

The Conway potential function of a graph link

BY DAVID CIMASONI

*Section de Mathématiques, Université de Genève,
2-4, rue du Lièvre, CP 240, CH-1211 Genève 24, Suisse
e-mail: David.Cimasoni@math.unige.ch*

(Received 15 July 2002)

Abstract

We give a closed formula for the multivariable Conway potential function of any graph link in a homology sphere. As corollaries, we answer three questions by Walter Neumann [8] about graph links.

1. Introduction

A link L in a homology 3-sphere is called a *graph link* if its exterior is a graph manifold. For example, a link in S^3 is a graph link if and only if it is *solvable*, that is, if it can be constructed by iterated cabling and connected sum operations from the unknot. This class of links is particularly interesting, since all the links that arise in complex algebraic geometry are of this type.

Eisenbud and Neumann [4] gave a classification of graph links by means of decorated trees called *splice diagrams*; they also found a formula for the multivariable Alexander polynomial Δ_L of graph links in terms of the splice diagram (see (3.1) below). Since the Conway potential function is determined up to a sign by Δ_L , the only problem is to compute this sign.

In 1999, Walter Neumann [8] succeeded in computing the Conway polynomial (that is: the one variable potential function) of any fibered solvable link. In this article, Neumann proves several formulas for fibered solvable links, and asks whether these equalities still hold for any graph link. In this paper, we compute the multivariable potential function of any (fibered or non-fibered) graph link in a homology sphere (Theorem 1). As a consequence, we give a positive answer to Neumann's questions (Corollaries 1, 2 and 3).

2. The Conway Potential Function

In 1970, Conway [3] introduced a new invariant of links called the *potential function*. Given an oriented ordered link $L = L_1 \cup \dots \cup L_n$ in S^3 , its potential function is a well defined rational function $\nabla_L(t_1, \dots, t_n)$ which satisfies

$$\nabla_L(t_1, \dots, t_n) \doteq \begin{cases} \frac{1}{t_1 - t_1^{-1}} \Delta_L(t_1^2) & \text{if } n = 1; \\ \Delta_L(t_1^2, \dots, t_n^2) & \text{if } n \geq 2, \end{cases} \quad (2.1)$$

where \doteq means equal up to a multiplication by $\pm t_1^{\nu_1} \dots t_n^{\nu_n}$. Thus, this invariant is basically the multivariable Alexander polynomial without the ambiguity concerning multipli-

cation by units of $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. As a particular case of the potential function, Conway defined what was later called the *Conway polynomial* of a non-ordered link L . It is given by

$$\Omega_L(t) = (t - t^{-1}) \nabla_L(t, \dots, t).$$

Unfortunately, Conway's paper contains neither a precise definition of the potential function, nor a proof of its uniqueness.

In 1981, Kauffman [7] found a very simple geometric construction of the Conway polynomial, namely

$$\Omega_L(t) = \det(t^{-1}A - tA^T),$$

where A is any Seifert matrix of the link L and A^T the transpose of A . Finally, in 1983, Hartley [6] gave a definition of the multivariable potential function ∇_L for any ordered oriented link in S^3 . This definition was later extended by Turaev [10] to links in a \mathbb{Z} -homology 3-sphere, and by Boyer and Lines [1] to links in a \mathbb{Q} -homology 3-sphere.

Let us now state several useful properties of the potential function; we refer to [1] for the proofs. Given an oriented ordered link $L = L_1 \cup \dots \cup L_n$ in a \mathbb{Z} -homology sphere, there exists a well defined invariant ∇_L related to the multivariable Alexander polynomial Δ_L of L by the equality (2.1). Furthermore, ∇_L satisfies the symmetry formula

$$\nabla_L(t_1^{-1}, \dots, t_n^{-1}) = (-1)^n \nabla_L(t_1, \dots, t_n). \quad (2.2)$$

Also, if $L' = (-L_1) \cup L_2 \cup \dots \cup L_n$, where $(-L_1)$ denotes L_1 with the opposite orientation, then

$$\nabla_{L'}(t_1, t_2, \dots, t_n) = -\nabla_L(t_1^{-1}, t_2, \dots, t_n). \quad (2.3)$$

Finally, if $L' = L_2 \cup \dots \cup L_n$, we have the following *Torres formula*:

$$\nabla_L(1, t_2, \dots, t_n) = (t_2^{\ell_{12}} \dots t_n^{\ell_{1n}} - t_2^{-\ell_{12}} \dots t_n^{-\ell_{1n}}) \nabla_{L'}(t_2, \dots, t_n), \quad (2.4)$$

where ℓ_{ij} stands for the linking number $\ell k(L_i, L_j)$.

