

ALGEBRAIC CONCORDANCE AND CASSON-GORDON INVARIANTS

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ABSTRACT. The following pages contain some notes on algebraic concordance and the Casson-Gordon invariants. No originality is claimed and all mistakes are my own.

INTRODUCTION: WHAT IS IN THESE NOTES AND WHAT IS NOT

The aim of these notes is to provide a very brief introduction to two topics in knot concordance: algebraic concordance and the Casson-Gordon invariants. In fact, these notes outgrew a reading group on knot concordance which was organized during the spring of 2017 in Geneva: the ten subsections below are expanded versions of the ten talks given at that occasion. Our overall goal was to study slice knots which are not algebraically slice.

These notes neither have the ambition of providing a detailed account of the subject (as in [20]), nor a survey of knot concordance (as in [19]). Furthermore, there is no mention of more recent topics such as the solvable filtration [5] or the invariants coming from knot homologies. Rather, these notes strive to provide a leisurely introduction to algebraic concordance and Casson-Gordon invariants, assuming only minimal background. In particular, technical tools are introduced as required by the exposition (often avoiding the greatest generality) and several points in the first part are intentionally slightly informal.

We conclude with a more technical remark. Although the goal is to discuss algebraic slice knots which are not slice, we avoided the shortest path to such examples (arguably, such a path might involve the integral of the Levine-Tristram signature on the circle, see e.g. these introductory notes [23]). Instead, we chose to follow the original approach of Casson and Gordon [3], discussing the invariants $\sigma(K, \chi)$ and $\tau(K, \chi)$. Note however that in place of the G-signature computation involved in [3], we use the surgery formula of [2] to compute $\sigma(K, \chi)$.

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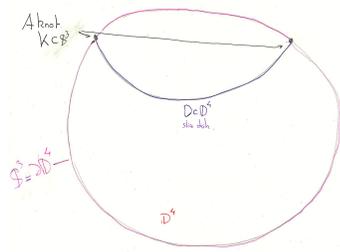
1. CONCORDANCE AND THE ALGEBRAIC CONCORDANCE GROUP

1.1. **Concordance and sliceness.** The aim of this first subsection is to introduce slice knots and the knot concordance group. References include [20] and [13, Chapter 12].

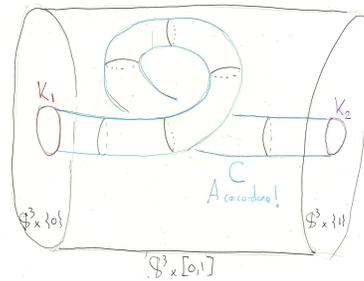
A *knot* consists of a smooth embedding of the circle S^1 into the 3-sphere S^3 . Although a knot bounds a smoothly embedded disk in S^3 if and only if it is the trivial knot, the situation in D^4 is quite different.

Definition 1. A knot $K \subset S^3$ is *slice* if it bounds a smoothly embedded 2-disk in the 4-ball D^4 . Two knots K_1 and K_2 are *concordant* if there exists a smoothly properly embedded annulus $S^1 \times [0, 1]$ in $S^3 \times [0, 1]$ whose boundary is $K_1 \sqcup -\overline{K_2}$.

A schematic picture of a slice disk can be seen in the dimensionally reduced picture on the left hand side of figure below, while a schematic illustration of a concordance is depicted on the right hand side:



(a) A schematic picture of a slice disk D for a knot K .



(b) A concordance C between two knots K_1 and K_2 .

The set of isotopy classes of knots forms a monoid under the connected sum. The goal is now to show that “the set of all knots modulo slice knots” forms a group.

Lemma 1.1. *Concordance is an equivalence relation on the set of isotopy classes of knots. Moreover, the connected sum is well-defined on the set of concordance classes of knots.*

Proof. The relation is reflexive since $K \times I$ provides a concordance from a knot K to itself. Symmetry is also clear: by turning upside down a concordance between K_1 and K_2 , one obtains a concordance from K_2 to K_1 . Finally, if K_1 is concordant to K_2 and K_2 is concordant to K_3 , then a concordance from K_1 to K_3 is obtained by stacking the concordances one on top of the other, see Figure 2.

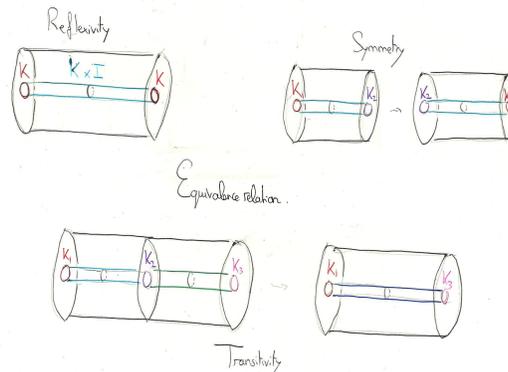


FIGURE 2. Concordance is an equivalence relation.

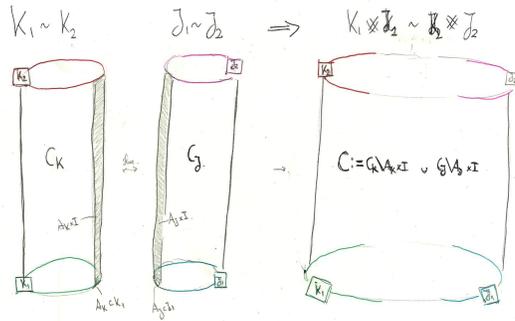


FIGURE 3. If K_1 is concordant to K_2 and J_1 is concordant to J_2 , then $K_1 \# J_1$ is concordant to $K_2 \# J_2$.

To deal with the second statement, assume that K_1 is concordant to K_2 via a concordance C_K and that J_1 is concordant to J_2 via a concordance C_J . Our aim is to provide a concordance C between $K_1 \# J_1$ and $K_2 \# J_2$. Picking intervals $A_K \subset K_1$ and $A_J \subset J_1$ as in Figure 3, the desired concordance C is $C_K \setminus (A_K \times I) \cup C_J \setminus (A_J \times I)$. Everything plays out well on the level of ambient manifolds, since the pair $(S^3 \times I, C)$ is homeomorphic to the pair $(S^3 \times I \setminus (\nu A_K \times I) \cup S^3 \times I \setminus (\nu A_J \times I), C_K \setminus (A_K \times I) \cup C_J \setminus (A_J \times I))$. This concludes the proof of the lemma. \square

The next proposition relates concordance to sliceness.

Proposition 1.2. *Given knots K_1, K_2 and K , the following assertions hold:*

- (1) K_1 is concordant to K_2 if and only if $K_1 \# -\overline{K_2}$ is slice.
- (2) K is slice if and only if it is concordant to the unknot.
- (3) $K \# -\overline{K}$ is slice.

Proof. Assume that K_1 and K_2 are concordant via a concordance C in $S^3 \times I$. Pick an interval A in K_1 such that $A \times I \subset S^3 \times I$ meets $K_2 \subset S^3 \times \{1\}$ in an interval and such that the 4-ball $B := \nu A \times I$ intersects $S^3 \times \{0\}, S^3 \times \{1\}$ in 3-balls. As one can see in Figure 4, the pair $(S^3 \times I \setminus B, C \setminus A \times I)$ is homeomorphic to a pair (D^4, D) , where D is a slice disk for $K_1 \# -\overline{K_2}$.

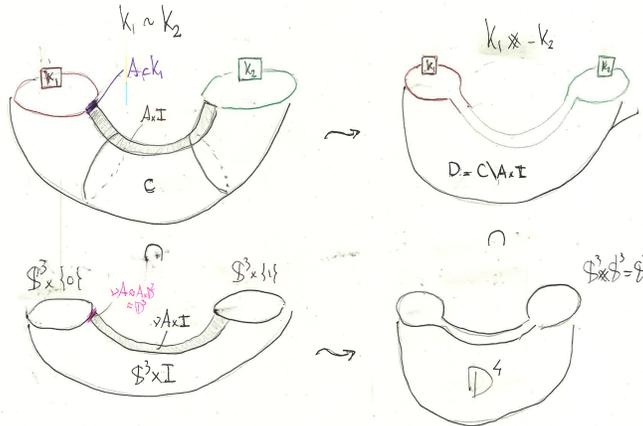


FIGURE 4. Going from a concordance in $S^3 \times I$ to a slice disk in D^4 .

Conversely, if D is a slice disk for $K_1 \# -\overline{K}_2$, then one may remove a small 4-ball neighborhood of the center of D yielding the pair $(S^3 \times I, C)$. The second statement follows immediately: denote the unknot by U , write K as $K \# -\overline{U}$ and use the fact we just proved.

To prove the last statement, let B be a small 3-ball in S^3 which intersects K in an arc A , see Figure 5. Removing the pair (B, A) from (S^3, K) and taking a product with the interval I gives a pair homeomorphic to (D^4, D) , where D is a slice disk for $K \# -\overline{K}$. This concludes the proof of the proposition. \square

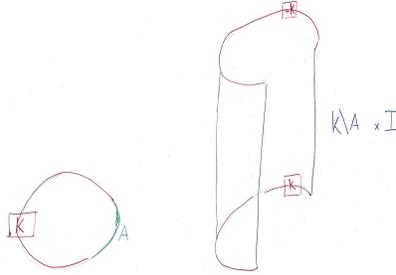


FIGURE 5. $K \# -K$ bounding a disk.

As a corollary, we are finally able to define the *knot concordance group* \mathcal{C} .

Corollary 1.3. *The set \mathcal{C} of concordance classes of knots forms an abelian group under the direct sum. The identity element is the class of the unknot while the additive inverse of $[K]$ is $[-\overline{K}]$.*

Proof. Using Lemma 1.1, we know that concordance is an equivalence relation over which connected sum is well-defined. Since connected sum is already associative and commutative on isotopy classes of knots, it is still associative and commutative on \mathcal{C} . The fact that the unknot is the identity element for the connected sum also descends immediately to \mathcal{C} . Finally the inverse of a class $[K]$ is given by $[-\overline{K}]$. Indeed, by the third point of Proposition 1.2, $K \# -\overline{K}$ is slice i.e. concordant to the trivial knot, by the second point of Proposition 1.2. \square

1.2. Classical invariants via Seifert matrices. The goal of this second subsection is to provide the first obstructions to sliceness. To this end, we introduce the Alexander polynomial and the signature by means of Seifert matrices. References include [17, Chapters 6 and 8], [13, Chapters 5 and 12] and [7, Chapters 5, 6, 7].

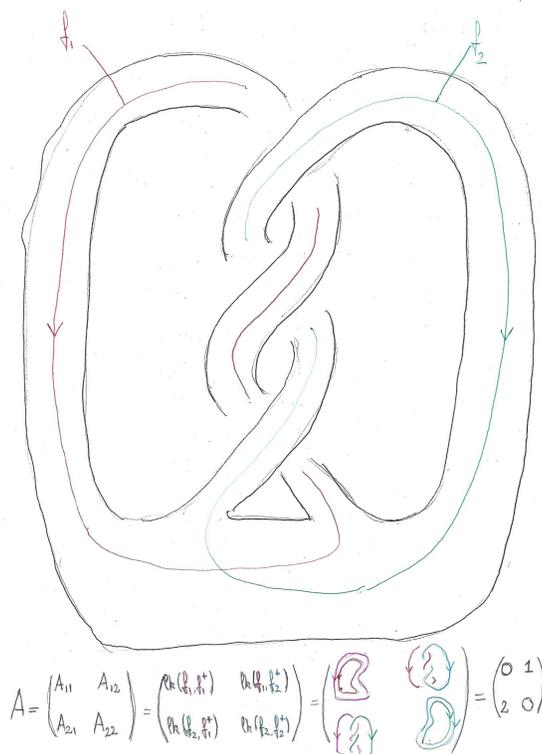
A *Seifert surface* for an oriented knot K is a compact connected oriented surface whose oriented boundary is K . Recall that the first homology group of such a surface F is free abelian of rank $2g$, where g is the genus of F . Since F is oriented, it admits a neighborhood $F \times [-1, 1] \subset S^3$. If x is a simple closed curve on F , then we denote by x^\pm the *pushed off* curves $x \times \{\pm 1\}$ in $F \times \{\pm 1\}$. One may then consider the *Seifert pairing*

$$\begin{aligned} \theta: H_1(F; \mathbb{Z}) \times H_1(F; \mathbb{Z}) &\rightarrow \mathbb{Z} \\ ([x], [y]) &\mapsto \ell k(x^-, y) = \ell k(x, y^+). \end{aligned}$$

This pairing is well defined on homology and a *Seifert matrix* for a knot K is a matrix representing the Seifert pairing. Here is a concrete example of a Seifert matrix.

Example 1.4. Figure 6 depicts a natural Seifert surface for the knot $K := 9_{46}$ and shows that K admits $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ as a Seifert matrix.

Isotopic knots need not have congruent Seifert matrices and we refer to [17, Chapter 8] for a discussion of *S-equivalence*. Knot invariants can nevertheless be extracted from these matrices.

FIGURE 6. A Seifert surface and a Seifert matrix for the knot 9_{46} .

Definition 2. The *Alexander polynomial* of a knot K is the Laurent polynomial $\Delta_K(t) = \det(tA - A^T)$, considered up to multiplication by $\pm t^n$ for $n \in \mathbb{Z}$. The *signature* $\sigma(K)$ of a knot K is the signature of the symmetric matrix $A + A^T$.

It is well known that the Alexander polynomial and signature are knot invariants, i.e. they are independent of the choice of Seifert surface [17, Chapter 8], [7, Chapter 7]. Here are two sample computations.

