A Lagrangian representation of tangles

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Abstract

We construct a functor from the category of oriented tangles in $\mathbb{R}^3$ to the category of Hermitian modules and Lagrangian relations over $\mathbb{Z}[t, t^{-1}]$. This functor extends the Burau representations of the braid groups and its generalization to string links due to Le Dimet.

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1. Introduction

The aim of this paper is to generalize the classical Burau representation of braid groups to tangles. The Burau representation is a homomorphism from the group of braids on $n$ strands to the group of $(n \times n)$-matrices over the ring $A = \mathbb{Z}[t, t^{-1}]$, where $n$ is a positive integer. This representation has been extensively studied by various authors since the foundational work of Burau [2]. In the last 15 years, new important representations of braid groups came to light, specifically those associated with the Jones knot polynomial, $R$-matrices, and ribbon categories. These latter representations do extend to tangles, so it is natural to ask whether the Burau representation has a similar property.

An extension of the Burau representation to a certain class of tangles was first pointed out by Le Dimet [5]. He considered so-called ‘string links’, which are tangles whose all components are intervals going from the bottom to the top but not necessarily monotonically. The string links on $n$ strands form a

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monoid with respect to the usual composition of tangles. Le Dimet’s work yields a homomorphism of this monoid into the group of \((n \times n)\)-matrices over the quotient field of \(A\). For braids, this gives the Burau representation. The construction of Le Dimet also applies to colored string links, giving a generalization of the Gassner representation of the pure braid group. These representations of Le Dimet were studied by Kirk et al. [4] (see also [6,9]).

To extend the Burau representation to arbitrary oriented tangles, we first observe that oriented tangles do not form a group or a monoid but rather a category \textbf{Tangles} whose objects are finite sequences of \(\pm 1\). An extension of the Burau representation to \textbf{Tangles} should be a functor from \textbf{Tangles} to some algebraically defined category. We show that the relevant algebraic category is the one of Hermitian \(A\)-modules and Lagrangian relations. Our principal result is a construction of a functor from \textbf{Tangles} to this category. For braids and string links, our constructions are equivalent to those of Burau and Le Dimet.

The appearance of Lagrangian relations rather than homomorphisms is parallel to the following well-known observations concerning cobordisms. Generally speaking, a cobordism \((W, M_-, M_+)\) does not induce a homomorphism from the homology (with any coefficients) of the bottom base \(M_-\) to the homology of the top base \(M_+\). However, the kernel of the inclusion homomorphism \(H_*(M_-) \oplus H_*(M_+) \to H_*(W)\) can be viewed as a morphism from \(H_*(M_-)\) to \(H_*(M_+)\) determined by \(W\). This kernel is Lagrangian with respect to the usual intersection form in homology. These observations suggest a definition of a Lagrangian category over any integral domain with involution. Applying these ideas to the infinite cyclic covering of the tangle exterior, we obtain our functor from the category of tangles to the category of Lagrangian relations over \(A\). Parallel constructions involving 2-fold coverings are studied in [7].

Note that recently, a most interesting representation of braid groups due to R. Lawrence was shown to be faithful by S. Bigelow and D. Krammer. We do not know whether this representation extends to tangles.

The organization of the paper is as follows. In Section 2, we introduce the category \textbf{Lagr}_\(A\) of Lagrangian relations over the ring \(A\). In Section 3, we define our functor \textbf{Tangles} \(\to\) \textbf{Lagr}_\(A\). Section 4 deals with the proof of three technical lemmas stated in the previous section. In Section 5, we discuss the case of braids and string links. Finally, Section 6 outlines a multivariable generalization of the theory as well as a high-dimensional version.

2. Category of Lagrangian relations

Fix throughout this section an integral domain \(A\) (i.e., a commutative ring with unit and without zero-divisors) with ring involution \(A \to A, \lambda \mapsto \bar{\lambda}\).

2.1. Hermitian modules

A \textit{skew-hermitian form} on a \(A\)-module \(H\) is a form \(\omega: H \times H \to A\) such that for all \(x, x', y \in H\) and all \(\lambda, \lambda' \in A\),

\begin{enumerate}
  \item \(\omega(\lambda x + \lambda' x', y) = \lambda \omega(x, y) + \lambda' \omega(x', y)\),
  \item \(\omega(x, y) = -\omega(y, x)\).
\end{enumerate}
Such a form is called *non-degenerate* when it satisfies:

(iii) If $\omega(x, y) = 0$ for all $y \in H$, then $x = 0$.

A *Hermitian $\Lambda$-module* is a finitely generated $\Lambda$-module $H$ endowed with a non-degenerate skew-hermitian form $\omega$. The same module $H$ with the opposite form $-\omega$ will be denoted by $-H$. Note that a Hermitian $\Lambda$-module is always torsion-free.

For a submodule $A \subset H$, denote by $\text{Ann}(A)$ the annihilator of $A$ with respect to $\omega$, that is, the module \[ \{ x \in H \mid \omega(x, a) = 0 \text{ for all } a \in A \}. \] We say that $A$ is *isotropic* if $A \subset \text{Ann}(A)$, and *Lagrangian* if $A = \text{Ann}(A)$.

Given a submodule $A$ of $H$, denote by \[ \overline{A} = \{ x \in H \mid \lambda x \in A \text{ for a non-zero } \lambda \in \Lambda \}. \] Clearly $A \subset \overline{A}$ and $\overline{\text{Ann}(A)} = \text{Ann}(A) = \text{Ann}(\overline{A})$. Note that for any Lagrangian $A \subset H$, we have $A = \overline{A}$.

**Lemma 2.1.** For any submodule $A$ of a Hermitian $\Lambda$-module $H$,

\[ \text{Ann}(\text{Ann}(A)) = \overline{A}. \]

**Proof.** Let $Q = Q(\Lambda)$ denote the field of fractions of $\Lambda$. Given a $\Lambda$-module $F$, denote by $F_Q$ the vector space $F \otimes_\Lambda Q$. Note that the kernel of the natural homomorphism $F \to F_Q$ is the $\Lambda$-torsion $\text{Tors}_\Lambda F \subset F$.

The form $\omega$ uniquely extends to a skew-hermitian form $H_Q \times H_Q \to Q$. Given a linear subspace $V$ of $H_Q$, let $\text{Ann}_Q(V)$ be the annihilator of $V$ with respect to the latter form. Observe that $\text{Ann}_Q(\text{Ann}_Q(V)) = V$. Indeed, one inclusion is trivial and the other one follows from dimension count, since $\text{dim}(\text{Ann}_Q(V)) = \text{dim}(H_Q) - \text{dim}(V)$.

The inclusion $A \hookrightarrow H$ induces an inclusion $A_Q \hookrightarrow H_Q$. Since $H$ is torsion-free, $H \subset H_Q$ (and $A \subset A_Q$). Clearly, $\overline{A} = A_Q \cap H$ and $\text{Ann}(A)_Q = \text{Ann}_Q(A_Q)$. Replacing in the latter formula $A$ with $\text{Ann}(A)$, we obtain

\[ \text{Ann}(\text{Ann}(A))_Q = \text{Ann}_Q(\text{Ann}(A)_Q) = \text{Ann}_Q(\text{Ann}_Q(A_Q)) = A_Q. \]

Therefore

\[ \overline{A} = A_Q \cap H = \text{Ann}(\text{Ann}(A))_Q \cap H = \overline{\text{Ann}(\text{Ann}(A))} = \text{Ann}(\text{Ann}(A)), \]

and the lemma is proved. $\square$

**Lemma 2.2.** For any submodules $A, B \subset H$,

\[ \text{Ann}(A + B) = \text{Ann}(A) \cap \text{Ann}(B), \quad \text{Ann}(A \cap B) = \overline{\text{Ann}(A) + \text{Ann}(B)}. \]

**Proof.** The first equality is obvious, and implies

\[ \text{Ann}(\text{Ann}(A) + \text{Ann}(B)) = \text{Ann}(\text{Ann}(A)) \cap \text{Ann}(\text{Ann}(B)) \]

\[ = \overline{A} \cap \overline{B} = \overline{A \cap B}. \]
Therefore

\[ \text{Ann}(A \cap B) = \text{Ann}(\overline{A \cap B}) = \text{Ann}(\text{Ann}(\text{Ann}(A) + \text{Ann}(B))), \]

which is equal to \( \text{Ann}(A) + \text{Ann}(B) \) by Lemma 2.1. □

**Lemma 2.3.** For any submodules \( A \subset B \subset H \), we have \( \overline{B}/A = \overline{B}/A \subset H/A \).

**Proof.** Consider the canonical projection \( \pi: H \rightarrow H/A \). Clearly,

\[ \pi(\overline{B}) = \{ \xi \in H/A \mid \lambda \xi \in B/A \text{ for a non-zero } \lambda \in A \} = \overline{B}/A. \]

Also ker(\( \pi|_{\overline{B}} \)) = ker(\( \pi \)) \cap \overline{B} = A \cap \overline{B} = A. Hence \( \overline{B}/A = \overline{B}/A \). □

### 2.2. Lagrangian contractions

The results above in hand, we can develop the theory of Lagrangian contractions and Lagrangian relations over \( A \) by mimicking the well-known theory over \( \mathbb{R} \) (see, for instance, [10, Section IV.3]).

Let \( (H, \omega) \) be a Hermitian \( A \)-module as above. Let \( A \) be an isotropic submodule of \( H \) such that \( A = \overline{A} \). Denote by \( H/A \) the quotient module \( \text{Ann}(A)/A \) with the skew-hermitian form

\[ (x \mod A, y \mod A) = \omega(x, y). \]

For a submodule \( L \subset H \), set

\[ L|A = ((L + A) \cap \text{Ann}(A))/A \subset H|A. \]

We say that \( L|A \) is obtained from \( L \) by **contraction along** \( A \).

