of loops based at v is a monoid with respect to the concatenation of paths, with neutral element 1 given by the constant path based at v. Let us call two loops based at v (or more generally, two paths with same source and same target) *homotopic* if one can be obtained from the other by removing or adding loops of the form $(e,\overline{e})$ along the path. Then, the set of homotopy classes of loops based at v forms a group, with the inverse of $\gamma=(e_1,\dots,e_n)$ given by $\overline{\gamma}=(\overline{e}_n,\dots,\overline{e}_1)$.

Definition 2.2. This group is the *fundamental group of the graph* Γ *based at* v, and is denoted by $\pi_1(\Gamma,v)$.

If Γ is connected, then the isomorphism class of $\pi_1(\Gamma, v)$ is easily seen not to depend on the base vertex v.

By a slight abuse of terminology, we define the *fundamental group of a directed graph* Γ as the fundamental group of the associated graph obtained by adding an inverse to each edge of Γ.

We will make use of the alternative definition of the fundamental group, based on a spanning tree. Recall that a *circuit* (of length $n \geq 1$) is a loop $\gamma = (e_1, \dots, e_n)$ such that $e_{i+1} \neq \overline{e}_i$ for $i \in \{1, \dots, n-1\}$, $e_1 \neq \overline{e}_n$, and such that the vertices $t(e_1), \dots, t(e_n)$ are all distinct. A *spanning tree* of Γ is a connected non-empty subgraph $T \subset \Gamma$ without circuit, such that the vertices of T coincide with the vertices of Γ. Note that the number of vertices and edges in a finite tree satisfy $|\mathsf{V}(T)| - |\mathsf{E}(T)| = 1$.

The *fundamental group of the graph* Γ *based at* T, denoted by $\pi_1(\Gamma, T)$, is defined as the quotient of the free group over D by the relations $\overline{e} = e^{-1}$ for all edges of Γ, and $e = 1$ for all edges of T. If Γ is connected, then it admits a spanning tree, and the groups $\pi_1(\Gamma, v)$ and $\pi_1(\Gamma, T)$ are easily seen to be isomorphic for all $v \in \mathsf{V}$ and all spanning trees T of Γ. As a consequence, if Γ is connected and finite, its fundamental group is free of rank $|\mathsf{E}| - |\mathsf{V}| + 1$.

2.3. **Covering graphs.** A *morphism of digraphs* p from $\widetilde{\Gamma} = (\widetilde{\mathsf{V}}, \widetilde{\mathsf{D}}, \tilde{s}, \tilde{t})$ to $\Gamma = (\mathsf{V}, \mathsf{D}, s, t)$ consists of two maps $p_0 : \widetilde{\mathsf{V}} \to \mathsf{V}$ and $p_1 : \widetilde{\mathsf{D}} \to \mathsf{D}$ such that $s \circ p_1 = p_0 \circ \tilde{s}$ and $t \circ p_1 = p_0 \circ \tilde{t}$. A *morphism of graphs* $p \colon \widetilde{\Gamma} \to \Gamma$ is a morphism of digraphs which also satisfies the equality $p_1(\overline{e}) = \overline{p_1(e)}$ for all $e \in \widetilde{\mathsf{D}}$.

As one easily checks, a morphism of graphs $p \colon \widetilde{\Gamma} \to \Gamma$ induces in the obvious way a homomorphism of groups $p_* \colon \pi_1(\widetilde{\Gamma}, v) \to \pi_1(\Gamma, p(v))$.

Definition 2.3. A *covering map* is a morphism of directed graphs $p \colon \widetilde{\Gamma} \to \Gamma$ with $p_0 \colon \widetilde{\mathsf{V}} \to \mathsf{V}$ surjective, such that for all $\tilde{v} \in \widetilde{\mathsf{V}}$, the restriction of p_1 defines bijections $\widetilde{\mathsf{D}}_{\tilde{v}} \to \mathsf{D}_{p(\tilde{v})}$ and $\widetilde{\mathsf{D}}^{\tilde{v}} \to \mathsf{D}^{p(\tilde{v})}$. In that case, $\widetilde{\Gamma}$ is called a *covering digraph* of Γ.

If Γ is a connected digraph, then the fibers $p_0^{-1}(v)$ and $p_1^{-1}(e)$ have the same cardinality for all $v \in \mathsf{V}$ and $e \in \mathsf{D}$. This is called the *degree* of the covering. From now on, we will drop the subscripts in p_0 and p_1 and denote both maps by p.

Note that any morphism of digraphs $p \colon \widetilde{\Gamma} \to \Gamma$ extends to a unique morphism between the associated graphs (obtained by adding an inverse to each edge). Moreover, if the morphism of digraphs is a covering map, then so is the associated morphism of graphs. In such a case, the graph $\widetilde{\Gamma}$ is called a *covering graph* of Γ.

The easy proof of the following lemma is left to the reader.

Lemma 2.4. *If* $p \colon \widetilde{\Gamma} \to \Gamma$ *is a covering map, then:*

(i) the homomorphism $p_* \colon \pi_1(\widetilde{\Gamma}, v) \to \pi_1(\Gamma, p(v))$ *is injective;*

(ii) for any $v \in \mathsf{V}$*, the following subsets of* $\widetilde{\mathsf{V}}$ *coincide:*

$$\{t(\tilde{e}) \in \widetilde{\mathsf{V}} \,|\, \tilde{e} \in \widetilde{\mathsf{D}}_{\tilde{v}} \text{ with } \tilde{v} \in p^{-1}(v)\} = \{\tilde{w} \in \widetilde{\mathsf{V}} \,|\, \text{there exists } e \in \mathsf{D}_v \text{ with } t(e) = p(\tilde{w})\}\,.$$

The following *path lifting property* is a direct consequence of the definitions, but nevertheless a fundamental feature of a covering map $p \colon \widetilde{\Gamma} \to \Gamma$. Given any path γ in Γ with $s(\gamma) = v_0$ and any $\tilde{v}_0 \in p^{-1}(v_0)$, there is a unique path $\tilde{\gamma}$ in $\widetilde{\Gamma}$ with $p(\tilde{\gamma}) = \gamma$ and $s(\tilde{\gamma}) = \tilde{v}_0$. Furthermore, the formula $[\gamma]\cdot\tilde{v}_0 = t(\tilde{\gamma})$ defines an action of $\pi_1(\Gamma, v_0)$ on $p^{-1}(v_0)$. If $\widetilde{\Gamma}$ is connected, then this action is easily seen to be transitive, with isotropy group of $\tilde{v}_0$ equal to $p_*(\pi_1(\widetilde{\Gamma}, \tilde{v}_0))$. As a consequence, the degree of the covering coincides with the index of $p_*(\pi_1(\widetilde{\Gamma}, \tilde{v}_0))$ in $\pi_1(\Gamma, v_0)$.

Let us finally recall that a covering map $p\colon \widetilde{\Gamma} \to \Gamma$ is said to be *normal* (or *regular*) if $p_*(\pi_1(\widetilde{\Gamma}, v))$ is a normal subgroup of $\pi_1(\Gamma, p(v))$. In such a case, we denote the quotient group by $G(\widetilde{\Gamma}/\Gamma)$. This is nothing but the group of covering transformations of this covering map, usually referred to as the *Galois group*.

2.4. **Connections on graphs.** Following [22, Section 3.1], let us fix a *vector bundle* on a graph Γ, i.e. a vector space W and the choice of a vector space W_v isomorphic to W for each $v \in \mathsf{V}$. Such a vector bundle can be identified with $W_\Gamma := \bigoplus_{v\in\mathsf{V}} W_v \simeq W^{\mathsf{V}}$.

Definition 2.5. A *connection* on a vector bundle W_Γ is the choice $\Phi = (\varphi_e)_{e\in\mathsf{D}}$ of an isomorphism $\varphi_e\colon W_{t(e)} \to W_{s(e)}$ for each $e \in \mathsf{D}$, such that $\varphi_{\overline{e}} = \varphi_e^{-1}$ for all $e \in \mathsf{D}$.

Two connections $\Phi = (\varphi_e)_{e\in\mathsf{D}}$ and $\Phi' = (\varphi'_e)_{e\in\mathsf{D}}$ are said to be *gauge-equivalent* if there is a family of automorphisms $\{\psi_v\colon W_v \to W_v\}_{v\in\mathsf{V}}$ such that $\psi_{s(e)} \circ \varphi_e = \varphi'_e \circ \psi_{t(e)}$ for all $e \in \mathsf{D}$.