Example 1.5. In the case of the knot $K = 9_{46}$ considered in Example 1.4, a short computation shows that $\Delta_K(t) = (2 - t)(2t - 1) \doteq (2 - t)(2 - t^{-1})$ and $\sigma(K) = 0$. As a second example, consider the twist knot J_k depicted in Figure 7 with Seifert matrix $\begin{bmatrix} -1 & 1 \\ 0 & k \end{bmatrix}$. Here the convention is that if $k > 0$ (respectively $k < 0$), then J_k has $2k$ positive (respectively negative) half twists. A short computation now shows that $\Delta_{J_k}(t) = -kt^2 + t(1 + 2k) - k$ and $\sigma(J_k) = 0$ (respectively $\sigma(J_k) = -2$) if $k \geq 0$ (respectively $k < 0$).

The next theorem provides our first obstructions to sliceness.

Theorem 1.6. *If K is a slice knot, then*

- (1) *the Alexander polynomial of K is a norm, i.e. $\Delta_K(t) = f(t)f(t^{-1})$ for some $f(t) \in \mathbb{Z}[t^{\pm 1}]$.*
- (2) *the signature of K vanishes.*

Observe that Theorem 1.6 already obstructs many knots from being slice:

Example 1.7. For $k < 0$, we saw in Example 1.5 that the twist knots J_k have non-vanishing signature and we therefore conclude that they are not slice. Even more concretely, the trefoil knot is not slice since its signature is equal to -2 . On the other hand, note that the figure eight knot is not slice since its Alexander polynomial is not a norm (even though its signature vanishes).

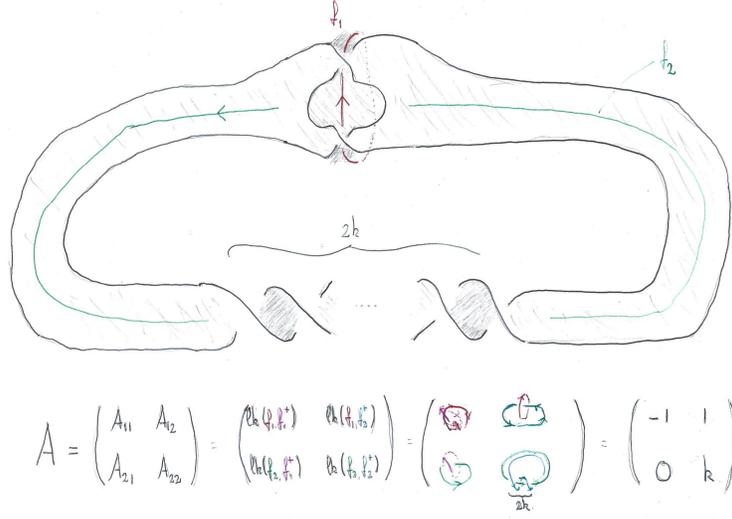


FIGURE 7. A Seifert surface and a Seifert matrix for the twist knot J_k .

In fact, Theorem 1.6 will follow from a stronger result which states that slice knots admit Seifert matrices of a particular type. To make this precise, we start with the following definition:

Definition 3. A $2g \times 2g$ matrix is *metabolic* if it is congruent to a bloc matrix of the form $\begin{bmatrix} 0 & B \\ C & D \end{bmatrix}$, where each block is a $g \times g$ matrix.

We shall discuss metabolic matrices in more detail in Subsection 1.3 but we already note that the signature of a nonsingular metabolic symmetric matrix vanishes (see e.g. [14, page 19]). Consequently, if we show that a slice knot K admits a metabolic Seifert matrix, then it will immediately follow that $\sigma(K) = 0$ (indeed $A + A^T$ is nonsingular since its determinant is equal modulo 2 to $\Delta_K(1) = \pm 1$). On the other hand, a direct computation shows that if A is metabolic, then $\Delta_K(t) = \det(tA - A^T)$ is a norm.

Summarizing, Theorem 1.6 will follow if one manages to show that slice knots admit metabolic Seifert matrices. This is the content of the next theorem whose proof is only sketched: in order to avoid introducing too much machinery at an early stage we prefer to refer the reader to [17, Proposition 8.17] for details.

Theorem 1.8. *If K is slice, then it admits a metabolic Seifert matrix.*

Proof. Let $D \subset D^4$ be a slice disk for K and let $F \subset S^3$ be a genus g Seifert surface for K . Using obstruction theory, there exists a compact connected oriented 3-manifold $M \subset D^4$ such that $\partial M = F \cup_{\partial} D$, see [17, Lemma 8.14]. A standard “half lives half dies” argument now shows that $\ker(H_1(\partial M; \mathbb{Q}) \rightarrow H_1(M; \mathbb{Q}))$ is a half dimensional subspace of $H_1(\partial M; \mathbb{Q})$ [17, Lemma 8.15]. It follows that there is a \mathbb{Z} -basis $[f_1], \dots, [f_{2g}]$ for $H_1(\partial M, \mathbb{Z})$ such that $[f_1], \dots, [f_g]$ are sent to zero in $H_1(M, \mathbb{Q})$, see [17, Corollary 8.16].

Since $\partial M = F \cup_{\partial} D$ and D is simply connected, the f_i can be homotoped to lie in F . We claim that $[f_1], \dots, [f_g]$ span the desired metabolizer of the Seifert form. Since each $[f_i]$ maps to zero in $H_1(M; \mathbb{Q})$, there exists integers n_i such that $n_i[f_i]$ vanishes in $H_1(M, \mathbb{Z})$. It follows that $n_i f_i$ bounds a surface N_i in M . Moreover, if f_i^- is a push-off of f_i , then $(n_i f_i)^-$ bounds a surface N_i^- which is disjoint from the N_j . Consequently, $\text{lk}(f_i^-, f_j) = N_i^- \cdot N_j = 0$ for $i, j = 1, \dots, g$, where $N_i \cdot N_j$ denotes the algebraic intersection of the surfaces in the 4-ball (which vanishes since N_i^- is disjoint from N_j). \square

We conclude this subsection with some terminology: a knot is *algebraically slice* if it admits a metabolic Seifert matrix. Summarizing, Theorem 1.8 showed that algebraically slice knots admit

metabolic Seifert surfaces, while a straightforward computation using Example 1.5 shows that the knots J_k are algebraically slice if and only if $k \geq 0$ and $4k + 1$ is a square.

1.3. The algebraic concordance group. The aim of this subsection is to build a surjection from the concordance group \mathcal{C} onto an algebraically defined group \mathcal{C}^{alg} . References for this subsection include [13, Chapter 12] and [20].

We begin with an algebraic characterization of Seifert matrices.

Proposition 1.9. *A square matrix A with integer coefficients is a Seifert matrix of a knot if and only if $\det(A - A^T) = \pm 1$.*

Proof. Given a Seifert form θ for a knot K , pick a basis for the first homology of the underlying Seifert surface F , and let A be the associated Seifert matrix. We shall show that $\det(A - A^T) = \pm 1$ by arguing that $A - A^T$ represents the intersection form ξ on F . Given two homology classes $[x], [y] \in H_1(F; \mathbb{Z})$, one obtains

$$\begin{aligned} \theta([x], [y]) - \theta([y], [x]) &= \ell k(x, y^+) - \ell k(y, x^+) = \ell k(x, y^+) - \ell k(y^-, x) = \ell k(x, y^+) - \ell k(x, y^-) \\ &= \ell k(x, y^+ - y^-). \end{aligned}$$

Now let C_y be the cylinder whose boundary is $y^+ - y^-$. Note that $\ell k(x, y^+ - y^-) = \ell k(x, \partial C_y)$ is equal to the algebraic intersection between x and C_y . Since the latter also computes $\xi(x, y)$, we conclude that $A - A^T$ has determinant ± 1 . Indeed ξ is nonsingular over the integers because F has a single boundary component.

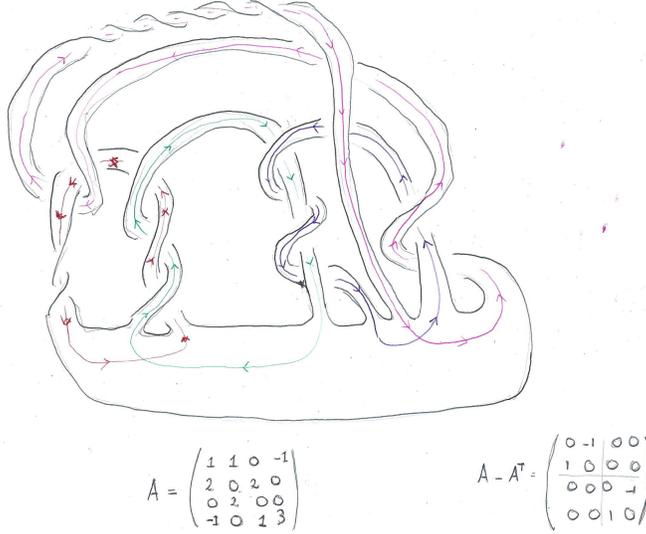


FIGURE 8. Realizing a matrix A which satisfies $\det(A - A^T) = \pm 1$ as a Seifert matrix.

To prove the converse, let A be an integral matrix which satisfies $\det(A - A^T) = \pm 1$. Since $A - A^T$ is congruent to a symplectic matrix, A must be of even size, say $2g$. Build a Seifert surface F of genus g by gluing g pairs of bands to a disk. Twisting the bands and linking them appropriately recovers the coefficients of A , as illustrated in Figure 8. \square

Proposition 1.9 naturally leads us to consider the set \mathcal{S} of integral matrices which satisfy $\det(A - A^T) = \pm 1$. We will often refer to elements of \mathcal{S} as *Seifert matrices*. The remainder of this section is devoted to turning \mathcal{S} into a group \mathcal{C}^{alg} by “moding out the metabolic matrices” (recall from Definition 3 that a $(2g \times 2g)$ matrix is metabolic if it is congruent to one with a $(g \times g)$ submatrix in its upper left corner).

Definition 4. Two Seifert matrices $A, B \in \mathcal{S}$ are *algebraically concordant* if $A \oplus -B$ is metabolic.

We have to check that algebraic concordance is an equivalence relation on \mathcal{S} . Before getting to that, we need a preliminary lemma.

Lemma 1.10. *Given Seifert matrices A, B, A', B', M_1, M_2 and M , the following statements hold:*

- (1) *if M_1 and M_2 are metabolic, then $M_1 \oplus M_2$ is metabolic;*
- (2) *if $A \oplus -A'$ and $B \oplus -B'$ are metabolic, then $(A \oplus B) \oplus -(A' \oplus B')$ is metabolic;*
- (3) *if M and $A \oplus M$ are metabolic, then A is metabolic.*

Proof. Let P_1 and P_2 be invertible matrices (of respective size $2g$ and $2h$) such that $P_1 M_1 P_1^T$ and $P_2 M_2 P_2^T$ have g and h dimensional blocks of zeros in their upper left corner. Let P be the $2(g+h)$ square permutation matrix which places the two aforementioned blocks of zeros next to each other in the basis. Clearly, $P(P_1 \oplus P_2)(M_1 \oplus M_2)(P_1 \oplus P_2)^T P^T$ contains a $g+h$ block of zeros in its upper left corner: an illustration is provided in (1) in the case where $g = h = 1$ and $P_1 = P_2$ are identity matrices:

$$(1) \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & y & 0 & 0 \\ z & t & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & c & d \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & y & 0 \\ 0 & 0 & 0 & b \\ z & 0 & t & 0 \\ 0 & c & 0 & d \end{bmatrix}.$$

The second assertion can be dealt with similarly, while the third is a version of the Witt cancellation lemma, see [20, Lemma 3.4.5] for details. \square

We are now ready to define the algebraic concordance group.

Theorem 1.11. *Algebraic concordance is an equivalence relation on the set \mathcal{S} of Seifert matrices. Moreover, the set \mathcal{C}^{alg} of algebraic concordance classes forms an abelian group called the algebraic concordance group: the group law is induced by the direct sum, the zero element is represented by metabolic matrices and the inverse of a class $[A]$ is $[-A]$.*

Proof. Given a $2m \times 2m$ square matrix A , in order to show reflexivity, we must prove that $A \oplus -A$ is metabolic. Let P be the $4m \times 4m$ dimensional matrix which adds the $(2m+i)$ -th row of a $4m \times 4m$ matrix to its i -th row, for $1 \leq i \leq 2m$. Since P is nonsingular and $P(A \oplus -A)P^T$ contains a $2m$ dimensional diagonal block of zeros on the upper left, $A \oplus -A$ is metabolic; a sample computation is provided in (2) in the case where $m = 2$:

$$(2) \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x & y & 0 & 0 \\ z & t & 0 & 0 \\ 0 & 0 & -x & -y \\ 0 & 0 & -z & -t \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & -x & -z \\ 0 & 0 & -z & -t \\ -x & -y & -x & -y \\ -z & -t & -z & -t \end{bmatrix}.$$

We refer to [20] for the proof of symmetry and prove transitivity instead: we suppose that $A \oplus -B$ and $B \oplus -C$ are metabolic and we wish to prove that $A \oplus -C$ is metabolic. Since $B \oplus -B$ is metabolic, using the third point of Lemma 1.10, it is enough to show that $(A \oplus -C) \oplus (B \oplus -B)$ is metabolic. Since $-C \oplus (B \oplus -B)$ is algebraically concordant to $(-B \oplus B) \oplus -C$, it follows that

$$(A \oplus -C) \oplus (B \oplus -B) \text{ is algebraically concordant to } (A \oplus -B) \oplus (B \oplus -C).$$

By assumption, we know that $(A \oplus -B)$ and $(B \oplus -C)$ are metabolic and, since we showed in the first point of Lemma 1.10 that the direct sum of metabolic matrices is metabolic, we deduce that $(A \oplus -C) \oplus (B \oplus -B)$ is indeed metabolic.