**Lemma 2.4.** \( H|A \) is a Hermitian \( A \)-module. If \( L \) is a Lagrangian submodule of \( H \), then \( L|A \) is a Lagrangian submodule of \( H|A \).

**Proof.** To check that the form on \( H|A \) is non-degenerate, pick \( x \in \text{Ann}(A) \) such that \( \omega(x, y) = 0 \) for all \( y \in \text{Ann}(A) \). Then, \( x \in \text{Ann}(\text{Ann}(A)) = \overline{A} = A \) so that \( x \mod A = 0 \).

To prove the second claim of the lemma, set \( B = (L + A) \cap \text{Ann}(A) \subset H \). We claim that \( \text{Ann}(B) = \overline{B} \). Since both \( A \) and \( L \) are isotropic, it is easy to check that \( B \subset \text{Ann}(B) \) and therefore \( \overline{B} \subset \text{Ann}(B) \). Let us verify the opposite inclusion. Lemmas 2.1 and 2.2 imply that

\[ \text{Ann}(B) = \text{Ann}((L + A) \cap \text{Ann}(A)) = \text{Ann}(L + A) + \text{Ann}(\text{Ann}(A)) \]

\[ \subset \text{Ann}(L) + \overline{A} = \overline{L + A}. \]

Since \( A \subset B \), we have \( \text{Ann}(B) \subset \text{Ann}(A) \) and therefore

\[ \text{Ann}(B) \subset \overline{L + A} \cap \text{Ann}(A) = (L + A) \cap \text{Ann}(A) = \overline{B}. \]

Thus \( \text{Ann}(B) = \overline{B} \). This implies that \( \text{Ann}(B/A) = \overline{B}/A \), which is equal to \( B/A \) by Lemma 2.3. So \( B/A \) is Lagrangian. □
2.3. Category of Lagrangian relations

Let $H_1, H_2$ be Hermitian $A$-modules. A Lagrangian relation between $H_1$ and $H_2$ is a Lagrangian submodule of $(-H_1) \oplus H_2$ (the latter is a Hermitian $A$-module in the obvious way). For a Lagrangian relation $N \subset (-H_1) \oplus H_2$, we shall use the notation $N: H_1 \Rightarrow H_2$.

For a Hermitian $A$-module $H$, the submodule of $H \oplus H$

$$\text{diag}_H = \{h \oplus h \in (-H) \oplus H \mid h \in H\}$$

is clearly a Lagrangian relation $H \Rightarrow H$. It is called the diagonal Lagrangian relation. Given two Lagrangian relations $N_1: H_1 \Rightarrow H_2$ and $N_2: H_2 \Rightarrow H_3$, their composition is defined by $N_2 \circ N_1 = \overline{N_2N_1}: H_1 \Rightarrow H_3$, where $N_2N_1$ denotes the following submodule of $(-H_1) \oplus H_3$:

$$N_2N_1 = \{h_1 \oplus h_3 \mid h_1 \oplus h_2 \in N_1 \text{ and } h_2 \oplus h_3 \in N_2 \text{ for a certain } h_2 \in H_2\}.$$

**Lemma 2.5.** The composition of two Lagrangian relations is a Lagrangian relation.

**Proof.** Given two Lagrangian relations $N_1: H_1 \Rightarrow H_2$ and $N_2: H_2 \Rightarrow H_3$, consider the Hermitian $A$-module $H = (-H_1) \oplus H_2 \oplus (-H_2) \oplus H_3$ and its isotropic submodule

$$A = 0 \oplus \text{diag}_{H_2} \oplus 0 = \{0 \oplus h \oplus h \oplus 0 \mid h \in H_2\}.$$

Note that $\overline{A} = A$. It follows from the non-degeneracy of $H_2$ that $\text{Ann}(A) = (-H_1) \oplus \text{diag}_{H_2} \oplus H_3$. Therefore $H|A = (-H_1) \oplus H_3$. Observe that $N_2N_1 = (N_1 \oplus N_2)|A$. Lemma 2.4 implies that $N_2 \circ N_1 = \overline{N_2N_1}$ is a Lagrangian submodule of $(-H_1) \oplus H_3$. □

**Lemma 2.6.** For any submodules $N_1 \subset H_1 \oplus H_2$ and $N_2 \subset H_2 \oplus H_3$, we have $\overline{N_2N_1} = \overline{N_2N_1}$.

**Proof.** Consider an element $h_1 \oplus h_3$ of $\overline{N_2N_1}$. By definition, $h_1 \oplus h_2 \in \overline{N_1}$ and $h_2 \oplus h_3 \in \overline{N_2}$ for some $h_2 \in H_2$, so $i_1(h_1 \oplus h_2) \in N_1$ and $i_2(h_2 \oplus h_3) \in N_2$ for some $i_1, i_2 \neq 0$. Then $i_1i_2(h_1 \oplus h_3) \in N_2N_1$, so $h_1 \oplus h_3 \in \overline{N_2N_1}$. Hence, $\overline{N_2N_1} \subset \overline{N_2N_1}$. Taking the closure on both sides, we get $\overline{N_2N_1} \subset \overline{N_2N_1}$. The opposite inclusion is obvious. □

**Theorem 2.7.** Hermitian $A$-modules, as objects, and Lagrangian relations, as morphisms, form a category.

**Proof.** The composition law is well-defined by Lemma 2.5; let us check that it is associative. Consider Lagrangian relations $N_1: H_1 \Rightarrow H_2$, $N_2: H_2 \Rightarrow H_3$, and $N_3: H_3 \Rightarrow H_4$. By Lemma 2.6,

$$N_3 \circ (N_2 \circ N_1) = \overline{N_3N_2N_1} = \overline{N_3N_2N_1} = \overline{N_3N_2N_1}.$$

Similarly, $(N_3 \circ N_2) \circ N_1 = (N_3N_2)N_1$. It follows from the definitions that $N_3(N_2N_1) = (N_3N_2)N_1$; this implies the associativity. The role of the identity morphisms is played by the diagonal Lagrangian relations. Indeed, for any Lagrangian relation $N: H_1 \Rightarrow H_2$,

$$\text{diag}_{H_2} \circ N = \overline{\text{diag}_{H_2}N} = \overline{N},$$

which is equal to $N$ since $N$ is Lagrangian. Similarly, $N \circ \text{diag}_{H_1} = N$. □
We shall call this category the \textit{category of Lagrangian relations over $\Lambda$}. It will be denoted by $\text{Lagr}_\Lambda$.

2.4. Lagrangian relations from unitary isomorphisms

By the \textit{graph} of a homomorphism $f: A \to B$ of abelian groups, we mean the set
$$\Gamma_f = \{a \oplus f(a) | a \in A\} \subset A \oplus B.$$  

Let $H_1$, $H_2$ be Hermitian $\Lambda$-modules. Consider the Hermitian $Q$-modules $H_1 \otimes Q$ and $H_2 \otimes Q$, where $Q = Q(\Lambda)$ is the field of fractions of $\Lambda$ and $\otimes = \otimes_\Lambda$. For a unitary $Q$-isomorphism $\varphi: H_1 \otimes Q \to H_2 \otimes Q$, we define its \textit{restricted graph} $\Gamma^0_{\varphi}$ by
$$\Gamma^0_{\varphi} = \Gamma_{\varphi} \cap (H_1 \oplus H_2) = \{h \oplus \varphi(h) | h \in H_1, \varphi(h) \in H_2\} \subset H_1 \oplus H_2.$$  

If $\varphi$ is induced by a unitary $\Lambda$-isomorphism $f: H_1 \to H_2$, then clearly $\Gamma^0_{\varphi} = \Gamma_f$.

\textbf{Lemma 2.8.} Given any unitary isomorphism $\varphi: H_1 \otimes Q \to H_2 \otimes Q$, the restricted graph $\Gamma^0_{\varphi}$ is a Lagrangian relation $H_1 \Rightarrow H_2$.

\textbf{Proof.} Denote by $\omega_1$ (resp. $\omega_2$, $\omega$) the skew-hermitian form on $H_1$ (resp. $H_2$, $(-H_1) \oplus H_2$), and pick $h, h' \in H_1$ such that $\varphi(h), \varphi(h') \in H_2$. Then,
$$\omega(h \oplus \varphi(h), h' \oplus \varphi(h')) = -\omega_1(h, h') + \omega_2(\varphi(h), \varphi(h')) = 0.$$  

Therefore, $\Gamma^0_{\varphi}$ is isotropic. To check that it is Lagrangian, consider an element $x = x_1 \oplus x_2$ of $\text{Ann}(\Gamma^0_{\varphi}) \subset (-H_1) \oplus H_2$. For all $h$ in $H_1$ such that $\varphi(h) \in H_2$,
$$0 = \omega(x, h \oplus \varphi(h)) = -\omega_1(x_1, h) + \omega_2(x_2, \varphi(h)) = -\omega_2(\varphi(x_1), \varphi(h)) + \omega_2(x_2, \varphi(h)) = \omega_2(x_2 - \varphi(x_1), \varphi(h)).$$  

Since $\varphi$ is an isomorphism, we have $H_2 \subset \{\varphi(h) | h \in H_1, \varphi(h) \in H_2\}$. Therefore, $\omega_2(x_2 - \varphi(x_1), h_2) = 0$ for all $h_2 \in H_2$. Since $\omega_2$ is non-degenerate, it follows that $x_2 = \varphi(x_1)$ so $x = x_1 \oplus \varphi(x_1) \in \Gamma^0_{\varphi}$ and the lemma is proved. \hfill $\square$

Therefore, Lagrangian relations can be understood as a generalization of unitary isomorphisms. More precisely, let $U_\Lambda$ be the category of Hermitian $\Lambda$-modules and unitary $\Lambda$-isomorphisms. Also, let $U^0_\Lambda$ be the category of Hermitian $\Lambda$-modules, where the morphisms between $H_1$ and $H_2$ are the unitary $Q$-isomorphisms between $H_1 \otimes Q$ and $H_2 \otimes Q$.