Let us fix a base vertex $v_0 \in \mathsf{V}$ and a connection Φ on a vector bundle W_Γ. Any loop $\gamma = (e_1, \dots, e_n)$ based at v_0 gives an automorphism $\varphi_{e_1} \circ \dots \circ \varphi_{e_n}$ of $W_{v_0} =: W$ called the *monodromy* of γ. This construction defines a homomorphism

$$\rho^\Phi\colon \pi_1(\Gamma, v_0) \longrightarrow \mathrm{GL}(W)\,,$$

i.e. a representation of the fundamental group of Γ in W.

Any representation $\rho\colon \pi_1(\Gamma, v_0) \to \mathrm{GL}(W)$ is of the form ρ^Φ for some connection Φ: indeed, one can fix a spanning tree $T \subset \Gamma$ (recall that $\pi_1(\Gamma, v_0) \simeq \pi_1(\Gamma, T)$), set $\varphi_e = \mathrm{id}_W$ for each edge of T and $\varphi_e = \rho_e$ for each of the remaining edges of Γ. Furthermore, given two connections Φ and Φ' on W_Γ, one easily checks that ρ^Φ and $\rho^{\Phi'}$ are conjugate representations if and only if Φ and Φ' are gauge-equivalent connections.

In other words, the $\mathrm{GL}(W)$-*character variety* of $\pi_1(\Gamma, v_0)$, i.e. the set of conjugation classes of homomorphisms $\pi_1(\Gamma, v_0) \to \mathrm{GL}(W)$, is given by the set of connections on W_Γ up to gauge-equivalence.

Remark 2.6. The definition of a connection as isomorphisms $\varphi_e\colon W_{s(e)} \to W_{t(e)}$ seems more natural, but leads to antihomomorphisms of $\pi_1(\Gamma, v_0)$. On the other hand, our convention yields homomorphisms, and is coherent with the definition of a local coefficient system for twisted homology, see e.g. [36, p. 255].

2.5. **Linear representations of groups.** We now recall the necessary notation and terminology of linear representations of groups, following [31]. Throughout this subsection, G denotes a group.

Let us first recall that the *degree* of a representation $\rho\colon G \to \mathrm{GL}(W)$, denoted by $\deg(\rho)$, is defined as the dimension of W, which we always assume to be finite. The only representation of degree 0 is written $\rho = 0$, while the degree 1 representation sending all elements of G to $1 \in \mathbb{C}^* = \mathrm{GL}(\mathbb{C})$ is denoted by $\rho = 1$.

Let us now fix two linear representations $\rho\colon G \to \mathrm{GL}(W)$ and $\rho'\colon G \to \mathrm{GL}(W')$. The *direct sum* of ρ and ρ' is the representation $\rho \oplus \rho'\colon G \to \mathrm{GL}(W \oplus W')$ given by $(\rho \oplus \rho')_g = \rho_g \oplus \rho_{g'}$. A representation of G is said to be *irreducible* if it is not the direct sum of two representations that are both not 0.

Now, fix a subgroup $H < G$ of finite index, and a representation $\rho\colon H \to \mathrm{GL}(W)$. There is a representation $\rho^{\#}\colon G \to \mathrm{GL}(Z)$ which is uniquely determined up to isomorphism by the following two properties. Let $R \subset G$ denote a *set of representatives* of G/H, i.e. each $g \in G$ can be written uniquely as $g = rh \in G$ with $r \in R$ and $h \in H$.

(i) We have $Z = \bigoplus_{r\in R} \rho^{\#}_r(W)$.
(ii) For any $g \in G$ and $w \in W$, we have $\rho^{\#}_g(\rho^{\#}_r(w)) = \rho^{\#}_{r'}(\rho_h(w))$ where $gr = r'h \in G$ with $r' \in R$ and $h \in H$.

This representation $\rho^{\#}\colon G \to \mathrm{GL}(Z)$ is said to be *induced* by $\rho\colon H \to \mathrm{GL}(W)$.

3. Twisted operators on graph coverings

This section contains the proof of our main result, Theorem 3.6, which relates twisted adjacency operators on directed graphs connected by a covering map. We start in Section 3.1 by defining the relevant twisted operators, while Section 3.2 deals with Theorem 3.6, its proof, and a couple of corollaries. Finally, Section 3.3 shows how this result can be interpreted as a combinatorial version of the Artin formalism for these operators, yielding consequences on associated L-series.

3.1. **Twisted weighted adjacency operators.** Fix a locally finite directed graph $\Gamma = (\mathsf{V}, \mathsf{D}, s, t)$. Let us assume that it is endowed with *edge-weights*, i.e. a collection $x = \{x_e\}_{e\in\mathsf{D}}$ of complex numbers attached to the edges. The associated *weighted adjacency operator* A_Γ acts on $\mathbb{C}^{\mathsf{V}}$ via

$$(\mathsf{A}_\Gamma f)(v) = \sum_{e\in\mathsf{D}_v} x_e\, f(t(e)) \quad \text{for all } f \in W^{\mathsf{V}} \text{ and } v \in \mathsf{V}\,.$$

Adapting [22, Section 3.2] to our purposes, this operator can be twisted by a representation $\rho\colon \pi_1(\Gamma, v_0) \to \mathrm{GL}(W)$ in the following way. Fix a vector bundle $W_\Gamma \simeq W^{\mathsf{V}}$ and a connection $\Phi = (\varphi_e)_{e\in\mathsf{D}}$ representing ρ.

Definition 3.1. The associated *twisted weighted adjacency operator* A^ρ_Γ is the operator on W^{V} given by

$$(\mathsf{A}^\rho_\Gamma f)(v) = \sum_{e\in\mathsf{D}_v} x_e\, \varphi_e(f(t(e))) \quad \text{for all } f \in W^{\mathsf{V}} \text{ and } v \in \mathsf{V}\,.$$

Several remarks are in order.

Remark 3.2. (i) By Section 2.4, conjugate representations are given by gauge equivalent connections. Furthermore, the corresponding twisted operators are conjugated by an automorphism of W^{V}. Therefore, the conjugacy class of A^ρ_Γ only depends on the conjugacy class of ρ.
(ii) If a representation ρ is given by the direct sum of ρ_1 and ρ_2, then the operator A^ρ_Γ is conjugate to $\mathsf{A}^{\rho_1}_\Gamma \oplus \mathsf{A}^{\rho_2}_\Gamma$.
(iii) The operator A^{1}_Γ is nothing but the untwisted operator A_Γ.

Obviously, the untwisted operator A_Γ is uniquely associated to a directed graph Γ, so our setting may seem quite restrictive. Nevertheless, there are many natural assignments $\mathsf{G} \mapsto \Gamma$ mapping a directed graph G to another directed graph Γ so that A_Γ provides a new operator on G. Moreover, if there is a natural homomorphism $\alpha\colon \pi_1(\Gamma) \to \pi_1(\mathsf{G})$, then a ρ-twisted version of this new operator can be understood as $\mathsf{A}_\Gamma^{\alpha\circ\rho}$. Finally, if the assignment $\mathsf{G}\mapsto\Gamma$ preserves covering maps, then our results apply to these new twisted operators as well.

We now give three explicit examples of such natural maps $\mathsf{G}\mapsto\Gamma$, claiming no exhaustivity. It is easy indeed to find additional interesting ones, e.g. the *Fisher correspondance* used in the study of the Ising model [12].

Example 3.3. Let $\mathsf{G}=(\mathsf{V}(\mathsf{G}),\mathsf{E}(\mathsf{G}))$ be a graph endowed with *symmetric* edge-weights, i.e. labels $\mathsf{x}=(\mathsf{x}_\mathsf{e})_{\mathsf{e}\in\mathsf{E}(\mathsf{G})}$ associated to its unoriented edges. Consider the associated graph $\Gamma=(\mathsf{V},\mathsf{E})$ defined by $\mathsf{V}=\mathsf{V}(\mathsf{G})$ and $\mathsf{E}=\mathsf{E}(\mathsf{G})\cup\mathsf{V}(\mathsf{G})$, with source, target, and involution maps of Γ given by extending the ones of G via $\overline{v}=v$ and $s(v)=t(v)=v$ for all $v\in\mathsf{V}$. (Concretely, the graph Γ is obtained from G by adding a loop at each vertex.) Also, extend the edge-weights on G to symmetric edge-weights on Γ via $\mathsf{x}_v=-\sum_{e\in\mathsf{D}(\mathsf{G})_v}\mathsf{x}_e$. Then, the corresponding weighted adjacency operator A_Γ is (the opposite of) the *Laplacian* Δ_G on G. It can be used to count spanning trees of G – this is the celebrated *matrix-tree theorem* – but also rooted spanning forests, see Section 4.1.

