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Invariants of colored links and generalizations of the Burau representation

THÈSE

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Abstract

The focus of this thesis lies in low dimensional topology and more specifically in knot theory. It decomposes into three parts. The first deals with invariants of colored links, the second discusses the Burau representation of the braid group and its generalizations. The third and final part relies on the previous two and studies the non-additivity of signatures. Let us now summarize each of these three parts.

A knot consists of an embedded circle in the 3-sphere. Knots are studied via knot invariants: one assigns an algebraic quantity (which is called a knot invariant) to each knot in such a way that isotopic knots have identical invariants. Classical invariants of knots include the Alexander polynomial, the Levine-Tristram signature and the Blanchfield pairing. In the case of links (i.e. disjoint union of knots), similar multivariable invariants are available although they are often harder to study. The first part of this thesis is concerned with this type of invariants and our results can be summarized as follows. We use the multivariable signature of Cimasoni-Florens in order to produce new lower bounds on the splitting number, we provide the first explicit computation of the Blanchfield pairing for arbitrary links, we improve the 4-dimensional understanding of the multivariable signature and, in particular, we show that it is invariant under 1-solvable cobordism.

A braid with n strands roughly consists of n monotonic intervals in the cylinder. The set of isotopy classes of n-stranded braids forms a group called the braid group. The (reduced) Burau representation of the braid group is known to be closely related to the Alexander polynomial. We generalize this relation to the context of twisted Alexander polynomials and L^2-Alexander invariants. Our results are obtained by defining and studying the notions of twisted Burau maps and L^2-Burau maps. Braids can also be understood as particular cases of tangles, which consist in a type of properly embedded 1-submanifolds of the cylinder. Tangles no longer form a group but are the morphisms of a category. Nevertheless, the Burau representation admits a generalization to tangles under the form of the so-called Lagrangian functor which is due to Cimasoni-Turaev. Using cospans, we show how this functor can be promoted to a (weak) 2-functor on the bicategory of tangles.

The third and final part of this thesis deals with the non-additivity of signature invariants. Consider an arbitrary link invariant $\mathscr{I}$ taking values in an abelian group. Precomposing this invariant with the braid closure $\alpha \mapsto \widehat{\alpha}$ defines maps from the braid group to this abelian group. Therefore, one might try to evaluate the homomorphism defect $\mathscr{I}(\widehat{\alpha\beta})-\mathscr{I}(\widehat{\alpha})-\mathscr{I}(\widehat{\beta})$. This program was carried out by Gambaudo and Ghys who expressed the homomorphism defect of the Levine-Tristram signature in terms of the reduced Burau representation. We extend this theorem to colored tangles. The result expresses the homomorphism defect of the multivariable signature using the Lagrangian functor and the Maslov index.

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Contents

Chapter 1

Introduction and overview of results

The focus of this thesis lies in low dimensional topology and more specifically in knot theory. It decomposes into three parts. The first deals with invariants of colored links, the second discusses the Burau representation of the braid group and its generalizations. The third and final part relies on the previous two and studies the non-additivity of signatures. The material exposed here expands on our papers [13, 38, 39, 40, 50, 51, 52, 53]. Let us now outline our work in more detail and state our main results.

1.1 Part I: Invariants of colored links

A *knot* is a smoothly embedded circle in the 3-sphere S^3. Given a knot K, we denote its exterior by $X_K = S^3 \setminus \nu K$, where νK is a tubular neighborhood of K. Since the first homology group $H_1(X_K)$ is infinite cyclic, the kernel of the abelianization map $\pi_1(X_K) \to \mathbb{Z}$ gives rise to a cover $\widehat{X}_K \to X_K$ upon which $\mathbb{Z}$ acts. Identifying, the group ring $\mathbb{Z}[\mathbb{Z}]$ with the ring $\mathbb{Z}[t^{\pm 1}]$ of Laurent polynomials, the homology groups of $\widehat{X}_K$ become $\mathbb{Z}[t^{\pm 1}]$-modules. Of particular interest is the *Alexander module* $H_1(\widehat{X}_K)$, from which the *Alexander polynomial* $\Delta_K(t)$ can be extracted [1]. Additionally, the Alexander module supports a nonsingular Hermitian pairing

$$\mathrm{Bl}(K)\colon H_1(\widehat{X}_K) \times H_1(\widehat{X}_K) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$

called the *Blanchfield pairing* [18]. The Alexander polynomial and the Blanchfield pairing can both be computed via Seifert matrices. These matrices, which arise from compact connected oriented surfaces whose boundary is K (i.e. *Seifert surfaces*), also give rise to the *Levine-Tristram signature* and *nullity*

$$\sigma_K, \eta_K \colon S^1 \to \mathbb{Z}.$$

As outlined in Chapter 2, similar considerations for n-component links (a *link* L is a disjoint union of knots) lead to analogous one variable invariants. However, since $H_1(X_L) = \mathbb{Z}^n$, other covering spaces of X_L are available and produce the multivariable invariants which are the main focus of the first part of this thesis.

As we shall recall in Chapter 3, a *μ-colored link* is an oriented link L in S^3 whose components are partitioned into μ sublinks $L_1 \cup \cdots \cup L_\mu$. This way, a 1-colored link is an oriented link while an n-component n-colored link is an ordered link. Associated to such a μ-colored link L is a regular $\mathbb{Z}^\mu$-cover $\widehat{X}_L \to X_L$. The homology of this cover is a module

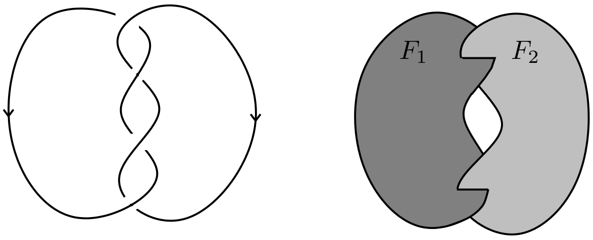

Figure 1.1: A C-complex for a colored link.

over $\Lambda_\mu := \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$ from which the *Alexander polynomial* $\Delta_L(t_1, \dots, t_\mu)$ *of the colored link* can be extracted. For 1-colored links, Δ_L agrees with the (one variable) Alexander polynomial of L, while for n-colored n-component links, it is the *multivariable Alexander polynomial* introduced by Alexander [1], see also [69].