\textbf{Theorem 2.9.} The maps $f \mapsto f \otimes \text{id}_Q$, $\varphi \mapsto \Gamma^0_{\varphi}$ and $f \mapsto \Gamma_f$ define embeddings of categories $U_\Lambda \subset U^0_\Lambda \subset \text{Lagr}_\Lambda$ and $U_\Lambda \subset \text{Lagr}_\Lambda$ which fit in the commutative diagram

$$
\begin{array}{ccc}
U_\Lambda & \xrightarrow{\text{id}_\Lambda} & U^0_\Lambda \\
\downarrow & & \downarrow \\
\text{Lagr}_\Lambda
\end{array}
$$
Proof. The first embedding being clear, we check the second one. By Lemma 2.8, $\Gamma^0_\phi$ is a Lagrangian relation. Also, note that $\Gamma^0_\phi \otimes Q = \Gamma_\phi$. Therefore, given two unitary $Q$-isomorphisms $\varphi_1$ and $\varphi_2$,

$$I^0_{\varphi_2 \circ \varphi_1} = \Gamma_{\varphi_2 \circ \varphi_1} \cap (H_1 \oplus H_3) = \Gamma_{\varphi_2} \circ \Gamma_{\varphi_1} \cap (H_1 \oplus H_3)$$

$$= (I^0_{\varphi_2} \otimes Q)(I^0_{\varphi_1} \otimes Q) \cap (H_1 \oplus H_3) = (I^0_{\varphi_2} \circ I^0_{\varphi_1} \otimes Q) \cap (H_1 \oplus H_3)$$

$$= \Gamma^0_{\varphi_2 \circ \varphi_1} = \Gamma^0_{\varphi_2} \circ \Gamma^0_{\varphi_1}.$$ 

It is clear that a $Q$-isomorphism $\phi$ is entirely determined by its restricted graph $I^0_\phi$. Finally, the graph $\Gamma_f$ of a unitary $A$-isomorphism $f$ is equal to the restricted graph of the induced unitary $Q$-isomorphism $f \otimes \text{id}_Q$. Therefore, the diagram commutes. The theorem follows.  

3. The Lagrangian representation

3.1. The category of oriented tangles

Let $D^2$ be the closed unit disk in $\mathbb{R}^2$. Given a positive integer $n$, denote by $x_i$ the point $((2i - n - 1)/n, 0)$ in $D^2$, for $i = 1, \ldots, n$. Let $\varepsilon$ and $\varepsilon'$ be sequences of $\pm 1$ of respective length $n$ and $n'$. An $(\varepsilon, \varepsilon')$-tangle is the pair consisting of the cylinder $D^2 \times [0, 1]$ and its oriented piecewise linear 1-submanifold $\tau$ whose oriented boundary $\partial \tau$ is $\sum_{j=1}^{n'} \varepsilon'_j (x'_j, 1) - \sum_{i=1}^{n} \varepsilon_i (x_i, 0)$. Note that for such a tangle to exist, we must have $\sum_i \varepsilon_i = \sum_j \varepsilon'_j$.

Two $(\varepsilon, \varepsilon')$-tangles $(D^2 \times [0, 1], \tau_1)$ and $(D^2 \times [0, 1], \tau_2)$ are isotopic if there exists an auto-homeomorphism $h$ of $D^2 \times [0, 1]$, keeping $D^2 \times [0, 1]$ fixed, such that $h(\tau_1) = \tau_2$ and $h|_{\tau_1}: \tau_1 \simeq \tau_2$ is orientation-preserving. We shall denote by $T(\varepsilon, \varepsilon')$ the set of isotopy classes of $(\varepsilon, \varepsilon')$-tangles, and by $\text{id}_\varepsilon$ the isotopy class of the trivial $(\varepsilon, \varepsilon)$-tangle $(D^2, \{x_1, \ldots, x_n\}) \times [0, 1]$.

Given an $(\varepsilon, \varepsilon')$-tangle $\tau_1$ and an $(\varepsilon', \varepsilon'')$-tangle $\tau_2$, their composition is the $(\varepsilon, \varepsilon'')$-tangle $\tau_2 \circ \tau_1$ obtained by gluing the two cylinders along the disk corresponding to $\varepsilon'$ and shrinking the length of the resulting cylinder by a factor 2 (see Fig. 1). Clearly, the composition of tangles induces a composition

$$T(\varepsilon, \varepsilon') \times T(\varepsilon', \varepsilon'') \to T(\varepsilon, \varepsilon'')$$

on the isotopy classes of tangles.

The category of oriented tangles $\text{Tangles}$ is defined as follows: the objects are the finite sequences $\varepsilon$ of $\pm 1$, and the morphisms are given by $\text{Hom} (\varepsilon, \varepsilon') = T(\varepsilon, \varepsilon')$. The composition is clearly associative, and the trivial tangle $\text{id}_\varepsilon$ plays the role of the identity endomorphism of $\varepsilon$. The aim of this section is to construct a functor $\text{Tangles} \to \text{Lagr}_A$.

3.2. Objects

Denote by $N'(\{x_1, \ldots, x_n\})$ an open tubular neighborhood of $\{x_1, \ldots, x_n\}$ in $D^2 \subset \mathbb{R}^2$, and by $S^2$ the 2-sphere $\mathbb{R}^2 \cup \{\infty\}$. Given a sequence $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ of $\pm 1$, let $\ell_\varepsilon$ be the sum $\sum_{i=1}^{n} \varepsilon_i$. We shall denote
Fig. 1. A tangle composition.

Fig. 2. The space $D_{\varepsilon}$ for $\varepsilon = (+1, +1, -1, +1)$.

by $D_{\varepsilon}$ the compact surface

$$D_{\varepsilon} = \begin{cases} D^2 \setminus \mathcal{N}([x_1, \ldots, x_n]) & \text{if } \ell_{\varepsilon} \neq 0, \\ S^2 \setminus \mathcal{N}([x_1, \ldots, x_n]) & \text{if } \ell_{\varepsilon} = 0, \end{cases}$$

endowed with the counterclockwise orientation, a base point $z$, and the generating family $\{e_1, \ldots, e_n\}$ of $\pi_1(D_{\varepsilon}, z)$, where $e_i$ is a simple loop turning once around $x_i$ counterclockwise if $\varepsilon_i = +1$, clockwise if $\varepsilon_i = -1$ (see Fig. 2). The same space with the clockwise orientation will be denoted by $-D_{\varepsilon}$.

The natural epimorphism $\pi_1(D_{\varepsilon}) \to \mathbb{Z}$, $e_i \mapsto 1$ gives an infinite cyclic covering $\hat{D}_{\varepsilon} \to D_{\varepsilon}$. Choosing a generator $t$ of the group of the covering transformations endows the homology $H_1(\hat{D}_{\varepsilon})$ with a structure of module over $A = \mathbb{Z}[t, t^{-1}]$. If $\ell_{\varepsilon} \neq 0$, then $D_{\varepsilon}$ retracts by deformation on the wedge of $n$ circles representing $e_1, \ldots, e_n$, and one easily checks that $H_1(\hat{D}_{\varepsilon})$ is a free $A$-module with basis $v_1 = \hat{e}_1 - \hat{e}_2, \ldots, v_{n-1} = \hat{e}_{R-1} - \hat{e}_n$, where $\hat{e}_i$ is the path in $\hat{D}_{\varepsilon}$ lifting $e_i$ starting at some fixed lift $\hat{z} \in \hat{D}_{\varepsilon}$ of $z$. If $\ell_{\varepsilon} = 0$, then $H_1(\hat{D}_{\varepsilon}) = \bigoplus_j A \gamma_j / A_{\gamma_j}$, where $\hat{\gamma}$ is a lift of $\gamma = e_1^{n_1} \cdots e_n^{n_n}$ to $\hat{D}_{\varepsilon}$. Note that in any case, $H_1(\hat{D}_{\varepsilon})$ is a free $A$-module.

Let $\langle \cdot, \cdot \rangle : H_1(\hat{D}_{\varepsilon}) \times H_1(\hat{D}_{\varepsilon}) \to \mathbb{Z}$ be the ($\mathbb{Z}$-bilinear, skew-symmetric) intersection form induced by the orientation of $D_{\varepsilon}$ lifted to $\hat{D}_{\varepsilon}$. Consider the pairing $\omega_{\varepsilon} : H_1(\hat{D}_{\varepsilon}) \times H_1(\hat{D}_{\varepsilon}) \to A$ given by

$$\omega_{\varepsilon}(x, y) = \sum_k (t^k x, y) t^{-k}.$$
Note that this form is well-defined since, for any given \(x, y \in H_1(\hat{D}_e)\), the intersection \(\langle t^k x, y \rangle\) vanishes for all but a finite number of integers \(k\). The multiplication by \(t\) being an isometry with respect to the intersection form, it is easy to check that \(\omega_e\) is skew-hermitian with respect to the involution \(\Lambda \rightarrow \Lambda\) induced by \(t \mapsto t^{-1}\).