Note that there is a natural homomorphism $\alpha\colon\pi_1(\Gamma,v_0)\to\pi_1(\mathsf{G},v_0)$ mapping all the newly introduced loops to the neutral element. Given any representation ρ of $\pi_1(\mathsf{G},v_0)$, the associated twisted operator $\mathsf{A}_\Gamma^{\alpha\circ\rho}$ is the *vector bundle Laplacian* Δ_G^ρ of [22]. When the representation ρ takes values in $\mathbb{C}^*$ or $\mathrm{SL}_2(\mathbb{C})$, then Δ_G^ρ can be used to study cycle-rooted spanning forests [13, 22], while representations of higher degree yield more involved combinatorial objects [19].

Example 3.4. Let Γ be a graph endowed with symmetric edge-weights $\mathsf{x}=(\mathsf{x}_\mathsf{e})_{\mathsf{e}\in\mathsf{E}}$. Fix an orientation of the edges of Γ and consider the same graph Γ endowed with the anti-symmetric edge-weights $x=\{x_e\}_{e\in\mathsf{D}}$ given by $x_e=\mathsf{x}_\mathsf{e}$ if the orientation of $e\in\mathsf{D}$ agrees with the fixed orientation, and $x_e=-\mathsf{x}_\mathsf{e}$ else. Then, the operator A_Γ is a weighted skew-adjacency operator that has been used by Kasteleyn [20, 21] and many others in the study of the 2-dimensional dimer and Ising models, see Section 4.2. Such operators twisted by $\mathrm{SL}_2(\mathbb{C})$-representations are also considered by Kenyon in his study of the double-dimer model [23].

Example 3.5. Let us start with a graph $\mathsf{G}=(\mathsf{V}(\mathsf{G}),\mathsf{D}(\mathsf{G}),s_\mathsf{G},t_\mathsf{G},i)$ endowed with symmetric edge-weights $\mathsf{x}=\{\mathsf{x}_\mathsf{e}\}_\mathsf{e}$, and consider the associated *directed line graph* $\Gamma=(\mathsf{V},\mathsf{D},s,t)$ defined by

$$\mathsf{V}=\mathsf{D}(\mathsf{G}),\quad \mathsf{D}=\{(e,e')\in\mathsf{V}\times\mathsf{V}\,|\,t_\mathsf{G}(e)=s_\mathsf{G}(e')\text{ but }e'\neq\overline{e}\},\quad s(e,e')=e,\quad t(e,e')=e',$$

and endowed with the edge-weights $x=\{x_{e,e'}\}_{(e,e')\in\mathsf{D}}$ defined by $x_{e,e'}=\mathsf{x}_\mathsf{e}$. Then, the operator $\mathrm{I}-\mathsf{A}_\Gamma$ is considered by Stark and Terras [33, 34] in their study of prime cycles (see Section 3.3), while a similar operator is defined by Kac and Ward [17] in their exploration of the planar Ising model (see Section 4.2). Note also that there is a natural homomomorphism $\alpha\colon\pi_1(\Gamma,e_0)\to\pi_1(\mathsf{G},s(e_0))$, so any representation $\rho\colon\pi_1(\mathsf{G},v_0)\to\mathrm{GL}(W)$ defines a twisted operator $\mathrm{I}-\mathsf{A}_\Gamma^{\rho\circ\alpha}$.

3.2. The main result.

We are finally ready to state and prove our main theorem.

Let $\Gamma=(\mathsf{V},\mathsf{D},s,t)$ be a locally finite connected directed graph with weights $x=(x_e)_{e\in\mathsf{D}}$, and let $p\colon\widetilde{\Gamma}\to\Gamma$ be a covering map of finite degree d, with $\widetilde{\Gamma}=(\widetilde{\mathsf{V}},\widetilde{\mathsf{D}},s,t)$ connected. The weights x on Γ lift to weights $\tilde{x}$ on $\widetilde{\Gamma}$ via $\tilde{x}_{\tilde{e}}:=x_{p(\tilde{e})}$ for all $\tilde{e}\in\widetilde{\mathsf{D}}$, so $\widetilde{\Gamma}$ is a weighted directed graph, which is locally finite.

Fix base vertices $v_0\in\mathsf{V}$ and $\tilde{v}_0\in p^{-1}(v_0)$, and recall from Lemma 2.4 that p induces an injection $p_*\colon\pi_1(\widetilde{\Gamma},\tilde{v}_0)\to\pi_1(\Gamma,v_0)$ between the fundamental groups of the associated graphs, so $\pi_1(\widetilde{\Gamma},\tilde{v}_0)$ can be considered as a subgroup of $\pi_1(\Gamma,v_0)$ of index d. Therefore, as explained in Section 2.5, any representation $\rho\colon \pi_1(\widetilde{\Gamma},\tilde{v}_0)\to \mathrm{GL}(W)$ induces a representation $\rho^{\#}\colon \pi_1(\Gamma,v_0)\to\mathrm{GL}(Z)$.

Theorem 3.6. *For any covering map of connected directed graphs $p\colon\widetilde{\Gamma}\to\Gamma$ as above and any representation $\rho\colon \pi_1(\widetilde{\Gamma},\tilde{v}_0)\to\mathrm{GL}(W)$, there is an isomorphism $\psi\colon W^{\widetilde{\mathsf{V}}}\to Z^{\mathsf{V}}$ such that $\psi\circ\mathsf{A}^{\rho}_{\widetilde{\Gamma}}=\mathsf{A}^{\rho^{\#}}_{\Gamma}\circ\psi$.*

Remark 3.7. The existence of an isomorphism $W^{\widetilde{\mathsf{V}}}\simeq Z^{\mathsf{V}}$ is a chain-complex version of the *Eckmann-Shapiro Lemma*, traditionally stated in the context of group (co)homology (see e.g. [6, p. 73]). Moreover, the tensor-product definition of the induced representation (see [31, Chapter 7]) makes the existence of this isomorphism a routine check. The interesting part of Theorem 3.6 is the explicit form of this isomorphism in this setting, which turns out to conjugate the relevant twisted adjacency operators.

Before giving the proof of Theorem 3.6, we present a couple of consequences.

Corollary 3.8. *If $\widetilde{\Gamma}$ is a connected covering digraph of Γ of finite degree, then $\mathsf{A}_{\widetilde{\Gamma}}$ is conjugate to $\mathsf{A}_\Gamma\oplus\mathsf{A}^{\rho'}_\Gamma$ for some representation ρ' of $\pi_1(\Gamma,v_0)$.*

Proof. Applying Theorem 3.6 to the trivial representation $\rho=1$ of $\pi_1(\widetilde{\Gamma},\tilde{v}_0)=:H$, we get that $\mathsf{A}^{\rho}_{\widetilde{\Gamma}}=\mathsf{A}_{\widetilde{\Gamma}}$ is conjugate to $\mathsf{A}^{\rho^{\#}}_{\Gamma}$, with $\rho^{\#}$ the induced representation of $\pi_1(\Gamma,v_0)=:G$. By definition, this representation is nothing but the action of G on the vector space Z with basis G/H, an action given by left multiplication of G on G/H. Since G acts by permutation on the set G/H, which is finite, the subspace of Z generated by the sum of these basis elements is fixed by this action. Therefore, the induced representation splits as $\rho^{\#}=1\oplus\rho'$ for some representation ρ' of $\pi_1(\Gamma,v_0)$. The statement now follows from the second and third points of Remark 3.2. □

Corollary 3.9. *If $\widetilde{\Gamma}\to\Gamma$ is a normal covering map of finite degree with $\widetilde{\Gamma}$ connected, then $\mathsf{A}_{\widetilde{\Gamma}}$ is conjugate to*

$$\bigoplus_{\rho\ \textit{irred.}}\left(\mathsf{A}^{\rho\circ\mathrm{pr}}_{\Gamma}\right)^{\oplus\deg(\rho)}\,,$$

where the direct sum is over all irreducible representations of $G(\widetilde{\Gamma}/\Gamma)$, and pr *stands for the canonical projection of $\pi_1(\Gamma,v_0)$ onto $\pi_1(\Gamma,v_0)/p_*(\pi_1(\widetilde{\Gamma},\tilde{v}_0))=G(\widetilde{\Gamma}/\Gamma)$.*

Proof. Applying Theorem 3.6 to the trivial representation $\rho=1$ of $\pi_1(\widetilde{\Gamma},\tilde{v}_0)$, we see that the induced representation can be written as $\rho^{\#}=\rho_{\mathrm{reg}}\circ\mathrm{pr}$, with pr as above and ρ_{reg} the so-called *regular representation* of $G(\widetilde{\Gamma}/\Gamma)$. Since this group is finite, this representation splits as

$$\rho_{\mathrm{reg}}=\bigoplus_{\rho\ \mathrm{irred.}}\rho^{\oplus\deg(\rho)}\,,$$

the sum being over all irreducible representations of $G(\widetilde{\Gamma}/\Gamma)$ (see [31, Section 2.4]). The statement now follows from the second point of Remark 3.2. □

Proof of Theorem 3.6. Let $p\colon\widetilde{\Gamma}\to\Gamma$ be a covering map of degree d sending the base vertex $\tilde{v}_0$ of $\widetilde{\Gamma}$ to the base vertex v_0 of Γ, with $\Gamma=(\mathsf{V},\mathsf{D},s,t)$ a locally finite and connected directed graph endowed with edge-weights $x=(x_e)_{e\in\mathsf{D}}$, and $\widetilde{\Gamma}=(\widetilde{\mathsf{V}},\widetilde{\mathsf{D}},s,t)$ a (locally finite) connected directed graph endowed with the lifted edge-weights $\tilde{x}=(\tilde{x}_{\tilde{e}})_{\tilde{e}\in\widetilde{\mathsf{D}}}$ defined by $\tilde{x}_{\tilde{e}}=x_{p(\tilde{e})}$. As always, we use the same notation $\widetilde{\Gamma},\Gamma$ for the directed graphs and for the associated graphs.