3. Graph Links

Eisenbud and Neumann [4] gave a classification of graph links using splice diagrams. Following Neumann [8], we will not recall the whole construction here, just several important features of these combinatorial objects. A splice diagram Γ for a graph link L is a tree decorated as follows:

- (i) some of its valency one vertices are drawn as arrowheads, and correspond to the components of L ;
- (ii) the arrowheads have weight $+1$ or -1 (depending on whether the corresponding component of L has "intrinsic" orientation or not);
- (iii) each edge has an integer weight at any end where it meets a *node* (vertex of valency greater than one), and all the edge weights around a node are pairwise coprime.

Given a non-arrowhead vertex v of Γ , there is a so-called *virtual component* L_v : this is the additional link component that would correspond to a single arrow at v with edge weight 1. It is very easy, given two vertices v and w , to compute the linking number $\ell k(L_v, L_w)$ of their corresponding components (virtual or genuine): if σ_{vw} denotes the

shortest path in Γ joining v and w (including v and w), then $\ell k(L_v, L_w)$ is the product of the edge weights adjacent to but not on σ_{vw} , and of the possible arrowhead weights.

Let us now recall Eisenbud and Neumann's formula for the multivariable Alexander polynomial of a graph link [4, Theorem 12.1]. If $L = L_1 \cup \dots \cup L_n$ is a graph link given by a splice diagram Γ , its multivariable Alexander polynomial is equal to

$$\Delta_L(t_1, \dots, t_n) \doteq \begin{cases} (t_1 - 1) \prod_v (t_1^{\ell_{1v}} - 1)^{\delta_v - 2} & \text{if } n = 1; \\ \prod_v (t_1^{\ell_{1v}} \dots t_n^{\ell_{nv}} - 1)^{\delta_v - 2} & \text{if } n \geq 2, \end{cases} \quad (3.1)$$

where the product is over all non-arrowhead vertices v of Γ , δ_v is the valency of the vertex v , and ℓ_{iv} denotes the linking number of L_i with L_v . In this equation, the terms of the form $(t_1^0 \dots t_n^0 - 1)^a$ should be formally cancelled against each other before being set equal to zero.

In proving this theorem, Eisenbud and Neumann also show another remarkable result [4, Theorem 12.2]. Let us call an n -component link *algebraically split* if, after possible renumbering, there is an index $1 \leq q < n$ such that $\ell k(L_i, L_j) = 0$ whenever $1 \leq i \leq q < j \leq n$. The theorem says that if L is a graph link, then

$$\Delta_L(t_1, \dots, t_n) = 0 \iff L \text{ is algebraically split.} \quad (3.2)$$

This is a very striking property of graph links. For example, it implies that the Alexander polynomial Δ_L of a 2-component graph link is zero if and only if the linking number ℓ of the components is zero. For general 2-component links L , if Δ_L vanishes, then $\ell = 0$ (by the Torres formula). But the converse is false: the Whitehead link has Alexander polynomial $(t_1 - 1)(t_2 - 1)$, although $\ell = 0$. As a matter of fact, it is still an open question how to characterize geometrically 2-component links with vanishing Alexander polynomial (see [5, Problem 16]).

Before stating and proving our results, let us finally recall very briefly Neumann's argument [8] for fibered solvable links. Let L be an oriented graph link given by a splice diagram Γ . Using the value of the one variable Alexander polynomial Δ_L of L , together with the equality $\Omega_L(t) = \pm t^{-d} \Delta_L(t^2)$, where d is the degree of Δ_L , we obtain the formula

$$\Omega_L(t) = \pm (t - t^{-1}) \prod_v (t^{\ell_v} - t^{-\ell_v})^{\delta_v - 2},$$

where the product is over all non-arrowhead vertices of Γ , δ_v is the valency of the vertex v , and ℓ_v denotes the linking number of L with L_v . Thus, the only issue is the determination of the sign of Ω_L .

If L is a fibered link, its Seifert matrix A is unimodular. Therefore, the leading coefficient of $\Omega_L(t) = \det(t^{-1}A - tA^T)$ is given by $\det(-A)$. For fibered links, Lee Rudolph defined an integer invariant λ called the *enhanced Milnor number*. If the ambient sphere is S^3 , this number is known to satisfy the following formula (see [9]):

$$(-1)^\lambda = \det(-A). \quad (\star)$$

Neumann proves that $\lambda \equiv k_- + j_- \pmod{2}$, where k_- is the number of (-1) -weighted arrowheads and j_- the number of non-arrowhead vertices v for which ℓ_v is negative and δ_v is odd. This leads to

$$\det(-A) = (-1)^{k_- + j_-}, \quad (\star\star)$$

giving the formula

$$\Omega_L(t) = (-1)^{k-} (t - t^{-1}) \prod_v (t^{\ell_v} - t^{-\ell_v})^{\delta_v - 2}. \quad (\star\star\star)$$

These results lead Neumann to the following questions:

- (i) is the formula $(\star\star\star)$ true for any graph link in a homology sphere ?
- (ii) does the equality $(\star\star)$ hold for fibered graph links in a homology sphere ?
- (iii) is the equation (\star) still valid for fibered graph links in a homology sphere ?

