THE ALEXANDER MODULE OF LINKS AT INFINITY

DAVID CIMASONI

Abstract. Walter Neumann [5] showed that the topology of a "regular" algebraic curve $V \subset \mathbb{C}^2$ is determined up to proper isotopy by some link in S^3 called the link at infinity of V. In this note, we compute the Alexander module over $\mathbb{C}[t^{\pm 1}]$ of any such link at infinity.

1. Introduction

The intersection of a reduced algebraic curve $V \subset \mathbb{C}^2$ with any sufficiently large sphere S^3 about the origin in $\mathbb{C}^2$ gives a well-defined link called the *link at infinity* of $V \subset \mathbb{C}^2$. This link at infinity was first introduced by Walter Neumann and Lee Rudolph [4] and studied further by Neumann [5]. In order to state several of their results, let us recall some terminology. The fiber $f^{-1}(c)$ of a polynomial map $f\colon \mathbb{C}^2 \to \mathbb{C}$ is called *regular* if there exists a neighborhood D of c in $\mathbb{C}$ such that $f|\colon f^{-1}(D) \to D$ is a locally trivial fibration. An algebraic curve $V \subset \mathbb{C}^2$ is *regular* if it is a regular fiber of its defining polynomial. One might think that if c is not a singular value of f, then $f^{-1}(c)$ is regular; this is wrong. In fact, the following additional condition is required: a fiber $f^{-1}(c)$ is *regular at infinity* if there exists a neighborhood D of c in $\mathbb{C}$ and a compact K in $\mathbb{C}^2$ such that f restricted to $f^{-1}(D) \setminus K$ is a locally trivial fibration. It can be proved that $f^{-1}(c)$ is regular if and only if it is non-singular and regular at infinity [3].

A first interesting result is that only finitely many fibers of a given f are irregular at infinity, and that the regular fibers all define the same link at infinity up to isotopy: it is called the *regular link at infinity* of f, and denoted by $\mathcal{L}(f,\infty)$. Furthermore, $\mathcal{L}(f,\infty)$ is a fibered link if and only if all the fibers of f are regular at infinity. Finally, Walter Neumann proved the following striking result: the topology of a regular algebraic curve $V \subset \mathbb{C}^2$ (as an embedded smooth manifold) is determined by its link at infinity. More precisely: up to isotopy in S^3, there exists a unique minimal Seifert surface F for $\mathcal{L}(f,\infty)$, and V is properly isotopic to the embedded surface obtained from F by attaching a collar out to infinity in $\mathbb{C}^2$ to the boundary of F.

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In the present note, we give a closed formula for the Alexander module over $\mathbb{C}[t^{\pm 1}]$ of the regular link at infinity of any polynomial map $f\colon \mathbb{C}^2 \to \mathbb{C}$ (Theorem 3.2). The decisive property of $\mathcal{L}(f,\infty)$ is that it can be seen as the boundary of the fiber F of a fibered multilink (see [5, Theorem 4]). Furthermore, this multilink can be constructed by iterated cabling and connected sum operations from the unknot, and the Alexander module over $\mathbb{C}[t^{\pm 1}]$ of this type of fibered multilinks is well-known (we recall this result of [2] in Theorem 3.1 below). Therefore, our method will be to consider a fibered multilink with fiber F and Alexander module A, and to compute the Alexander module of the oriented link $\mathcal{L} = \partial F$ from the module A (Proposition 2.5). This is achieved by introducing "generalized Seifert forms" for the multilink, and comparing them with the traditional Seifert form for $\mathcal{L}$. The result is then applied to $\mathcal{L}(f,\infty)$ (Theorem 3.2), and an example concludes the paper.

2. Fibered multilinks

A *multilink* [2] is an oriented link $L = L_1 \cup \ldots \cup L_n$ in S^3 together with an integer m_i associated with each component L_i, with the convention that a component L_i with multiplicity m_i is the same as $-L_i$ (L_i with reversed orientation) with mutliplicity $-m_i$. Throughout this paper, we will write $\underline{m}$ for the integers $(m_1, \ldots, m_n)$, d for their greatest common divisor, and $L(\underline{m})$ for the multilink. Of course, a set of multiplicities $\underline{m}$ can be thought of as an element of $\mathrm{H}_1(L)$. If X denotes the exterior of L, several classical theorems implie that $\mathrm{H}_1(L)$ is isomorphic to $[X, S^1]$, the group of homotopy classes of maps $X \to S^1$. As a consequence, assigning a set of multiplicities to an oriented link is a way to specify a preferred infinite cyclic covering $\widetilde{X}(\underline{m}) \stackrel{p}{\to} X$: it is the pullback $\mathbb{Z}$-bundle $\phi^* \exp$, where $\mathbb{R} \stackrel{\exp}{\to} S^1$ is the universal $\mathbb{Z}$-bundle and $X \stackrel{\phi}{\to} S^1$ any map in the homotopy class $\underline{m}$:

$$\begin{array}{ccc} \widetilde{X}(\underline{m}) & \longrightarrow & \mathbb{R} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \exp} \\ X & \xrightarrow{\ \phi\ } & S^1. \end{array}$$