For colored links, the role of Seifert surfaces is played by the *C-complexes* which were defined by Cooper [54]. Roughly speaking, a C-complex F for a μ-colored link L consists of Seifert surfaces $F_1, \dots, F_\mu$ that intersect pairwise in clasp intersections, see Figure 1.1 for an example. The choice of a C-complex gives rise to 2^μ *generalized Seifert matrices* which can be used to compute the Alexander polynomial [36] and define the *multivariable signature and nullity*

$$\sigma_L, \eta_L \colon \mathbb{T}^\mu \to \mathbb{Z}\,,$$

where $\mathbb{T}^\mu$ denotes the μ-dimensional torus. These latter invariants which, in this form, are due to Cimasoni and Florens [41], recover the Levine-Tristram signature and nullity in the case $\mu = 1$.

The remainder of the first part of this thesis deals with the study and applications of invariants of colored links.

1.1.1 Splitting numbers

Any link L can be turned into the split union of its components by a sequence of crossing changes between different components. Following Batson and Seed [10], the *splitting number* $\mathrm{sp}(L)$ of L is the minimal number of crossing changes in such a sequence. Since upper bounds on $\mathrm{sp}(L)$ can be found by inspection of diagrams, the difficulty in computing it is to find lower bounds. The aim of Chapter 4 is to provide such bounds by using the multivariable signature and nullity.

Figure 1.2: The Whitehead link L has $\mathrm{sp}(L) \leq 2$.

Let us first briefly outline the previous work in the field. As observed in [10], the linking numbers provide an elementary lower bound on the splitting number. However, since this *linking number bound* is not always sharp, Batson and Seed used Khovanov homology to obtain a new lower bound on $\mathrm{sp}(L)$ [10]. Testing it on links with up to 12 crossings, they found only 17 examples where this *Batson-Seed bound* is strictly stronger than the linking number bound. This enabled them to compute the splitting number of 7 of these links, while the remaining ones were left undetermined.

In [32], Cha, Friedl and Powell introduced two new techniques for computing splitting numbers. The first one is based on *covering link calculus*, while the second provides an obstruction in terms of the multivariable Alexander polynomial. These two techniques together with the linking number bound allowed these authors to determine the splitting numbers of the 130 prime links with up to 9 crossings and to compute the splitting numbers of all of the 17 links in the Batson-Seed list.

In a different direction, Borodzik and Gorsky found a Heegaard Floer theoretical criterion for bounding the splitting number [22]. As an application, they showed that for any positive a, the 2-bridge link with Conway normal form $C(2a, 1, 2a)$ has splitting number $2a$, even though the linking number of the two components vanishes, see Figure 1.3 for an example of a 2-bridge link.

The main result of Chapter 4 (which was proved together with David Cimasoni and Kleopatra Zacharova [40]) is a new lower bound on the splitting number of a link in terms of its multivariable signature and nullity.

Theorem 4.1.1. *If $L = K_1 \cup \cdots \cup K_n$ is an ordered link, then*

$$\left|\sigma_L(\omega_1, \ldots, \omega_n) - \sum_{i=1}^{n} \sigma_{K_i}(\omega_i)\right| + \left|n - 1 - \eta_L(\omega_1, \ldots, \omega_n) + \sum_{i=1}^{n} \eta_{K_i}(\omega_i)\right| \leq \mathrm{sp}(L)$$

for all $(\omega_1, \ldots, \omega_n) \in \mathbb{T}^n$.

As we shall observe in Section 4.3, our bound is sharp for 127 out of the 130 prime links with up to 9 crossings, and two of the remaining splitting numbers can be determined with the linking number bound. Also, our method gives the splitting number of all but one of the 17 links in the Batson-Seed list. Our bound also implies the following generalization of [22, Theorem 7.12]: for any $n \geq 1$ and positive $a_1, \ldots, a_n, b_1, \ldots, b_{n-1}$, the splitting number of the 2-bridge link with Conway normal form $C(2a_1, b_1, 2a_2, b_2, \ldots, 2a_{n-1}, b_{n-1}, 2a_n)$ is equal to $a_1 + \cdots + a_n$, see Figure 1.3 and Theorem 4.3.6.

Chapter 4 concludes with the study of the so-called *weak splitting number* and includes some new results, see in particular Propositions 4.4.3 and 4.4.5.

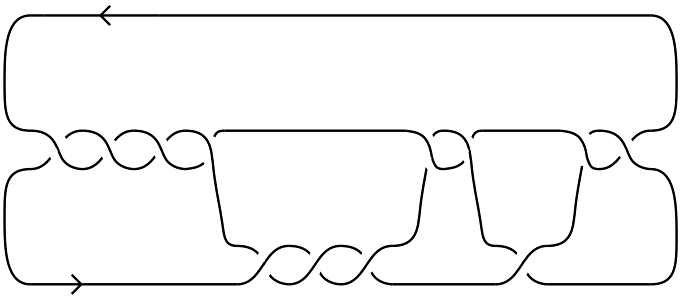

Figure 1.3: The 2-bridge link $C(4, 3, 2, 1, 2)$.

1.1.2 Twisted homology

A major part of this thesis is based on the concept of twisted homology. Chapter 5 collects several results on the subject which are scattered across the literature. Since these tools appear in many of our statements, we take the time to introduce some notations.

Let X be a CW complex and let $Y \subset X$ be a possibly empty subcomplex. Denote by $p\colon \widetilde{X} \to X$ the universal cover of X and set $\widetilde{Y} := p^{-1}(X)$. The left action of $\pi_1(X)$ on $\widetilde{X}$ endows the chain complex $C_*(\widetilde{X}, \widetilde{Y})$ with the structure of a left $\mathbb{Z}[\pi_1(X)]$-module. Moreover, let R be a ring and let M be a $(R, \mathbb{Z}[\pi_1(X)])$-bimodule. The *twisted homology* left R-modules of the pair (X, Y) are defined as

$$H_*(X, Y; M) := H_*(M \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X}, \widetilde{Y})).$$

These modules generalize ordinary cellular homology in the following sense. If $R = \mathbb{Z}$ and $M = \mathbb{Z}$ is given the trivial right $\mathbb{Z}[\pi_1(X)]$-module structure $n \cdot \gamma = n$ and the left $\mathbb{Z}$-module structure induced by multiplication, then $H_*(X; \mathbb{Z})$ is nothing but the usual cellular homology of X. More generally, if $\psi\colon \pi_1(X) \to G$ is a surjective group homomorphism and $R = M = \mathbb{Z}[G]$, then $H_*(X; \mathbb{Z}[G])$ is equal to $H_*(\widehat{X})$, where $\widehat{X}$ is the cover of X corresponding to $\ker(\psi)$. For this reason, the Alexander module of a μ-colored link L will often be written as $H_1(X_L; \Lambda_\mu)$, where $\Lambda_\mu = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$.