Corollaries 1, 2 and 3 provide an affirmative answer to these three questions.

4. Results

We are now ready to state and prove our results. Using (3.1) along with the equations (2.1) and (2.2), it is easy to compute the Conway potential function up to a sign.

LEMMA 1. *The potential function of a graph link L in a homology sphere is given by*

$$\nabla_L(t_1, \dots, t_n) = \epsilon(L) \prod_v (t_1^{\ell_{1v}} \dots t_n^{\ell_{nv}} - t_1^{-\ell_{1v}} \dots t_n^{-\ell_{nv}})^{\delta_v - 2},$$

where $\epsilon(L)$ is equal to $+1$ or -1 .

Proof. By (3.1) and equation (2.1), $\nabla_L(t_1, \dots, t_n)$ is equal to

$$\begin{aligned} \nabla_L &= \begin{cases} \frac{\epsilon(L)}{t_1 - t_1^{-1}} t_1^{\nu_1} (t_1 - t_1^{-1}) \prod_v (t_1^{\ell_{1v}} - t_1^{-\ell_{1v}})^{\delta_v - 2} & \text{if } n = 1; \\ \epsilon(L) t_1^{\nu_1} \dots t_n^{\nu_n} \prod_v (t_1^{\ell_{1v}} \dots t_n^{\ell_{nv}} - t_1^{-\ell_{1v}} \dots t_n^{-\ell_{nv}})^{\delta_v - 2} & \text{if } n \geq 2, \end{cases} \\ &= \epsilon(L) t_1^{\nu_1} \dots t_n^{\nu_n} \prod_v (t_1^{\ell_{1v}} \dots t_n^{\ell_{nv}} - t_1^{-\ell_{1v}} \dots t_n^{-\ell_{nv}})^{\delta_v - 2}, \end{aligned}$$

for some integers ν_1, \dots, ν_n and some sign $\epsilon(L) = \pm 1$. The symmetry formula (2.2) implies that $\nu_1 = \dots = \nu_n = 0$, giving the lemma.

LEMMA 2. *Let K be a graph knot with $(+1)$ -weighted arrowhead. Then $\epsilon(K) = +1$, that is:*

$$\nabla_K(t) = \prod_v (t^{\ell_v} - t^{-\ell_v})^{\delta_v - 2}.$$

Proof. Let us use Kauffman's construction of the reduced potential function

$$\Omega_K(t) = (t - t^{-1}) \nabla_K(t) = \det(t^{-1}A - tA^T),$$

where A is any Seifert matrix for K . It is easy to check that the Seifert matrix for a graph knot K is equal to the Seifert matrix for the knot K' obtained from K by deleting every non-fibered splice component. (In other words: if Γ is a splice diagram for K , a splice diagram Γ' for K' is obtained from Γ by deleting every vertex v such that $\ell_v = 0$.) Therefore, we can assume that K is fibered. In that case, A is unimodular. In particular, $\det(-A)$ is non-zero, so that $\det(-A)$ is the leading coefficient of $\Omega_L(t) = \det(t^{-1}A - tA^T)$. There is an explicit computation of A for graph knots in [4]; it can be used to check the lemma. Details can be found in [2].

LEMMA 3. *Consider $L = L_1 \cup \dots \cup L_n$. If $L - L_i$ is algebraically split for all i , then L is algebraically split.*

Proof. Let us associate to L a graph G_L as follows: the vertices of G_L correspond to the components of L , and two vertices are linked with an edge if the linking number of the corresponding components is not equal to zero. Clearly, L is algebraically split if and only if G_L is not connected. Given a vertex v of a graph G , let us denote by $G - v$ the subgraph obtained by deleting the vertex v and every edge adjacent to v . We are left with the proof of the following assertion: given a graph G , if $G - v$ is not connected for any vertex v of G , then G is not connected. In other words: if G is a connected graph, there exists a vertex v such that $G - v$ is connected. This last statement is very easy to prove: given G a connected graph, let T be a maximal subtree of G . Since T is a tree, it has at least one vertex v of degree one. Then, $T - v$ is connected, as well as $G - v$.