Let $\rho\colon\pi_1(\widetilde{\Gamma},\tilde{v}_0)\to\mathrm{GL}(W)$ be a representation, and let $\rho^{\#}\colon\pi_1(\Gamma,v_0)\to\mathrm{GL}(Z)$ be the induced representation. The aim is to find connections $\widetilde{\Phi}$ on $W^{\widetilde{\mathsf{V}}}$ and Φ on Z^{V} such that $\rho^{\widetilde{\Phi}}=\rho$ and $\rho^{\Phi}=\rho^{\#}$, together with a natural isomorphism $\psi\colon W^{\widetilde{\mathsf{V}}}\to Z^{\mathsf{V}}$ such that the following diagram commutes:

$$\begin{array}{ccc} W^{\widetilde{\mathsf{V}}} & \xrightarrow{\ \psi\ } & Z^{\mathsf{V}} \\ {\scriptstyle \mathsf{A}_{\widetilde{\Gamma}}^{\widetilde{\Phi}}}\downarrow & & \downarrow{\scriptstyle \mathsf{A}_{\Gamma}^{\Phi}} \\ W^{\widetilde{\mathsf{V}}} & \xrightarrow{\ \psi\ } & Z^{\mathsf{V}}. \end{array}$$

Note that here, we use the more precise notation A_Γ^{Φ} (instead of $\mathsf{A}_\Gamma^{\rho^{\#}}$) for the operator A_Γ twisted by the connection Φ, and similarly for $\mathsf{A}_{\widetilde{\Gamma}}^{\widetilde{\Phi}}$ instead of $\mathsf{A}_{\widetilde{\Gamma}}^{\rho}$.

Since Γ is connected, it contains a spanning tree T. Note that for any $v\in\mathsf{V}$, there is a path γ_v in T from v_0 to v which is unique up to homotopy. Set $\Phi=(\varphi_e)_{e\in\mathsf{D}}$ with $\varphi_e=\rho^{\#}_{[e]}\in\mathrm{GL}(Z)$ and $[e]\in\pi_1(\Gamma,v_0)$ defined by

$$[e]=[\gamma_{s(e)}\,e\,\overline{\gamma}_{t(e)}]\,,$$

where $\overline{\gamma}$ denotes the inverse of the path γ. This is illustrated in Figure 1. The lift $p^{-1}(T)$ of T is a subgraph of $\widetilde{\Gamma}$ which does not contain any circuit and spans all vertices of $\widetilde{\Gamma}$, i.e. it is a *spanning forest*. Since $\widetilde{\Gamma}$ is connected, this spanning forest can be completed to a spanning tree $\widetilde{T}$ of $\widetilde{\Gamma}$. Here again, for any $\tilde{v}\in\widetilde{\mathsf{V}}$, there is a path $\gamma_{\tilde{v}}$ in $\widetilde{T}$ from $\tilde{v}_0$ to $\tilde{v}$ which is unique up to homotopy. Set $\tilde{\varphi}_{\tilde{e}}=\rho_{[\tilde{e}]}\in\mathrm{GL}(W)$ with $[\tilde{e}]=[\gamma_{s(\tilde{e})}\,\tilde{e}\,\overline{\gamma}_{t(\tilde{e})}]\in\pi_1(\widetilde{\Gamma},\tilde{v}_0)$. By construction, the connections Φ and $\widetilde{\Phi}$ represent $\rho^{\#}$ and ρ, respectively.

We now come to the definition of the map $\psi\colon W^{\widetilde{\mathsf{V}}}\to Z^{\mathsf{V}}$. For $f\in W^{\widetilde{\mathsf{V}}}$ and $v\in\mathsf{V}$, set

$$\psi(f)(v)=\sum_{\tilde{v}\in p^{-1}(v)}\rho^{\#}_{g_{\tilde{v}}}(f(\tilde{v}))\in Z,$$

where for any $\tilde{v}\in p^{-1}(v)$, the element $g_{\tilde{v}}\in\pi_1(\Gamma,v_0)$ is defined by

$$g_{\tilde{v}}=[\gamma_v\,p(\overline{\gamma}_{\tilde{v}})]\,.$$

To show that ψ is an isomorphism, first note that it splits as the direct sum $\psi=\bigoplus_{v\in\mathsf{V}}\psi_v$, with ψ_v the restriction of ψ to $W^{p^{-1}(v)}$. Therefore, we only need to check that for all $v\in\mathsf{V}$, the map

$$\psi_v\colon\bigoplus_{\tilde{v}\in p^{-1}(v)}W\longrightarrow Z,\quad (f(\tilde{v}))_{\tilde{v}}\longmapsto\sum_{\tilde{v}\in p^{-1}(v)}\rho^{\#}_{g_{\tilde{v}}}(f(\tilde{v}))$$

is an isomorphism. By definition of the induced representation, we have $Z=\bigoplus_{r\in R}\rho^{\#}_r(W)$, where $R\subset\pi_1(\Gamma,v_0)$ is a set of representatives of the cosets $\pi_1(\Gamma,v_0)/p_*(\pi_1(\widetilde{\Gamma},\tilde{v}_0))$. Therefore, we are left with the proof that for all $v\in\mathsf{V}$, the set $\{g_{\tilde{v}}\,|\,\tilde{v}\in p^{-1}(v)\}$ is a set of representatives of these cosets. In other words, we need to show that for any $g\in\pi_1(\Gamma,v_0)$, there is a unique $\tilde{v}\in p^{-1}(v)$ such that $g=g_{\tilde{v}}\,h$ with $h\in p_*(\pi_1(\widetilde{\Gamma},\tilde{v}_0))$.

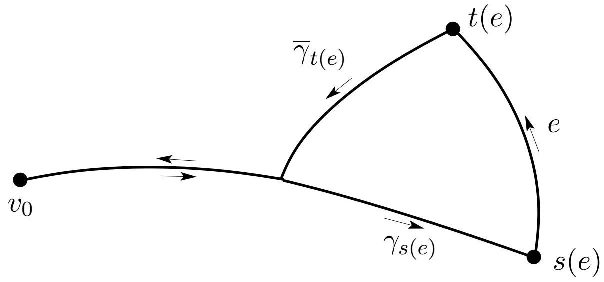

FIGURE 1. The composition of the three paths $\gamma_{s(e)}$, e and $\overline{\gamma}_{t(e)}$ forms a loop based at v_0 whose homotopy class $[\gamma_{s(e)}\, e\, \overline{\gamma}_{t(e)}] \in \pi_1(\Gamma, v_0)$ corresponds to $[e] \in \pi_1(\Gamma, T)$.

To check this claim, note that the path $\gamma_{\tilde{v}}$ can be written as $\gamma_{\tilde{v}} = \alpha\, \tilde{\gamma}_v$, with α a path in $\widetilde{T}$ from $\tilde{v}_0$ to $t(\alpha) = [p(\alpha)] \cdot \tilde{v}_0 \in p^{-1}(v_0)$ (recall the transitive action of $\pi_1(\Gamma, v_0)$ on $p^{-1}(v_0)$ from Section 2.3), and $\tilde{\gamma}_v$ the unique lift of γ_v with $s(\tilde{\gamma}_v) = t(\alpha)$. As a consequence, we have $g_{\tilde{v}} = [\gamma_v\, p(\overline{\gamma}_{\tilde{v}})] = [\gamma_v\, \overline{\gamma}_v\, p(\overline{\alpha})] = [p(\alpha)]^{-1}$. Hence, given any $g \in \pi_1(\Gamma, v_0)$, let $\tilde{v}$ be the unique element in $p^{-1}(v)$ such that the corresponding path $\gamma_{\tilde{v}} = \alpha\, \tilde{\gamma}_v$ satisfies $[p(\alpha)]^{-1} \cdot \tilde{v}_0 = g \cdot \tilde{v}_0$. By construction, we now have $g^{-1} g_{\tilde{v}}$ belonging to the isotropy subgroup of $\tilde{v}_0$, i.e. to $p_*(\pi_1(\widetilde{\Gamma}, \tilde{v}_0))$. This completes the proof that ψ is an isomorphism.