We conclude with another key example. Let $\Lambda_S := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}, (1-t_1)^{-1}, \ldots, (1-t_\mu)^{-1}]$ denote the localization of the ring $\Lambda_\mu = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$ with respect to the multiplicative system generated by $\{1 - t_1, \ldots, 1 - t_\mu\}$. We will frequently consider the localized Alexander module $H_1(X_L; \Lambda_S)$. Note that since Λ_S is flat over Λ_μ, the Λ_S-module $H_1(X_L; \Lambda_S)$ is in fact canonically isomorphic to $\Lambda_S \otimes_{\Lambda_\mu} H_1(X_L; \Lambda_\mu)$.

1.1.3 The Blanchfield pairing of colored links

As we mentioned above, the Alexander module of a knot supports a nonsingular Hermitian pairing called the Blanchfield pairing. Denoting the field of fractions $\mathbb{Q}(t_1, \ldots, t_\mu)$ of Λ_μ by Q_μ, the Blanchfield pairing generalizes to a sesquilinear pairing

$$Bl(L)\colon TH_1(X_L; \Lambda_S) \times TH_1(X_L; \Lambda_S) \to Q_\mu/\Lambda_S$$

on the Λ_S-torsion submodule $TH_1(X_L; \Lambda_S)$ of the localized Alexander module $H_1(X_L; \Lambda_S)$ [87, 133]. Since both the multivariable Alexander polynomial and signatures can be computed using C-complexes, one might suspect that the Blanchfield pairing should be computable using generalized Seifert matrices. This turns out to be true and shall be the focus of Chapter 6.

First, note that if K is a knot, then $\mathrm{Bl}(K)$ can be computed using Seifert matrices [96]. More precisely, given a Seifert matrix A for K of size $2g$, the Blanchfield pairing of K is isometric to the pairing

$$\begin{aligned} \Lambda^{2g}/(tA - A^T)\Lambda^{2g} \times \Lambda^{2g}/(tA - A^T)\Lambda^{2g} &\to Q/\Lambda \\ (a, b) &\mapsto a^T(t-1)(A - tA^T)^{-1}\overline{b}. \end{aligned} \tag{1.1}$$

Implicit in this statement is the fact that the Alexander module of a knot admits a square presentation matrix. In general however, $\mathrm{Bl}(L)$ is defined on $TH_1(X_L; \Lambda_S)$ and to the best of our knowledge, the latter Λ_S-module has no reason of admitting a square presentation matrix.

Thus a direct generalization of (1.1) seems out of reach. On the other hand, if we consider links whose Alexander polynomial is nonzero, then $TH_1(X_L;\Lambda_S)$ is equal to $H_1(X_L;\Lambda_S)$ which does admit a square presentation matrix. Restricting to this setting, a computation of $\mathrm{Bl}(L)$ in the spirit of (1.1) once again seems within reach.

In fact, when the Alexander polynomial of L is nonzero, we obtained such a result together with Stefan Friedl and Enrico Toffoli [52]. In order to give a precise statement, recall that given a C-complex and a sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_\mu)$ of ± 1's, there are 2^μ generalized Seifert matrices A^ε which extend the usual Seifert matrix. The associated *C-complex matrix* is the Λ_μ-valued Hermitian matrix

$$H \;:=\; \sum_{\varepsilon} \prod_{i=1}^{\mu} (1 - t_i^{\varepsilon_i})\, A^\varepsilon,$$

where the sum is on all sequences $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_\mu)$ of ± 1's. The main theorem of [52] reads as follows.

Theorem 6.1.1. *Let L be a μ-colored link and let H be a C-complex matrix for L. If $H_1(X_L;\Lambda_S)$ is Λ_S-torsion, then the Blanchfield pairing $\mathrm{Bl}(L)$ is isometric to the pairing*

$$\begin{aligned} \Lambda_S^n/H^T\Lambda_S^n \times \Lambda_S^n/H^T\Lambda_S^n \;&\to\; Q_\mu/\Lambda_S \\ (a,b) \;&\mapsto\; -a^T H^{-1}\overline{b}. \end{aligned} \tag{1.2}$$

Restricting to knots, Theorem 6.1.1 recovers (1.1); however, for arbitrary links, the torsion assumption is somewhat restrictive. Fortunately, in further work (carried out in our paper [51]), this assumption was removed. Let us describe this improvement.

Let Δ denote the order of $\mathrm{Tor}_{\Lambda_S}(\Lambda_S^n/H^T\Lambda_S^n)$, the Λ_S-torsion submodule of $\Lambda_S^n/H^T\Lambda_S^n$. Note that for any class $[x]$ in $\mathrm{Tor}_{\Lambda_S}(\Lambda_S^n/H^T\Lambda_S^n)$, there exists an x_0 in Λ_S^n such that $\Delta x = H^T x_0$. As we shall see in Proposition 6.6.2, the assignment $(v,w) \mapsto \frac{1}{\Delta^2} v_0^T H \overline{w_0}$ induces a well-defined pairing

$$\lambda_H \colon \mathrm{Tor}_{\Lambda_S}\big(\Lambda_S^n/H^T\Lambda_S^n\big) \times \mathrm{Tor}_{\Lambda_S}\big(\Lambda_S^n/H^T\Lambda_S^n\big) \to Q_\mu/\Lambda_S,$$

which recovers the pairing described in (1.2) when $\det(H) \neq 0$. The main theorem of Chapter 6 reads as follows.

Theorem 6.1.2. *The Blanchfield pairing of a colored link L is isometric to the pairing $-\lambda_H$, where H is any C-complex matrix for L.*

Theorem 6.1.2 generalizes Theorem 6.1.1 to links whose Alexander module $H_1(X_L;\Lambda_S)$ is not torsion and recovers it if $H_1(X_L;\Lambda_S)$ is torsion. Furthermore Theorem 6.1.2 also recovers known computations of $\mathrm{Bl}(L)$ when L is a boundary link, see Theorem 6.6.7. Finally, note that to the best of our knowledge, Theorem 6.1.2 was not even known in the case of oriented links (i.e. $\mu = 1$).

While the Blanchfield pairing of a knot is known to be Hermitian and nonsingular, the corresponding statements for links require some more work [87, 133]. On the other hand, Theorem 6.1.2 reduces these questions to algebraic considerations on the pairing λ_H. Our results on this issue are summarized in the following corollary, where Δ_L^{tor} denotes the first non-vanishing Alexander polynomial of L over Λ_S.

Corollary 6.1.3. *The Blanchfield pairing of a link L is Hermitian and takes values in $\Delta_L^{\mathrm{tor}^{-1}}\Lambda_S/\Lambda_S$.*

Since the definition of the pairing λ_H is quite manageable, we also use Theorem 6.1.2 to obtain quick proofs regarding the behavior of $\mathrm{Bl}(L)$ under connected sums, disjoint unions, band claspings, mirror images and orientation reversals, see Proposition 6.6.4, Proposition 6.6.5 and Proposition 6.6.6.