THEOREM 1. *Let L be a graph link with n components given by a splice diagram Γ . Then, its Conway potential function is equal to*

$$\nabla_L(t_1, \dots, t_n) = (-1)^{k_-} \prod_v (t_1^{\ell_{1v}} \dots t_n^{\ell_{nv}} - t_1^{-\ell_{1v}} \dots t_n^{-\ell_{nv}})^{\delta_v - 2},$$

where the product is over all non-arrowhead vertices v of Γ , δ_v is the valency of the vertex v , $\ell_{iv} = \ell k(L_i, L_v)$, and k_- is equal to the number of (-1) -weighted arrowheads.

Proof. By formula (2.3), it may be assumed that all the arrowheads have weight $(+1)$. Using the notation of Lemma 1, we have to check that if $k_- = 0$, then $\epsilon(L) = +1$. Let us do this by induction on $n \geq 1$.

The case $n = 1$ is settled by Lemma 2. Let us fix some $n \geq 2$, and assume that $\epsilon(L') = +1$ for all graph link L' with $n' \leq n - 1$ arrowheads, all $(+1)$ -weighted. Let $L = L_1 \cup \dots \cup L_n$ be a graph link with $k_- = 0$. If $\nabla_{L-L_i} = 0$ for all i , then (by (2.1) and (3.2)) $L - L_i$ is algebraically split for all i . By Lemma 3, it follows that L is algebraically split, so that $\nabla_L = 0$. In this case, there is nothing to prove. Therefore, it may be assumed (after possible renumbering) that $\nabla_{L-L_1} \neq 0$. Let us note $L' = L - L_1$. If $\ell_{12} = \dots = \ell_{1n} = 0$, then $L = L_1 \sqcup L'$ is algebraically split, so $\nabla_L = 0$. Hence, we can assume without loss of generality that $\ell_{12} \dots \ell_{1n} \neq 0$. To summarize, it may be assumed that

$$(t_2^{\ell_{12}} \dots t_n^{\ell_{1n}} - t_2^{-\ell_{12}} \dots t_n^{-\ell_{1n}}) \nabla_{L'}(t_2, \dots, t_n) \neq 0.$$

The key ingredient of the induction step is the Torres formula (2.4). By Lemma 1, we have

$$\nabla_L(1, t_2, \dots, t_n) = \epsilon(L) \prod_{v \in \mathcal{V}} (t_2^{\ell_{2v}} \dots t_n^{\ell_{nv}} - t_2^{-\ell_{2v}} \dots t_n^{-\ell_{nv}})^{\delta_v - 2}.$$

On the other hand, by (2.4),

$$\nabla_L(1, t_2, \dots, t_n) = (t_2^{\ell_{12}} \dots t_n^{\ell_{1n}} - t_2^{-\ell_{12}} \dots t_n^{-\ell_{1n}}) \nabla_{L'}(t_2, \dots, t_n),$$

which, by induction, is equal to

$$(t_2^{\ell_{12}} \dots t_n^{\ell_{1n}} - t_2^{-\ell_{12}} \dots t_n^{-\ell_{1n}}) \prod_{v \in \mathcal{V}'} (t_2^{\ell_{2v}} \dots t_n^{\ell_{nv}} - t_2^{-\ell_{2v}} \dots t_n^{-\ell_{nv}})^{\delta_v - 2}.$$

Now, the set of non-arrowhead vertices \mathcal{V}' of L' is equal to $\mathcal{V} \cup \{v_1\}$, where v_1 is the vertex corresponding to the component L_1 . Therefore, $\delta_{v_1} = 1$, $\ell_{v_1 v} = \ell_{1v}$, and we have the equality

$$\epsilon(L) \prod_{v \in \mathcal{V}} (t_2^{\ell_{2v}} \dots t_n^{\ell_{nv}} - t_2^{-\ell_{2v}} \dots t_n^{-\ell_{nv}})^{\delta_v - 2} = \prod_{v \in \mathcal{V}} (t_2^{\ell_{2v}} \dots t_n^{\ell_{nv}} - t_2^{-\ell_{2v}} \dots t_n^{-\ell_{nv}})^{\delta_v - 2}.$$

Since

$$\prod_{v \in \mathcal{V}} (t_2^{\ell_{2v}} \cdots t_n^{\ell_{nv}} - t_2^{-\ell_{2v}} \cdots t_n^{-\ell_{nv}})^{\delta_v - 2} = (t_2^{\ell_{12}} \cdots t_n^{\ell_{1n}} - t_2^{-\ell_{12}} \cdots t_n^{-\ell_{1n}}) \nabla_{L'}(t_2, \dots, t_n),$$

which is assumed to be non-zero, $\epsilon(L)$ is equal to $+1$.