Let us now check that the diagram commutes, i.e. that the equality

$$(\psi \circ \mathsf{A}_{\widetilde{\Gamma}}^{\widetilde{\Phi}})(f)(v) = (\mathsf{A}_{\Gamma}^{\Phi} \circ \psi)(f)(v)$$

holds for all $f \in W^{\widetilde{\mathsf{V}}}$ and all $v \in \mathsf{V}$. Expanding the left-hand side using the definition of ψ, of $\mathsf{A}_{\widetilde{\Gamma}}^{\widetilde{\Phi}}$ and of $\widetilde{\Phi}$, the linearity of $\rho^{\#}_g$, the equality $\rho_{[\tilde{e}]} = \rho^{\#}_{p_*([\tilde{e}])}$ together with the fact that $\rho^{\#}$ is a group homomorphism, we get

$$(\psi \circ \mathsf{A}_{\widetilde{\Gamma}}^{\widetilde{\Phi}})(f)(v) = \sum_{\tilde{v} \in p^{-1}(v)} \sum_{\tilde{e} \in \widetilde{\mathsf{D}}_{\tilde{v}}} \tilde{x}_{\tilde{e}}\, \rho^{\#}_{g_{\tilde{v}}\, p_*([\tilde{e}])}(f(t(\tilde{e}))) \,.$$

Expanding the right-hand side in a similar way, we get

$$(\mathsf{A}_{\Gamma}^{\Phi} \circ \psi)(f)(v) = \sum_{e \in \mathsf{D}_v} \sum_{\tilde{w} \in p^{-1}(t(e))} x_e\, \rho^{\#}_{[e]\, g_{\tilde{w}}}(f(\tilde{w})) \,.$$

Since $p\colon \widetilde{\Gamma} \to \Gamma$ is a covering map, we can apply the second point of Lemma 2.4, which states that the two families of evaluations of f displayed above are over the same subset of $\widetilde{\mathsf{V}}$. Furthermore, the equality $\tilde{x}_{\tilde{e}} = x_e$ holds by definition. Therefore, we are left with the proof that for any $\tilde{e} \in \widetilde{\mathsf{D}}$ with source vertex $\tilde{v}$, target vertex $\tilde{w}$ and image $p(\tilde{e}) = e$, the equality $[e]\, g_{\tilde{w}} = g_{\tilde{v}}\, p_*([\tilde{e}])$ holds in $\pi_1(\Gamma, v_0)$. Writing $p(\tilde{v}) = v$ and $p(\tilde{w}) = w$, we have

$$[e]\, g_{\tilde{w}} = [\gamma_v\, e\, \overline{\gamma}_w]\, [\gamma_w\, p(\overline{\gamma}_{\tilde{w}})] = [\gamma_v\, e\, p(\overline{\gamma}_{\tilde{w}})] = [\gamma_v\, p(\overline{\gamma}_{\tilde{v}})]\, [p(\gamma_{\tilde{v}})\, e\, p(\overline{\gamma}_{\tilde{w}})] = g_{\tilde{v}}\, p_*([\tilde{e}]) \,.$$

This concludes the proof. □

3.3. **The Artin formalism for graphs.** In his foundational work in algebraic number theory [2, 3], Artin associates an L-series to any Galois field extension endowed with a representation of its Galois group. He shows that these L-series satisfy 4 axioms, the so-called *Artin formalism* (see [28, Chapter XII.2] for a modern account). Since then, analogous axioms have been shown to hold for L-series in topology [27], in analysis [16], and for some L-series associated to finite graphs [34].

The aim of this subsection is to explain how Theorem 3.6 can be interpreted as (the non-trivial part) of an Artin formalism for graphs. We also show that our approach allows for wide generalisations of the results of Stark and Terras [33, 34].

Recall from Section 3.1 that to any weighted locally finite directed graph $\Gamma = (\mathsf{V}, \mathsf{D}, s, t)$ endowed with a representation $\rho\colon \pi_1(\Gamma, v_0) =: \pi_1(\Gamma) \to \mathrm{GL}(W)$, we associate a twisted weighted adjacency operator A_Γ^ρ in $\mathrm{End}(W^\vee)$, well-defined up to conjugation. Hence, given a normal covering $p\colon \widetilde{\Gamma} \to \Gamma$ and a representation $\rho\colon G \to \mathrm{GL}(W)$ of its Galois group $G := G(\widetilde{\Gamma}/\Gamma) = \pi_1(\Gamma)/\pi_1(\widetilde{\Gamma})$, one can consider the representation $\rho\circ\mathrm{pr}$ of $\pi_1(\Gamma)$, where $\mathrm{pr}\colon \pi_1(\Gamma) \to G$ denotes the canonical projection.

Let $\mathcal{O}(\widetilde{\Gamma}/\Gamma, \rho)$ be the associated conjugacy class $\big[\mathsf{A}_\Gamma^{\rho\circ\mathrm{pr}}\big] \in \mathrm{End}(W^\vee)/\!\sim$.

Proposition 3.10. *This association satisfies the following 4 axioms.*

1. $\mathcal{O}(\widetilde{\Gamma}/\Gamma, 1) = [\mathsf{A}_\Gamma]$*, the untwisted adjacency operator on* Γ*.*

2. Given any two representations ρ_1 *and* ρ_2 *of* G*, we have*

$$\mathcal{O}(\widetilde{\Gamma}/\Gamma, \rho_1 \oplus \rho_2) = \mathcal{O}(\widetilde{\Gamma}/\Gamma, \rho_1) \oplus \mathcal{O}(\widetilde{\Gamma}/\Gamma, \rho_2)\,.$$

3. If H *is a normal subgroup of* G *and* $\overline{\Gamma} = H\backslash\widetilde{\Gamma}$ *denotes the corresponding covering of* Γ*, then for any representation* ρ *of* G/H*, we have*

$$\mathcal{O}(\overline{\Gamma}/\Gamma, \rho) = \mathcal{O}(\widetilde{\Gamma}/\Gamma, \rho\circ\pi)\,,$$

where $\pi\colon G \to G/H$ *denotes the canonical projection.*

4. If H *is a subgroup of* G *and* $\overline{\Gamma} = H\backslash\widetilde{\Gamma}$*, then for any representation* ρ *of* H*, we have*

$$\mathcal{O}(\widetilde{\Gamma}/\overline{\Gamma}, \rho) = \mathcal{O}(\widetilde{\Gamma}/\Gamma, \rho^\#)\,,$$

where $\rho^\#$ *is the representation of* G *induced by* ρ*.*

Proof. The first and second points are reformulations of the trivial Remarks 3.2 (ii) and (iii), while the third point follows from the fact that the composition $\pi\circ\mathrm{pr}\colon \pi_1(\Gamma) \to G/H$ coincides with the canonical projection of $\pi_1(\Gamma)$ onto $\pi_1(\Gamma)/\pi_1(\overline{\Gamma})$. As for the last point, let $\overline{p}\colon \overline{\Gamma} \to \Gamma$ denote the relevant covering map, and $\overline{\mathrm{pr}}$ the canonical projection of $\pi_1(\overline{\Gamma})$ onto $\pi_1(\overline{\Gamma})/\pi_1(\widetilde{\Gamma}) = H$. By naturality, the composition of $\overline{\mathrm{pr}}$ with the inclusion of H in G coincides with $\mathrm{pr}\circ\overline{p}_*$. Therefore, the representation induced by $\rho\circ\overline{\mathrm{pr}}$ coincides with $\rho^\#\circ\mathrm{pr}$. The fourth point is now a formal consequence of Theorem 3.6:

$$\mathcal{O}(\widetilde{\Gamma}/\overline{\Gamma}, \rho) = [\mathsf{A}_{\overline{\Gamma}}^{\rho\circ\overline{\mathrm{pr}}}] = [\mathsf{A}_\Gamma^{(\rho\circ\overline{\mathrm{pr}})^\#}] = [\mathsf{A}_\Gamma^{\rho^\#\circ\mathrm{pr}}] = \mathcal{O}(\widetilde{\Gamma}/\Gamma, \rho^\#)\,. \qquad \square$$

With the L-series of [34] in mind, it is natural to consider $\det(\mathrm{I} - \mathsf{A}_\Gamma^{\rho\circ\mathrm{pr}})^{-1}$ as the object of study. The fact that these L-series satisfy the Artin formalism follows from the proposition above.