Choosing a generator t of the infinite cyclic group of the covering endows $\mathrm{H}_*(\widetilde{X}(\underline{m}))$ with a structure of module over $\mathbb{Z} < t > = \mathbb{Z}[t^{\pm 1}]$, the ring of Laurent polynomials with integer coefficients. Most of these invariants are not interesting: it is easy to prove that $\mathrm{H}_0(\widetilde{X}(\underline{m})) \simeq \mathbb{Z}[t^{\pm 1}]/(t^d - 1)$, that $\mathrm{H}_2(\widetilde{X}(\underline{m}))$ is a free module with the same rank as $\mathrm{H}_1(\widetilde{X}(\underline{m}))$, and of course, that $\mathrm{H}_i(\widetilde{X}(\underline{m})) = 0$ for all $i \geq 3$. Therefore, the only interesting module is

$H_1(\widetilde{X}(\underline{m}))$: it is called the *Alexander module* of the multilink $L(\underline{m})$, and we will denote it by $A(L(\underline{m}))$. Also, we will write $A(L(\underline{m});\mathbb{K})$ for the $\mathbb{K}[t^{\pm 1}]$-module $A(L(\underline{m}))\otimes\mathbb{K}[t^{\pm 1}]$, where $\mathbb{K}=\mathbb{Q}$ or $\mathbb{C}$. Given $\mathcal{P}$ an $m\times n$ presentation matrix of $A(L(\underline{m}))$ (that is, the matrix corresponding to a finite presentation of $A(L(\underline{m}))$ with n generators and m relations), the greatest common divisor of the $n\times n$ minor determinants of $\mathcal{P}$ is called the *Alexander polynomial* of $L(\underline{m})$. This Laurent polynomial, denoted by $\Delta_{L(\underline{m})}$, is only defined up to multiplication by units of $\mathbb{Z}[t^{\pm 1}]$. Of course, if a multilink has multiplicities ± 1, it is just an oriented link and these Alexander invariants coincide with the usual Alexander invariants of the corresponding oriented link.

Let us now recall the definition of a very interesting class of multilinks that generalizes the notion of fibered link: a *fibered multilink* is a multilink $L(\underline{m})$ such that there exists a locally trivial fibration $X \xrightarrow{\varphi} S^1$ in the homotopy class $\underline{m}\in[X,S^1]$. The oriented surface $F=\varphi^{-1}(1)$ is called the *fiber* of $L(\underline{m})$. The diagram

$$\begin{array}{ccc} \widetilde{X}(\underline{m}) & \xrightarrow{\Phi} & \mathbb{R} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \exp} \\ X & \xrightarrow{\varphi} & S^1 \end{array}$$

can now be understood as defining the pullback fibration $\Phi=\exp^*\varphi$. Since $\mathbb{R}$ is contractible, there exists a homeomorphism $F\times\mathbb{R}\to\widetilde{X}(\underline{m})$ such that the following diagram commutes:

$$\begin{array}{ccc} F\times\mathbb{R} & \longrightarrow & \widetilde{X}(\underline{m}) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \mathbb{R} & = & \mathbb{R}. \end{array}$$

Hence, the generator $\widetilde{X}(\underline{m}) \xrightarrow{t} \widetilde{X}(\underline{m})$ of the infinite cyclic group of the covering p can be seen as the transformation

$$\begin{array}{ccc} F\times\mathbb{R} & \longrightarrow & F\times\mathbb{R} \\ (x,z) & \longmapsto & (h(x),z+1), \end{array}$$

where $F\xrightarrow{h} F$ is some homeomorphism, unique up to isotopy, called the *monodromy* of the multilink $L(\underline{m})$. We will use the same terminology for the induced automorphism $H_1(F)\xrightarrow{h_*}H_1(F)$.

PROPOSITION 2.1. *A presentation matrix of the Alexander module of a fibered multilink is given by* H^T-tI, *where* H *is any matrix of the monodromy. In*

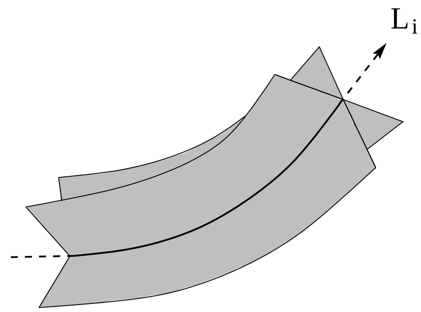


Figure 1: $\overline{F}$ near the component L_i of a multilink, with $m_i = 4$.

particular, the Alexander polynomial of a fibered multilink is the characteristic polynomial of the monodromy.

Proof. As seen in the above discussion, there is an isomorphism of $\mathbb{Z}$-modules $\mathrm{H}_1(F) \xrightarrow{f} \mathrm{H}_1(\widetilde{X}(\underline{m}))$ such that $t \cdot f(x) = f(h_*(x))$. Choosing a $\mathbb{Z}$-basis $e_1, \ldots, e_\mu$ of $\mathrm{H}_1(F)$, this gives an exact sequence of $\mathbb{Z}[t^{\pm 1}]$-modules

$$\bigoplus_{i=1}^{\mu} \mathbb{Z}[t^{\pm 1}]\, e_i \xrightarrow{h_* - t} \bigoplus_{i=1}^{\mu} \mathbb{Z}[t^{\pm 1}]\, e_i \xrightarrow{f_*} \mathrm{H}_1(\widetilde{X}(\underline{m})) \longrightarrow 0,$$

where f_* denotes the $\mathbb{Z}[t^{\pm 1}]$-linear extension of f. This is a finite presentation of $\mathrm{H}_1(\widetilde{X}(\underline{m}))$, so $(H - tI)^T$ is a presentation matrix of this module. □