COROLLARY 1. *The Conway polynomial of a fibered graph link L is given by*

$$\Omega_L(t) = (-1)^{k_-} (t - t^{-1}) \prod_v (t^{\ell_v} - t^{-\ell_v})^{\delta_v - 2}.$$

Proof. Use the fact that $\Omega_L(t) = (t - t^{-1}) \nabla_L(t, \dots, t)$.

COROLLARY 2. *Let A be a Seifert matrix for a fibered graph link L ; then*

$$\det(-A) = (-1)^{k_- + j_-},$$

where j_- is the number of non-arrowhead vertices v with ℓ_v negative and δ_v odd.

Proof. If L is a fibered link, A is unimodular, so $\det(-A)$ is the leading coefficient of $\Omega_L(t) = \det(t^{-1}A - tA^T)$. By Corollary 1, $\det(-A)$ is the leading coefficient of $(-1)^{k_-} \prod_v (t^{\ell_v} - t^{-\ell_v})^{\delta_v - 2}$, which is clearly $(-1)^{k_- + j_-}$.

COROLLARY 3. *Let A be a Seifert matrix for a fibered graph link L ; then*

$$\det(-A) = (-1)^\lambda,$$

where λ is the enhanced Milnor number of L .

Proof. By Theorem 6.1 and §10 in [9], we have the equality

$$\lambda = \ell k(P \cup L_+, N \cup L_-),$$

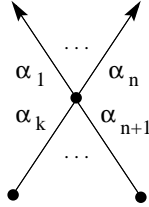
where the notations are as in [8]. By Corollary 2, it remains to show that

$$\ell k(P \cup L_+, N \cup L_-) \equiv k_- + j_- \pmod{2}.$$

The argument is exactly as in [8].

Of course, Corollaries 2 and 3 are false for non-fibered graph links. For example, a Seifert matrix A for the trivial 2-component link satisfies $\det(-A) = 0$.

EXAMPLE. Consider the graph link L given by the splice diagram illustrated below,



and let us note $\alpha = \alpha_1 \cdots \alpha_k$. Then, the potential function of L is equal to

$$\nabla_L(t_1, \dots, t_n) = \frac{\left(t_1^{\alpha/\alpha_1} \cdots t_n^{\alpha/\alpha_n} - t_1^{-\alpha/\alpha_1} \cdots t_n^{-\alpha/\alpha_n} \right)^{k-2}}{\prod_{i=n+1}^k \left(t_1^{\alpha/\alpha_1 \alpha_i} \cdots t_n^{\alpha/\alpha_n \alpha_i} - t_1^{-\alpha/\alpha_1 \alpha_i} \cdots t_n^{-\alpha/\alpha_n \alpha_i} \right)}.$$

Acknowledgements. The author wishes to express his thanks to Françoise Michel, Claude Weber and Mathieu Baillif.

REFERENCES

- [1] S. BOYER and D. LINES. Conway potential functions for links in \mathbb{Q} -homology 3-spheres. *Proc. Edinburgh Math. Soc.* (2) **35** (1992), 53–69.
- [2] D. CIMASONI. Alexander invariants of multilinks. *PhD. thesis*. University of Geneva (2002).
- [3] J. CONWAY. An enumeration of knots and links, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Oxford, 1967)*, Proc. Conf. (Pergamon Press, Oxford, 1967), pp. 329–358.
- [4] D. EISENBUD and W. NEUMANN. Three-dimensional link theory and invariants of plane curve singularities. *Annals of Mathematics Studies*, vol. 110 (Princeton University Press, 1985).
- [5] R. H. FOX. Some problems in knot theory. In *Topology of 3-manifolds and related topics (1961)*, Proc. The Univ. of Georgia Institute (Prentice-Hall, Englewood Cliffs N.J., 1962), pp. 168–176.
- [6] R. HARTLEY. The Conway potential function for links. *Comment. Math. Helv.* **58** (1983), 365–378.
- [7] L. H. KAUFFMAN. The Conway polynomial. *Topology* **20** (1981), 101–108.
- [8] W. NEUMANN. Conway polynomial of a fibered solvable link. *J. Knot Theory Ramifications* **8** (1999), 505–509.
- [9] W. NEUMANN and L. RUDOLPH. Difference index of vectorfields and the enhanced Milnor number. *Topology* **29** (1990), 83–100.
- [10] V. TURAEV. Reidemeister torsion in knot theory. *Russian Math. Surveys* **14** (1986), 119–182.