A major part of the proof of Theorem 6.1.2 requires 4-dimensional considerations. Namely, it involves pushing a C-complex for L into the 4-ball and studying the algebraic topology of the exterior. Notably, we will describe the fundamental group of this space in Proposition 6.3.3 and its twisted homology Λ_S-modules in Proposition 6.4.2 and Corollary 6.3.4. Our main technical result however is the computation of the twisted intersection form of this space. Since this result (which was obtained with Stefan Friedl and Enrico Toffoli [52]) plays an important role in this thesis, we state it as follows:

Theorem 6.1.4. *Let L be a colored link, let F be a C-complex for L and let W be the exterior of a push-in of F into the 4-ball D^4. Then the intersection pairing on $H_2(W;\Lambda_S)$ is represented by a C-complex matrix H associated to F.*

In fact, pushed-in C-complexes are particular types of objects which we shall frequently encounter in Chapters 7 and 8. Anticipating this, we introduce some terminology. A *colored bounding surface* F for a colored link L consists of a union $F_1 \cup \cdots \cup F_\mu \subset D^4$ of properly embedded, locally flat, oriented surfaces which only intersect each other transversally in double points and whose boundary is L.

1.1.4 The multivariable signature via local coefficients

For ω in $\mathbb{T}^\mu$ of finite order, Cimasoni and Florens showed that the multivariable signature $\sigma_L(\omega)$ can be understood through finite branched covers of the 4-ball [41]. The aim of Chapter 8 is twofold: firstly we show how twisted homology allows for a 4-dimensional interpretation at *all* ω in $\mathbb{T}^\mu_* := (S^1 \setminus \{1\})^\mu$; secondly we use this framework to study cobordisms between links and link concordance. The work described here was carried out with Matthias Nagel and Enrico Toffoli [53].

Let us start by briefly outlining the 4-dimensional construction of Cimasoni-Florens. In what follows, F will be a colored bounding surface whose exterior in the 4-ball will be denoted by W_F. Given roots of unity $\omega_1, \ldots, \omega_\mu$ of respective orders $k_1, \ldots, k_\mu$, one can form a covering $\overline{W}_F$ of the 4-ball D^4 branched along F, with group $G := \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_\mu}$. Restricting the intersection form on $H_2(\overline{W}_F, \mathbb{C})$ to a certain "generalized eigenspace" $H_2(\overline{W}_F, \mathbb{C})_\omega$ produces a Hermitian pairing whose signature we denote by $\sigma_\omega(\overline{W}_F)$. Cimasoni and Florens [41, Theorem 6.1] showed that

$$\sigma_L(\omega) = \sigma_\omega(\overline{W}_F). \tag{1.3}$$

The restriction to ω's of finite order has a clear cause: the use of finite branched covers. Consequently if an equation similar to (1.3) were to hold on the whole of $\mathbb{T}^\mu_*$, different methods ought to appear. The solution to this issue involves twisted homology, as we now briefly outline.

Since $H_1(W_F)$ is free abelian on the meridians of F, mapping each meridian of F_i to some ω_i produces a homomorphism $\mathbb{Z}[\pi_1(W_F)] \to \mathbb{C}$. This leads to the twisted homology complex vector space $H_2(W_F;\mathbb{C}^\omega) := H_2(\mathbb{C}\otimes_{\mathbb{Z}[\pi_1(W_F)]} C_*(\widetilde{W}_F))$. Furthermore, this latter vector space is endowed with a twisted intersection form $\lambda_{\mathbb{C}^\omega}(W_F)$. While Viro [153] showed that $\operatorname{sign}(\lambda_{\mathbb{C}^\omega}(W_F))$ does not depend on the choice of a colored bounding surface, our first result of Chapter 8 (which makes heavy use of Theorem 6.1.4) identifies $\operatorname{sign}(\lambda_{\mathbb{C}^\omega}(W_F))$ with the multivariable signature.

Theorem 8.1.1. *Let L be a μ-colored link. For every colored bounding surface F and for all $\omega \in \mathbb{T}^\mu_*$, we have the equality*

$$\sigma_L(\omega) = \operatorname{sign}(\lambda_{\mathbb{C}^\omega}(W_F)).$$

Owing to the restrictions which appear in (1.3), Cimasoni-Florens' subsequent 4-dimensional results are proved for a certain subset $\mathbb{T}^\mu_P$ of roots of unity [41, Theorem 7.1 and 7.2]. Theorem 8.1.1 allows for generalizations of these results whose proofs make no use of branched covers. Building on work of Nagel-Powell [129], a key step in the process relies on the set $\mathbb{T}^\mu_!$ of so-called *non-concordance roots*

$$\mathbb{T}^\mu_P \subsetneq \mathbb{T}^\mu_! \subsetneq \mathbb{T}^\mu$$

and on some additional terminology which we now introduce. A *colored cobordism* between two μ-colored links L and L' is a collection of properly embedded locally flat surfaces $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_\mu$ in $S^3 \times [0,1]$ which have the following properties: the surfaces only intersect each other transversally in double points, each surface Σ_i has boundary $L_i \sqcup -L'_i$, and each connected component of Σ_i has at least one boundary component in $S^3 \times \{0\}$ and one in $S^3 \times \{1\}$. The second main result of Chapter 8 is a bound on the Euler characteristic and number of double points in such a cobordism.

Theorem 8.1.2. *If $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_\mu$ is a colored cobordism between two μ-colored links L and L' with c double points, then, for all $\omega \in \mathbb{T}^\mu_!$, we have*

$$|\sigma_L(\omega) - \sigma_{L'}(\omega)| + |\eta_L(\omega) - \eta_{L'}(\omega)| \leq c - \sum_{i=1}^{\mu} \chi(\Sigma_i).$$

Before describing two corollaries of Theorem 8.1.2, we wish to emphasize that our results hold in the topological category whereas Cimasoni-Florens worked in the smooth setting. First, we obtain a generalization of [41, Theorem 7.2] and of [153, Theorem 4.C], both of which extend the classical Murasugi-Tristram inequality [128, 147].

Corollary 8.1.4. *Let $F = F_1 \cup \cdots \cup F_\mu$ be a colored bounding surface for a μ-colored link L such that $F_1, \ldots, F_\mu$ have a total number of m connected components, intersecting in c double points. Set $\beta_1 = \sum_{i=1}^{\mu} \operatorname{rk} H_1(F_i)$. Then, for all $\omega \in \mathbb{T}^\mu_!$, we have*

$$|\sigma_L(\omega)| + |\eta_L(\omega) - m + 1| \leq \beta_1 + c.$$

Two μ-colored links L and L' are *concordant* if there exists a μ-colored cobordism between L and L' which has no intersection points and consists exclusively of annuli. The following theorem is a generalization of [41, Theorem 7.1].