The proof that \mathcal{C}^{alg} is an abelian group is now straightforward: using the second point of Lemma 1.10, the direct sum induces a well defined operation on \mathcal{C}^{alg} , the zero element is represented by metabolic matrices, the inverse of $[A]$ is $[-A]$ (since we know that $A \oplus -A$ is metabolic), associativity and commutativity also follow promptly. \square

In order to prove that \mathcal{C} surjects onto \mathcal{C}^{alg} , we need one last lemma.

Lemma 1.12. *Given Seifert matrices V_1, V_2 and V for knots K_1, K_2 and K , the following statements hold:*

- (1) $V_1 \oplus V_2$ is a Seifert matrix for $K_1 \# K_2$.
- (2) $-V^T$ is a Seifert matrix for \overline{K} .
- (3) V^T is a Seifert matrix for $-K$.

Proof. If F_1 and F_2 are Seifert surfaces for K_1 and K_2 , a Seifert surface F for $K_1 \# K_2$ is obtained by connecting F_1 and F_2 by the obvious band. A Mayer-Vietoris argument shows that $H_1(F) \cong H_1(F_1) \oplus H_1(F_2)$ and the first assertion follows readily. The two last points are immediate. \square

We can now prove the main theorem of this subsection.

Theorem 1.13. *The map that sends the concordance class of a knot to the algebraic concordance class of any of its Seifert matrices gives rise to a well-defined group homomorphism $\Phi: \mathcal{C} \rightarrow \mathcal{C}^{alg}$.*

Proof. Given concordant knots K_1 and K_2 , we pick Seifert matrices V_1 and V_2 . In order to show that Φ is well-defined, we must prove that $V_1 \oplus -V_2$ is metabolic. Using Lemma 1.12, $K_1 \oplus -\overline{K_2}$ admits $V_1 \oplus -V_2$ as a Seifert matrix. Since K_1 and K_2 are concordant, Proposition 1.2 implies that $K_1 \oplus -\overline{K_2}$ is slice. Since we showed in Theorem 1.8 that slice knots admit metabolic Seifert matrices, $V_1 \oplus -V_2$ must be metabolic. This proves that Φ is well-defined.

The remaining assertions are also straightforward. Since we saw in Lemma 1.12 that a Seifert matrix for a connected sum of knots is given by the direct sum of the individual matrices, Φ is a homomorphism. As Proposition 1.11 implies that every matrix in \mathcal{S} can be realized as the Seifert matrix of a knot, surjectivity follows. \square

2. CASSON-GORDON INVARIANTS

2.1. Finite branched covers. In this subsection, we review finite covers branched along a knot or a slice disk. The homology of these spaces will be used during our study of the Casson-Gordon obstructions. References include [2, 3, 11, 17, 20].

The exterior $X_K := S^3 \setminus \nu K$ of a knot K is the 3-manifold with boundary obtained by removing a tubular neighborhood $\nu K \approx K \times D^2$ of K from S^3 . Similarly, if K is slice with slice disk D , we let $N_D = D^4 \setminus \nu D$ be the slice disk exterior, where $\nu D \approx K \times D^2$ is a tubular neighborhood of D .

A Mayer-Vietoris argument (or Alexander duality) immediately yields the following result.

Lemma 2.1. *The exterior X_K of a knot K has the homology of a circle. Furthermore, if K is sliced by a disk $D \subset D^4$, then the same conclusion holds for N_D .*

Since the homology of N_D is not enlightening, we shall study its covering spaces.

Remark 2.2. Recall that covering spaces of a (connected, locally connected, semi-locally-simply connected) space X are classified by subgroups of $\pi_1(X)$. In particular, specifying a surjective map $\varphi: \pi_1(X) \rightarrow G$ gives rise to a regular cover: take the cover \widehat{X} corresponding to the normal subgroup $\ker(\varphi)$. Note furthermore that the deck transformation group of \widehat{X} is G . When $X = N_D$ or X_K , we use Lemma 2.1 and consider the following composition:

$$\pi_1(X) \rightarrow H_1(X) \cong \mathbb{Z} \rightarrow \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}.$$

We shall denote the resulting n -fold cyclic covers by N_n and X_n . As we mentioned above, the cyclic group \mathbb{Z}_n acts upon these spaces via deck transformations.

Observe that the boundary of X_K is a torus. Since the Euler characteristic is multiplicative under finite covers, the restriction of the finite cover $X_n \rightarrow X_K$ to ∂X_K is also a torus. The next remark outlines another way of seeing this.

Remark 2.3. By definition of the cover $X_n \rightarrow X_K$, the longitude of K lifts to a loop in X_n , but the meridian μ of K does not. On the other hand, μ^n does lift to a loop $\widetilde{\mu}^n$. Put differently, the covering map $\partial X_n \rightarrow \partial X_K$ is given by $(z_1, z_2) \mapsto (z_1^n, z_2)$.

Using Remark 2.3, it is possible to attach a solid torus to X_n , leading to the following definition.

Definition 5. The n -fold cover of S^3 branched along K is the closed 3-manifold L_n obtained by gluing a solid torus $D^2 \times S^1$ to ∂X_n via a homeomorphism mapping the meridian of the solid torus to μ^n .

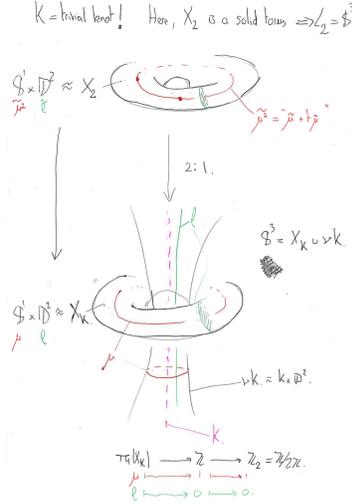


FIGURE 9. The case of the trivial knot.

We now explain the terminology used in Definition 5. Using Remark 2.3, the covering map $X_n \rightarrow X_K$ nearly extends to a cover $p: L_n = X_n \cup_{\partial} (D^2 \times S^1) \rightarrow S^3 = X_K \cup_{\partial} (D^2 \times S^1)$. Indeed, since the cover $(z_1, z_2) \mapsto (z_1^n, z_2)$ does not extend to a cover $D^2 \times S^1 \rightarrow D^2 \times S^1$ on $0 \times S^1$, the map p is a covering everywhere except along $K = 0 \times K$.

Before giving examples of this construction, we make a remark regarding the choice of the homeomorphism $\partial T \rightarrow \partial X_n$ which appears in Definition 5.

Remark 2.4. Let M be a 3-manifold whose boundary is a torus. Suppose one wants to obtain a closed 3-manifold M' by attaching a solid torus $T = D^2 \times S^1$ to M . This construction requires that we specify a homeomorphism $f: \partial T \rightarrow \partial M$, i.e. a homeomorphism of the torus. Let μ denote the meridian of T . It is a standard fact that the homeomorphism type of M' is entirely determined by the class $f_*(\mu)$ in $H_1(\partial M; \mathbb{Z}) \cong \mathbb{Z}^2$, i.e. by two integers p and q , see [26] or [27, Chapter 2].

Remark 2.4 justifies why in Definition 5 it was enough to specify the image of the meridian of the torus T . Returning to more concrete matters, let us discuss some examples of branched covers, starting with the case of the trivial knot.

Example 2.5. Let K be the trivial knot. Observe that X_K is a solid torus and that the meridian μ of K is the longitude of X_K , while the longitude of X_K is the meridian of X_K , see Figure 9. Since X_K is a solid torus, X_n is also a solid torus and therefore L_n is obtained by gluing a solid torus $T = D^2 \times S^1$ to X_n , identifying the meridian of T with μ^n . Since μ^n is the longitude of X_n , we deduce that the 2-fold cover of S^3 branched along the trivial knot is $L_2 = S^3$.

We now move on to a class of links whose 2-fold branched cover is also obtained by gluing two solid tori together. Generalizing Example 2.5, suppose we wish to glue a solid torus T to another solid torus T' along their boundaries. Using Remark 2.4, such a manifold is determined by the homology class of the meridian μ_T under a homeomorphism $f: \partial T \rightarrow \partial T'$. Since such a class is specified by two integers p and q (i.e. $f_*([\mu_T]) = p[\mu_{T'}] + q[\ell_{T'}]$), we can define the *lens space* $L(p, q)$ as the resulting identification space $T \cup_f T'$.

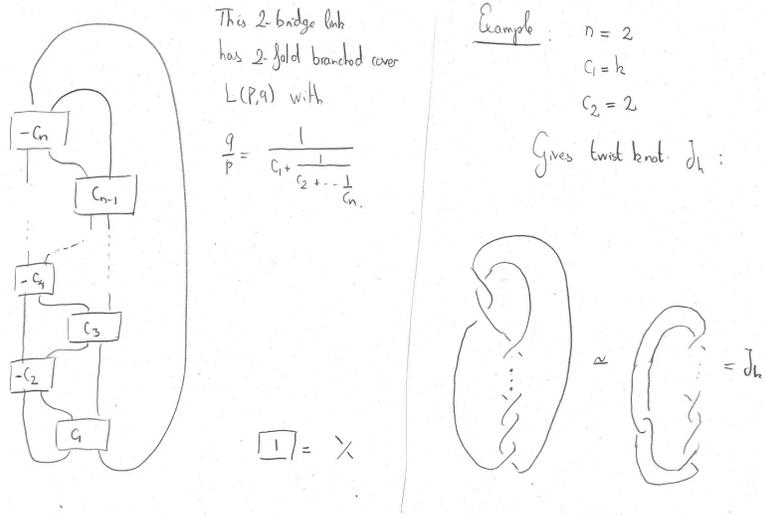


FIGURE 10. Two bridge links include twist knots as a particular case.

Example 2.6. Consider the *two bridge link* $L = L_{[c_1, \dots, c_n]}$ depicted in Figure 10 (here we are using the convention from [3, page 2] which is not the same as the one of Rolfsen [26]). The 2-fold cover of S^3 branched along the 2-bridge link $L_{[c_1, \dots, c_n]}$ is the lens space $L(p, q)$, where q/p is obtained by the continuous fraction $[c_1, \dots, c_n]$ whose definition is:

$$[c_1, c_2, \dots, c_n] = p/q = c_1 + \frac{1}{c_2 + \frac{1}{\dots + \frac{1}{c_n}}}.$$

We shall only prove that the 2-fold cover of S^3 branched along $L_{[a_1, \dots, a_n]}$ is a lens space (the interested reader is referred to [26] for the proof of the full statement). The pair (S^3, L) can be written as $(S^3, L) = (D^3, A) \cup_f (D^3, A)$, where $A = A_1 \sqcup A_2$ consists in two disjoint properly embedded arcs, and $f: \partial D^3 \rightarrow \partial D^3$ is an orientation preserving homeomorphism which fixes (globally) the 4-punctures $A \cap D^3 \subset \partial D^3$, see Figure 11. Since the 2-fold cover of D^3 branched along A is a solid torus, lifting the decomposition $(S^3, L) = (D^3, A) \cup_f (D^3, A)$ shows that the 2-fold cover branched along L is a union of two solid tori, i.e. a lens space, see Figure 11.

One can define covers $V_n \rightarrow D^4$ branched along a slice disk D in a similar fashion. Observe that the cover $N_n \rightarrow N_D$ restricts to $X_n \rightarrow X_K$ on the knot exterior and to $D \times \widehat{S}^1 \rightarrow D \times S^1$ on $D \times S^1$, where $\pi: \widehat{S}^1 \rightarrow S^1$ is the k -fold cover. Extending π to a cover $D^2 \rightarrow D^2$ branched along 0, we deduce that $V_n := N_n \cup_{\partial} (D \times D^2) \rightarrow N_D \cup_{\partial} (D \times D^2) = D^4$ is branched along $D = D \times 0$. It also follows that the boundary of the branched cover $V_n \rightarrow D^4$ is precisely the branched cover $L_n \rightarrow S^3$ along the knot $K = \partial D$.

The next proposition shows that the homology of branched covers remains manageable.

Proposition 2.7. *Let p be a prime. The p^r -branched cover L_{p^r} is a rational homology 3-sphere (i.e. $H_*(L_{p^r}, \mathbb{Q}) = H_*(S^3, \mathbb{Q})$) and the p^r -branched cover V_{p^r} is a rational homology 4-ball (i.e. $H_*(V_{p^r}, \mathbb{Q}) = H_*(D^4, \mathbb{Q})$).*

Proof. Set $n = p^r$. We only deal with the case of L_n , since the computation for V_n is practically the same. Since L_n is a closed connected 3-manifold, $H_3(L_n) = \mathbb{Z}$ and $H_0(L_n) = \mathbb{Z}$. In order to compute the remaining homology groups, we shall first study the unbranched covers X_n , where n may be infinite. The Milnor exact sequence for the cover $X_\infty \rightarrow X_n$ (see [11]) gives

$$\dots \rightarrow H_1(X_\infty; \mathbb{Z}_p) \xrightarrow{t^n - 1} H_1(X_\infty; \mathbb{Z}_p) \rightarrow H_1(X_n; \mathbb{Z}_p) \rightarrow H_0(X_\infty; \mathbb{Z}_p) \rightarrow 0.$$

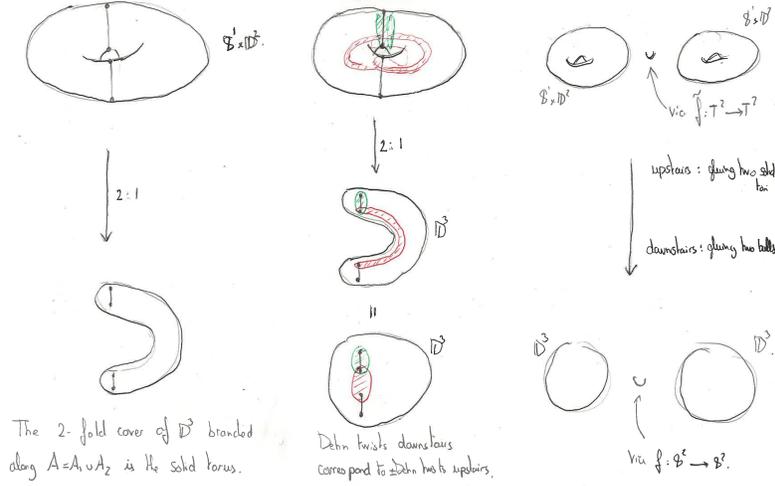


FIGURE 11. The torus is a 2-fold cover over D^3 branched along two disjoint intervals.