Actually, our method easily yields results on more general L-series, as follows. Let us fix a map associating to a weighted graph (G, x) a weighted directed graph (Γ, x), as in Examples 3.3–3.5. Formally, we want this assignment to preserve the ingredients of Theorem 3.6: a covering map $\widetilde{\mathsf{G}} \to \mathsf{G}$ of locally finite connected graphs is sent to a covering map $\widetilde{\Gamma} \to \Gamma$ of locally finite connected digraphs, and there is a natural group homomorphism $\alpha\colon \pi_1(\Gamma) \to \pi_1(\mathsf{G})$.

Given any representation ρ of $\pi_1(\mathsf{G})$, we can now consider the L-series

$$L(\mathsf{G},\mathsf{x},\rho)=\det(\mathrm{I}-\mathsf{A}_\Gamma^{\rho\circ\alpha})^{-1}\in\mathbb{C}[\![\mathsf{x}]\!]\,.$$

By the Amitsur formula (see [1, 30]), it can be written as

$$L(\mathsf{G},\mathsf{x},\rho)=\prod_{[\gamma]}\det(1-x(\gamma)\rho_\gamma)^{-1}\,,$$

where the product is over all loops γ in Γ that cannot be expressed as δ^ℓ for some path δ and integer $\ell>1$, loops considered up to change of base vertex. Also, $x(\gamma)$ denotes the product of the weights of the edges of γ, while ρ_γ is the monodromy of the loop γ. (Note that changing the base point yields a conjugate monodromy, so $\det(1-x(\gamma)\rho_\gamma)$ is well defined.) Of course, these loops in Γ correspond to some class of loops in G, a class which depends on the way Γ is obtained from G. But for any such assignment, the results of Section 3.2 have straightforward implications on the corresponding L-series, and on the corresponding class of loops in G.

For concreteness, let us focus on the directed line graph assignment $\mathsf{G}\mapsto\Gamma$ described in Example 3.5. The corresponding twisted weighted operator $\mathrm{I}-\mathsf{A}_\Gamma^{\rho\circ\alpha}$ coincides with the operator considered in [34, Theorem 7], where the authors restrict themselves to representations of a finite quotient of $\pi_1(\mathsf{G})$, i.e. representations of the Galois group of a finite cover of G. In the expression displayed above, the product is over so-called *prime cycles* in G, i.e. equivalence classes of cyclic loops in G that do not contain a subpath of the form $(e,\overline{e})$ and that cannot be expressed as the power of a shorter loop. In the special case when ρ factorises through a finite quotient of $\pi_1(\mathsf{G})$, this is what Stark and Terras define as the *multiedge Artin L-function* of G, an object extending several other functions introduced in [33, 34].

The theory of Section 3 applied to $\mathsf{A}_\Gamma^{\rho\circ\alpha}$ now allow us to easily extend their results to this more general L-function. For example, our Corollary 3.8 shows that if $\widetilde{\mathsf{G}}$ is a finite connected covering graph of a connected graph G, then $L(\mathsf{G},\mathsf{x},1)^{-1}$ divides $L(\widetilde{\mathsf{G}},\tilde{\mathsf{x}},1)^{-1}$, extending Corollary 1 of [33, Theorem 3]. Also, our Corollary 3.9 recovers the corollary of [34, Proposition 3], while our Theorem 3.6 extends Theorem 8 of [34]. Finally, expanding the equality $\log L(\widetilde{\mathsf{G}},\tilde{\mathsf{x}},\rho)=\log L(\mathsf{G},\mathsf{x},\rho^\#)$ yields an extension of the technical Lemma 7 of [34] to more general covers and representations.

We conclude this section by recalling that our approach immediately yields similar results for any assignment $\mathsf{G}\mapsto\Gamma$ preserving covering maps.

4. Combinatorial applications

Each time the determinant of an operator counts combinatorial objects, Theorem 3.6 and Corollaries 3.8 and 3.9 have combinatorial implications. This is the case of the operators given in Examples 3.3 and 3.4, whose determinants count spanning trees and perfect matchings, respectively. We explain these applications in Sections 4.1 and 4.2. We also briefly enumerate additional applications in Section 4.3.

4.1. **Spanning trees and rooted spanning forests.**

Our first combinatorial application relies on a slightly generalised version of the matrix-tree theorem, that we now recall.

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a finite graph endowed with symmetric weights $\mathsf{x} = \{\mathsf{x_e}\}_{\mathsf{e}\in\mathsf{E}}$, that we consider as formal variables. Let Δ_{G} be the associated Laplacian, acting on $\mathbb{C}^{\mathsf{V}}$ via

$$\Delta_{\mathsf{G}} f(v) = \sum_{e\in\mathsf{D}_v} x_e(f(v) - f(t(e)))$$

for $f \in \mathbb{C}^{\mathsf{V}}$ and $v \in \mathsf{V}$. Set $n := |\mathsf{V}|$, and consider the characteristic polynomial in $|\mathsf{E}| + 1$ variables

$$P_{\mathsf{G}}(\lambda) := \det(\lambda\,\mathrm{I} - \Delta_{\mathsf{G}}) = \sum_{i=0}^{n} c_i\,\lambda^i \in \mathbb{Z}[\mathsf{x}, \lambda]\,.$$

Then, the coefficient $c_i \in \mathbb{Z}[\mathsf{x}]$ admits the combinatorial interpretation

$$(-1)^{n-i}\,c_i = \sum_{F\subset\mathsf{G},\,|\pi_0(F)|=i} \phi(F) \prod_{\mathsf{e}\in\mathsf{E}(F)} \mathsf{x_e}\,,$$

where the sum is over all spanning forests F in G with i connected components (or equivalently, with $n - i$ edges), and $\phi(F) \in \mathbb{Z}_+$ denotes the number of possible *roots* of F: if $F = \bigsqcup_j T_j$ denotes the decomposition of F into connected components, then $\phi(F) = \prod_j |\mathsf{V}(T_j)|$.

For example, there is a unique spanning forest F in G with n connected components (given by the vertices of G), it admits a unique root, leading to the expected value $c_n = 1$. As additional reality checks, we have the values $-c_{n-1} = 2\sum_{\mathsf{e}\in\mathsf{E}} \mathsf{x_e}$ and $c_0 = \det(\Delta_{\mathsf{G}}) = 0$. Finally, since connected spanning forests coincide with spanning trees, and all spanning trees admit exactly n roots, we have

$$(-1)^{n-1}\,c_1 = n \sum_{T\subset\mathsf{G}} \prod_{\mathsf{e}\in\mathsf{E}(T)} \mathsf{x_e}\,,$$

the sum being over all spanning trees of G. This latter result is nothing but Kirchoff's matrix-tree theorem.

Remark 4.1. This result can be derived from the (usual version of the) matrix-tree theorem applied to the graph obtained from G by adding one vertex connected to each vertex of G by an edge of weight $-\lambda$.

Let us also mention that this result was obtained by Chung and Langlands in the context of graphs endowed with vertex-weights rather than edge-weights [7]. Theorem 3.6 trivially extends to graphs endowed with vertex-weights (in addition to edge-weights), and it is a routine task to adapt the results of the present subsection to this more general case.

Definition 4.2. The *spanning tree partition function* of a weighted graph (G, x) is

$$\mathsf{Z}_{ST}(\mathsf{G}, \mathsf{x}) := \sum_{T\subset\mathsf{G}} \prod_{\mathsf{e}\in\mathsf{E}(T)} \mathsf{x_e}\,,$$

the sum being over all spanning trees in G. Similarly, the *rooted spanning forest partition function* of (G, x) is

$$\mathsf{Z}_{RSF}(\mathsf{G}, \mathsf{x}) := \sum_{F\subset\mathsf{G}} \phi(F) \prod_{\mathsf{e}\in\mathsf{E}(F)} \mathsf{x_e}\,,$$

the sum being over all spanning forests in Γ.

Note that if one sets all the weights to 1, then $\mathsf{Z}_{ST}(\mathsf{G}, 1)$ is the number of spanning trees in G, while $\mathsf{Z}_{RSF}(\mathsf{G}, 1)$ counts the number of rooted spanning forests in G.

Theorem 4.3. *Let $\widetilde{\mathsf{G}}$ be a finite covering graph of a finite connected graph G endowed with edge-weights $\mathsf{x} = \{\mathsf{x}_\mathsf{e}\}_{\mathsf{e}\in\mathsf{E}}$, and let $\tilde{\mathsf{x}}$ denote these weights lifted to the edges of $\widetilde{\mathsf{G}}$. Then $\mathsf{Z}_{ST}(\mathsf{G},\mathsf{x})$ divides $\mathsf{Z}_{ST}(\widetilde{\mathsf{G}},\tilde{\mathsf{x}})$ and $\mathsf{Z}_{RSF}(\mathsf{G},\mathsf{x})$ divides $\mathsf{Z}_{RSF}(\widetilde{\mathsf{G}},\tilde{\mathsf{x}})$ in the ring $\mathbb{Z}[\mathsf{x}]$.*

This immediately leads to the following corollary. The first point is known since the work of Berman (see [4, Theorem 5.7]), while the second one appears to be new.