Let $F \subset S^3 \setminus L$ be the fiber of a fibered multilink $L(\underline{m})$, and let us denote by $\overline{F}$ the union $F \cup L$ (see Figure 1 for an illustration of F and $\overline{F}$ near a component of the multilink). The *Seifert forms* associated to F are the bilinear forms

$$\alpha_+, \alpha_- \colon \mathrm{H}_1(F) \times \mathrm{H}_1(\overline{F}) \longrightarrow \mathbb{Z}$$

given by $\alpha_+(x, y) = \ell k(i_+ x, y)$ and $\alpha_-(x, y) = \ell k(i_- x, y)$, where ℓk denotes the linking number and $i_+, i_- \colon \mathrm{H}_1(F) \to \mathrm{H}_1(S^3 \setminus \overline{F})$ the morphisms induced by the push in the positive or negative normal direction off F. We will use the notation V_+, V_- for matrices of these forms.

As in the usual case of a fibered oriented link, the monodromy can be recovered from the Seifert forms.

Proposition 2.2. *If a multilink is fibered with fiber F, the matrices V_+ and V_- are square and unimodular. Furthermore, a matrix of the monodromy is given by $H = (V_+ V_-^{-1})^T$.*

Proof. The bilinear form $\alpha_+ \colon \mathrm{H}_1(F) \times \mathrm{H}_1(\overline{F}) \to \mathbb{Z}$ can be understood as a homomorphism $\mathrm{H}_1(F) \to \mathrm{Hom}(\mathrm{H}_1(\overline{F}), \mathbb{Z}) \simeq \mathrm{H}^1(\overline{F})$. The composition of this morphism with the Alexander isomorphism $\mathrm{H}^1(\overline{F}) \simeq \mathrm{H}_1(S^3 \setminus \overline{F})$ is nothing but $i_+ \colon \mathrm{H}_1(F) \to \mathrm{H}_1(S^3 \setminus \overline{F})$. The same holds for α_- and i_-. As a consequence, the Seifert matrix V_+ (resp. V_-) with respect to basis $\mathcal{A}$ of $\mathrm{H}_1(F)$ and $\overline{\mathcal{A}}$ of $\mathrm{H}_1(\overline{F})$ is equal to the transposed matrix of i_+ (resp. i_-) with respect to the basis $\mathcal{A}$ and $\overline{\mathcal{A}}^*$, where $\overline{\mathcal{A}}^*$ is the dual basis of $\overline{\mathcal{A}}$ via Alexander duality.

Now, the fibration $S^3 \setminus L \to S^1$ yields a fibration $S^3 \setminus \overline{F} \to (0,1)$, so $S^3 \setminus \overline{F}$ is homeomorphic to $F \times (0,1)$. Hence, the maps $i_+, i_- \colon F \to S^3 \setminus \overline{F}$ are homotopy equivalences, and $i_+, i_- \colon \mathrm{H}_1(F) \to \mathrm{H}_1(S^3 \setminus \overline{F})$ are isomorphisms. Therefore, the matrices V_+, V_- are unimodular. Finally, the monodromy of a fibered multilink can be defined as the composition $(i_-)^{-1} \circ (i_+)$. Therefore, a matrix of the monodromy is given by $H = (V_-^{-1})^T V_+^T = (V_+ V_-^{-1})^T$. □

As an immediate consequence of this proposition, $\mathrm{H}_1(F)$ and $\mathrm{H}_1(\overline{F})$ have the same rank. We need some more information about these modules.

Lemma 2.3. *Let $L(\underline{m})$ be a fibered multilink with fiber F of genus g. For $i = 1, \ldots, n$, F has $d_i = \gcd(m_i, \sum_{j \neq i} m_j \ell k(L_i, L_j))$ boundary components near L_i. Furthermore, the homology of F has the form*

$$\mathrm{H}_1(F) \;=\; G \,\oplus \bigoplus_{\substack{i=1\ldots n-1 \\ j=1\ldots d_i}} \mathbb{Z}T_i^j \,\oplus\, \bigoplus_{j=1}^{d_n - d} \mathbb{Z}T_n^j,$$

where G is a free $\mathbb{Z}$-module of rank $2dg$, and $T_i^1, \ldots, T_i^{d_i}$ are the boundary components of F near L_i. Finally,

$$\mathrm{H}_1(\overline{F}) \;=\; G \,\oplus \left(\bigoplus_{i=1}^{n} \mathbb{Z}L_i \Big/ \sum_{i=1}^{n} \frac{m_i}{d} L_i \right) \oplus\, \overline{B}\,,$$

where $\overline{B}$ is a free $\mathbb{Z}$-module of rank $1 - n - d + \sum_{i=1}^{n} d_i$.

Proof. The fact that $F \cap \mathcal{N}(L_i)$ is a link with d_i components is very easy to check and well-known (see [2, p. 30]). Since F consists of d parallel copies of the fiber of the multilink $L(\frac{\underline{m}}{d})$, it may be assumed that $d = 1$. In this case, F is a connected oriented surface of genus g with $\sum_{i=1}^{n} d_i$ boundary components and the result holds.