**Example 3.1.** Consider \(\varepsilon\) of length 2. If \(\varepsilon_1 + \varepsilon_2 = 0\), then \(\hat{D}_e\) is contractible so \(H_1(\hat{D}_e) = 0\). If \(\varepsilon_1 + \varepsilon_2 \neq 0\), then \(H_1(\hat{D}_e) = \Lambda v\) with \(v = \hat{e}_1 - \hat{e}_2\), and \(\omega_e(v, v) = \frac{\varepsilon_1 + \varepsilon_2}{2}(t - t^{-1})\), cf. Fig. 3.

We shall give a proof of the following result in Section 4.

**Lemma 3.2.** For any \(\varepsilon\), the form \(\omega_e: H_1(\hat{D}_e) \times H_1(\hat{D}_e) \rightarrow \Lambda\) is non-degenerate.

### 3.3. Morphisms

Given an \((\varepsilon, \varepsilon')\)-tangle \(\tau \subset D^2 \times [0, 1]\), denote by \(\mathcal{N}(\tau)\) an open tubular neighborhood of \(\tau\) and by \(X_\tau\) its exterior

\[
X_\tau = \begin{cases} 
  (D^2 \times [0, 1]) \setminus \mathcal{N}(\tau) & \text{if } \ell_\varepsilon \neq 0, \\
  (S^2 \times [0, 1]) \setminus \mathcal{N}(\tau) & \text{if } \ell_\varepsilon = 0.
\end{cases}
\]

Note that \(\ell_\varepsilon = \ell_{\varepsilon'}\). We shall orient \(X_\tau\) so that the induced orientation on \(\partial X_\tau\) extends the orientation on \((-D_\varepsilon) \sqcup D_{\varepsilon'}\). If \(\ell_\varepsilon \neq 0\), then the exact sequence of the pair \((D^2 \times [0, 1], X_\tau)\) and the excision isomorphism...
Theorem 3.5. Given a sequence \( H \) of \( 1 \)-tangles, the Laga-

\[ H_1(X_\tau) = H_2(D^2 \times [0, 1], X_\tau) = H_2(\mathcal{N}(\tau), \mathcal{N}(\tau) \cap X_\tau), \]

\[ = \bigoplus_{j=1}^\mu H_2(\mathcal{N}(\tau_j), \mathcal{N}(\tau_j) \cap X_\tau), \]

where \( \tau_1, \ldots, \tau_\mu \) are the connected components of \( \tau \). Since \( (\mathcal{N}(\tau_j), \mathcal{N}(\tau_j) \cap X_\tau) \) is homeomorph to \((\tau_j \times D^2, \tau_j \times S^1)\), we have \( H_2(\mathcal{N}(\tau_j), \mathcal{N}(\tau_j) \cap X_\tau) = \mathbb{Z}m_j \), where \( m_j \) is a meridian of \( \tau_j \) oriented so that its linking number with \( \tau_j \) is 1. Hence, \( H_1(X_\tau) = \bigoplus_{j=1}^\mu \mathbb{Z}m_j \). If \( e_\ell = 0 \), then \( H_1(X_\tau) \cong \bigoplus_{j=1}^\mu \mathbb{Z}m_j \sum_{i=1}^n e_i \).

The composition of the Hurewicz homomorphism and the homomorphism \( H_1(\hat{X}_\tau) \to \mathbb{Z}, m_j \mapsto 1 \) gives an epimorphism \( \pi_1(X_\tau) \to \mathbb{Z} \) which extends the previously defined homomorphisms \( \pi_1(D_\ell) \to \mathbb{Z} \) and \( \pi_1(D_{\ell'}) \to \mathbb{Z} \). As before, it determines an infinite cyclic covering \( \hat{X}_\tau \to X_\tau \), so the homology of \( \hat{X}_\tau \) is endowed with a natural structure of module over \( A = \mathbb{Z}[t, t^{-1}] \).

Let \( i_\tau: H_1(\hat{D}_\ell) \to H_1(\hat{X}_\tau) \) and \( i'_\tau: H_1(\hat{D}_{\ell'}) \to H_1(\hat{X}_\tau) \) be the homomorphisms induced by the obvious inclusion \( \hat{D}_\ell \cup \hat{D}_{\ell'} \subset \hat{X}_\tau \). Denote by \( j_\tau \) the homomorphism \( H_1(\hat{D}_\ell) \oplus H_1(\hat{D}_{\ell'}) \to H_1(\hat{X}_\tau) \) given by \( j_\tau(x, x') = i'_\tau(x') - i_\tau(x) \). Finally, set

\[ N(\tau) = \ker(j_\tau) \subset H_1(\hat{D}_\ell) \oplus H_1(\hat{D}_{\ell'}). \]

Note that if \( \tau \) and \( \tau' \) are two isotopic \((\ell, \ell')\)-tangles, then \( N(\tau) = N(\tau') \).

Lemma 3.3. \( N(\tau) \) is a Lagrangian submodule of \((-H_1(\hat{D}_\ell)) \oplus H_1(\hat{D}_{\ell'}) \).

Lemma 3.4. If \( \tau_1 \in T(\ell, \ell') \) and \( \tau_2 \in T(\ell', \ell'') \), then \( N(\tau_2 \circ \tau_1) = N(\tau_2) \circ N(\tau_1) \).

We postpone the proof of these lemmas to the next section, and summarize our results in the following theorem.

Theorem 3.5. Given a sequence \( \epsilon \) of \( \pm 1 \), denote by \( \mathcal{H}(\epsilon) \) the Hermitian \( \Lambda \)-module \((H_1(\hat{D}_\ell), \omega_\epsilon)\). For \( \tau \in T(\ell, \ell') \), let \( \mathcal{H}(\tau) \) be the Lagrangian relation \( N(\tau): H_1(\hat{D}_\ell) \Rightarrow H_1(\hat{D}_{\ell'}). \) Then, \( \mathcal{H} \) is a functor \( Tangles \to \text{Lagr}_\Lambda \).

The usual notions of cobordism and \( I \)-equivalence for links generalize to tangles in the obvious way. (The surface in \( D^2 \times [0, 1] \times [0, 1] \) interpolating between two tangles \( \tau_1, \tau_2 \subset D^2 \times [0, 1] \) should be standard on \( D^2 \times [0, 1] \times [0, 1] \) and homeomorphic to \( \tau_1 \times [0, 1] \)) It is easy to see (cf. [4, Theorem 5.1 and the proof of Proposition 5.3]) that the Lagrangian relation \( N(\tau) \) is an \( I \)-equivalence invariant of \( \tau \).

The usual computation of the Alexander module of a link \( L \) from a diagram of \( L \) extends to our setting. This gives a computation of \( H_1(\hat{D}_\ell) \oplus H_1(\hat{D}_{\ell'}) \xrightarrow{j_\tau} H_1(\hat{X}_\tau) \) (cf. [4, Proposition 4.4]). Hence, it is possible to compute \( N(\tau) \) from a diagram of \( \tau \).

Finally, given an \((\ell, \ell')\)-tangle \( \tau \), one can construct an oriented link \( \hat{\tau} \subset S^3 \) by ‘closing’ \( \tau \) in the obvious way. Although we shall not discuss it here, note that the Lagrangian submodule \( N(\tau) \) is closely related to the Alexander polynomial of \( \hat{\tau} \).
3.4. Freeness of \( N(\tau) \)

As pointed out in Section 3.2, the functor \( \overline{\mathcal{C}}: \text{Tangles} \to \text{Lagr}_A \) maps the objects to free modules over the ring \( A = \mathbb{Z}[t, t^{-1}] \). What about the morphisms? Given an oriented tangle \( \tau \), is the \( A \)-module \( N(\tau) \) free? The following theorem answers this question.

**Theorem 3.6.** Given any tangle \( \tau \in T(\varepsilon, \varepsilon') \), the \( A \)-module \( N(\tau) \) is free. Its rank is given by

\[
\text{rk}_A N(\tau) = \begin{cases} 
0 & \text{if } n = n' = 0, \\
\frac{n + n'}{2} - 1 & \text{if } \ell_\varepsilon \neq 0 \text{ or } nn' = 0 \text{ and } (n, n') \neq (0, 0), \\
\frac{n + n'}{2} - 2 & \text{if } \ell_\varepsilon = 0 \text{ and } nn' > 0,
\end{cases}
\]

where \( n \) and \( n' \) denote the length of \( \varepsilon \) and \( \varepsilon' \).

In order to prove this result, we shall need several notions of homological algebra, that we recall now. Let \( A \) be a commutative ring with unit. The *projective dimension* \( \text{pd}(A) \) of a \( A \)-module \( A \) is the minimum integer \( n \) (if it exists) such that there is a projective resolution of length \( n \) of \( A \), that is, an exact sequence

\[
0 \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0,
\]

where all the \( P_i \)'s are projective modules. It is a well-known fact that if \( 0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0 \) is any resolution of \( A \) with \( \text{pd}(A) \leq n \) and all the \( P_i \)'s projective, then \( K_n \) is projective as well (see, for instance, [11, Lemma 4.1.6]). The *global dimension* of a ring \( A \) is the (possibly infinite) number \( \sup \{ \text{pd}(A) \mid A \text{ is a } A\text{-module} \} \). For example, the global dimension of \( A \) is zero if \( A \) is a field, and at most one if \( A \) is a principal ideal domain.

**Lemma 3.7.** Let \( A = \mathbb{Z}[t, t^{-1}] \). Consider an exact sequence of \( A \)-modules

\[
0 \to K \to P \to F,
\]

where \( P \) and \( F \) are free \( A \)-modules. Then \( K \) is free.