Since n is a prime power and since we are using \mathbb{Z}_p coefficients, $(t^n - 1) = (t - 1)^n$ would be an isomorphism if $(t - 1)$ were an isomorphism on $H_1(X_\infty)$. Temporarily taking this fact for granted, one gets $H_1(X_n; \mathbb{Z}_p) \cong H_0(X_\infty; \mathbb{Z}_p) = \mathbb{Z}_p$. Passing to the branched cover, it then follows that $H_1(L_n, \mathbb{Z}_p) = 0$ which implies that $H_1(L_n, \mathbb{Q}) = 0$. An Euler characteristic argument now shows that $H_2(L_n, \mathbb{Q}) = 0$. Summarizing, we have shown that L_n is a rational homology sphere modulo the fact that $t - 1$ induces an isomorphism on the *Alexander module* $H_1(X_\infty; \mathbb{Z})$. The proof of this fact follows readily from the Milnor exact sequence for the cover $X_\infty \rightarrow X_K$, see also Levine [15, Proposition 1.2] for a different proof. \square

2.2. Cohomology and intersection forms. In order to extract information from coverings of the slice disk exterior, we review cohomology and intersection forms. An accessible reference for cohomology is Hatcher’s textbook [12, Chapter 3]. A classical reference for the intersection form is [1, Chapter 6.11] while a treatment more in the spirit of this section will be given in [8].

Given a pair (X, Y) of spaces and an abelian group G , the n -th *cochain group* $C^n(X, Y; G)$ is defined as $\text{Hom}_{\mathbb{Z}}(C_n(X, Y), G)$. The dual of the boundary map $\partial: C_n(X, Y) \rightarrow C_{n-1}(X, Y)$ gives rise to a (co)boundary map $\delta: C^{n-1}(X, Y; G) \rightarrow C^n(X, Y; G)$. The homology $H^n(X, Y; G)$ of the resulting chain complex is called the n -th *cohomology group* of the pair (X, Y) .

In order to relate homology and cohomology, we start by considering the bilinear pairing $\langle -, - \rangle: C^k(X, Y; G) \times C_k(X, Y) \rightarrow G, (\varphi, \sigma) \rightarrow \varphi(\sigma)$. Since $\langle -, - \rangle$ descends to (co)homology, it gives rise to an *evaluation map*

$$\text{ev}: H^k(X, Y; G) \rightarrow \text{Hom}_{\mathbb{Z}}(H_k(X, Y; \mathbb{Z}), G).$$

Although ev is surjective, it is generally not an isomorphism. More precisely, the *universal coefficient theorem* states that

$$H^n(X, Y; G) \cong \text{Hom}_{\mathbb{Z}}(H_n(X, Y; \mathbb{Z}), G) \oplus \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(X, Y; \mathbb{Z}), G).$$

Here $\text{Ext}_{\mathbb{Z}}^1(-, -)$ is the so-called *Ext-functor* and all we need to know is that $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_n, \mathbb{Z}) = \mathbb{Z}_n$ and $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}, G) = 0$ for every abelian group G . On the other hand, if M is a compact oriented manifold, there are *Poincaré duality isomorphisms*

$$H_k(M; G) \cong H^{n-k}(M, \partial M; G), \quad H_k(M, \partial M; G) \cong H^{n-k}(M; G).$$

The following technical lemma illustrates these results and will be useful in the next sections.

Lemma 2.8. *Let V be a rational homology 4-ball. If the image of $H_1(\partial V; \mathbb{Z}) \xrightarrow{i} H_1(V; \mathbb{Z})$ has order m , then $|H_1(\partial V; \mathbb{Z})| = m^2$.*

Proof. We first claim that $H_2(\partial V; \mathbb{Z}) = 0$. To see this, start by noting that ∂V is a rational homology sphere: use that V is a rational homology 4-ball together with duality and the universal coefficient theorem to obtain that $H_*(V, \partial V; \mathbb{Q}) = 0$; conclude with the long exact sequence of the pair $(V, \partial V)$. Since duality gives $H_2(\partial V; \mathbb{Z}) \cong H^1(\partial V; \mathbb{Z})$, the claim will follow once we show that $H^1(\partial V; \mathbb{Z})$ vanishes. By the universal coefficient theorem, this latter group is isomorphic to $\text{Hom}_{\mathbb{Z}}(H_1(\partial V; \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}_{\mathbb{Z}}^1(H_0(\partial V; \mathbb{Z}), \mathbb{Z})$. Now $\text{Hom}_{\mathbb{Z}}(H_1(\partial V; \mathbb{Z}), \mathbb{Z}) = 0$ (indeed since ∂V is a rational homology sphere, $H_1(\partial V; \mathbb{Z})$ is finite) and $\text{Ext}_{\mathbb{Z}}^1(H_0(\partial V; \mathbb{Z}), \mathbb{Z}) = \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}, \mathbb{Z}) = 0$, which proves our claim.

Using the claim, and since both V and ∂V are connected, the long exact of the pair $(V, \partial V)$ boils down to the following exact sequence:

$$(3) \quad 0 \rightarrow H_2(V; \mathbb{Z}) \rightarrow H_2(V, \partial V; \mathbb{Z}) \rightarrow H_1(\partial V; \mathbb{Z}) \xrightarrow{i} H_1(V; \mathbb{Z}) \xrightarrow{j} H_1(V, \partial V; \mathbb{Z}) \rightarrow 0.$$

Using once again duality and universal coefficients, $|H_2(V; \mathbb{Z})| = |H_1(V, \partial V; \mathbb{Z})| =: a$ and similarly $|H_2(V, \partial V; \mathbb{Z})| = |H_1(V; \mathbb{Z})| =: b$. Since the alternating product of the orders of an exact sequence of finite groups is equal to 1, we deduce that $|H_1(\partial V; \mathbb{Z})| = b^2/a^2$. To conclude the proof, it therefore remains to show that $\text{im}(i)$ has order b/a . Using the exactness of the sequence displayed in (3), we observe that $|\text{im}(i)| = |\ker(j)| = |H_1(V; \mathbb{Z})|/|H_1(V, \partial V; \mathbb{Z})|$. But now, by definition of a and b , this is equal to b/a , which concludes the proof of the proposition. \square

We now briefly review the intersection form of a 4-manifold W . Let $i: H_2(W; \mathbb{Z}) \rightarrow H_2(W, \partial W; \mathbb{Z})$ denote the map induced by the inclusion $(W, \emptyset) \rightarrow (W, \partial W)$. Further composing this map with the Poincaré duality isomorphism $\text{PD}: H_2(W, \partial W; \mathbb{Z}) \rightarrow H^2(W; \mathbb{Z})$ and evaluation yields the map

$$\Phi: H_2(W; \mathbb{Z}) \xrightarrow{i} H_2(W, \partial W; \mathbb{Z}) \xrightarrow{\text{PD}} H^2(W; \mathbb{Z}) \xrightarrow{\text{ev}} \text{Hom}(H_2(W; \mathbb{Z}), \mathbb{Z}).$$

Set $\lambda_{\mathbb{Z}}(x, y) := \Phi(x)(y)$. It turns out that $\lambda_{\mathbb{Z}}$ is symmetric, see Remark 2.10 below.

Definition 6. The symmetric bilinear pairing $\lambda_{\mathbb{Z}}: H_2(W; \mathbb{Z}) \times H_2(W; \mathbb{Z}) \rightarrow \mathbb{Z}$ is called the *intersection form* of W . The signature of $\lambda_{\mathbb{Z}}$ is called the *signature* of W and is denoted by $\text{sign}(W)$.

As a first example, observe that if $H_2(W; \mathbb{Z}) = 0$, then $\text{sign}(W) = 0$. For instance $\text{sign}(S^4) = 0$.

Remark 2.9. Some remarks are in order.

- (1) The intersection pairing has no reason of being non-degenerate. Indeed, the evaluation map may not be injective and, furthermore, using the exact sequence of the pair $(W, \partial W)$, one sees that $\lambda_{\mathbb{Z}}$ vanishes on $\text{im}(H_2(\partial W; \mathbb{Z}) \rightarrow H_2(W; \mathbb{Z}))$.
- (2) Definition 6 can be adapted to rational coefficients. This yields a pairing $\lambda_{\mathbb{Q}}$ on $H_2(W; \mathbb{Q})$ whose signature coincides with the signature of $\lambda_{\mathbb{Z}}$. This also implies that if W is a rational homology ball, then $\text{sign}(W) = 0$.
- (3) Combining the previous two observations, observe that if $H_2(\partial W; \mathbb{Q}) = 0$, then $\lambda_{\mathbb{Q}}$ is non-degenerate (and in fact nonsingular since \mathbb{Q} is a field).
- (4) If M_1 and M_2 are two compact oriented 4-manifolds with homeomorphic boundaries, then $\text{sign}(M_1 \cup_{\partial} M_2) = \text{sign}(M_1) + \text{sign}(M_2)$. This result is frequently referred to as the *Novikov additivity* of the signature.

We now briefly recall the more geometric interpretation of the intersection form (in the *closed* case for simplicity). This definition will not be used later on, but is the one which can usually be found in textbooks. Given a closed oriented n -manifold M , the group $H_n(M; \mathbb{Z})$ is infinite cyclic and is generated by the *fundamental class* $[M]$ of M , see [12, Section 3.3].

The following remark requires some knowledge of the cup and cap products [1, 12].

Remark 2.10. Most textbooks define the intersection pairing as $\lambda_{\mathbb{Z}}(x, y) = \langle \text{PD}(x) \cup \text{PD}(y), [M] \rangle$, where $\langle -, - \rangle$ is evaluation and \cup denotes the cup product. Using $\text{PD}^{-1}(x) = x \cap [M]$, the relation $(a \cup b) \cap c = a \cap (b \cap c)$ between the cup and cap product, and the isomorphism $H_0(W; \mathbb{Z}) \cong \mathbb{Z}$, this definition is seen to be equivalent to Definition 6. The properties of the cup product immediately imply that $\lambda_{\mathbb{Z}}$ is symmetric. On the other hand, the (non-) degeneracy is less immediate.

Assuming W to be smooth, a class $x \in H_2(W; \mathbb{Z})$ can always be *represented* by a smoothly embedded surface $A \subset W$, i.e. $i_*[A] = x$, where $i_*: H_2(A; \mathbb{Z}) \rightarrow H_2(W; \mathbb{Z})$ is the inclusion induced map. Now, if A, B are surfaces which respectively represent $x, y \in H_2(W; \mathbb{Z})$, one can arrange the intersections to occur transversally. Counting these intersections with sign yields an integer $A \cdot B$ and it turns out that $\lambda_{\mathbb{Z}}(x, y) = A \cdot B$, see [1, Chapter 6.11] for details.

Example 2.11. Using the above discussion, we outline the reason for which the intersection form of $S^2 \times S^2$ is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. As a first step, observe that a Künneth formula implies that $H_2(S^2 \times S^2; \mathbb{Z})$ is freely generated by the classes represented by $A = \{a\} \times S^2$ and $B = S^2 \times \{b\}$. Since A and B intersect in a single point, paying attention to the orientations, one obtains $\lambda(x, y) = A \cdot B = 1$. Similar considerations show that $\lambda(x, x) = \lambda(y, y) = 0$.

2.3. The Casson-Gordon invariant $\sigma(K, \chi)$. We describe the invariant $\sigma(M, \chi)$ of Casson-Gordon and study how it obstructs sliceness. The description of this invariant appears at the beginning of [3] and in [2, Section 2]. In order to quickly give the main definition, we temporarily take twisted intersection forms as a blackbox, delaying a thorough discussion to Section 2.5.

Given a compact oriented 4-manifold W and a morphism $\psi: \pi_1(W) \rightarrow \mathbb{Z}_m$, we use $W_m \rightarrow W$ to denote the associated \mathbb{Z}_m -cover. In particular, $H_2(W_m; \mathbb{Z})$ is a $\mathbb{Z}[\mathbb{Z}_m]$ -module. Mapping the generator of \mathbb{Z}_m to $\omega := e^{\frac{2\pi i}{m}}$ gives rise to a map $\mathbb{Z}[\mathbb{Z}_m] \rightarrow \mathbb{Q}(\omega)$. This endows $\mathbb{Q}(\omega)$ with a $(\mathbb{Q}(\omega), \mathbb{Z}[\mathbb{Z}_m])$ -bimodule structure and we set

$$H_*(W; \mathbb{Q}(\omega)) = \mathbb{Q}(\omega) \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_*(W_m; \mathbb{Z}).$$

The justification for this notation will be given in Subsection 2.5 We temporarily take for granted the fact that the $\mathbb{Q}(\omega)$ -vector space $H_2(W; \mathbb{Q}(\omega))$ is endowed with a $\mathbb{Q}(\omega)$ -valued Hermitian form $\lambda_{\mathbb{Q}(\omega)}$ (the definition is essentially the same as Definition 6.)