Corollary 4.4. *Let $\widetilde{\mathsf{G}}$ be a finite covering graph of a finite connected graph G.*

(i) The number of spanning trees in G divides the number of spanning trees in $\widetilde{\mathsf{G}}$.
(ii) The number of rooted spanning forests in G divides the number of rooted spanning forests in $\widetilde{\mathsf{G}}$. □

Proof of Theorem 4.3. First note that $\mathsf{Z}_{RSF}(\widetilde{\mathsf{G}},\tilde{\mathsf{x}})$ is multiplicative with respect to connected sum while $\mathsf{Z}_{ST}(\widetilde{\mathsf{G}},\tilde{\mathsf{x}})$ vanishes for $\widetilde{\mathsf{G}}$ not connected. Therefore, it can be assumed that $\widetilde{\mathsf{G}}$ is connected. Let $\widetilde{\mathsf{G}} \to \mathsf{G}$ be a covering map between two finite connected graphs, with edge-weights x on G inducing lifted edge-weights $\tilde{\mathsf{x}}$ on $\widetilde{\mathsf{G}}$. Let $\widetilde{\Gamma}$ (resp. Γ) be the graph associated with $\widetilde{\mathsf{G}}$ (resp. G) as in Example 3.3. Note that the graphs $\widetilde{\Gamma}$ and Γ remain finite and connected, and the covering map $\widetilde{\mathsf{G}} \to \mathsf{G}$ trivially extends to a covering map $\widetilde{\Gamma} \to \Gamma$. By Example 3.3 and Corollary 3.8, we know that $\Delta_{\widetilde{\mathsf{G}}} = \mathsf{A}_{\widetilde{\Gamma}}$ is conjugate to $\mathsf{A}_\Gamma \oplus \mathsf{A}^\rho_\Gamma = \Delta_\mathsf{G} \oplus \Delta^\rho_\mathsf{G}$ for some representation ρ of $\pi_1(\Gamma, v_0)$. Therefore, setting $P^\rho_\mathsf{G}(\lambda) := \det(\lambda\,\mathrm{I} - \Delta^\rho_\mathsf{G}) \in \mathbb{C}[\mathsf{x},\lambda]$, we have the equality

$$P_{\widetilde{\mathsf{G}}}(\lambda) = P_\mathsf{G}(\lambda) \cdot P^\rho_\mathsf{G}(\lambda) \in \mathbb{C}[\mathsf{x},\lambda]\,.$$

Observe that $P_{\widetilde{\mathsf{G}}}(\lambda)$ and $P_\mathsf{G}(\lambda)$ belong to $\mathbb{Z}[\mathsf{x},\lambda]$, so $P^\rho_\mathsf{G}(\lambda)$ belongs to the intersection of $\mathbb{C}[\mathsf{x},\lambda]$ with the field of fractions $Q(\mathbb{Z}[\mathsf{x},\lambda]) = \mathbb{Q}(\mathsf{x},\lambda)$, i.e. it belongs to the ring $\mathbb{Q}[\mathsf{x},\lambda]$. Since the leading λ-coefficient of $P_\mathsf{G}(\lambda)$ is equal to 1, the greatest common divisor of its coefficients is 1. An application of Gauss's lemma (see e.g. [29, Corollary 2.2]) now implies that $P^\rho_\mathsf{G}(\lambda)$ belongs to $\mathbb{Z}[\mathsf{x},\lambda]$. In conclusion, we have that $P_\mathsf{G}(\lambda)$ divides $P_{\widetilde{\mathsf{G}}}(\lambda)$ in $\mathbb{Z}[\mathsf{x},\lambda]$.

By the extended matrix-tree theorem stated above, we have that $P_\mathsf{G}(-1) = \pm\mathsf{Z}_{RSF}(\mathsf{G},\mathsf{x})$ divides $P_{\widetilde{\mathsf{G}}}(-1) = \pm\mathsf{Z}_{RSF}(\widetilde{\mathsf{G}},\tilde{\mathsf{x}})$ in $\mathbb{Z}[\mathsf{x}]$, proving the second claim.

To show the first one, consider again the equation $P_{\widetilde{\mathsf{G}}}(\lambda) = P_\mathsf{G}(\lambda) \cdot P^\rho_\mathsf{G}(\lambda)$ in $\mathbb{Z}[\mathsf{x},\lambda]$, and observe that $P_{\widetilde{\mathsf{G}}}(\lambda)$ and $P_\mathsf{G}(\lambda)$ are both multiples of λ. Dividing both sides by λ and setting $\lambda = 0$, the matrix-tree theorem (in the form stated above) implies

$$|\mathsf{V}(\widetilde{\mathsf{G}})| \cdot \mathsf{Z}_{ST}(\widetilde{\mathsf{G}},\tilde{\mathsf{x}}) = \pm|\mathsf{V}(\mathsf{G})| \cdot \mathsf{Z}_{ST}(\mathsf{G},\mathsf{x}) \cdot P^\rho_\mathsf{G}(0)\,,$$

i.e. $\mathsf{Z}_{ST}(\widetilde{\mathsf{G}},\tilde{\mathsf{x}}) = \mathsf{Z}_{ST}(\mathsf{G},\mathsf{x}) \cdot g(\mathsf{x})$, with $g(\mathsf{x}) = \frac{\pm 1}{\deg(\widetilde{\mathsf{G}}/\mathsf{G})} P^\rho_\mathsf{G}(0) \in \mathbb{Q}[\mathsf{x}]$. Since both $\mathsf{Z}_{ST}(\widetilde{\mathsf{G}},\tilde{\mathsf{x}})$ and $\mathsf{Z}_{ST}(\mathsf{G},\mathsf{x})$ belong to $\mathbb{Z}[\mathsf{x}]$ and the greatest common divisor of the coefficients of $\mathsf{Z}_{ST}(\mathsf{G},\mathsf{x})$ is 1, one more application of Gauss's lemma yields that $g(\mathsf{x})$ lies in $\mathbb{Z}[\mathsf{x}]$, and concludes the proof. □

4.2. **Perfect matchings.** In this subsection, we review some applications of Theorem 3.6 to perfect matchings, and more generally to the dimer model.

Recall that a *perfect matching* (or *dimer configuration*) in a graph Γ is a family of edges $M \subset \mathsf{E}$ such that each vertex of Γ is adjacent to a unique element of M. If Γ is finite and endowed with symmetric edge-weights $\mathsf{x} = \{\mathsf{x}_\mathsf{e}\}_{\mathsf{e}\in\mathsf{E}}$, then one defines the *dimer partition function* of Γ

as

$$\mathsf{Z}_{\mathrm{dimer}}(\Gamma, \mathsf{x}) = \sum_{M} \prod_{\mathsf{e} \in M} \mathsf{x}_{\mathsf{e}} \,,$$

the sum being over all perfect matchings in Γ. Note that if all the weights are equal to 1, then $\mathsf{Z}_{\mathrm{dimer}}(\Gamma, 1)$ simply counts the number of perfect matchings in Γ.

Now, assume that Γ is embedded in the plane, and endowed with an orientation of its edges so that around each face of $\Gamma \subset \mathbb{R}^2$, there is an odd number of edges oriented clockwise. Let $x = \{x_e\}_{e \in \mathsf{D}}$ be the anti-symmetric edge-weights obtained as in Example 3.4, and let A_Γ be the associated weighted skew-adjacency operator. By Kasteleyn's celebrated theorem [20, 21], the Pfaffian of A_Γ is equal to $\pm \mathsf{Z}_{\mathrm{dimer}}(\Gamma, \mathsf{x})$.

With this powerful method in hand, we can try to use Theorem 3.6 in studying the dimer model on symmetric graphs. Quite unsurprisingly, the straightforward applications of our theory are not new. Indeed, the only divisibility statement that we obtain via Corollary 3.8 is the following known result (see Theorem 3 of [15] for the bipartite case, and Section IV.C of [25] for a general discussion).