**Proof.** Note that the ring \( A \) has global dimension 2 (see e.g. [11, Theorem 4.3.7]). We shall also need the fact that all projective \( A \)-modules are free [8, Chapter 3.3]. Let \( A \) be the image of the homomorphism \( P \to F \). We claim that the projective dimension of \( A \) is at most 1. Indeed, since the global dimension of \( A \) is at most two, there is a projective resolution \( 0 \to P_2 \to P_1 \to P_0 \to A \to 0 \) of \( A \). Splicing this resolution with the exact sequence \( 0 \to A \to F \to F/A \to 0 \), we get a resolution of \( F/A \)

\[
0 \to P_1/\partial P_2 \to P_0 \to F \to F/A \to 0,
\]

where \( P_0 \) and \( F \) are projective. Since the global dimension of \( A \) is 2, we have \( \text{pd}(F/A) \leq 2 \). Hence, \( P_1/\partial P_2 \) is projective as well. Therefore, the resolution of \( A \)

\[
0 \to P_1/\partial P_2 \to P_0 \to A \to 0
\]

is projective, so \( \text{pd}(A) \leq 1 \). Now, the exact sequence \( 0 \to K \to P \to A \to 0 \) together with the fact that \( P \) is free and \( \text{pd}(A) \leq 1 \), implies that \( K \) is projective. Therefore, it is free. \( \square \)
Proof of Theorem 3.6. Consider the exact sequence

\[ 0 \to N(\tau) \to H_1(\hat{D}_\varepsilon) \oplus H_1(\hat{D}'_\varepsilon) \to (H_1(\hat{D}_\varepsilon) \oplus H_1(\hat{D}'_\varepsilon))/N(\tau) \to 0. \]

Clearly, the latter module is finitely generated and torsion free. Since \( A \) is a noetherian ring, such a module embeds in a free \( A \)-module \( F \), giving an exact sequence

\[ 0 \to N(\tau) \to H_1(\hat{D}_\varepsilon) \oplus H_1(\hat{D}'_\varepsilon) \to F. \]

By Lemma 3.7, \( N(\tau) \) is free. Since \( N(\tau) \) is a Lagrangian submodule of \( H_1(\hat{D}_\varepsilon) \oplus H_1(\hat{D}'_\varepsilon) \), we have \( \text{rk}_A N(\tau) = \frac{1}{2} \text{rk}_A (H_1(\hat{D}_\varepsilon) \oplus H_1(\hat{D}'_\varepsilon)) \). If \( \varepsilon \) has length \( n \), we know from Section 3.2 that

\[ \text{rk}_A H_1(\hat{D}_\varepsilon) = \begin{cases} 0 & \text{if } n = 0, \\ n - 1 & \text{if } \ell_\varepsilon \neq 0, \\ n - 2 & \text{if } \ell_\varepsilon = 0 \text{ and } n > 0. \end{cases} \]

The result follows. \( \square \)

4. Proof of the lemmas

The proof of Lemmas 3.2 and 3.3 rely on the Blanchfield duality theorem. We recall this fundamental result referring for a proof and further details to [3, Appendix E].

Let \( M \) be a piecewise linear compact connected oriented \( m \)-dimensional manifold possibly with boundary. Consider an epimorphism of \( \pi_1(M) \) onto a finitely generated free abelian group \( G \). It induces a \( G \)-covering \( \tilde{M} \to M \), so the homology modules of \( \tilde{M} \) are modules over \( A = \mathbb{Z} G \). For any integer \( q \), let \( \langle \cdot, \cdot \rangle : H_q(\tilde{M}) \times H_{m-q}(\tilde{M}, \partial \tilde{M}) \to \mathbb{Z} \) be the \( \mathbb{Z} \)-bilinear intersection form induced by the orientation of \( M \) lifted to \( \tilde{M} \). The Blanchfield pairing is the form \( S : H_q(\tilde{M}) \times H_{m-q}(\tilde{M}, \partial \tilde{M}) \to A \) given by

\[ S(x, y) = \sum_{g \in G} \langle gx, y \rangle g^{-1}. \]

Note that \( S \) is \( A \)-sesquilinear with respect to the involution of \( A \) given by \( \sum_{g \in G} n g g^{-1} \mapsto \sum_{g \in G} n g^{-1} g \). The form \( S \) induces a \( A \)-sesquilinear form

\[ H_q(\tilde{M})/\text{Tors}_A H_q(\tilde{M}) \times H_{m-q}(\tilde{M}, \partial \tilde{M})/\text{Tors}_A H_{m-q}(\tilde{M}, \partial \tilde{M}) \to A. \]

Theorem 4.1 (Blanchfield). The latter form is non-degenerate.

Let us now prove the lemmas stated in the previous section.

Proof of Lemma 3.2. Consider the Blanchfield pairing

\[ S_\varepsilon : H_1(\hat{D}_\varepsilon) \times H_1(\hat{D}_\varepsilon, \partial \hat{D}_\varepsilon) \to A. \]

It follows from the definitions that \( \omega_\varepsilon(x, y) = S_\varepsilon(x, j_\varepsilon(y)), \) where \( j_\varepsilon : H_1(\hat{D}_\varepsilon) \to H_1(\hat{D}_\varepsilon, \partial \hat{D}_\varepsilon) \) is the inclusion homomorphism. Note that \( \partial \hat{D}_\varepsilon \) consists of a finite number of copies of \( \mathbb{R} \), so \( H_1(\partial \hat{D}_\varepsilon) = 0 \) and \( j_\varepsilon \) is injective. Pick \( y \in H_1(\hat{D}_\varepsilon) \) and assume that for all \( x \in H_1(\hat{D}_\varepsilon) \), \( 0 = \omega_\varepsilon(x, y) = S_\varepsilon(x, j_\varepsilon(y)) \). By
the Blanchfield duality theorem. \( j_e(y) \in \text{Tors}_A(H_1(\hat{D}_e, \partial \hat{D}_e)) \), so \( 0 = \lambda j_e(y) = j_e(\lambda y) \) for some \( \lambda \in A \), \( \lambda \neq 0 \). Since \( j_e \) is injective, \( \lambda y = 0 \). As \( H_1(\hat{D}_e) \) is torsion-free, \( y = 0 \), so \( \omega_e \) is non-degenerate. \( \square \)

**Proof of Lemma 3.3.** Let \( H_1(\hat{D}_e) \oplus H_1(\hat{D}_e') \xrightarrow{i} H_1(\partial \hat{X}_\tau) \) be the inclusion homomorphism, and denote by

\[
H_2(\hat{X}_\tau, \partial \hat{X}_\tau) \xrightarrow{\partial} H_1(\partial \hat{X}_\tau) \xrightarrow{j} H_1(\hat{X}_\tau)
\]

the homomorphisms appearing in the exact sequence of the pair \((\hat{X}_\tau, \partial \hat{X}_\tau)\). Also, denote by \( \omega \) the pairing \((-\omega_e) \oplus \omega_e \)' on \((-H_1(\hat{D}_e)) \oplus H_1(\hat{D}_e') \) and by

\[
S_{eX}: H_1(\partial \hat{X}_\tau) \times H_1(\partial \hat{X}_\tau) \to A, \quad S_X: H_1(\hat{X}_\tau) \times H_2(\hat{X}_\tau, \partial \hat{X}_\tau) \to A
\]

the Blanchfield pairings. Clearly, \( N(\tau) = \langle \text{ann} (\partial) \rangle (\bar{L}) \), where \( L = \ker (j \circ i) \) and \( \text{id} \) (resp. \( \text{id}' \)) is the identity endomorphism of \( H_1(\hat{D}_e) \) (resp. \( H_1(\hat{D}_e') \)). Then, \( \text{ann}(N(\tau)) = \langle \text{ann} (\partial) \rangle \text{ann}(L) \) and we just need to check that \( \text{ann}(L) = \text{ann} \). Therefore

\[
\text{Ann}_{\partial X}(K) = \{ x \in H_1(\partial \hat{X}_\tau) | S_{eX}(x, K) = 0 \}
\]

\[
= \{ x \in H_1(\partial \hat{X}_\tau) | S_X(j(x), H_2(\hat{X}_\tau, \partial \hat{X}_\tau)) = 0 \}.
\]

By the Blanchfield duality, the latter set is just \( j^{-1}(\text{Tors}_A(H_1(\hat{X}_\tau))) = \text{ann} \). Clearly, \( \partial(L) \subset K \). The exact sequence of the pair \((\partial \hat{X}_\tau, \hat{D}_e \sqcup \hat{D}_e')\) gives

\[
H_1(\hat{D}_e) \oplus H_1(\hat{D}_e') \xrightarrow{i} H_1(\partial \hat{X}_\tau) \to T,
\]

where \( T \) is a torsion \( A \)-module. This implies that \( K \subset \text{ann}(L) \) and therefore \( \text{ann}(L) = \text{ann} \). Since the forms \( \omega \) and \( S_{eX} \) are compatible under \( i \),

\[
\text{ann}(L) = i^{-1}(\text{ann}_{\partial X}(i(L))) = i^{-1}(\text{ann}_{\partial X}(\text{ann}(L))) = i^{-1}(\text{ann}_{\partial X}(\text{ann}(K)))
\]

\[
= i^{-1}(\text{ann}_{\partial X}(K)) = i^{-1}(\text{ann}(K)) = \text{ann}(L) = \text{ann}(L),
\]

and the lemma is proved. \( \square \)

**Proof of Lemma 3.4.** Denote by \( \tau \) the composition \( \tau_2 \circ \tau_1 \). Note that it is sufficient to check the equality \( \ker(j_\tau) = \ker(j_{\tau_2}) \ker(j_{\tau_1}) \). Indeed, Lemma 2.6 then implies

\[
N(\tau) = \ker(j_\tau) = \ker(j_{\tau_2}) \ker(j_{\tau_1})
\]

\[
= \ker(j_{\tau_2}) \ker(j_{\tau_1}) = N(\tau_2)N(\tau_1) = N(\tau_2) \circ N(\tau_1).
\]

Since \( X_\tau = X_{\tau_1} \cup X_{\tau_2} \) and \( X_{\tau_1} \cap X_{\tau_2} = D_{\tau'} \), we get the following Mayer–Vietoris exact sequence of \( A \)-modules:

\[
H_1(\hat{D}_{\tau'}) \xrightarrow{\alpha} H_1(\hat{X}_{\tau_1}) \oplus H_1(\hat{X}_{\tau_2}) \xrightarrow{\beta} H_1(\hat{X}_\tau) \to H_0(\hat{D}_{\tau'}) \xrightarrow{\gamma} H_0(\hat{X}_{\tau_1}) \oplus H_0(\hat{X}_{\tau_2}).
\]
The finite sequences of $\pm a f_i$, $i = 1, \ldots, n$ are just needed to check that $\phi(x, x', x'') = (j_{t_1}(x, x'), j_{t_2}(x', x''))$. Clearly, $\pi(\ker(\phi)) = \{x \otimes x'' \mid \phi(x, x', x'') = 0 \text{ for some } x' \in H_{\epsilon'}\} = \ker(j_{t_2}) \ker(j_{t_1})$.