Definition 7. Given a 4-manifold and a homomorphism $\psi: \pi_1(W) \rightarrow \mathbb{Z}_m$, we refer to the signature of $\lambda_{\mathbb{Q}(\omega)}$ as the *signature of W twisted by ψ* :

$$\text{sign}^{\psi}(W) := \text{sign}(\lambda_{\mathbb{Q}(\omega)}(W)).$$

This twisted signature shares several similarities with the ordinary signature. In particular it also satisfies the Novikov-additivity discussed in the fourth point of Remark 2.9. Furthermore, note that if W is simply connected, then $\text{sign}^{\psi}(W) = \text{sign}(W)$. The connection to 3-manifolds is made through the following proposition whose proof is delayed to the end of this section.

Proposition 2.12. *Let M be a closed 3-manifold and let $\chi: \pi_1(M) \rightarrow \mathbb{Z}_m$ be a homomorphism. There exists a non-negative integer r , a 4-manifold W and a map $\psi: \pi_1(W) \rightarrow \mathbb{Z}_m$ such that*

- (1) *the boundary of W consists of the disjoint of r copies of M ,*
- (2) *the restriction of ψ to ∂W coincides with χ on each copy of M .*

In these conditions are satisfied, we write $\partial(W, \psi) = r(M, \chi)$ for brevity.

Combining Definition 7 with Proposition 2.12, we can give the main definition of this section.

Definition 8. Let M be a closed 3-manifold and let $\chi: \pi_1(M) \rightarrow \mathbb{Z}_m$ be a homomorphism. Assume that $r(M, \chi) = \partial(W, \psi)$ for some integer r , some 4-manifold W and some homomorphism ψ . The *Casson-Gordon σ -invariant* is

$$\sigma(M, \chi) := \frac{1}{r} (\text{sign}^{\psi}(W) - \text{sign}(W)) \in \mathbb{Q}.$$

In Lemma 2.15 below, we shall show that $\sigma(M, \chi)$ is well-defined, i.e. that it is independent of the choice of r, W and ψ . However, in order to carry this out, we need to introduce bordism groups. Fix a space X . An X -manifold consists of a pair (M, f) , where M is a compact oriented n -manifold and $f: M \rightarrow X$ is a continuous map. Two closed X -manifolds $(M_1, f_1), (M_2, f_2)$ are X -bordant if there exists an X -manifold (W, g) of dimension $n + 1$ such that

$$\partial(W, g) = (M_1 \sqcup -M_2, f_1 \sqcup -f_2).$$

One can check that X -bordism is an equivalence relation, leading to the following definition.

Definition 9. The *bordism group* $\Omega_n(X)$ consists of X -bordism classes of n -dimensional X -manifolds.

The group law in $\Omega_n(X)$ is induced by the disjoint union, and an X -manifold (M, f) represents the zero class if and only if there exists an X -manifold (W, g) such that $\partial(W, g) = (M, f)$. It can also be checked that the bordism class of (M, f) only depends on the homotopy class of f .

Example 2.13. When X is a point, one simply writes Ω_n . Note that an n -manifold M represents the zero class in Ω_n if and only if it bounds an $n + 1$ -manifold. It follows that $\Omega_1 = \Omega_2 = 0$. It can also be shown that $\Omega_3 = 0$ [16], while $\Omega_4 \cong \mathbb{Z}$ is generated by the bordism class of the complex projective plane $\mathbb{C}P^2$ [6]. In fact, the signature is a bordism invariant and the aforementioned isomorphism is obtained by mapping the bordism class of a 4-manifold to its signature.

A second example is crucial in our study of the Casson-Gordon σ -invariant: the case where G is a discrete group and X is a $K(G, 1)$ i.e. a CW-complex whose only non-trivial homotopy group is $\pi_1(X) = G$. It is well known that such *Eilenberg-MacLane spaces* exist and are unique up to homotopy equivalence [12].

Definition 10. An element of $\Omega_n(G) := \Omega_n(K(G, 1))$ is called a G -manifold.

Using properties of $K(G, 1)$, it can then be checked that G -manifolds are represented by pairs (M, φ) , where $\varphi: \pi_1(M) \rightarrow G$ is a homomorphism. This way, a pair (M, φ) represents the zero class in $\Omega_n(G)$ if and only if M bounds and the homomorphism φ extends. In other words, Proposition 2.12 states that $\Omega_3(\mathbb{Z}_m)$ is a finite group. This latter fact can then be proved using the so-called Atiyah-Hirzerbruch spectral sequence [6].

Example 2.14. In fact, using this spectral sequence, it can be shown that $\Omega_4(\mathbb{Z}_m) \cong \Omega_4$. Since Ω_4 is generated by $\mathbb{C}P^2$, we deduce that every closed 4-dimensional \mathbb{Z}_m -manifold is bordant to a disjoint union of $\mathbb{C}P^2$'s. Since $\mathbb{C}P^2$ is simply-connected, its twisted signature is equal to its ordinary signature. Since signatures are bordism invariants, we deduce that every closed 4-dimensional \mathbb{Z}_m -manifold W has vanishing ‘signature defect’, i.e. the following quantity vanishes on such closed 4-manifolds:

$$\text{dsign}^\psi(W) := \text{sign}^\psi(W) - \text{sign}(W).$$

We can now show that $\sigma(M, \chi)$ depends neither on W nor on the extension ψ of χ .

Lemma 2.15. *The Casson-Gordon σ -invariant $\sigma(M, \chi)$ is well-defined.*

Proof. Assume that (W_1, ψ_1) and (W_2, ψ_2) admit $r(M, \chi)$ as a common boundary. We claim that $\text{dsign}^{\psi_1}(W_1) = \text{dsign}^{\psi_2}(W_2)$. Consider the closed manifold $W := W_1 \cup_{\partial} (-W_2)$. Since ψ_1 and ψ_2 agree on rM , they induce a homomorphism ψ on $\pi_1(W)$. The Novikov additivity of signatures that we mentioned in the fourth point of Remark 2.9 then provides the following equality:

$$\text{dsign}^\psi(W) = \text{dsign}^{\psi_1}(W_1) - \text{dsign}^{\psi_2}(W_2).$$

Since (W, ψ) is a closed \mathbb{Z}_m -manifold, we saw in Example 2.14 that $\text{dsign}^\psi(W)$ vanishes. We therefore deduce that the signature defects of (W_1, ψ_1) and (W_2, ψ_2) agree, proving the claim. To conclude the proof the proposition, we now suppose that (W_1, ψ_1) (resp. (W_2, ψ_2)) bounds r_1M (resp. r_2M) and strive to show that $\frac{1}{r_1} \text{dsign}^{\psi_1}(W_1) = \frac{1}{r_2} \text{dsign}^{\psi_2}(W_2)$. Taking r_2 copies of W_1 and r_1 copies of W_2 , we obtain two manifolds with boundary r_1r_2M and we can now apply the claim. This concludes the proof of the lemma. \square

If $L_2(K)$ denotes the 2-fold cover of S^3 branched along K , then Casson and Gordon introduce the following notation:

$$\sigma(K, \chi) := \sigma(L_2(K), \chi).$$

The statement of the next theorem is obtained by combining Theorem 2 and 3 of [3] and then specializing to knots whose 2-fold branched cover is a lens space.

Theorem 2.16. *If K is a slice knot whose double branched cover is a lens space L , then*

- (1) $|H_1(L)|$ is a square, say m^2 ,
- (2) if ℓ is a prime power dividing m and $\chi: \pi_1(L) \rightarrow \mathbb{Z}_\ell$ is a non-trivial character, then

$$|\sigma(K, \chi)| \leq 1.$$

At this stage, the proof of Theorem 2.16 is out of our reach for slice knots. On the other hand, as observed in [3, Theorem 1], the proof is much more accessible for ribbon knots, i.e. for knots which bound immersed disks D^2 in S^3 with only ribbon singularities (see Figure 12).

Proof of Theorem 2.16 for ribbon knots. Let D be the ribbon disk for the knot K and let W be the 2-fold cover of D^4 branched along D . Recall from Subsection 2.1 that ∂W is precisely the 2-fold cover of S^3 branched along K , i.e., by hypothesis, a lens space L . Let $i: H_1(L; \mathbb{Z}) \rightarrow H_1(W; \mathbb{Z})$ denote the inclusion induced map and let m be the order of $|\text{im}(i)|$. Using Lemma 2.8, we have $H_1(L) = m^2$ and the first statement is proved.

We now show that W can be used to compute $\sigma(M, \chi)$. As L is a lens space, the previous paragraph shows that $\pi_1(L) = \mathbb{Z}_{m^2}$. Since K is ribbon, $\pi_1(X_K) \rightarrow \pi_1(N_D)$ is surjective [3, Lemma 1] and it follows that $\pi_1(W) \cong \mathbb{Z}_m$.¹ As a consequence $\ker(i)$ must also be of order m . Since the order of χ divides m , it therefore vanishes on $\ker(i)$ and thus extends to a map $\psi: H_1(W; \mathbb{Z}) \rightarrow \mathbb{Z}_m$. Consequently, we can use (W, ψ) to compute $\sigma(K, \chi)$, where χ is now understood as \mathbb{Z}_m -valued.

We finally show that $|\sigma(K, \chi)| = 1$. Since $\pi_1(W) = \mathbb{Z}_m$, the \mathbb{Z}_m -cover W_m of W induced by χ is simply connected and thus $H_1(W_m; \mathbb{Z}) = 0$. Set $k := \mathbb{Q}(\omega)$. Since, we defined $H_i(W; k)$ as $k \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_i(W_m; \mathbb{Z})$, we deduce that $H_1(W; k) = 0$. Using duality and the universal coefficient theorem, it follows that $H_3(W_m; k) = 0$. Since χ is non-constant, $H_0(W; k)$ vanishes and thus so does $H_4(W; k)$. Using Proposition 2.7, W is a homology 4-ball and thus its Euler characteristic is equal to 1. Since the Euler characteristic can be computed with any coefficients, it follows that $H_2(W; k)$ has dimension 1 over k . As $H_1(L; k) = 0^2$, duality and the universal coefficient theorem imply that $H_2(L; k) = 0$. It follows from Remark 2.9 that the twisted intersection pairing λ_k is nonsingular. Combining the previous statements, $|\text{sign}^\psi(W)| = \dim_k H_2(W; k) = 1$ and $\text{sign}(W) = 0$, proving the theorem. \square

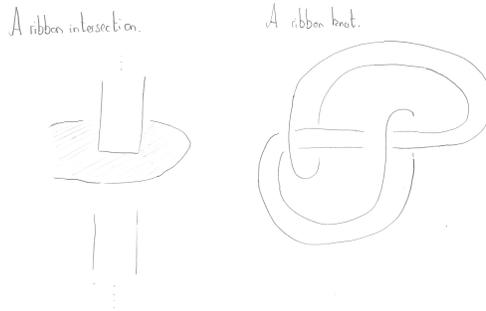


FIGURE 12. A ribbon intersection and an example of a ribbon knot.

¹why? Does the surjectivity of $\pi_1(X_K) \rightarrow \pi_1(N_D)$ imply the one of $\pi_1(L) \rightarrow \pi_1(W)$?

²why?

2.4. Non-slice knots which are algebraically slice. In this subsection, following [3], we shall give examples of non-slice knots which are algebraically slice. Along the way, we review the notion of surgery along a knot; here references include [26, Section 9F], [24, Chapter 6] and [27, Chapter 2].

Let K be a knot in S^3 and let $r = p/q$ be a rational number. We use X_K to denote the exterior of K ; this is a 3-manifold whose boundary is torus. The meridian-longitude pair of the latter is denoted by (λ, μ) . The *Dehn surgery* along K with *framing* $r = p/q$ consists of the closed 3-manifold $S_r^3(K)$ obtained by gluing a solid torus $D^2 \times S^1$ to X_K , identifying the meridian of $D^2 \times S^1$ with $p\mu + q\lambda$:

$$S_r^3(K) := X_K \cup_r (D^2 \times S^1)$$

Here, we are implicitly using the fact that to glue a solid torus by a homeomorphism, it is enough to specify the image of its meridian, recall Remark 2.4. Surgery can be generalized to framed links (i.e. each component of the link comes with a framing). A theorem proved independently by Lickorish [16] and Wallace [28] ensures that every closed 3-manifold can be obtained by surgery on a framed link.

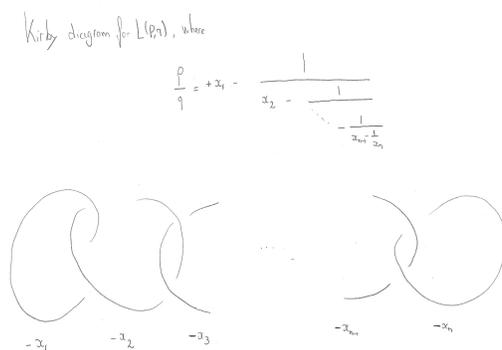


FIGURE 13. The lens space $L(p, q)$ is obtained by surgery on the link above.

Example 2.17. As we saw in Subsection 2.1, the lens space $L(p, q)$ is obtained as a union of two tori. In fact, since the exterior of the unknot is a solid torus, this means that $L(p, q)$ is obtained by p/q surgery on the unknot. In fact, as explained in [27, Theorem 2.3], $L(p, q)$ is also obtained by surgery on the n -component framed link depicted in Figure 13, whose (integer) framing coefficients $-x_1, \dots, -x_n$ satisfy

$$\frac{p}{q} = x_1 - \frac{1}{x_2 - \frac{1}{\dots - \frac{1}{x_n}}}$$

For instance, consider the twist knot J_k . Writing J_k as a 2-bridge knot $K_{[2k, 2]}$ (recall Figure 10), we see that its 2-fold branched cover is $L(p, q)$, where $q/p = \frac{1}{2k + \frac{1}{2}} = \frac{2}{4k + 1}$. We deduce that $p = 4k + 1$ and $q = 2$. Inverting this fraction, we see that

$$\frac{p}{q} = \frac{4k + 1}{2} = 2k - \frac{1}{-2}$$

Setting $x_1 = 2k$ and $x_2 = -2$, we deduce that the two-fold cover of J_k is obtained by surgery on the Hopf link $L = K_1 \cup K_2$ with framing coefficients $-x_1 = -2k$ and $-x_2 = 2$.