We will now compute $\mathrm{H}_1(\overline{F})$ by induction on $d \geq 1$. Let us assume that $d = 1$. The Mayer-Vietoris exact sequence associated with the decomposition $\overline{F} = F \cup (\overline{F} \cap \mathcal{N}(L))$ gives

$$0 \to \mathrm{H}_1(\partial F) \stackrel{\varphi_1}{\to} \mathrm{H}_1(F) \oplus \mathrm{H}_1(L) \to \mathrm{H}_1(\overline{F}) \to \widetilde{\mathrm{H}}_0(\partial F) \stackrel{\varphi_0}{\to} \widetilde{\mathrm{H}}_0(L),$$

where $\varphi_1(T_i^j) = (T_i^j, \frac{m_i}{d_i} L_i)$. Using the value of $\mathrm{H}_1(F)$, it follows that $(\mathrm{H}_1(F) \oplus \mathrm{H}_1(L))/\mathrm{Im}\,\varphi_1 = G \oplus (\bigoplus_{i=1}^n \mathbb{Z} L_i / \sum_i m_i L_i)$. Since the module $\mathrm{Ker}\,\varphi_0$ is free of rank $\sum_{i=1}^n (d_i - 1)$, this concludes the case $d = 1$. Let us now consider a fibered multilink $L(\underline{m})$ with $\gcd(m_1, \ldots, m_n) = d > 1$. Clearly, $\overline{F} = \overline{F}' \cup \overline{F}''$, where F' (resp. F'') is the fiber of $L(\frac{\underline{m}}{d})$ (resp. $L(\frac{d-1}{d}\underline{m})$). The associated Mayer-Vietoris sequence together with the case $d = 1$ and the induction hypothesis give the result. □

Proposition 2.4. *Let $L(\underline{m})$ be a fibered multilink. For $i = 1, \ldots, n$, let us note $D_i = \gcd(d_1, \ldots, d_i)$ with d_i as above. Then, the Alexander module of $L(\underline{m})$ naturally factors into $A(L(\underline{m})) = A_G \oplus A_B$, where*

$$A_B = \bigoplus_{i=1}^{n-1} \mathbb{Z}[t^{\pm 1}] \Big/ \Big(\frac{(t^{D_i} - 1)(t^{d_{i+1}} - 1)}{(t^{D_{i+1}} - 1)} \Big).$$

Proof. As seen above, the fiber F is given by d parallel copies of a connected surface $\widetilde{F}$ with $\sum_{i=1}^n \frac{d_i}{d}$ boundary components. Let us write $\widetilde{F} = \widetilde{G} \cup \widetilde{B}$, where $\widetilde{G}$ is a closed surface with a single boundary component, and $\widetilde{B}$ a planar surface with $1 + \sum_{i=1}^n \frac{d_i}{d}$ boundary components. The Mayer-Vietoris sequence gives $\mathrm{H}_1(\widetilde{F}) = \mathrm{H}_1(\widetilde{G}) \oplus \mathrm{H}_1(\widetilde{B})$. Therefore, $\mathrm{H}_1(F) = \mathrm{H}_1(G) \oplus \mathrm{H}_1(B)$, where G (resp. B) consists of d parallel copies of $\widetilde{G}$ (resp. $\widetilde{B}$). Since the monodromy $F \stackrel{h}{\to} F$ of $L(\underline{m})$ is a homeomorphism, the monodromy $\mathrm{H}_1(F) \stackrel{h_*}{\to} \mathrm{H}_1(F)$ splits into $h_G \oplus h_B$, where $h_G = (h|_G)_*$ and $h_B = (h|_B)_*$. Therefore, a matrix H of h_* with respect to some basis $\mathcal{A} = \mathcal{A}_G \cup \mathcal{A}_B$ of $\mathrm{H}_1(F) = \mathrm{H}_1(G) \oplus \mathrm{H}_1(B)$ can be written $H = H_G \oplus H_B$. By Proposition 2.1, $A(L(\underline{m}))$ is presented by

$$H^T - tI \;=\; H_G^T \oplus H_B^T - tI \;=\; (H_G^T - tI) \oplus (H_B^T - tI).$$

Let us denote by A_G (resp. A_B) the $\mathbb{Z}[t^{\pm 1}]$-module presented by $H_G^T - tI$ (resp. $H_B^T - tI$). It remains to compute the module A_B.

As seen in Lemma 2.3, a basis of $\mathrm{H}_1(B)$ is given by

$$\mathcal{A}_B = \left\langle T_1^1, \ldots, T_1^{d_1}, \ldots, T_{n-1}^1, \ldots, T_{n-1}^{d_{n-1}}, T_n^1, \ldots, T_n^{d_n - d} \right\rangle,$$

where $T_i^1, \dots, T_i^{d_i}$ are the boundary components of F near L_i. Clearly, the monodromy cyclically permutes these components, that is: $h_*(T_i^j) = T_i^{j+1}$ for $1 \leq j \leq d_i - 1$ and $h_*(T_i^{d_i}) = T_i^1$. Note that

$$h_*(T_n^{d_n-d}) = T_n^{d_n-d+1} = -\sum_{\substack{i=1\dots n-1\\ j\equiv 1 \pmod d}} T_i^j - \sum_{\substack{1\leq j\leq d_n-d\\ j\equiv 1 \pmod d}} T_n^j$$

in $\mathrm{H}_1(B)$, since $\partial\widetilde{F} = \bigcup_{i=1\dots n}\bigcup_{j\equiv 1 \pmod d} T_i^j$. Therefore, the matrix of h_B with respect to $\mathcal{A}_B$ is given by

$$H_B = \begin{pmatrix} P_1 & & & \\ & \ddots & & v \\ & & P_{n-1} & \\ & & & Q_n \end{pmatrix},$$