Corollary 8.1.3. *The multivariable signature and nullity are topological concordance invariants at all $\omega \in \mathbb{T}^\mu_!$.*

In fact, Corollary 8.1.3 can be vastly generalized. To appreciate this statement, we briefly recall some notions regarding concordance, referring to [115] for a survey. A knot is (topologically) *slice* if it bounds a locally flat 2-disk in the 4-ball, or equivalently if it is concordant to the unknot. The monoid of all knots modulo concordance gives rise to an abelian group $\mathscr{C}$ called the *knot concordance group*.

Arguably, the study of $\mathscr{C}$ (which goes back to Fox and Milnor [70]) has undergone three large steps. Firstly, in the sixties, Levine [108] showed that slice knots admit metabolic Seifert matrices (such knots are called *algebraically slice*) and computed the resulting *algebraic concordance group*. Secondly, in the late seventies, Casson-Gordon [28, 29] produced examples of algebraically slice knots which are not slice. Thirdly, Cochran-Orr-Teichner [47] vastly improved the understanding of $\mathscr{C}$ by introducing an infinite filtration

$$\ldots \subset \mathscr{F}_{1.5} \subset \mathscr{F}_1 \subset \mathscr{F}_{0.5} \subset \mathscr{F}_0 \subset \mathscr{C}.$$

Following [47], knots belonging to $\mathscr{F}_n$ are called *n-solvable*. Cochran-Orr-Teichner also showed that a knot is 0.5-solvable if and only if it is algebraically slice and that 1.5-solvable knots have vanishing Casson-Gordon invariants. In particular, 0.5-solvable knots have vanishing Levine-Tristram signature.

While Cochran-Orr-Teichner's set-up also filters (strongly) slice links, we shall focus on a relative version due to Cha [30]. We refer to Section 8.5 for the precise definition of n-solvable cobordant links, but we note that concordant links are n-solvable cobordant for all n. As in the knot case, abelian link invariants are not expected to distinguish 1-solvable cobordant links. For instance, if two links are 1-solvable cobordant, then their first non-zero Alexander polynomials agree up to norms and their Blanchfield pairings are Witt equivalent [99, Theorems *B* and *C*].

The last main result of Chapter 8 is the corresponding statement for the multivariable signature and nullity.

Theorem 8.1.5. *If two μ-colored links L and L' are 1-solvable cobordant, then, for all $\omega \in \mathbb{T}^\mu_!$, we have*

$$\eta_L(\omega) = \eta_{L'}(\omega) \quad \textit{and} \quad \sigma_L(\omega) = \sigma_{L'}(\omega).$$

Since concordant links are n-solvable cobordant for all n, Theorem 8.1.5 can be viewed as a vast refinement of Corollary 8.1.3. We also believe that Theorem 8.1.5 would be difficult to prove using branched coverings. Finally, note that Theorem 8.1.5 relies on a somewhat technical result (Proposition 7.1.1) which shall be proved in Chapter 7.

1.2 Part II: The Burau representation and its generalizations

The second part of this thesis is concerned with the Burau representation of the braid group and its generalizations. Roughly speaking, a *braid* consists of n monotonic disjoint strands in the cylinder $D^2 \times [0,1]$. Given two braids β_1, β_2, one can form their *composition* $\beta_1\beta_2$ by stacking β_1 on top of β_2, see Figure 1.4. Using this operation, isotopy classes of braids form a group B_n which is referred to as the *braid group*. Among the most studied representations of the braid group is the *reduced Burau representation*

$$\overline{\mathscr{B}}_t \colon B_n \to GL_{n-1}(\mathbb{Z}[t^{\pm 1}]),$$

which we shall review in Chapter 9. Since the work of Burau [25], it is known that an important feature of $\overline{\mathscr{B}}_t$ lies in its close relation to the Alexander polynomial. Indeed, writing a link L as the closure $\widehat{\alpha}$ of an n-stranded braid α (this is always possible thanks to Alexander's theorem [1], see Figure 1.4), the Alexander polynomial of L satisfies

$$\Delta_L(t)(t^n - 1) = (t-1)\det(\overline{\mathscr{B}}_t(\alpha) - I_{n-1}).$$

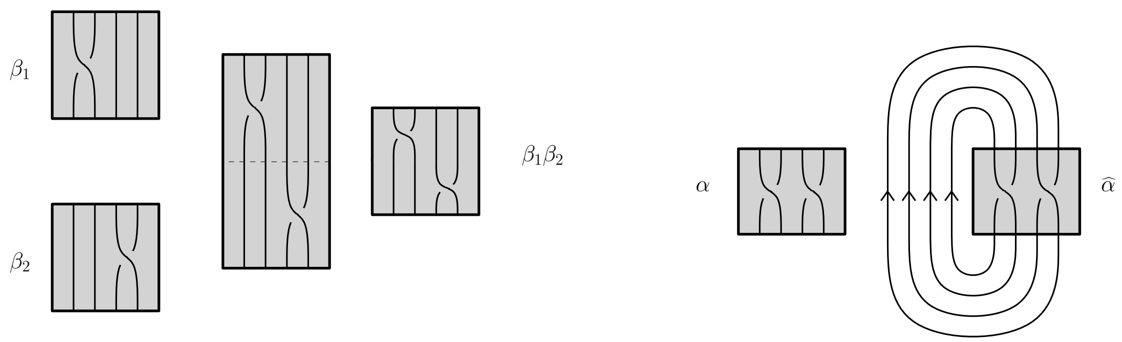

Figure 1.4: On the left-hand side: the composition of two braids. On the right-hand side: the closure of a braid.

An analogous result for the multivariable Alexander polynomial was later obtained by Birman via the somewhat lesser known *reduced Gassner representation* of the pure braid group P_n [17]. In order to interpolate between the Gassner representation and the Burau representation, we consider colored braids.

An n-stranded braid $\beta \subset D^2 \times [0,1]$ is *μ-colored* if each of its components is assigned an element in $\{1, 2, \ldots, \mu\}$. A μ-colored braid induces a coloring on the punctures of $D^2 \times \{0,1\}$. For emphasis, we shall denote the resulting punctured disks by D_c and $D_{c'}$, and call a μ-colored braid a (c, c')-braid. Isotopy classes of (c, c)-braids form a subgroup of the braid group, called the *c-colored braid group*. We then consider the *reduced colored Gassner representation*

$$B_c \to GL_{n-1}(\mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]),$$

which interpolates between the reduced Burau representation (for $\mu = 1$) and the reduced Gassner representation (for $\mu = n$). Note that while the Burau representation has been studied in depth, there seems to be little literature on the colored Gassner representation. Consequently, Section 9.4 contains a thorough study of this latter representation and several new results, some of which might be folklore, see in particular Proposition 9.4.3, Remark 9.4.10 and Section 9.5.