Given a link $L = K_1 \cup \dots \cup K_n$ with integer framing coefficients a_1, \dots, a_n , we define the *linking matrix* Λ of L by $\Lambda_{ij} = \ell k(K_i, K_j)$ for $i \neq j$ and $\Lambda_{ii} = a_i$. The next lemma describes generators for the first homology of a 3-manifold obtained by surgery on a link.

Lemma 2.18. *If M is obtained by surgery on a framed link L , then the abelian group $H_1(M)$ is generated by the meridians of L . Furthermore, if Λ is the linking matrix for the surgery, then one obtains the following exact sequence:*

$$\mathbb{Z}^n \xrightarrow{\Lambda} \mathbb{Z}^n \rightarrow H_1(M; \mathbb{Z}) \rightarrow 0.$$

Proof. Throughout this proof, the coefficients in homology will be understood to be in \mathbb{Z} . We will use the Mayer-Vietoris sequence on the decomposition $M = X_L \cup (\sqcup_{i=1}^n D^2 \times S^1)$. The map $i_0: H_0(\sqcup_{i=1}^n S^1 \times S^1) \rightarrow H_0(X_L) \oplus H_0(\sqcup_{i=1}^n D^2 \times S^1)$ is injective and consequently the aforementioned Mayer-Vietoris exact sequence for M yields

$$(4) \quad H_1(\sqcup_{i=1}^n S^1 \times S^1) \xrightarrow{i_1} H_1(X_L) \oplus H_1(\sqcup_{i=1}^n S^1 \times D^2) \rightarrow H_1(M) \rightarrow 0$$

Use j_L and j_0 to denote the maps induced on homology by the inclusions $\sqcup_{i=1}^n S^1 \times S^1 \rightarrow X_L$ and $\sqcup_{i=1}^n S^1 \times S^1 \rightarrow \sqcup_{i=1}^n S^1 \times D^2$. This way, the map i_1 which appears in (4) can be described as $i_1 = (j_K, j_0)^T$. Let m_i and l_i be the meridian and longitude of the i -th $S^1 \times S^1 = \partial(D^2 \times S^1)$, so that l_i generates $H_1(D^2 \times S^1)$. Since $j_0(l_i) = 1_i$ (the i -longitude generates the i -th copy of $H_1(S^1 \times D^2)$), we deduce that j_0 is surjective. Therefore the exact sequence (4) reduces to

$$\ker(j_0) \xrightarrow{j_L} H_1(X_L) \rightarrow H_1(M) \rightarrow 0.$$

Since $H_1(X_L)$ is generated by the meridians of X_L , so is $H_1(M)$, proving the first assertion. Since $\ker(j_0)$ is generated by the meridians of the tori $S^1 \times S^1$, the map j_L is represented by the linking matrix Λ . This concludes the proof of the Lemma. \square

Recall from Subsection 1.2 that the signature of a link L is the signature of $A + A^T$, where A is any Seifert matrix for L . Given $\omega \in S^1$, the *Levine-Tristram signature* $\sigma_L(\omega)$ of L is the signature of the Hermitian matrix $(1 - \omega)A + (1 - \bar{\omega})A^T$, where A is any Seifert matrix for L [17].

The following lemma is due to Casson and Gordon; its proof can be found in [2, Lemma 3.1].

Proposition 2.19. *Let M be the 3-manifold obtained by surgery on a framed link L with linking matrix Λ . Assume that m is a prime power and let $\chi: H_1(M; \mathbb{Z}) \rightarrow \mathbb{Z}_m \subset \mathbb{C}^*$ be the character mapping each meridian of L to α^r , where $\alpha = e^{\frac{2\pi i}{m}}$ and $0 < r < m$. Then the Casson-Gordon σ -invariant satisfies*

$$\sigma(M, \chi) = \sigma_L(\alpha^r) - \text{sign}(\Lambda) + \frac{2r(m-r)}{m^2} \sum_{i,j} \Lambda_{ij}.$$

As we shall see below, requiring that χ map each meridian to the same root of unity is a restrictive assumption. In order to use more general characters, we make use of the multivariable signature of Cimasoni-Florens [4] instead of the Levine-Tristram signature. Since the precise description of this invariant would take us too far astray, we only mention that for an n -component ordered link L and for $\omega_1, \dots, \omega_n \in S^1$, the multivariable signature $\sigma_L(\omega_1, \dots, \omega_n)$ is an integer. In fact, we need only to know that the multivariable signature of the Hopf link is identically zero and the following generalization of Proposition 2.19 whose proof can be found in [4, Theorem 6.7].

Proposition 2.20. *Let M be the 3-manifold obtained by surgery on an n -component ordered link L with linking matrix Λ . Let $m = p^s$ be a prime power and let $\chi: H_1(M; \mathbb{Z}) \rightarrow \mathbb{Z}_m \subset \mathbb{C}^*$ be the character mapping the meridian of the j -th component of L to α^{r_j} , where $\alpha = e^{\frac{2\pi i}{m}}$ and $0 < r_j < m$ is coprime to p for $j = 1, \dots, n$. Then the Casson-Gordon σ -invariant satisfies*

$$\sigma(M, \chi) = \left(\sigma_L(\alpha^{r_1}, \dots, \alpha^{r_n}) - \sum_{i < j} \Lambda_{ij} \right) - \text{sign}(\Lambda) + \frac{2}{m^2} \sum_{i,j} (m - r_i) r_j \Lambda_{ij}.$$

At the end of [3], Casson and Gordon prove that the twist knot J_k is slice if and only if $k = 0, 2$. This produces an infinite number of knots which are algebraically slice but not slice: indeed, we saw at the end of Subsection 1.2 that J_k is algebraically slice if and only if $4k + 1$ is a square.

Theorem 2.21. *The twist knot J_k is slice if and only if $k = 0, 2$.*

Proof. If $k = 0, 2$, then J_k is either the unknot or 6_1 both of which are slice. Conversely, if J_k is slice, then it must be algebraically slice by Theorem 1.8. Using the end of Subsection 1.2, it follows that $4k + 1$ must be a square, say ℓ^2 . On the other hand, we also saw in Example 2.17 that the 2-fold cover of S^3 branched along J_k is obtained by surgery on the Hopf link $L = K_1 \cup K_2$ with framing coefficients $a_1 = -2k$ and $a_2 = 2$, see Figure 14 below.

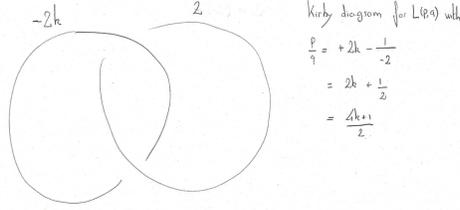


FIGURE 14. A Kirby diagram for the lens space $L(4k + 1, 2)$.

In order to compute a Casson-Gordon σ -invariant for $M := L(4k + 1, 2)$, we first define a character $\chi: H_1(M) \rightarrow \mathbb{C}^*$. Thanks to Lemma 2.18, it is enough to specify an element of \mathbb{C}^* for each meridian of L . In fact Lemma 2.18 also tells that a presentation matrix for $H_1(M)$ is $\Lambda = \begin{pmatrix} -2k & -1 \\ -1 & 2 \end{pmatrix}$. Consequently using μ_1 and μ_2 to denote the meridians of L , we have the relations $\mu_1 = 2\mu_2$ and $\mu_2 = -2k\mu_1$. Write $\ell^2 = mn$, where m is a prime power and let $\alpha = e^{\frac{2\pi i}{m}}$. Given $0 < s < m$, define the character $\chi_s: H_1(M) \rightarrow \mathbb{Z}_m = \langle \alpha \rangle \subset \mathbb{C}^*$ by setting $\chi_s(\mu_1) = \alpha^{2s}$ and $\chi_s(\mu_2) = \alpha^s$. We now check that this character is well defined. Since χ_s clearly preserves the relation $\mu_1 = 2\mu_2$, we only have to check that $\chi_s(\mu_2) = \chi_s(-2k\mu_1)$. Using the definition of χ_s , this is equivalent to $\alpha^s = \alpha^{-4ks}$ i.e. to $\alpha^{(4k+1)s} = 1$. Since $4k + 1 = mn$ and α is an m -th root of unity, the verification is completed.

We shall now compute $\sigma(M, \chi_s)$ by relying on Proposition 2.20. First of all, as we mentioned before Proposition 2.20, the multivariable signature of the Hopf link vanishes identically. Since the signature of the linking matrix $\Lambda = \begin{pmatrix} -2k & -1 \\ -1 & 2 \end{pmatrix}$ also vanishes, Proposition 2.20 yields

$$\begin{aligned} \sigma(K, \chi_s) &= \left(\sigma_L(\alpha^{2s}, \alpha^s) - \sum_{i < j} \Lambda_{ij} \right) - \text{sign}(\Lambda) + \frac{2}{m^2} \sum_{i,j} (m - r_i) r_j \Lambda_{ij} \\ &= (0 - (-1)) - 0 + \frac{2}{m^2} ((m - 2s)(-4sk - s) + (m - s)(-2s + 2s)) \\ &= 1 - \frac{2s(m - 2s)(4k + 1)}{m^2}. \end{aligned}$$

Expanding this expression and using that $\ell^2 = 4k + 1$, we obtain the same result as the one computed in Casson-Gordon's original paper [3]:

$$\sigma(K, \chi_s) = 1 - \frac{2s(m - 2s)\ell^2}{m^2}.$$

Our aim is to show that $k = 0, 2$ or equivalently that $\ell = 1, 3$. Suppose that $\ell > 3$. Since ℓ is odd, it must be greater or equal to 5 and must have a prime-power factor $m \geq 5$. Taking $s = 1$, we see that for $\ell \geq 5$ and $m \geq 5$, we have $|\sigma(K, \chi_1)| = \left| 1 - \frac{2(m-2)\ell^2}{m^2} \right| > 1$, contradicting Theorem 2.16. This concludes the proof of the theorem. \square

2.5. Homology with twisted coefficients. We review some features of homology with twisted coefficients. We also give details on the twisted signature which appeared in the definition of $\sigma(M, \chi)$. References include [9, Section 2] and [10, Section 2].

Fix a ring R endowed with involution. Given a left R -module M , we use \overline{M} to denote the right R -module which has the same underlying abelian group as M but whose R -module structure is given by $mr := \overline{r}m$.

Let X be a connected CW-complex and let $Y \subset X$ be a possibly empty subcomplex. Denote by $p: \tilde{X} \rightarrow X$ the universal cover of X and set $\tilde{Y} := p^{-1}(Y)$, so that $C(\tilde{X}, \tilde{Y})$ is a left $\mathbb{Z}[\pi_1(X)]$ -module. The choice of a homomorphism $\varphi: \mathbb{Z}[\pi_1(X)] \rightarrow R$ now endows R with the structure of a $(R, \mathbb{Z}[\pi_1(X)])$ -bimodule.

Definition 11. The *twisted homology* of the pair (X, Y) with coefficients in R is given by

$$\begin{aligned} H_*(X, Y; R) &= H_* \left(R \otimes_{\mathbb{Z}[\pi_1(X)]} C(\tilde{X}, \tilde{Y}) \right), \\ H^*(X, Y; R) &= H_* \left(\text{Hom}_{\text{right-}\mathbb{Z}[\pi_1(X)]}(\overline{C(\tilde{X}, \tilde{Y})}, R) \right). \end{aligned}$$

In order to understand this definition, it is helpful to observe that $C_k(\tilde{X})$ is freely generated by lifts of the k -cells of X to \tilde{X} . Here is a simple example:

Example 2.22. Assume $X = S^1$ so that the universal cover \tilde{X} of X is nothing but the real line \mathbb{R} . The fundamental group of S^1 is \mathbb{Z} and acts on \mathbb{R} by $n \cdot r = r + n$. From now on however, we shall think of \mathbb{Z} as the multiplicative group generated by t . This way $C_*(\tilde{X})$ is a chain complex of $\mathbb{Z}[t^{\pm 1}]$ -modules.

Next, assume that X is given its natural cell structure with one zero cell v and one one cell e . Pick a lift \tilde{v} of v to \tilde{X} and a lift \tilde{e} of e which starts at \tilde{v} . Now $C_0(\tilde{X})$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]\tilde{v}$ and $C_1(\tilde{X})$ is isomorphic to $\mathbb{Z}[t^{\pm 1}]\tilde{e}$. Furthermore, the differential is given by $d(\tilde{e}) = (t - 1)\tilde{e}$. Tensoring with R , we see that the twisted chain complex $R \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\tilde{X})$ takes the form

$$0 \rightarrow R \xrightarrow{\varphi(t)-1} R \rightarrow 0.$$

In particular passing to homology, we deduce that $H_0(X; R)$ is the quotient of R by the ideal generated by $\varphi(t) - 1$ while $H_1(X; R)$ is equal to the kernel of $\varphi(t) - 1$. For instance if $R = \mathbb{Z}$ and $\varphi(t) = 1$, then $H_0(X; \mathbb{Z}) = \mathbb{Z}$ and $H_1(X; \mathbb{Z}) = \mathbb{Z}$. On the other hand, if $R = \mathbb{Z}[\pi_1(X)]$ and $\varphi(t) = t$, then $H_0(X; \mathbb{Z}[\pi_1(X)]) \cong \mathbb{Z}[t^{\pm 1}]/(t - 1) \cong \mathbb{Z}$ and $H_1(X; \mathbb{Z}[\pi_1(X)]) \cong 0$. Observe that this latter case recovers the homology of the universal cover $\tilde{X} \cong \mathbb{R}$ of X .