Therefore, we just need to check that $\pi(\ker(\phi)) = \ker(j_\tau)$, which is an easy diagram chasing exercise using the surjectivity of $x: H_{\epsilon'} \to \ker(\beta)$. $\square$

5. Examples

5.1. Braids

An $(\epsilon, \epsilon')$-tangle $\tau = \tau_1 \cup \cdots \cup \tau_n \subset D^2 \times [0, 1]$ is called an oriented braid if every component $\tau_j$ of $\tau$ is strictly increasing or strictly decreasing with respect to the projection to $[0, 1]$. Note that for such an oriented braid to exist, we must have $\sharp\{i \mid \epsilon_i = 1\} = \sharp\{j \mid \epsilon_j' = 1\}$ and $\sharp\{i \mid \epsilon_i = -1\} = \sharp\{j \mid \epsilon_j' = -1\}$.

The finite sequences of $\pm 1$, as objects, and the isotopy classes of oriented braids, as morphisms, form a subcategory $\textbf{Braids}$ of the category of oriented tangles. We shall now investigate the restriction of the functor $\mathcal{X}$ to this subcategory.

Consider an oriented braid $\beta = \beta_1 \cup \cdots \cup \beta_n \subset D^2 \times [0, 1]$. Clearly, there exists an isotopy $H_\beta: D^2 \times [0, 1] \to D^2 \times [0, 1]$ with $H_\beta(x, t) = (x, t)$ for $(x, t) \in (D^2 \times \{0\}) \cup (\partial D^2 \times [0, 1])$, such that $t \mapsto H_\beta(x_i, t)$ is a homeomorphism of $[0, 1]$ onto the arc $\beta_i$ for $i = 1, \ldots, n$. Let $h_\beta: D_\varepsilon \to D_{\epsilon'}$ be the homeomorphism given by $x \mapsto H_\beta(x, 1)$, and by the identity on $S^2 \setminus D^2$ if $\epsilon_1 + \cdots + \epsilon_n = 0$. It is a standard result that the isotopy class of $\partial D^2$ of $h_\beta$ only depends on the isotopy class of $\beta$. Consider the lift $\hat{h}_\beta: \hat{D}_\varepsilon \to \hat{D}_{\epsilon'}$ of $h_\beta$ fixing $\partial \hat{D}^2$ pointwise, and denote by $f_\beta$ the induced unitary isomorphism $(\hat{h}_\beta)_*: H_1(\hat{D}_\varepsilon) \to H_1(\hat{D}_{\epsilon'})$.

The isotopy $H_\beta$ provides a deformation retraction of $X_\beta$ to $D_{\epsilon'}$: let us identify $H_1(\hat{X}_\beta)$ and $H_1(\hat{D}_{\epsilon'})$ via this deformation. Clearly, the homomorphism $j_\beta: H_1(\hat{D}_\varepsilon) \oplus H_1(\hat{D}_{\epsilon'}) \to H_1(\hat{X}_\beta)$ is given by $j_\beta(x, y) = y - f_\beta(x)$. Therefore,

$$N(\beta) = \ker(j_\beta) = \ker(j_\beta) = \{x \otimes f_\beta(x) \mid x \in H_1(\hat{D}_\varepsilon)\} = \Gamma_{f_\beta},$$

the graph of the unitary isomorphism $f_\beta$. We have proved:

**Proposition 5.1.** The restriction of $\mathcal{X}$ to the subcategory of oriented braids gives a functor $\textbf{Braids} \to U_\Lambda$.

Consider an $(\epsilon, \epsilon')$-tangle $\tau = \tau_1 \cup \cdots \cup \tau_n \subset D^2 \times [0, 1]$ such that every component $\tau_j$ of $\tau$ is strictly increasing with respect to the projection to $[0, 1]$. Here, $\epsilon = \epsilon' = (1, \ldots, 1)$. We will simply call $\tau$ a braid,
Fig. 4. The action of $h_i$ on the loops $e_{i-1}, \ldots, e_{i+2}$.

or an $n$-strand braid. As usual, we will denote by $B_n$ the group of isotopy classes of $n$-strand braids, and by $\sigma_1, \ldots, \sigma_{n-1}$ its standard set of generators (see Fig. 5). Recall that the Burau representation $B_n \to \text{GL}_n(A)$ maps the generator $\sigma_i$ to the matrix

$$I_{i-1} \oplus \begin{pmatrix} 1 - t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1},$$

where $I_k$ denotes the identity ($k \times k$)-matrix. This representation is reducible: it splits into the direct sum of an $(n-1)$-dimensional representation $\rho$ and the trivial one-dimensional representation (see e.g. [1]). Using the Artin presentation of $B_n$, one easily checks that the map $\sigma_i \mapsto \rho(\sigma_i)^T$, where $T$ denotes the transposition, also defines a representation $\rho^T: B_n \to \text{GL}_{n-1}(A)$.

**Proposition 5.2.** The restriction of the functor $\mathfrak{F}$ to $B_n$ gives a linear anti-representation $B_n \to \text{GL}_{n-1}(A)$ which is the dual of $\rho^T$.

**Proof.** Consider two braids $\alpha, \beta \in B_n$. By Proposition 5.1, $N(\alpha)$ (resp. $N(\beta)$, $N(\alpha \beta)$) is the graph of a unitary automorphism $f_\alpha$ (resp. $f_\beta$, $f_{\alpha \beta}$) of $H_1(\hat{D}_\varepsilon)$. Note that the product $x \beta \in B_n$ represents the composition $\beta \circ x$ in the category of tangles. Clearly, $f_{x \beta} = f_\beta \circ f_x$. Therefore, $\mathfrak{F}$ restricted to $B_n$ is an anti-representation. In order to check that it corresponds to the dual of $\rho^T$, we just need to verify that these anti-representations coincide on the generators $\sigma_i$ of $B_n$.

Denote by $f_i$ the unitary isomorphism corresponding to $\sigma_i$. We shall now compute the matrix of $f_i$ with respect to the basis $v_1, \ldots, v_{n-1}$ of $H_1(\hat{D}_\varepsilon)$. Consider the homeomorphism $h_i$ of $D_\varepsilon$ associated with $\sigma_i$. As shown in Fig. 4, its action on the loops $e_j$ is given by

$$h_i(e_j) = \begin{cases} e_i e_{i+1} e^{-1}_i & \text{if } j = i, \\ e_i & \text{if } j = i + 1, \\ e_j & \text{else}. \end{cases}$$

Therefore, the lift $\hat{h}_i$ of $h_i$ satisfies

$$\hat{h}_i(\hat{e}_j) = \begin{cases} \hat{e}_i - t(\hat{e}_i - \hat{e}_{i+1}) & \text{if } j = i, \\ \hat{e}_i & \text{if } j = i + 1, \\ \hat{e}_j & \text{else}, \end{cases}$$
and the matrix of \( f_i = (\hat{h}_i)_* \) with respect to the basis \( v_j = \hat{e}_j - \hat{e}_{j+1} \) is

\[
M_{f_i} = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{n-3}, \quad M_{f_{n-1}} = I_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix},
\]

\[
M_{f_i} = I_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-i-2} \quad \text{for } 2 \leq i \leq n - 2.
\]

This is exactly \( \rho(\sigma_i) \) (see, for instance, [1, p. 121]). \( \square \)

### 5.2. String links

An \( (\varepsilon, \varepsilon') \)-tangle \( \tau = \tau_1 \cup \cdots \cup \tau_n \subset D^2 \times [0, 1] \) is called an **oriented string link** if every component \( \tau_j \) of \( \tau \) joins \( D^2 \times \{0\} \) and \( D^2 \times \{1\} \). Oriented string links clearly form a category \( \text{Strings} \) which satisfies

\[
\text{Braids} \subset \text{Strings} \subset \text{Tangles},
\]

where all the inclusions denote embeddings of categories.

**Proposition 5.3.** The restriction of \( \mathcal{H} \) to the subcategory of oriented string links gives a functor \( \text{Strings} \to U^0_{\Lambda} \).