The remainder of Part II deals with several extensions of the Burau and colored Gassner representations.

1.2.1 Twisted invariants

As outlined above, the multivariable Alexander polynomial Δ_L of an ordered link L is extracted from a free abelian cover of the exterior X_L. Reformulating, Δ_L is closely related to the abelianization homomorphism $\pi_1(X_L) \to H_1(X_L)$. *Twisted Alexander polynomials* keep further track of the group $\pi_1(X_L)$ by means of representations $\rho\colon \pi_1(X_L) \to GL_k(R)$ with R a (Noetherian factorial) integral domain. The aim of Chapter 11 is to show how these invariants can be computed using braids; the exposition is based on our paper [50].

Let $Q(R[t_1^{\pm 1}, \dots, t_n^{\pm 1}])$ denote the field of fractions of the integral domain $R[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. In practice, we shall mostly work with the twisted torsion of L:

$$\tau^\rho(L) \in Q(R[t_1^{\pm 1}, \dots, t_n^{\pm 1}]).$$

This torsion invariant generalizes the classical Reidemeister torsion (which we review in Chapter 10) and the resulting theory shares many similarities with the classical "abelian" theory; we refer to [77] for a survey. Consequently it is natural to ask whether twisted Alexander polynomials (or equivalently twisted torsion invariants) can be computed via braids.

Chapter 11 answers this question positively by introducing *twisted Burau maps*. More precisely, given a representation $\rho : F_n \to GL_k(R)$ of the free group, we define a *twisted Burau map* and a *reduced twisted Burau map*

$$\overline{\mathscr{B}}_\rho\colon B_c \to GL_{(n-1)k}(R[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}])$$

and study their properties. Although these maps are not representations, they remain computable via Fox calculus, see Propositions 11.3.3 and 11.3.4. The main theorem of Chapter 11 relates the reduced twisted Burau map $\overline{\mathscr{B}}_\rho$ of a braid β to the twisted torsion $\tau^\rho_{\hat{\beta}}$ of its closure.

Theorem 11.1.1. *Let F_n be the free group on $x_1, \dots, x_n$ and let $\beta \in B_c$ be a μ-colored braid with n strands. If $\rho\colon F_n \to GL_k(R)$ is a representation which extends to $\pi_1(S^3 \setminus \hat{\beta})$, then*

$$\tau^\rho(\hat{\beta})(t_1, t_2, \dots, t_\mu) \det\left(\rho(x_1 x_2 \cdots x_n) t_{c_1} t_{c_2} \cdots t_{c_n} - I_k\right) \doteq \det(\overline{\mathscr{B}}_\rho(\beta) - I_{(n-1)k}).$$

In particular, taking ρ to be the trivial one-dimensional representation, Theorem 11.1.1 recovers the classical theorems of Burau and Birman.

1.2.2 L^2 invariants

As we just saw, twisted invariants of a space X aim to refine classical invariants via the additional data of a *finite dimensional* representation of the fundamental group $\pi_1(X)$. In the infinite dimensional case, one could therefore expect the action of $\pi_1(X)$ on the Hilbert space $\ell^2(\pi_1(X))$ to lead to new L^2*-invariants*. Note that L^2 theory has a rich history which goes back to Atiyah [6]. However we shall mostly focus on applications to low dimensional topology and refer to [117] for broader perspectives. Chapter 12 (which is based on joint work with Fathi Ben Aribi [13]) defines an L^2-version of the Burau representation.

Returning to knots, Li and Zhang were the first to define an L^2-Alexander "polynomial" via Fox calculus [111]. Even more recently, Dubois-Friedl-Luck [63] introduced the so-called L^2*-Alexander torsion* which generalizes the L^2-Alexander polynomial in the same way as the Reidemeister torsion generalizes the classical Alexander polynomial. More precisely, if one fixes a homomorphism $\phi\colon \pi_1(X) \to \mathbb{Z}$ and a second homomorphism $\gamma\colon \pi_1(X) \to G$ through which ϕ factors, then the L^2-Alexander torsion is a real valued function (defined up to some indeterminacy):

$$T^{(2)}(X, \phi, \gamma)\colon \mathbb{R}_{>0} \to \mathbb{R}.$$

With this set-up, there are several similarities between L^2 invariants and classical invariants [62, 63]. Consequently, it is natural to wonder whether some L^2-Alexander torsions can be computed from braids. The aim of Chapter 12 is to develop a theory of L^2-Burau representations and to provide such relations.

Let us briefly outline the data required to define the *L^2-Burau maps*. First denote by F_n the free group on $x_1, \ldots, x_n$. Next, fix $t > 0$ and a homomorphism $\gamma\colon F_n \to G$ through which the epimorphism $\phi\colon F_n \to \mathbb{Z}, x_i \mapsto 1$ factors. Denoting by $B(\ell^2(G))$ the algebra of bounded operators on $\ell^2(G)$, we define an *unreduced L^2-Burau map* and a *reduced-L^2 Burau map*

$$\overline{\mathscr{B}}^{(2)}_{t,\gamma}\colon B_n \to M_{n,n}(B(\ell^2(G)))$$

and study their properties, see for instance Propositions 12.4.3, 12.4.4 and 12.4.7. To state the relation with the L^2-Alexander torsion, we introduce some further notation.

Let β be a braid with closure L. Identifying the free group F_n with $\pi_1(D_n)$, there is a canonical epimorphism $F_n \to \pi_1(X_L)$ which we denote by γ_L. In particular one may consider the corresponding reduced L^2-Burau map $\overline{\mathscr{B}}^{(2)}_{t,\gamma_L}$. Finally, let $\phi\colon \pi_1(X_L) \to \mathbb{Z}$ be the homomorphism which sends each meridian to one. The main result of Chapter 12 reads as follows.

Theorem 12.1.1. *Given an oriented link L obtained as the closure of a braid $\beta \in B_n$, one has*

$$T^{(2)}_{L,(1,\ldots,1)}(id)(t) \cdot \max(1,t)^n \doteq \det{}^r_{\mathscr{N}(\pi_1(X_L))}\left(\overline{\mathscr{B}}^{(2)}_{t,\gamma_L}(\beta) - Id^{\oplus(n-1)}\right)$$

for all $t > 0$, where $\det{}^r_{\mathscr{N}(\pi_1(X_L))}$ denotes the regular Fuglede-Kadison determinant associated to $\pi_1(X_L)$.