Proceeding as in Example 2.22, one can show that twisted homology of X generalizes both the usual cellular homology of X and the cellular homology of \tilde{X} . Indeed, if $R = \mathbb{Z}$ is endowed with the trivial $\mathbb{Z}[\pi_1(X)]$ -module structure (i.e. $\gamma r = r$ for all $r \in R$), then one recovers the ordinary homology of X . On the other hand, if $R = \mathbb{Z}[\pi_1(X)]$ is endowed with the $\mathbb{Z}[\pi_1(X)]$ -module structure given by right multiplication, $H_*(X; R)$ is the homology of the universal cover of X .

More generally twisted homology encompasses the homology groups of all regular covering spaces of X :

Example 2.23. Given an epimorphism $\varphi: \pi_1(X) \rightarrow G$, we use \widehat{X} to denote the G -cover of X corresponding to $\ker(\varphi)$. Note furthermore that φ endows $\mathbb{Z}[G]$ with a right $\mathbb{Z}[\pi_1(X)]$ -module structure. Generalizing the observations of Example 2.22, it can be shown that the covering map $\tilde{X} \rightarrow \widehat{X}$ induces a chain isomorphism

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\tilde{X}) \cong C_*(\widehat{X}).$$

In particular, $H_*(X; \mathbb{Z}[G])$ and $H_*(\widehat{X}; \mathbb{Z})$ are canonically isomorphic. The cases $G = \{1\}$ and $G = \mathbb{Z}[\pi_1(X)]$ were described in the previous paragraph: they respectively recover the cellular homology of X and \tilde{X} .

In fact, we already encountered twisted homology in Subsection 2.3:

Example 2.24. Let us briefly recall definition of the Casson-Gordon σ -invariant (recall Definition 8). Given a closed 3-manifold M and a character $\chi: H_1(M; \mathbb{Z}) \rightarrow \mathbb{C}^*$ of finite order m ,

Proposition 2.12 ensures the existence of a non-negative integer r , a 4-manifold W and a map $\psi: \pi_1(W) \rightarrow \mathbb{Z}_m$ such that $\partial(W, \psi) = r(M, \chi)$. The kernel of ψ then gives rise to a \mathbb{Z}_m -cover $W_m \rightarrow W$ whose homology has the structure of a left $\mathbb{Z}[\mathbb{Z}_m]$ -module. Setting $\omega = e^{\frac{2\pi i}{m}}$, the left $\mathbb{Q}(\omega)$ -vector space $\mathbb{Q}(\omega)$ can then also be viewed as a right $\mathbb{Z}[\mathbb{Z}_m]$ -module, leading to the following vector space for which the notation $H_*(W; \mathbb{Q}(\omega))$ was used:

$$(5) \quad \mathbb{Q}(\omega) \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_*(W_m).$$

This notation is justified because $\mathbb{Q}(\omega) \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_*(W_m)$ is equal to the homology of a certain $\mathbb{Q}(\omega)$ -twisted chain complex. More precisely, we endow $\mathbb{Q}(\omega)$ with the $\mathbb{Z}[\pi_1(W)]$ -module structure arising from the composition $\mathbb{Z}[\pi_1(W)] \xrightarrow{\psi} \mathbb{Z}[\mathbb{Z}_m] \rightarrow \mathbb{Q}(\omega)$ and we claim that there is a canonical isomorphism

$$H_*(\mathbb{Q}(\omega) \otimes_{\mathbb{Z}[\pi_1(W)]} C_*(\widetilde{W})) \cong \mathbb{Q}(\omega) \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_*(W_m).$$

Firstly, Example 2.23 provides a canonical chain isomorphism $C_*(W_m) \cong \mathbb{Z}[\mathbb{Z}_m] \otimes_{\mathbb{Z}[\pi_1(W)]} C_*(\widetilde{W})$. Using the associativity of the tensor product, we deduce that

$$(6) \quad \begin{aligned} \mathbb{Q}(\omega) \otimes_{\mathbb{Z}[\mathbb{Z}_m]} C_*(W_m) &\cong \mathbb{Q}(\omega) \otimes_{\mathbb{Z}[\mathbb{Z}_m]} (\mathbb{Z}[\mathbb{Z}_m] \otimes_{\mathbb{Z}[\pi_1(W)]} C_*(\widetilde{W})) \\ &\cong \mathbb{Q}(\omega) \otimes_{\mathbb{Z}[\pi_1(W)]} C_*(\widetilde{W}). \end{aligned}$$

Secondly, note that since $\mathbb{Q}(\omega)$ is flat over $\mathbb{Z}[\mathbb{Z}_m]$, the vector space in (5) can be obtained by taking the homology of the chain complex $\mathbb{Q}(\omega) \otimes_{\mathbb{Z}[\mathbb{Z}_m]} C_*(W_m)$. Taking homology in (6) now concludes the proof of the claim.

Next, we describe twisted intersection forms by proceeding word for word as in Subsection 2.2. From now on, we assume that the homomorphism $\varphi: \mathbb{Z}[\pi_1(X)] \rightarrow R$ preserves the involutions, meaning that $\varphi(g^{-1}) = \overline{\varphi(g)}$ for all $g \in \pi_1(X)$. Given a compact oriented k -dimensional manifold W , there are Poincaré duality isomorphisms PD: $H_k(W, \partial W; R) \cong H^{n-k}(W; R)$ and PD: $H_k(W; R) \cong H^{n-k}(W, \partial W; R)$. Composing the map induced by the inclusion $(W, \emptyset) \rightarrow (W, \partial W)$ with duality and an evaluation homomorphism (see [9, Lemma 2.3]) produces the map

$$\Phi: H_k(W; R) \rightarrow H_k(W, \partial W; R) \xrightarrow{\text{PD}} H^k(W; R) \xrightarrow{\text{ev}} \overline{\text{Hom}_{\text{left-}R}(H_k(W; R), R)}.$$

The main definition of this section is the following.

Definition 12. The R -twisted intersection pairing $\lambda_R: H_i(W; R) \times H_i(W; R) \rightarrow R$ is defined by setting $\lambda_R(x, y) = \Phi(y)(x)$.

Note that when $R = \mathbb{Z}$, one recovers the ordinary intersection form described in Definition 6. Next, we observe that in 2.3, we already used a twisted intersection form in order to define the Casson-Gordon σ -invariant.

Example 2.25. Continuing with the notations of Example 2.24, in 2.3, we claimed that $\mathbb{Q}(\omega) \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_2(W_m)$ is endowed with an intersection form $\lambda_{\mathbb{Q}(\omega)}$. Using Example 2.24, we can identify this latter vector space with $H_2(W; \mathbb{Q}(\omega))$ and $\lambda_{\mathbb{Q}(\omega)}$ is the twisted intersection form described in Definition 12.

2.6. Witt groups. In this section, we describe the Witt group $W(R)$ of a ring R with involution. In the case where $R = \mathbb{C}(t)$ and $\omega \in S^1$, we also describe the (averaged) signature $\sigma_\omega([A(t)])$ of a Witt class $[A(t)]$. References include [22, Chapter 1] and [21, Chapitre 2].

Let R be a ring with involution. Given a projective R -module H , a map $b: H \times H \rightarrow R$ is *sesquilinear* if $b(px, qy) = pb(x, y)\bar{q}$. A sesquilinear map is *hermitian* if $b(y, x) = \overline{b(x, y)}$. Furthermore b is *non-degenerate* if $\overline{b(x, y)} = 0$ for all y implies $x = 0$. Equivalently, b is non-degenerate if the *adjoint map* $M \rightarrow \overline{\text{Hom}_R(M, R)}$, $x \mapsto (y \mapsto b(x, y))$ is injective. Finally b is *nonsingular* if its adjoint is an isomorphism. A *hermitian form* is a pair (M, b) , where M is a projective R -module and b is a nonsingular hermitian pairing.

Example 2.26. In practice, all our Hermitian forms will arise as intersection forms of 4-manifolds. Namely, given a 4-manifold W and a ring with involution R , the twisted intersection pairing λ_R described in Definition 12 is a hermitian pairing on the twisted homology R -module $H_2(W; R)$. It may however be singular.

The *direct sum* $(H_1, b_1) \oplus (H_2, b_2)$ of two hermitian forms is defined in the obvious way and a hermitian form (H, b) is *metabolic* if there is a submodule $P \subset M$ which is a direct summand and such that the orthogonal complement $P^\perp := \{x \in M \mid b(x, y) = 0 \text{ for all } y \in P\}$ is equal to P . Two (isomorphism classes of) hermitian forms (H_1, b_1) and (H_2, b_2) are *Witt equivalent* if $(H_1, b_1) \oplus -(H_2, b_2)$ is metabolic.

Theorem 2.27. *Let R be an integral domain with involution whose field of fraction is not of characteristic 2. Then the following facts hold:*

- (1) *Witt equivalence is an equivalence relation and the set of equivalence classes of hermitian forms is an abelian group $W(R)$ called the Witt group of R .*
- (2) *A Hermitian form represents zero in the Witt group if and only if it is metabolic.*
- (3) *The additive inverse of $[(H, b)]$ is given by $[(H, -b)]$.*

Proof. Witt equivalence, which we denote by \sim is defined on the commutative monoid $\widehat{W}(R)$ of isomorphism classes of hermitian forms. Consequently, if we manage to show that \sim is an equivalence relation which respects the direct sum, then $W(R)$ will automatically be a commutative monoid; it will then only remain to deal with inverses. We leave the proof that \sim respects the direct sum to the reader.

To show that $(H, b) \sim (H, b)$, it is enough to observe that the diagonal $\Delta_H \subset H \oplus H$ is a metabolizer for $b \oplus -b$. It is clear that $\Delta_H \subset \Delta_H^\perp$. To show the converse, assume that $y \oplus z$ satisfies $0 = (b \oplus -b)(x \oplus x, y \oplus z) = b(x, y) - b(x, z) = b(x, y - z)$ for all x . Since b is nondegenerate, it follows that $y = z$ and thus $y \oplus z$ belongs to Δ_H . Symmetry is clear since $\widehat{W}(R)$ is a commutative monoid. To show transitivity, assume there are metabolic forms m_1, m_2 such that $a \oplus -b \cong m_1$ and $b \oplus -c \cong m_2$. Summing these isomorphisms, and using that $b \oplus -b$ is metabolic, we observe that $a \oplus -c$ is *stably metabolic*, i.e. there is a metabolic form m for which $a \oplus -c \oplus m$ is metabolic. Since we must show that $a \oplus -c$ is metabolic, we are therefore led to the following question:

“When does *stably metabolic* imply *metabolic* ?”

Some additional work shows that the assumptions on the ring R are sufficient to guarantee this cancellation result (combine [22, Chapter 1 Theorem 4.4 and Lemma 6.3] with [29, Lemma 5.4]). As we mentioned above, we now know that $W(R)$ is an abelian monoid with the zero element being the class of $0 \in \widehat{W}(R)$. To conclude that $W(R)$ is a group, we must show that $[(H, b)] \oplus [(H, -b)] = [0]$. First, we observe that $(H, b) \sim 0$ if and only if $(H, b) \oplus -0$ is metabolic i.e. if and only if (H, b) is metabolic (this proves the second assertion of the proposition). But since we already observed that $(H, b) \oplus (H, -b)$ is metabolic, the proof is concluded. \square

We shall now provide several examples of computations of Witt groups. In each case, the signature will play an important role.

Example 2.28. Endowing \mathbb{R} with the trivial involution and \mathbb{C} with the involution given by conjugation, the signature induces isomorphisms $W(\mathbb{R}) \cong \mathbb{Z}$ and $W(\mathbb{C}) \cong \mathbb{Z}$. Surjectivity is clear: any integer can be realized as the signature of an appropriate diagonal matrix. Injectivity follows because over \mathbb{R} and \mathbb{C} , a Hermitian matrix is metabolic if and only if its signature vanishes.

The signature also leads to an isomorphism $W(\mathbb{Z}) \cong \mathbb{Z}$ but the proof is somewhat more complicated, see [22]. To provide further examples, we first note that ring homomorphisms induce a group homomorphisms on the level of Witt groups.

Example 2.29. A homomorphism $f: R \rightarrow S$ of rings with involution induces a homomorphism between the corresponding Witt groups. Assume (M, b) is a Hermitian form over R . The formula

$r \cdot s = f(r)s$ endows S with an R -module structure and the S -module $M \otimes_R S$ is endowed with an S -valued Hermitian form $f_*(b)$ defined as follows:

$$(m_1 \otimes s_1, m_2 \otimes s_2) \mapsto \overline{s_1} f(b(m_1, m_2)) s_2.$$

Now $f_*(b)$ is clearly sesquilinear and the fact that it is Hermitian follows from the fact that b is Hermitian and f preserves involutions. To see that $f_*(b)$ is nonsingular, observe that since M is projective, its adjoint is given by the isomorphism $M \otimes_R S \xrightarrow{\cong} \overline{\text{Hom}}_R(M, R) \otimes_R S \xrightarrow{\cong} \overline{\text{Hom}}_S(M \otimes_R S, S)$, where the latter isomorphism sends $\varphi \otimes s_1$ to $(x \otimes s_2 \mapsto \overline{s_1} f(\varphi(x)) s_2)$ see [25, Section 2.2].

The next example outlines a more involved computation of a familiar Witt group.

Example 2.30. If \mathbb{F} denotes either \mathbb{R} or \mathbb{C} , then $W(\mathbb{F}[t^{\pm 1}])$ is isomorphic to \mathbb{Z} . To prove this fact, it is sufficient to show that the inclusion induced map $i: W(\mathbb{F}) \rightarrow W(\mathbb{F}[t^{\pm 1}])$ is an isomorphism: indeed, the required result then follows from Example 2.28. To prove that i is an isomorphism, one shows that the map $\varphi: \mathbb{F}[t^{\pm 1}] \rightarrow \mathbb{F}, p(t) \mapsto p(1)$ induces a map $W(\mathbb{F}[t^{\pm 1}]) \rightarrow W(\mathbb{F})$ which is a left inverse for i . Some additional work shows that φ is injective proving that i is an isomorphism.