**Proof.** Since \( \tau \) is an oriented string link, the inclusions \( D_\varepsilon \subset X_\tau \) and \( D_{\varepsilon'} \subset X_\tau \) induce isomorphisms in integral homology. Therefore, the induced homomorphisms \( H_1(D_{\varepsilon}; Q) \overset{i_\tau}{\to} H_1(X_\tau; Q) \) and \( H_1(D_{\varepsilon'}; Q) \overset{i'_{\tau}}{\to} H_1(X_\tau; Q) \) are isomorphisms (see e.g. [4, Proposition 2.3]). Since \( Q = Q(A) \) is a flat \( \Lambda \)-module, \( \ker(j_\tau) \otimes Q \) is the kernel of

\[
H_1(D_{\varepsilon}; Q) \oplus H_1(D_{\varepsilon'}; Q) \overset{i'_{\tau} - i_\tau}{\to} H_1(X_\tau; Q).
\]

Hence,

\[
\mathcal{H}(\tau) = \ker(j_\tau) = (\ker(j_\tau) \otimes Q) \cap (H_1(D_{\varepsilon}) \oplus H_1(D_{\varepsilon'})) = I^0_{\phi},
\]

the restricted graph of the unitary \( Q \)-isomorphism \( \varphi = (i'_{\tau})^{-1} \circ i_\tau \). \( \square \)

If all the components of an oriented string link \( \tau \) are oriented from bottom to top, we will simply speak of \( \tau \) as a **string link**. By Proposition 5.3, the restriction of \( \mathcal{H} \) to the category of string links gives a functor to the category \( U^0_{\Lambda} \). This functor is due to Le Dimet [5] and was studied further in [4].

### 5.3. Elementary tangles

Every tangle \( \tau \in T(\varepsilon, \varepsilon') \) can be expressed as a composition of the **elementary tangles** given in Fig. 5, where the orientation of the strands is determined by the signs \( \varepsilon \) and \( \varepsilon' \). We shall now compute explicitly the functor \( \mathcal{H} \) on these tangles, assuming that \( \ell_\varepsilon \neq 0 \).

Let us start with the tangle \( u \in T(\varepsilon, \varepsilon') \). Here, \( H_1(\hat{D}_{\varepsilon}) = \bigoplus_{i=1}^{n-3} A v_i \) and \( H_1(\hat{D}_{\varepsilon'}) = \bigoplus_{i=1}^{n-1} A v'_i \) where \( v_i = \hat{e}_i - \hat{e}_{i+1} \) and \( v'_i = \hat{e}'_i - \hat{e}'_{i+1} \). Moreover, \( X_u \) is homeomorphic to the exterior of the trivial \( (\varepsilon'', \varepsilon'') \)-tangle,
where \( e'' = (-e'_1, e_1, \ldots, e_{n-2}) = (e'_2, \ldots, e'_n) \). Hence, \( H_1(\hat{X}_u) = \bigoplus_{i=1}^{n-2} A v''_i \) with \( v''_i = \hat{e}'_i - \hat{e}'_{i+1} \) and the homomorphism \( j_u: H_1(\hat{D}_e) \oplus H_1(\hat{D}'_e) \to H_1(\hat{X}_u) \) is given by \( j_u(v_i) = -v''_{i+1} \) for \( i = 1, \ldots, n-3 \), \( j_u(v'_1) = 0 \) and \( j_u(v'_i) = v''_{i-1} \) for \( i = 2, \ldots, n-1 \). Therefore,

\[
N(u) = \ker(j_u) = \ker(j_u) = A v'_1 \oplus \bigoplus_{i=1}^{n-3} A (v_i \oplus v'_{i+2}).
\]

Similarly, we easily compute

\[
N(\eta) = A v_1 \oplus \bigoplus_{i=1}^{n-3} A (v_{i+2} \oplus v'_i).
\]

Now, consider the oriented braid \( \sigma_i \in T(e, e') \) given in Fig. 5. Then, \( N(\sigma_i) \) is equal to the graph \( \Gamma_{fi} \) of a unitary isomorphism \( f_i: H_1(\hat{D}_e) \to H_1(\hat{D}'_e) \). As in the proof of Proposition 5.2, we can compute the matrix \( M_{fi} \) of \( f_i \) with respect to the bases \( v_1, \ldots, v_{n-1} \) of \( H_1(\hat{D}_e) \) and \( v'_1, \ldots, v'_{n-1} \) of \( H_1(\hat{D}'_e) \):

\[
M_{fi} = \begin{pmatrix}
-t^{e_2} & 1 \\
0 & 1
\end{pmatrix} \oplus I_{n-3},
\]

\[
M_{fi-1} = I_{n-3} \oplus \begin{pmatrix}
1 & 0 \\
t^{e_n} & -t^{e_n}
\end{pmatrix},
\]

\[
M_{fi} = I_{i-2} \oplus \begin{pmatrix}
1 & 0 & 0 \\
t^{e_{i+1}} & -t^{e_{i+1}} & 1 \\
0 & 0 & 1
\end{pmatrix} \oplus I_{n-i-2} \quad \text{for } 2 \leq i \leq n-2.
\]
Finally, consider the tangle $\sigma_i^{-1}$ given in Fig. 5. Since it is an oriented braid, $N(\sigma_i^{-1})$ is equal to the graph of a unitary isomorphism $g_i: H_1(\hat{D}_\varepsilon) \to H_1(\hat{D}_{\varepsilon'})$. Furthermore, we have
\[
\text{diag}_{H_1(\hat{D}_\varepsilon)} = N(\text{id}_\varepsilon) = N(\sigma_i^{-1} \circ \sigma_i) = N(\sigma_i^{-1}) \circ N(\sigma_i) = \Gamma_{g_i} \circ \Gamma_{f_i} = \Gamma_{g_i \circ f_i}.
\]
Therefore, $g_i \circ f_i$ is the identity endomorphism of $H_1(\hat{D}_\varepsilon)$, so the matrix of $g_i$ with respect to the basis given above is equal to $M_{g_i} = M_{f_i}^{-1}$.

With these elementary tangles, we can sketch an alternative proof of Lemma 3.3 which does not make use of the Blanchfield duality. Indeed, any tangle $\tau \in T(\varepsilon, \varepsilon')$ can be written as a composition of $\sigma_i$, $\sigma_i^{-1}$, $u$ and $\eta$. By Lemmas 2.5 and 3.4, we just need to check that $N(\sigma_i)$, $N(\sigma_i^{-1})$, $N(u)$ and $N(\eta)$ are Lagrangian. For $N(\sigma_i)$ and $N(\sigma_i^{-1})$, this follows from Proposition 5.1, Lemma 3.2 and Lemma 2.8. For $N(u)$ and $N(\eta)$, it can be verified by a direct computation of $\omega_e$.

Using the results above, it is possible to compute $N(\tau_2 \circ \tau_1)$ from $N(\tau_1)$ for any elementary tangle $\tau_2$ and a tangle $\tau_1$. This leads to a recursive computation of $N(\tau)$ for $(\varepsilon, \varepsilon')$-tangles with no closed components and at least one strand joining $D_\varepsilon$ with $D_{\varepsilon'}$.

6. Generalizations

6.1. The category of $m$-colored tangles

Fix throughout this section a positive integer $m$. An $m$-colored tangle is an oriented tangle $\tau$ together with a map $c$ assigning to each component $\tau_j$ of $\tau$ a color $c(i) \in \{1, \ldots, m\}$. The composition of two $m$-colored tangles is defined if and only if it is compatible with the coloring of each component. Finally, we say that an $m$-colored tangle is an oriented $m$-colored braid (resp. an oriented $m$-colored string link) if the underlying tangle is a braid (resp. a string link).

More formally, $m$-colored tangles can be understood as morphisms of a category in the following way. Consider two maps $\varphi: \{1, \ldots, n\} \to \{\pm 1, \ldots, \pm m\}$ and $\varphi': \{1, \ldots, n'\} \to \{\pm 1, \ldots, \pm m\}$, where $n$ and $n'$ are non-negative integers. We will say that an $m$-colored tangle $(\tau, c)$ is a $(\varphi, \varphi')$-tangle if the following conditions hold:

- $\tau$ is an $(\varepsilon, \varepsilon')$-tangle, where $\varepsilon = \varphi/|\varphi|$ and $\varepsilon' = \varphi'/|\varphi'|$;
- if $x_i \in D^2 \times \{0\}$ (resp. $x_i' \in D^2 \times \{1\}$) is an endpoint of a component $\tau_j$ of $\tau$, then $|\varphi(i)| = c(j)$ (resp. $|\varphi'(i)| = c(j)$).

Two $(\varphi, \varphi')$-tangles are isotopic if they are isotopic as $(\varepsilon, \varepsilon')$-tangles under an isotopy that respects the color of each component. We denote by $T(\varphi, \varphi')$ the set of isotopy classes of $(\varphi, \varphi')$-tangles. The composition of oriented tangles induces a composition $T(\varphi, \varphi') \times T(\varphi', \varphi'') \to T(\varphi, \varphi'')$ for any $\varphi, \varphi'$ and $\varphi''$.

This allows us to define the category of $m$-colored tangles $\text{Tangles}_m$. Its objects are the maps $\varphi: \{1, \ldots, n\} \to \{\pm 1, \ldots, \pm m\}$ with $n \geq 0$, and its morphisms are given by $\text{Hom}(\varphi, \varphi') = T(\varphi, \varphi')$. Clearly, oriented $m$-colored braids and oriented $m$-colored string links form categories $\text{Braids}_m$ and $\text{Strings}_m$ such that

$$\text{Braids}_m \subset \text{Strings}_m \subset \text{Tangles}_m.$$
6.2. The multivariable Lagrangian representation

We now define a functor \( \mathfrak{F}_m : \text{Tangles}_m \to \text{Lagr}_{A_m} \), where \( A_m \) denotes the ring \( \mathbb{Z} [ t_1^{\pm 1}, \ldots, t_m^{\pm 1} ] \). This construction generalizes the functor of Theorem 3.5, which corresponds to the case \( m = 1 \). It also extends the works of Gassner for pure braids and Le Dimet for pure string links.