Proposition 4.5. *Fix a planar, finite, connected weighted graph* $(\widetilde{\Gamma}, \tilde{\mathsf{x}})$ *invariant under rotation around a point in the complement of* $\widetilde{\Gamma}$*, of angle* $\frac{2\pi}{d}$ *for some odd integer* d*. Let* (Γ, x) *be the resulting quotient weighted graph. Then, the partition function* $\mathsf{Z}_{\mathrm{dimer}}(\Gamma, \mathsf{x})$ *divides* $\mathsf{Z}_{\mathrm{dimer}}(\widetilde{\Gamma}, \tilde{\mathsf{x}})$ *in the ring* $\mathbb{Z}[\mathsf{x}]$*.*

Proof. Let us fix an orientation of the edges of Γ satisfying the clockwise-odd condition. It lifts to an orientation of $\widetilde{\Gamma}$ which trivially satisfies the same condition around all faces except possibly the face containing the center of rotation; for this later face, it does satisfy the condition since d is odd. Hence, we have a d-fold cyclic covering of connected weighted graphs $(\widetilde{\Gamma}, \tilde{x}) \to (\Gamma, x)$, and Corollary 3.8 can be applied. Together with Kasteleyn's theorem, it yields the following equality in $\mathbb{C}[\mathsf{x}]$:

$$\mathsf{Z}_{\mathrm{dimer}}(\widetilde{\Gamma}, \tilde{\mathsf{x}})^2 = \det(\mathsf{A}_{\widetilde{\Gamma}}) = \det(\mathsf{A}_\Gamma) \det(\mathsf{A}_\Gamma^\rho) = \mathsf{Z}_{\mathrm{dimer}}(\Gamma, \mathsf{x})^2 \cdot \det(\mathsf{A}_\Gamma^\rho) \,.$$

This ring being factorial, it follows that $\mathsf{Z}_{\mathrm{dimer}}(\widetilde{\Gamma}, \tilde{\mathsf{x}}) = \mathsf{Z}_{\mathrm{dimer}}(\Gamma, \mathsf{x}) \cdot g$ for some $g \in \mathbb{C}[\mathsf{x}]$. The fact that g belongs to $\mathbb{Z}[\mathsf{x}]$ follows from Gauss's lemma as in the proof of Theorem 4.3. □

Our approach is limited by the fact that we consider graph coverings $\widetilde{\Gamma} \to \Gamma$ which, in the case of normal coverings, correspond to free actions of $G(\widetilde{\Gamma}/\Gamma)$ on $\widetilde{\Gamma}$. For this specific question of enumerating dimers on symmetric planar graphs, the discussion of Section IV of [25] is more complete, as non-free actions are also considered.

However, our approach is quite powerful when applied to non-planar graphs. Indeed, recall that Kasteleyn's theorem can be extended to weighted graphs embedded in a closed (possibly non-orientable) surface Σ, but the computation of the dimer partition function requires the Pfaffians of $2^{2-\chi(\Sigma)}$ different (possibly complex-valued) skew-adjacency matrices [35, 11, 8]. In particular, the partition function of any graph embedded in the torus $\mathbb{T}^2$ is given by 4 Pfaffians. For the Klein bottle $\mathcal{K}$, we also need 4 Pfaffians, which turn out to be two pairs of conjugate complex numbers, so 2 well-chosen Pfaffians are sufficient. We now illustrate the use of Theorem 3.6 in these two cases.

Let us first consider a toric graph $\Gamma \subset \mathbb{T}^2$, and let $\widetilde{\Gamma} = \Gamma_{mn}$ denote the lift of Γ by the natural $m \times n$ covering of the torus by itself. This covering is normal with Galois group $G(\widetilde{\Gamma}/\Gamma) \simeq \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. This group being abelian, all the irreducible representations are of degree 1;

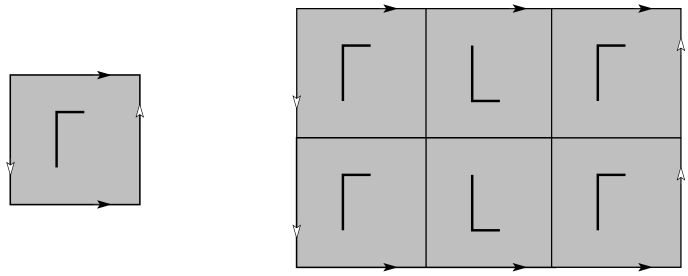

FIGURE 2. A graph Γ embedded in the Klein bottle $\mathcal{K}$ (pictured as a square with opposite sides identified according to the arrows), and the lift $\Gamma_{mn} \subset \mathcal{K}$, here with $m = 2$ and $n = 3$.

more precisely, they are given by $\{\rho(z,w)\}_{w^m=1,\, z^n=1}$, where $\rho(z,w)$ maps a fixed generator of $\mathbb{Z}/m\mathbb{Z}$ (resp. $\mathbb{Z}/n\mathbb{Z}$) to $w \in \mathbb{C}^*$ (resp. $z \in \mathbb{C}^*$). Writing $P_{mn}(z,w) = \det(\mathsf{A}^{\rho(z,w)}_{\Gamma_{mn}})$ and $P_{1,1} = P$, Corollary 3.9 immediately yields the equality

$$P_{mn}(1,1) = \prod_{z^n=1} \prod_{w^m=1} P(z,w)\,.$$

This is the well-known Theorem 3.3 of [24], a result of fundamental importance in the study of the dimer model on biperiodic graphs.

Let us now consider a weighted graph Γ embedded in the Klein bottle $\mathcal{K}$, and let $\widetilde{\Gamma} = \Gamma_{mn}$ denote the lift of Γ by the natural $m \times n$ cover $\mathcal{K}_{mn} \to \mathcal{K}$ of the Klein bottle by itself (with n odd), as illustrated in Figure 2. Now, we can interpret the two skew-adjacency matrices of $\widetilde{\Gamma} = \Gamma_{mn}$ used in the computation of the corresponding dimer partition function as weighted adjacency operators twisted by 1-dimensional representations ρ, ρ' of $\pi_1(\mathcal{K}_{mn}) < \pi_1(\mathcal{K})$. Using Theorem 3.6, we see that these matrices are conjugate to the skew-adjacency operators on $\Gamma \subset \mathcal{K}$ twisted by the corresponding induced representations $\rho^\#, (\rho')^\#$ of $\pi_1(\mathcal{K})$. Unlike that of the torus, the fundamental group of the Klein bottle is not abelian, so the representations $\rho^\#, (\rho')^\#$ need not split as products of 1-dimensional representations. It turns out that they split as products of representations of degree 1 and 2, yielding a closed formula for $\mathsf{Z}_{\text{dimer}}(\Gamma_{mn}, x)$ in terms of determinants of A^τ_Γ, with τ of degree 1 and 2. This result is at the core of the study of the dimer model on Klein bottles of the first-named author [9].

As a final remark, let us note that all the considerations of this subsection can be applied equally well to the Ising model, either via the use of Kac-Ward matrices [17], or via skew-adjacency matrices on the associated Fisher graph [12].

4.3. **Further combinatorial applications.** We conclude this article with a very brief and informal description of additional applications of our results.

As discovered by Forman [13], the determinant of Δ^ρ_{G} with $\deg(\rho) = 1$ can be expressed as a sum over *cycle-rooted spanning forests* (CRSFs) in G, each forest being counted with a complex weight depending on ρ. If there is a finite connected covering $\widetilde{\mathsf{G}} \to \mathsf{G}$ and a degree 1 representation of $\pi_1(\widetilde{\mathsf{G}}, \tilde{v}_0)$ such that the induced representation of $\pi_1(\mathsf{G}, v_0)$ admits a degree 1 subrepresentation ρ', then the CRSF partition function on G twisted by ρ' divides the partition function on $\widetilde{\mathsf{G}}$ twisted by ρ, in the ring $\mathbb{C}[\mathrm{x}]$. Furthermore, in the case of a normal abelian covering of degree d, Corollary 3.9 gives a factorisation of the CRSF partition function of $\widetilde{\mathsf{G}}$ in terms of d CRSF partition functions of G.

Finally, let X be a finite CW-complex of dimension r with weights $\mathsf{x} = (\mathsf{x}_\mathsf{e})_\mathsf{e}$ associated to the cells of top dimension. Let G be the weighted graph with vertex set given by the $(r-1)$-dimensional cells of X, two such vertices being connected by an unoriented edge of G each time they are in the boundary of an r-dimensional cell. (Note that if $r = 1$, then the 1-dimensional cell complex X is nothing but the geometric realisation of the graph G.) Finally, let Γ denote the weighted graph obtained from G as in Example 3.3. Then, the resulting operator A_Γ is the *Laplacian* Δ_X acting on r-cells of X. This operator can be used to count so-called *higher dimensional rooted forests* in X, see [18, 5] and references therein. Using Corollary 3.8, it is now straightforward to prove that, given any finite cover $\widetilde{X} \to X$, the corresponding rooted forest partition function of X divides the rooted forest partition function of $\widetilde{X}$, extending Theorem 4.3 to higher dimensional objects.

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