Example 2.30 shows that the Witt group $W(\mathbb{C}[t^{\pm 1}])$ is not more complicated than $W(\mathbb{C})$. On the other hand, the structure of $W(\mathbb{C}(t))$ is vastly more involved: it is isomorphic to a direct sum of an infinite number of copies of \mathbb{Z} [18, Appendix A]. Instead of delving into the details of this computation, for each $\omega \in S^1$, we describe instead the (averaged) signatures homomorphism

$$\text{sign}_\omega^{\text{av}} W(\mathbb{C}(t)) \rightarrow \mathbb{Q}.$$

Let $A(t)$ be a matrix over $\mathbb{C}(t)$. Observe that $\text{sign}_\omega(A(t)) := \text{sign}(A(\omega))$ is constant in a neighborhood of ω unless $\det(A(\omega))$ vanishes or some entry of $A(\omega)$ is infinite. Therefore $\text{sign}_\omega(A(t))$ is a step-function with finitely many discontinuities. At each discontinuity ω , we redefine $\text{sign}_\omega(A(t))$ to be the average of the one-sided limits of $\text{sign}_\eta(A(t))$ as η tends to ω . We use $\text{sign}_\omega^{\text{av}}(A(t))$ to denote the resulting rational number. As explained in [3, discussion preceding Theorem 3], one then obtains well-defined homomorphism $\text{sign}_\omega^{\text{av}}: W(\mathbb{C}(t)) \rightarrow \mathbb{Q}$ by setting

$$\text{sign}_\omega^{\text{av}}([A(t)]) := \text{sign}_\omega^{\text{av}}(A(t)).$$

It can be checked that without this averaging process, the signature function is not be invariant under Witt equivalence.

2.7. The Casson-Gordon invariant $\tau(K, \chi)$. Following [3], we review the definition of the Witt class $\tau(K, \chi)$ which lies in $W(\mathbb{C}(t)) \otimes \mathbb{Q}$. We then state [3, Theorem 2 and 3] whose combination recovers the statement of Theorem 2.16.

The *0-framed surgery* along a knot K is the closed 3-manifold M_K obtained from the knot exterior X_K by attaching a solid torus $T = S^1 \times D^2$, identifying the meridian of T with the longitude of X_K . In the notation of 2.4, M_K corresponds to $S_{0/1}^3(K)$ and, in particular, invoking Lemma 2.18, $H_1(M_K; \mathbb{Z}) \cong \mathbb{Z}$ is freely generated by the meridian of K . Just as in 2.1, the kernel of the composition

$$\pi_1(M_K) \rightarrow H_1(M_K; \mathbb{Z}) \cong \mathbb{Z} \rightarrow \mathbb{Z}/\mathbb{Z}_n$$

gives rise to finite cyclic covers $p: M_n \rightarrow M_K$. Furthermore, recall from 2.1, that $L_n \rightarrow S^3$ denotes the n -fold branched cover of S^3 branched along K . Casson and Gordon's construction associates to M_n and to a character $\chi: H_1(L_n; \mathbb{Z}) \rightarrow \mathbb{C}^*$ a Witt class $\tau(M_n, \chi)$ in $W(\mathbb{C}(t)) \otimes \mathbb{Q}$. The first step in this construction consists in relating the homology of M_n to the homology of L_n .

Lemma 2.31. *Let X_n be the n -fold cyclic cover of the exterior X_K of a knot K , let L_n be the corresponding branched cover and let M_n be the n -fold cover of the 0-framed surgery M_K . One has the following isomorphisms:*

- (1) $H_1(M_n; \mathbb{Z}) \cong H_1(X_n; \mathbb{Z})$,
- (2) $H_1(X_n; \mathbb{Z}) \cong H_1(L_n; \mathbb{Z}) \oplus \mathbb{Z}$.

Proof. Let μ_K and λ_K denote the meridian and longitude of K . By definition, M_K is obtained from X_K by attaching a solid torus $T = S^1 \times D^2$ to ∂X_K , identifying μ_K with the longitude of $S^1 \times D^2$ and λ_K with the meridian of T . Arguing word for word as in the proof of Lemma 2.18, we deduce that the natural map $H_1(X_K; \mathbb{Z}) \rightarrow H_1(M_K; \mathbb{Z})$ is surjective and its kernel is generated by λ_K . Since λ_K is nullhomologous in X_K , it follows that $H_1(M_K; \mathbb{Z})$ is freely generated by μ_K . Since M_n is obtained from the kernel of the map which reduces $\mu_K \bmod n$, X_n sits as a subspace of M_n and the solid torus T lifts to $T_n := S_n^1 \times D^2$, where $S_n^1 \rightarrow S^1$ is the n -fold cover $z \mapsto z^n$. Consequently, $M_n = X_n \cup T_n$ and the exact same Mayer-Vietoris argument shows that $H_1(X_n; \mathbb{Z}) \rightarrow H_1(M_n; \mathbb{Z})$ is a surjective map whose kernel is generated by a lift λ_n of λ_K . Since λ_n is still nullhomologous (e.g. lift a Seifert surface bounding λ_K), the first claim follows.

To prove the second claim, recall that L_n is obtained by gluing a solid torus $D^2 \times S^1$ to ∂X_n identifying a meridian of the solid torus with the lift of μ_K^n . A standard Mayer-Vietoris argument produces the split short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow H_1(X_n; \mathbb{Z}) \rightarrow H_1(L_n; \mathbb{Z}) \rightarrow 0$, where the \mathbb{Z} -summand is generated by the lift of μ_K^n . This concludes the proof of the proposition. \square

We can now use a character $\chi: H_1(L_n; \mathbb{Z}) \rightarrow \mathbb{C}^*$ of order m to define a $\mathbb{Z} \times \mathbb{Z}_m$ -valued character on $H_1(M_n; \mathbb{Z})$. Recall that $p: M_n \rightarrow M_K$ denotes the covering map and consider the composition $\pi_1(M_n) \xrightarrow{p_*} \pi_1(M_K) \rightarrow H_1(M_K; \mathbb{Z}) \cong \mathbb{Z}$. Since the image of this map is isomorphic to $n\mathbb{Z}$, mapping to this image produces a surjective map $\alpha: \pi_1(M_n) \rightarrow n\mathbb{Z}$. On the other hand, using Lemma 2.31, a character $\chi: H_1(L_n; \mathbb{Z}) \rightarrow \mathbb{C}^*$ induces a character on $H_1(M_n; \mathbb{Z})$ for which we use the same notation. Assuming χ to be of order m , we have therefore obtained a map

$$\alpha \times \chi: \pi_1(M_n) \rightarrow n\mathbb{Z} \times \mathbb{Z}_m.$$

Similarly to Proposition 2.12, there is an r for which r copies of $(M_n, \alpha \times \chi)$ bound (V_n, ψ) for some 4-manifold V_n and some $\psi: \pi_1(V_n) \rightarrow \mathbb{Z} \times \mathbb{Z}_m$. Endow $\mathbb{C}(t)$ with the $(\mathbb{C}(t), \mathbb{Z}[\pi_1(W)])$ -bimodule structure which arises from the composition $\mathbb{Z}[\pi_1(W)] \xrightarrow{\psi} \mathbb{Q}[\mathbb{Z} \times \mathbb{Z}_m] \rightarrow \mathbb{C}(t)$, where the second map sends the generator of \mathbb{Z}_m to $\omega := \exp(2\pi i/m)$ and the generator of \mathbb{Z} to t . This gives rise to twisted homology groups $H_*(V_n; \mathbb{C}(t))$ and, in particular, we may consider the $\mathbb{C}(t)$ -valued intersection form $\lambda_{\mathbb{C}(t), V_n}$ on $H_2(V_n; \mathbb{C}(t))$ as described in Definition 12.

The proof of the following lemma can be found in [3, Corollary following Lemma 4].

Lemma 2.32. *If χ is a character of prime power order, then the hermitian form $\lambda_{\mathbb{C}(t), V_n}$ is nonsingular.*

Using Lemma 2.32, $\lambda_{\mathbb{C}(t), V_n}$ defines an element in the Witt group $W(\mathbb{C}(t))$. On the other hand, the standard intersection form $\lambda_{\mathbb{Q}, V_n}$ of V_n (which was described in Definition 6) gives rise to an element in $W(\mathbb{Q})$. More precisely, although $\lambda_{\mathbb{Q}, V_n}$ may be singular, it is possible to consider the nonsingular form $\lambda_{\mathbb{Q}, V_n}^{\text{nonsing}}$ it induces on $H_2(V_n; \mathbb{Q})/\text{im}(H_2(\partial V_n; \mathbb{Q}) \rightarrow H_2(V_n; \mathbb{Q}))$.

As we saw in Example 2.29, the canonical inclusion $\mathbb{Q} \rightarrow \mathbb{C}(t)$ induces a group homomorphism $i: W(\mathbb{Q}) \rightarrow W(\mathbb{C}(t))$. Consequently the standard \mathbb{Q} -valued intersection form on V_n gives rise to an element $i(\lambda_{\mathbb{Q}, V_n}^{\text{nonsing}})$ in $W(\mathbb{C}(t))$.

Definition 13. Let n be a positive integer. Given a knot K , let M_n denote the n -fold cover of the 0-framed surgery M_K , let L_n be the cover of S^3 branched along K , and let $\chi: H_1(L_n; \mathbb{Z}) \rightarrow \mathbb{C}^*$ be a character of prime power order. The Casson-Gordon τ -invariant $\tau(M_n, \chi)$ is the Witt class

$$\tau(M_n, \chi) := [\lambda_{\mathbb{C}(t), V_n}] - i([\lambda_{\mathbb{Q}, V_n}^{\text{nonsing}}]) \otimes \frac{1}{r} \in W(\mathbb{C}(t)) \otimes \mathbb{Q}.$$

Arguing as in Lemma 2.15, it can be shown that $\tau(M_n, \chi)$ is independent of the choice of the 4-manifold V_n , of the rational number r and of the extension of $\alpha \times \chi$ to $\pi_1(V_n)$. One of the main theorems in the theory of Casson-Gordon invariants is the following theorem; its proof can be found in [3, Theorem 2].

Theorem 2.33. *Let K be a slice knot and let q be a prime power. There exists a subgroup G of $H_1(L_q)$ which satisfies:*

- (1) $|G|^2 = |H_1(L_q; \mathbb{Z})|$,³
 (2) if $\chi: H_1(L_q; \mathbb{Z}) \rightarrow \mathbb{C}^*$ is a character of prime power order satisfying $\chi(G) = \{0\}$, then

$$\tau(M_q, \chi) = 0.$$

Theorem 2.33 therefore provides an obstruction to a knot being slice. Since the τ -invariant is quite difficult to compute, the idea of Casson and Gordon is to approximate it using the easier σ invariant described in Definition 8. To make this more precise, recall from Subsection 2.6 that for $\omega \in S^1$, one may take the averaged signature $\text{sign}_\omega^{\text{av}}([A(t)])$ of a Witt class $[A(t)] \in W(\mathbb{C}(t))$. In particular, it makes sense to consider the rational number $\text{sign}_1^{\text{av}}(\tau(K, \chi))$.

We now state a third result of Casson and Gordon, referring to [3, Theorem 3] for the proof.

Theorem 2.34. *Let L_n be the n -fold branched cover of a knot K , let $\chi: H_1(L_n) \rightarrow \mathbb{C}$ be a non-trivial character of prime power order m . If the induced cover L_n^X satisfies $H_1(L_n^X) \otimes_{\mathbb{Z}[\mathbb{Z}_m]} \mathbb{C} = 0$, then*

$$|\sigma(L_n, \chi) - \text{sign}_1^{\text{av}}(\tau(M_n, \chi))| \leq 1.$$

To conclude these notes, we show how the combination of Theorem 2.33 and Theorem 2.34 recover Theorem 2.16. Recall the statement of this latter theorem: if K is a slice knot whose double branched cover is a lens space L , then $|H_1(L; \mathbb{Z})|$ is a square, say m^2 and if ℓ is a prime power dividing m and $\chi: \pi_1(L) \rightarrow \mathbb{Z}_\ell$ is a non-constant character, then

$$|\sigma(K, \chi)| \leq 1.$$

proof of Theorem 2.16 assuming Theorem 2.33 and Theorem 2.34. By assumption, the branched double cover of K is a lens space $L(p, q)$. Using the first statement of Theorem 2.33, we know that $H_1(L(p, q); \mathbb{Z}) = \mathbb{Z}_p = \mathbb{Z}_{m^2}$ for some m . Since this group is cyclic, it admits a unique subgroup G whose order is m . According to Theorem 2.33, we must now check that χ vanishes on G . By assumption, we have a prime power ℓ dividing m and so we write $m = \ell n$ for some n . By uniqueness, we know that G is given by

$$G = \{0, \ell, 2\ell, \dots, \ell n\} \cong \mathbb{Z}_{\ell n}.$$

Since any non-trivial character $\chi: \mathbb{Z}_{\ell^2 n^2} \rightarrow \mathbb{Z}_\ell$ vanishes on G , we deduce from Theorem 2.33 that for any such χ , the invariant $\tau(K, \chi)$ is zero. Using once again that χ is non-trivial, one may then check that $H_1(L_n^X) \otimes_{\mathbb{Z}[\mathbb{Z}_m]} \mathbb{C}$ vanishes and therefore Theorem 2.34 implies that $|\sigma(K, \chi)| \leq 1$. \square

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³Note that G is actually a metabolizer of the linking form $H_1(L_q) \times H_1(L_q) \rightarrow \mathbb{Q}/\mathbb{Z}$.

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