As an application of Theorem 12.1.1, we provide an example of two braids indistinguishable under the Burau representation but which can be told apart by the L^2 version, see Corollary 12.5.1.

1.2.3 The Lagrangian functor

In another direction, braids can also be viewed as particular examples of *tangles*, see Figure 1.5. Roughly speaking, a tangle is a particular type of properly embedded 1-dimensional submanifold of the cylinder $D^2 \times [0,1]$. Tangles no longer form a group but are the morphisms of a category **Tangles**. Consequently, if an extension of the reduced Burau representation $\overline{\mathscr{B}}_t\colon B_n \to GL_{n-1}(\Lambda)$ from braid to tangles were to exist, it could take the form of a functor defined on **Tangles**. As we shall review in Chapter 13, such a construction was produced by Cimasoni and Turaev [42], resulting in the *Lagrangian functor*

$$\mathscr{F}\colon \textbf{Tangles} \to \textbf{Lagr}_\Lambda,$$

where $\textbf{Lagr}_\Lambda$ is a category which we now briefly describe. The objects of $\textbf{Lagr}_\Lambda$ are pairs (H, λ), where H is a finitely generated Λ-module and λ is a Λ-valued non-degenerate skew-Hermitian form on H. A morphism from (H_1, λ_1) to (H_2, λ_2) is a Lagrangian submodule of $(H_1 \oplus H_2, -\lambda_1 \oplus \lambda_2)$. Lagrangian relations can be understood as a generalization of unitary Λ-isomorphisms. Namely, if $f\colon (H_1, \lambda_1) \to (H_2, \lambda_2)$ is a unitary isomorphism, then its graph $\Gamma_f \subset H_1 \oplus H_2$ is an example of a Lagrangian relation.

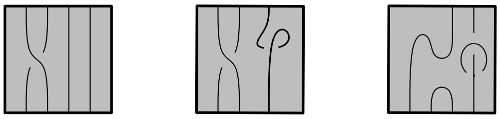

Figure 1.5: Only the tangle on the left is a braid.

The Lagrangian functor generalizes the reduced Burau representation in the following sense. The image of a braid β by $\mathscr{F}$ is the graph $\Gamma_{\overline{\mathscr{B}}_t(\beta)}$ of the reduced Burau representation $\overline{\mathscr{B}}_t(\beta)$. In this setting, a key observation is that $\overline{\mathscr{B}}_t$ is unitary [144]. In other words, for each braid β, the automorphism $\overline{\mathscr{B}}_t(\beta)$ preserves a skew-Hermitian form defined on Λ^{n-1}.

Finally, note that the Lagrangian functor is also defined in the multivariable case. Namely, there is a Lagrangian functor $\mathscr{F}\colon \mathbf{Tangles}_\mu \to \mathbf{Lagr}_{\Lambda_\mu}$ whose domain is the category of μ-colored tangles and whose target is defined analogously as in the one-variable case [42, Section 6]. This time, the multivariable Lagrangian functor generalizes the reduced colored Gassner representation.

1.2.4 A Burau-Alexander 2 functor

In fact, the category **Tangles** comes with additional structure. Indeed, roughly speaking, oriented surfaces between oriented tangles turn **Tangles** into a bicategory, i.e. a category in which there are “morphisms between morphisms”. Thus we are led to the following question: can the Lagrangian functor $\mathscr{F}$ be promoted to a (weak) 2-functor? This is in fact possible and the construction, based on joint work with David Cimasoni [38], is the subject of Chapter 14.

The first roadblock lies in the target category: *a priori*, the Lagrangian category $\mathbf{Lagr}_\Lambda$ does not admit the structure of a bicategory. In order to circumvent this issue, the idea is to consider *cospans* of Λ-modules, i.e. diagrams of the form $H \to T \leftarrow H'$. As we shall see, there is a category $\mathbf{L}_\Lambda$ of *Lagrangian cospans* which should be understood as a generalization of the category of Lagrangian relations, in the sense that there is a full (non-faithful) functor $F\colon \mathbf{L}_\Lambda \to \mathbf{Lagr}_\Lambda$ which is the identity on objects, see Proposition 14.2.4. Naturally, the point of considering $\mathbf{L}_\Lambda$ instead of $\mathbf{Lagr}_\Lambda$ lies in the fact that (a slight modification of) the former is endowed with the structure of a bicategory.

There is, however, a key point which we have been glossing over. A bicategory $\mathscr{C}$ roughly consists of a set of objects and, for each pair of objects (X, Y), a category $\mathscr{C}(X, Y)$ whose objects are called 1-morphisms and whose morphisms are called 2-morphisms. Furthermore, there is a composition law for 1-morphisms and two composition laws for 2-morphisms. The subtlety is that on the level of 1-morphisms, the composition is only associative up to isomorphism. In particular the objects and 1-morphisms of a bicategory do not provide the data for a category. The same remark goes for functors: restricting a weak 2-functor to objects and morphisms does not produce a functor.

For this reason, we will construct our *Burau-Alexander 2-functor* in two steps. First we will define a functor $\overline{\mathscr{B}}\colon \mathbf{Tangles} \to \mathbf{L}_\Lambda$ which generalizes the Lagrangian functor. Although a precise formulation can be found in Section 14.4, an approximate statement goes as follows:

Theorem 14.1.1. *There is a functor $\overline{\mathscr{B}}\colon \mathbf{Tangles} \to \mathbf{L}_\Lambda$ which satisfies $F \circ \overline{\mathscr{B}} = \mathscr{F}$, where $\mathscr{F}$ is the Lagrangian functor, and $F\colon L_\Lambda \to \mathbf{Lagr}_\Lambda$ is a functor described in Section 14.2.*

Furthermore, if τ is an oriented link, then $\overline{\mathscr{B}}(\tau)$ is nothing but its Alexander module.

Our Burau-Alexander 2-functor is then built from $\overline{\mathscr{B}}$ by taking into account the subtlety described above. Namely, slight modifications of the categories **Tangles** and $\mathbf{L}_\Lambda$ produce bicategories of tangles and Lagrangian 2-cospans, and additional work gives rise to our weak 2-functor. Once again, we delay a precise formulation to Section 14.4 but a short perusal of the following statement should give the reader the gist of the main result of Chapter 14.