where P_i is the $d_i \times d_i$-matrix

$$P_i = \begin{pmatrix} & & 1 \\ 1 & & \\ & \ddots & \\ & & 1 & \end{pmatrix},$$

Q_n the $(d_n - d) \times (d_n - d - 1)$-matrix

$$Q_n = \begin{pmatrix} 0 & \dots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix},$$

and $v = (v_j)$ a vector such that $v_j = -1$ if $j \equiv 1 \pmod d$, $v_j = 0$ else. It is easy to show that $H_B^T - tI$ is equivalent to

$$\begin{pmatrix} t^{d_1}-1 & & & & \\ & t^{d_2}-1 & & & \\ & & \ddots & & \\ & & & t^{d_{n-1}}-1 & \\ \frac{t^{d_1}-1}{t^d-1} & \frac{t^{d_2}-1}{t^d-1} & \dots & \frac{t^{d_{n-1}}-1}{t^d-1} & \frac{t^{d_n}-1}{t^d-1} \end{pmatrix}$$

as a presentation matrix. It is then an exercise to check that the module A_B presented by $H_B^T - tI$ is equal to $\bigoplus_{i=1}^{n-1}\mathbb{Z}[t^{\pm 1}]\Big/\Big(\frac{(t^{D_i}-1)(t^{d_{i+1}}-1)}{(t^{D_{i+1}}-1)}\Big)$. □

We are finally ready to prove the main result of this paragraph.

Proposition 2.5. *Let $L(\underline{m})$ be a fibered multilink with fiber F and multiplicities $m_i \neq 0$ for all i, and let us denote by L' the oriented link given by the boundary ∂F of F. Then, the Alexander module of L' over $\mathbb{Q}[t^{\pm 1}]$ is given by*

$$A(L';\mathbb{Q}) \;=\; (A_G \otimes \mathbb{Q}[t^{\pm 1}]) \oplus \left(\mathbb{Q}[t^{\pm 1}]/(t-1)\right)^{n-1} \oplus \left(\mathbb{Q}[t^{\pm 1}]\right)^{\sum_{i=1}^{n}(d_i-1)},$$

where A_G is the direct factor of $A(L(\underline{m}))$ given in Proposition 2.4, and $d_i = \gcd(m_i, \sum_{j\neq i} m_j \ell k(L_i, L_j))$.

Proof. Since $m_i \neq 0$, it follows that $d_i \neq 0$ for all i. By Lemma 2.3, one can write

$$\begin{aligned}
\mathrm{H}_1(F;\mathbb{Q}) \;&=\; (G\otimes\mathbb{Q}) \,\oplus\, \bigoplus_{i=1}^{n-1} \mathbb{Q} \sum_{j=1}^{d_i} T_i^j \,\oplus \bigoplus_{\substack{i=1\ldots n-1\\ j=1\ldots d_i-1}} \mathbb{Q}T_i^j \,\oplus\, \bigoplus_{j=1}^{d_n-d} \mathbb{Q}T_n^j\\
\mathrm{H}_1(\overline{F};\mathbb{Q}) \;&=\; (G\otimes\mathbb{Q}) \,\oplus\, \bigoplus_{i=1}^{n-1} \mathbb{Q}\, m_i L_i \,\oplus\, (\overline{B}\otimes\mathbb{Q}).
\end{aligned}$$

The matrices V_+ and V_- with respect to these basis of $\mathrm{H}_1(F;\mathbb{Q})$ and $\mathrm{H}_1(\overline{F};\mathbb{Q})$ are of the form

$$V_+ \;=\; \begin{array}{r} {\scriptstyle 2dg}\{ \\ {\scriptstyle n-1}\{ \\ \end{array}\!\!\begin{pmatrix} \overbrace{N}^{2dg} & \overbrace{M^T}^{n-1} & * \\ M & \ell & * \\ * & * & * \end{pmatrix} \qquad V_- \;=\; \begin{array}{r} {\scriptstyle 2dg}\{ \\ {\scriptstyle n-1}\{ \\ \end{array}\!\!\begin{pmatrix} \overbrace{N^T}^{2dg} & \overbrace{M^T}^{n-1} & * \\ M & \ell^T & * \\ * & * & * \end{pmatrix}.$$

As seen in the proof of Proposition 2.4, a matrix H of the monodromy splits into $H_G \oplus H_B$. Furthermore, the basis of $\mathrm{H}_1(F;\mathbb{Q})$ was chosen such that

$$H_B = \begin{pmatrix} I_{n-1} & * \\ 0 & * \end{pmatrix},$$

where I_{n-1} denotes the identity matrix of dimension $n-1$. By Proposition 2.2, $V_+ = H^T V_-$, that is,

$$\begin{pmatrix} N & M^T & * \\ M & \ell & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} H_G^T & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & * & * \end{pmatrix} \begin{pmatrix} N^T & M^T & * \\ M & \ell^T & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} H_G^T N^T & H_G^T M^T & * \\ M & \ell^T & * \\ * & * & * \end{pmatrix}.$$

Therefore, we have the equalities

$$N^T = NH_G\,, \quad M = MH_G\,, \quad \text{and} \quad \ell = \ell^T. \tag{$*$}$$