Consider an object of \( \text{Tangles}_m \) that is, a map \( \varphi : \{ 1, \ldots, n \} \to \{ \pm 1, \ldots, \pm m \} \) with \( n \geq 0 \). Set \( \ell(\varphi) = (\ell_1(\varphi), \ldots, \ell_m(\varphi)) \in \mathbb{Z}^m \), where \( \ell_j(\varphi) = \sum_{i | \varphi(i) = \pm j} \text{sign}(\varphi(i)) \) for \( j = 1, \ldots, m \). Using the notation of Section 3.2, we define

\[
D_\varphi = \begin{cases} 
D^2 \setminus \mathcal{N}'(\{x_1, \ldots, x_n\}) & \text{if } \ell(\varphi) \neq (0, \ldots, 0), \\
S^2 \setminus \mathcal{N}'(\{x_1, \ldots, x_n\}) & \text{if } \ell(\varphi) = (0, \ldots, 0).
\end{cases}
\]

As in the case of oriented tangles, we endow \( D_\varphi \) with the counterclockwise orientation, a base point \( z \), and generators \( e_1, \ldots, e_n \) of \( \pi_1(D_\varphi, z) \). Consider the homomorphism from \( \pi_1(D_\varphi) \) to the free abelian group \( G \cong \mathbb{Z}^m \) with basis \( t_1, \ldots, t_m \) given by \( e_i \mapsto t_i(\varphi(i)) \). It defines a regular \( G \)-covering \( \hat{D}_\varphi \to D_\varphi \), so the homology \( H_1(\hat{D}_\varphi) \) is a module over \( \mathbb{Z} G = A_m \). Finally, let \( \omega_\varphi : H_1(\hat{D}_\varphi) \times H_1(\hat{D}_\varphi) \to A_m \) be the skew-hermitian pairing given by

\[
\omega_\varphi(x, y) = \sum_{g \in G} (gx, y)g^{-1},
\]

where \( \langle , \rangle : H_1(\hat{D}_\varphi) \times H_1(\hat{D}_\varphi) \to \mathbb{Z} \) is the intersection form induced by the orientation of \( D_\varphi \) lifted to \( \hat{D}_\varphi \).

Consider now a \((\varphi, \varphi')\)-tangle \((\tau, c)\). Note that \( \ell(\varphi) = \ell(\varphi') \). Let \( X_\tau \) be the compact manifold

\[
X_\tau = \begin{cases} 
(D^2 \times [0, 1]) \setminus \mathcal{N}'(\tau) & \text{if } \ell(\varphi) \neq (0, \ldots, 0), \\
(S^2 \times [0, 1]) \setminus \mathcal{N}'(\tau) & \text{if } \ell(\varphi) = (0, \ldots, 0),
\end{cases}
\]

oriented so that the induced orientation on \( \partial X_\tau \) extends the orientation on \((-D_\varphi) \cup D_\varphi'\). We know from Section 3.3 that \( H_1(X_\tau) = \bigoplus_{j=1}^m \mathbb{Z} m_j \) if \( \ell(\varphi) \neq (0, \ldots, 0) \), and \( H_1(X_\tau) = \bigoplus_{j=1}^m \mathbb{Z} m_j / \sum_{i=1}^n \text{sign}(\varphi(i)) e_i \) otherwise. Hence, the coloring of \( \tau \) defines a homomorphism \( H_1(X_\tau) \to G, m_j \mapsto t_{\tau(j)} \) which induces a homomorphism \( \pi_1(X_\tau) \to G \) extending the homomorphisms \( \pi_1(D_\varphi) \to G \) and \( \pi_1(D_\varphi') \to G \). It gives a \( G \)-covering \( \hat{X}_\tau \to X_\tau \).

Consider the inclusion homomorphisms \( i_\tau : H_1(\hat{D}_\varphi) \to H_1(\hat{X}_\tau) \) and \( i'_\tau : H_1(\hat{D}_\varphi') \to H_1(\hat{X}_\tau) \). Denote by \( j_\tau \) the homomorphism \( H_1(\hat{D}_\varphi) \oplus H_1(\hat{D}_\varphi') \to H_1(\hat{X}_\tau) \) given by \( j_\tau(x, x') = i'_\tau(x') - i_\tau(x) \). Set

\[
\mathfrak{F}_m(\tau) = \ker(j_\tau) \subset H_1(\hat{D}_\varphi) \oplus H_1(\hat{D}_\varphi').
\]

**Theorem 6.1.** Let \( \mathfrak{F}_m \) assign to each map \( \varphi : \{ 1, \ldots, n \} \to \{ \pm 1, \ldots, \pm m \} \) the pair \( (H_1(\hat{D}_\varphi), \omega_\varphi) \) and to each \( \tau \in T(\varphi, \varphi') \) the submodule \( \mathfrak{F}_m(\tau) \) of \( H_1(\hat{D}_\varphi) \oplus H_1(\hat{D}_\varphi') \). Then, \( \mathfrak{F}_m \) is a functor
The definition of \( DM \) is a submodule of the free \( M \) follows easily from the definitions and the excision theorem. □

\[ H \]

by the homomorphism \( \subset \)

\[ M \]

on \( BH \)

\[ H \]

high-dimensional manifolds. We conclude the paper with a brief sketch of this construction.

6.3. High-dimensional Lagrangian representations

The Lagrangian representation of Theorem 3.5 can be generalized in another direction by considering high-dimensional manifolds. We conclude the paper with a brief sketch of this construction.

Fix throughout this section an integer \( n \geq 1 \). In the sequel, all the manifolds are assumed piecewise linear, compact and oriented. Consider a homology \( 2n \)-sphere \( D \). To this manifold, we associate a category \( \mathcal{C}_D \) as follows. Its objects are codimension-2 submanifolds \( M \) of \( D \) such that \( H_n(M) = 0 \). The morphisms between \( M \subset D \) and \( M' \subset D \) are given by properly embedded codimension-2 submanifolds \( T \) of \( D \times [0, 1] \) such that the oriented boundary \( \partial T \) of \( T \) satisfies \( \partial T \cap (D \times \{0\}) = -M \) and \( \partial T \cap (D \times \{1\}) = M' \), where \( -M \) denotes \( M \) with the opposite orientation. The composition is defined in the obvious way.

If \( D_M \) is the complement of an open tubular neighborhood of \( M \) in \( D \), we easily check that \( H_1(D_M) \cong H_0(M) \). Therefore, the epimorphism \( H_0(M) \to \mathbb{Z} \) which sends every generator to 1 determines a \( \mathbb{Z} \)-covering \( \hat{D}_M \to D_M \). The lift of the orientation of \( D_M \) to \( \hat{D}_M \) defines a \( \mathbb{Z} \)-bilinear intersection form on \( H_n(\hat{D}_M) \). This gives a \( \Lambda \)-sesquilinear form on \( H_n(\hat{D}_M) \), which in turn induces a \( \Lambda \)-sesquilinear form \( \omega_M \) on \( BH_n(\hat{D}_M) \), where \( BH = H/Tors_H \) for a \( \Lambda \)-module \( H \). (Note that \( \omega_M \) is skew-hermitian if \( n \) is odd, and Hermitian if \( n \) is even.)

Using the fact that \( H_n(M) = 0 \), the proof of Lemma 3.2 can be applied to this setting, showing that \( \omega_M \) is non-degenerate. Let \( \mathfrak{H}_D(M) \) denote the \( \Lambda \)-module \( BH_n(\hat{D}_M) \) endowed with the non-degenerate \( \Lambda \)-sesquilinear form \( \omega_M \).

Given a codimension-2 submanifold \( T \) of \( D \times [0, 1] \), denote by \( X_T \) the complement of an open tubular neighborhood of \( T \) in \( D \times [0, 1] \). Since \( H_1(X_T) \cong H_0(T) \), we have a \( \mathbb{Z} \)-covering \( \hat{X}_T \to X_T \) given by the homomorphism \( H_0(T) \to \mathbb{Z} \) which sends every generator to 1. There are obvious inclusions \( \hat{D}_M \subset \hat{X}_T \) and \( \hat{D}_M' \subset \hat{X}_T \) which induce homomorphisms \( i \) and \( i' \) in \( n \)-dimensional homology. Let \( j: H_n(\hat{D}_M) \oplus H_n(\hat{D}_M') \to H_n(\hat{X}_T) \) be the homomorphism given by \( j(x, x') = i'(x') - i(x) \). It induces
a homomorphism

\[ BH_n(\hat{D}_M) \oplus BH_n(\hat{D}_M') \xrightarrow{j_T} BH_n(\hat{X}_T). \]

Set \( \mathcal{F}_D(T) = \ker(j_T) \). The proof of Lemma 3.3 can be applied to check that \( \mathcal{F}_D(T) \) is a Lagrangian submodule of \( (-BH_n(\hat{D}_M)) \oplus BH_n(\hat{D}_M') \). Lemma 3.4 can also be adapted to our setting to show that \( \mathcal{F}_D(T_2 \circ T_1) = \mathcal{F}_D(T_2) \circ \mathcal{F}_D(T_1) \). Therefore, \( \mathcal{F}_D \) is a functor from \( \mathcal{C}_D \) to the Lagrangian category \( \text{Lagr}_A \) amended as follows: the non-degenerate form is Hermitian if \( n \) is even, skew-hermitian if \( n \) is odd.

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