Let us keep them in mind, and turn to the computation of the Alexander module of L'. Since F has d connected components, a connected Seifert surface F' for L' is obtained from F via $d-1$ handle attachments. Since $d_i \neq 0$ for all i, we can write

$$\mathrm{H}_1(F';\mathbb{Q}) \;=\; (G\otimes\mathbb{Q}) \;\oplus \bigoplus_{\substack{i=1\dots n-1\\ j=1\dots d_i}} \mathbb{Q}(d_iT_i^j) \;\oplus\; \bigoplus_{j=1}^{d_n-1}\mathbb{Q}(d_nT_n^j)\,.$$

The Seifert matrix of L' with respect to this basis has the form

$$V' = \begin{matrix} 2dg\{ \\ d_1\{ \\ \\ \\ d_n-1\{ \end{matrix}\begin{pmatrix} N & * & * \\ & \widetilde{\ell}_1 & \\ & \vdots & \\ & \widetilde{\ell}_n & \end{pmatrix} = \begin{pmatrix} \overbrace{N}^{2dg} & \overbrace{}^{d_1} & & \overbrace{}^{d_n-1} \\ * & \widetilde{\ell}_1^{\,T} & \dots & \widetilde{\ell}_n^{\,T} \\ * & & & \end{pmatrix},$$

where $\widetilde{\ell}_i$ denotes d_i copies of the same line ℓ_i (d_n-1 copies if $i=n$). A presentation matrix of $A(L';\mathbb{Q})$ is given by $\mathcal{P}' = V' - t(V')^T$. Since $\sum_{i,j}T_i^j = 0$ in $\mathrm{H}_1(F')$, it follows that $\ell_n = -\sum_{i=1}^{n-1}\ell_i$. As a presentation matrix, $\mathcal{P}'$ is therefore equivalent to

$$\begin{pmatrix} N-tN^T & * & * \\ & \ell_1(1-t) & \\ & \vdots & \\ & \ell_{n-1}(1-t) & \end{pmatrix} = \begin{pmatrix} N-tN^T & & \\ * & \ell_1^T(1-t)\dots\ell_{n-1}^T(1-t) & 0\dots0 \\ * & & \end{pmatrix},$$

where the number of zero columns is equal to

$$\sum_{i=1}^{n-1}(d_i-1) + (d_n-1) = \sum_{i=1}^{n}(d_i-1).$$

With the notations used above for V_+ and V_-, this matrix is nothing but

$$\begin{pmatrix} N-tN^T & M^T(1-t) & 0\dots0 \\ M(1-t) & \ell(1-t) & 0\dots0 \end{pmatrix}.$$

Let us note $\widetilde{\mathcal{P}}' = \widetilde{V} - t\widetilde{V}^T$, where $\widetilde{V} = \begin{pmatrix} N & M^T \\ M & \ell \end{pmatrix}$. The computation above shows that $\operatorname{rk} A(L'; \mathbb{Q}) \geq \sum_{i=1}^n (d_i - 1)$. The fact that the rank of $A(L'; \mathbb{Q})$ is equal to $\sum_{i=1}^n (d_i - 1)$ can be proved by (at least) two distinct methods. By a more subtle analysis of $V_\pm$, one can check that $\Delta_{L(\underline{m})} = \det \widetilde{\mathcal{P}}' \cdot \Delta'$ with some factor Δ'; since $L(\underline{m})$ is fibered, $\Delta_{L(\underline{m})} \neq 0$ so $\det \widetilde{\mathcal{P}}' \neq 0$ and $\operatorname{rk} A(L'; \mathbb{Q}) = \sum_{i=1}^n (d_i - 1)$. A more conceptual proof goes as follows: L' can be thought of as the result of the "splicing" of $L(\underline{m})$ with multilinks $L^{(1)}(\underline{m}^{(1)}), \ldots, L^{(n)}(\underline{m}^{(n)})$ (see [2, 5]). It can be showed that $\operatorname{rk} A(L^{(i)}(\underline{m}^{(i)})) = d_i - 1$ for $i = 1, \ldots, n$, and that the rank of the Alexander module is additive under splicing (see [1, Theorem 4.3.1 and Proposition 3.2.4]). Since $L(\underline{m})$ is fibered, $\operatorname{rk} A(L(\underline{m})) = 0$ and we get the result.

As a consequence, $\widetilde{\mathcal{P}}'$ is a presentation matrix of the torsion submodule of $A(L'; \mathbb{Q})$. Now, note that

$$(H_G^T \oplus I_{n-1})\widetilde{V}^T = \begin{pmatrix} H_G^T & 0 \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} N^T & M^T \\ M & \ell^T \end{pmatrix} = \begin{pmatrix} H_G^T N^T & H_G^T M^T \\ M & \ell^T \end{pmatrix}.$$

By the equations $(*)$, this is exactly the matrix $\widetilde{V}$. Hence, the torsion submodule of $A(L'; \mathbb{Q})$ is presented by

$$\widetilde{\mathcal{P}}' = \widetilde{V} - t\widetilde{V}^T = (H_G^T \oplus I_{n-1})\widetilde{V}^T - t\widetilde{V}^T = ((H_G^T \oplus I_{n-1}) - tI)\widetilde{V}^T.$$

Since $\det \widetilde{\mathcal{P}}' \neq 0$, $\widetilde{V}^T$ is unimodular. Therefore, $\widetilde{\mathcal{P}}'$ is equivalent as a presentation matrix to $(H_G^T \oplus I_{n-1}) - tI = (H_G^T - tI) \oplus (1-t)I_{n-1}$. This concludes the proof. □

3. Application to the Alexander module of links at infinity

In this paragraph, we use Propositions 2.4 and 2.5 to give a closed formula for the Alexander module over $\mathbb{C}[t^{\pm 1}]$ of the regular link at infinity $\mathcal{L} = \mathcal{L}(f, \infty)$ of any polynomial map $f \colon \mathbb{C}^2 \to \mathbb{C}$. Given such an f, there exists a fibered multilink with multiplicities $m_i \neq 0$ and fiber F such that $\mathcal{L} = \partial F$. Furthermore, this multilink is an iterated torus multilink: it can be constructed by iterated cabling and connected sum operations from the unknot. Since the Alexander module over $\mathbb{C}[t^{\pm 1}]$ of iterated torus fibered multilinks is known, the result for $\mathcal{L}$ will follow directly from Propositions 2.4 and 2.5.

To state our result, we must assume that the reader is familiar with splice diagrams (see [2]). Recall that a splice diagram representing a multilink $L(\underline{m})$ is a tree Γ decorated as follows:

- Some of its leaves (valency one vertices) are drawn as arrowheads and represent components of L; they are endowed with the multiplicity m_i of the corresponding component L_i of L.
- Each edge has an integer weight at any end where it meets a node (vertex of valency greater than one), and these edge-weights around a fixed node are pairwise coprime.

Associated to each non-arrowhead vertex v of Γ is a so-called "virtual component": this is the additional link component that would be represented by a single arrow at that vertex v with edge-weight 1. Splice diagrams are very convenient to compute linking numbers: given two vertices v and w of Γ, the linking number of the corresponding components (virtual or "real") is the product of all the edge-weights adjacent to but not on the shortest path in Γ connecting v and w.

General splice diagrams as described here encode graph multilinks (that is: multilinks in homology sphere with graph manifold exterior). A multilink in S^3 is a graph multilink if and only if it is an iterated torus multilink, so the multilink associated with a polynomial map is encoded by such a splice diagram. Furthermore, Eisenbud and Neumann succeeded in computing the Alexander module over $\mathbb{C}[t^{\pm 1}]$ of any fibered graph multilink $L(\underline{m})$ from its splice diagram Γ. If $L(\underline{m})$ has "uniform twists" (this is the case of the multilink associated with a polynomial map), the result goes as follows.

Let us denote by $\mathcal{N}$ the set of nodes of Γ, by $\mathcal{E}$ the set of edges connecting two nodes and by $\mathcal{V}$ the set of non-arrowhead vertices of Γ. By cutting an edge $E \in \mathcal{E}$ in two, one gets two splice diagrams representing two multilinks; let us denote by d_E the greatest common divisor of the linking numbers of these two multilinks with the vitual component corresponding to the middle of the edge E. For every $v \in \mathcal{V}$, let δ_v denote its valency and $\underline{m}(v)$ the linking number of $L(\underline{m})$ with the virtual component corresponding to v. Finally, for every node $v \in \mathcal{N}$, let d_v be the greatest common divisor of the d_E's of edges $E \in \mathcal{E}$ which meet v, and of all the m_i's of arrowheads adjacent to v.

THEOREM 3.1 (Eisenbud-Neumann [2, Theorem 14.1]). *Let $L(\underline{m})$ be a fibered graph multilink with monodromy h and uniform twists, given by a splice diagram Γ. The Alexander module $A(L(\underline{m}); \mathbb{C})$ is determined by the following properties:*

- *The Jordan normal form of h_* consists of 1×1 and 2×2 Jordan blocks.*

- *The characteristic polynomial of* h_* *is equal to*

$$\Delta(t) = (t^d - 1) \prod_{v \in \mathcal{V}} (t^{|\underline{m}(v)|} - 1)^{\delta_v - 2}.$$

- *The eigenvalues corresponding to the* 2×2 *Jordan blocks are the roots of*

$$\Delta'(t) = (t^d - 1) \frac{\prod_{E \in \mathcal{E}} (t^{d_E} - 1)}{\prod_{v \in \mathcal{N}} (t^{d_v} - 1)}.$$

Let us now state and prove our final result.

THEOREM 3.2. *Let* $f\colon \mathbb{C}^2 \to \mathbb{C}$ *be a polynomial map with regular link at infinity* $\mathcal{L} = \mathcal{L}(f, \infty)$. *If* $L(\underline{m}) = L(m_1, \ldots, m_n)$ *denotes the multilink associated with* $\mathcal{L}$, *let* d *be the greatest common divisor of* $m_1, \ldots, m_n$, *and* $d_i = \gcd(m_i, \sum_{j \neq i} m_j \ell k(L_i, L_j))$ *for* $i = 1, \ldots, n$. *Also, let* $\Delta(t)$ *be the characteristic polynomial of the monodromy of* $L(\underline{m})$, *and* $\Delta'(t)$ *the polynomial corresponding to the* 2×2 *Jordan blocks (as in Theorem 3.1). Then, the Alexander module* $A(\mathcal{L}; \mathbb{C})$ *of* $\mathcal{L}$ *over* $\mathbb{C}[t^{\pm 1}]$ *is given by the following properties:*

- *The rank of* $A(\mathcal{L}; \mathbb{C})$ *is equal to* $\sum_{i=1}^n (d_i - 1)$.
- *The Jordan normal form of* t *restricted to the torsion submodule of* $A(\mathcal{L}; \mathbb{C})$ *consists of* 1×1 *and* 2×2 *Jordan blocks.*
- *The order ideal of the torsion submodule of* $A(\mathcal{L}; \mathbb{C})$ *is generated by*

$$\widetilde{\Delta}(t) = (t - 1)^{n-1} \frac{(t^d - 1)\Delta(t)}{\prod_{i=1}^n (t^{d_i} - 1)}.$$

- *The eigenvalues corresponding to the* 2×2 *Jordan blocks are the roots of* $\Delta'(t)$.

Proof. The regular link at infinity $\mathcal{L}$ is given by the boundary ∂F of the fiber of $L(\underline{m})$, which has non-zero multiplicities. By Propositions 2.4 and 2.5,

$$A(\mathcal{L}; \mathbb{C}) \;=\; (A_G \otimes \mathbb{C}[t^{\pm 1}]) \oplus (\mathbb{C}[t^{\pm 1}]/(t-1))^{n-1} \oplus (\mathbb{C}[t^{\pm 1}])^{\sum_{i=1}^n (d_i - 1)},$$

where $A(L(\underline{m}); \mathbb{C}) = (A_G \otimes \mathbb{C}[t^{\pm 1}]) \oplus \bigoplus_{i=1}^{n-1} \mathbb{C}[t^{\pm 1}] \Big/ \Big(\frac{(t^{D_i} - 1)(t^{d_{i+1}} - 1)}{(t^{D_{i+1}} - 1)} \Big)$. Therefore, the rank of $A(\mathcal{L}; \mathbb{C})$ is $\sum_{i=1}^n (d_i - 1)$ and the order ideal of its torsion submodule is generated by

$$(t-1)^{n-1}\frac{\Delta(t)}{\prod_{i=1}^{n-1}\frac{(t^{D_i}-1)(t^{d_{i+1}}-1)}{(t^{D_{i+1}}-1)}} = (t-1)^{n-1}\frac{(t^d-1)\Delta(t)}{\prod_{i=1}^{n}(t^{d_i}-1)}\,,$$

since $D_1 = d_1$ and $D_n = \gcd(d_1,\dots,d_n) = \gcd(m_1,\dots,m_n) = d$. Furthermore, $A_G \otimes \mathbb{C}[t^{\pm 1}]$ contributes to Jordan blocks of dimension at most two (by Theorem 3.1), and $\bigoplus_{i=1}^{n-1}\mathbb{C}[t^{\pm1}]\Big/\Big(\frac{(t^{D_i}-1)(t^{d_{i+1}}-1)}{(t^{D_{i+1}}-1)}\Big)$ to Jordan blocks of dimension one, since the polynomial $\frac{(t^{D_i}-1)(t^{d_{i+1}}-1)}{(t^{D_{i+1}}-1)}$ has only simple roots. □

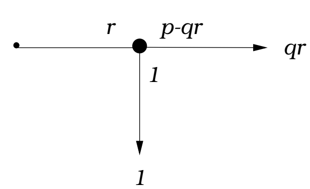
We refer to [1, § 5.6] for a different proof of this result. Note that Propositions 2.4 and 2.5 give the Alexander module $A(\mathcal{L};\mathbb{Q})$ from the module $A(L(\underline{m});\mathbb{Q})$. The problem is that a closed formula for the Alexander module over $\mathbb{Q}[t^{\pm1}]$ of a fibered graph multilink remains unknown.

Let us conclude this note with an example.

Example. Let p, q, r be positive integers with $\gcd(p,r) = 1$ and $p < (q+1)r$. Consider the polynomial map $f\colon \mathbb{C}^2 \to \mathbb{C}$ given by

$$f(x,y) = (x^q y + 1)^r + x^p\,.$$

As described in [5, p. 451], the associated multilink $L(\underline{m}) = L(1, qr)$ is given by the following splice diagram.



Using Theorem 3.1, one easily computes $\Delta(t) = (t-1)\frac{(t^{pr}-1)}{(t^p-1)}$ and $\Delta'(t) = 1$. Hence

$$A(L(\underline{m});\mathbb{C}) = \mathbb{C}[t^{\pm1}]\Big/\left(\frac{(t-1)(t^{pr}-1)}{(t^p-1)}\right).$$

Furthermore, $d_1 = \gcd(1, qr^2) = 1$ and $d_2 = \gcd(qr, r) = r$. Therefore,

$$A(\mathcal{L};\mathbb{C}) = \mathbb{C}[t^{\pm1}]\Big/\left(\frac{(t-1)^2(t^{pr}-1)}{(t^p-1)(t^r-1)}\right) \oplus \left(\mathbb{C}[t^{\pm1}]\right)^{r-1}.$$

Note that $L(\underline{m})$ is nothing but a torus multilink; on this simple example, it is possible to compute the Alexander modules over $\mathbb{Z}[t^{\pm 1}]$. Using methods described in [1], one can show that the Alexander module of $L(\underline{m})$ is $\mathbb{Z}[t^{\pm 1}] \Big/ \left(\frac{(t^{pr}-1)(t-1)}{(t^p-1)}\right)$, and that $A(\mathcal{L}) = \left(\mathbb{Z}[t^{\pm 1}]\right)^{r-1} \oplus \widetilde{A}(\mathcal{L})$, where $\widetilde{A}(\mathcal{L})$ is presented by the matrix

$$\begin{pmatrix} \frac{(t-1)(t^{pr}-1)}{(t^p-1)(t^r-1)} & q \\ 0 & q(t-1) \end{pmatrix}.$$

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