Theorem 14.1.2. *The functor $\overline{\mathscr{B}}$ of Theorem 14.1.1 gives rise to a weak 2-functor from the bicategory of oriented tangles to the bicategory of Lagrangian cospans, whose restriction to oriented surfaces is given by the Alexander module.*

Along the way, we shall discuss the *core* of the various categories we have introduced, i.e. their maximal subgroupoid. More precisely, as an additional feature of our theory, we shall observe that the restriction of $\overline{\mathscr{B}}$ to **Braids** (the core of **Tangle**) actually takes value in the core of $\mathbf{L}_\Lambda$. Moreover $core(\mathbf{L}_\Lambda)$ turns out to be equivalent to the category $\mathbf{U}_\Lambda$ of unitary morphisms, see Proposition 14.2.5.

1.3 Part III: Non-addivity of classical link invariants

Consider an arbitrary link invariant $\mathscr{I}$ taking values in an abelian group. Precomposing this invariant with the braid closure defines maps $\alpha \mapsto \mathscr{I}(\widehat{\alpha})$ from the braid groups B_n to this abelian group, and one might wonder whether these maps are group homomorphisms. In other words, one can ask whether

$$\mathscr{I}(\widehat{\alpha\beta}) - \mathscr{I}(\widehat{\alpha}) - \mathscr{I}(\widehat{\beta})$$

vanishes for all $\alpha, \beta \in B_n$. This question has an easy answer: the only invariant with this property is the trivial one. However, one can ask the more refined question of "evaluating" the homomorphism defect displayed above. This program was carried out by Gambaudo and Ghys [78] who expressed the homomorphism defect of the Levine-Tristram signature in terms the reduced Burau representation $\overline{\mathscr{B}}_t$. The aim of the last part of this thesis is to extend this result to colored tangles.

We start by giving some more details regarding the result of Gambaudo-Ghys. First, recall that the reduced Burau representation is unitary. Therefore, given two braids $\alpha, \beta \in B_n$ and a root of unity ω, one can consider the Meyer cocycle of the two unitary matrices $\overline{\mathscr{B}}_\omega(\alpha)$ and $\overline{\mathscr{B}}_\omega(\beta)$. Denoting by $\operatorname{sign}_\omega(L)$ the Levine-Tristram signature of an oriented link L, the main theorem of [78] is the equality

$$\operatorname{sign}_\omega(\widehat{\alpha\beta}) - \operatorname{sign}_\omega(\widehat{\alpha}) - \operatorname{sign}_\omega(\widehat{\beta}) = -\mathit{Meyer}(\overline{\mathscr{B}}_\omega(\alpha), \overline{\mathscr{B}}_\omega(\beta)) \tag{1.4}$$

for all $\alpha, \beta \in B_n$ and $\omega \in S^1$ of order coprime to n.

Recall that the Levine-Tristram signature admits a generalization, the *multivariable signature* which associates to a μ-colored link L a map on the μ-dimensional torus $\mathbb{T}^\mu$ (we now write $\operatorname{sign}_\omega(L)$ instead of $\sigma_L(\omega)$). We also know that the reduced Burau representation has a multivariable extension, called the reduced colored Gassner representation, which is unitary, and it is natural to wonder if (15.1) holds in this multivariable setting.

On the other hand, as outlined above, braids are special kind of tangles. Moreover, the tangles that are endomorphims of a given object can not only be composed, but also closed up to give oriented links, just like braids. Therefore, it makes sense to ask the same question as above, i.e. try to evaluate the defect of additivity of the signature on tangles. Since the Lagrangian functor

$$\mathscr{F}\colon \mathbf{Tangles}_\mu \to \mathbf{Lagr}_{\Lambda_\mu}$$

generalizes the reduced Burau representation, one might expect it to appear in the signature defect formula. However, one cannot consider the Meyer cocycle of (pairs of) objects in the Lagrangian category, but it makes sense to consider the *Maslov index* of three objects in this category, evaluated at some $t = \omega \in S^1$. Therefore, one can ask whether the additivity defect of the signature of tangles is related to the Maslov index of the image by $\mathscr{F}$ of these tangles, evaluated at $t = \omega$.

The last result of this thesis (which is based on joint work with David Cimasoni [39]). answers both these questions simultaneously. The precise statement will be given in Theorem 16.4.1 and Theorem 17.3.2 below, but in a nutshell, it can be phrased as follows.

Theorem 15.0.1. *Given an object c of the category of μ-colored tangles and two endomorphisms τ_1, τ_2 of this object, the equality*

$$\operatorname{sign}_\omega(\widehat{\tau_1\tau_2}) - \operatorname{sign}_\omega(\widehat{\tau_1}) - \operatorname{sign}_\omega(\widehat{\tau_2}) = Maslov(\mathscr{F}_\omega(\overline{\tau}_1), \mathscr{F}_\omega(id_c), \mathscr{F}_\omega(\tau_2))$$

holds for all $\omega = (\omega_1, \ldots, \omega_\mu)$ in an open dense subset of $\mathbb{T}^\mu$, where $\overline{\tau}$ denotes the horizontal reflection of the tangle τ, and $\mathscr{F}_\omega$ is the evaluation at $t = \omega$ of the multivariable extension of the Lagrangian functor $\mathscr{F}$.

In the case of colored braids, this functor gives back the graph of the reduced colored Gassner representation, the horizontal reflection of a braid is its inverse, and the Maslov index of the graphs of unitary automorphisms γ_1^{-1}, id and γ_2 is one possible definition of the Meyer cocycle of γ_1 and γ_2. Therefore, in the case of μ-colored braids, our theorem is exactly the expected multivariable extension of (15.1), see Corollary 16.4.4. On other hand, the restriction of Theorem 15.0.1 to oriented tangles is stated in Corollary 16.4.3. Finally note, that even the intersection of these latter corollaries (i.e. the case of oriented braids and the Levine-Tristram signature) is slightly more general than the main theorem of Gambaudo and Ghys, as we allow the strands of the braids to be oriented in different directions.

Part I

Invariants of colored links

Chapter 2

Classical invariants of links

2.1 Introduction

This introductory chapter deals with (single variable) classical invariants of oriented links. More precisely, we shall review the Alexander polynomial, the Levine-Tristram signature and the Blanchfield pairing. Let X_L denote the exterior of an oriented link $L = K_1 \cup \ldots \cup K_n$. A common feature of the aforementioned invariants is their relation to the cover $\widehat{X}_L \to X_L$ corresponding to the kernel of the homomorphism $\pi_1(X_L) \to \mathbb{Z}$ given by $\gamma \mapsto \ell k(\gamma, K_1) + \ldots + \ell k(\gamma, K_n)$. Keeping this in mind, let us outline the structure of this chapter.

First of all, Section 2.2 will review the Alexander polynomial. Namely, we shall recall the definition of the *Alexander module* $H_1(\widehat{X}_L)$ and orders of modules. However, the Alexander polynomial, which was first introduced by Alexander [1], can also be understood using other tools. We will focus on Seifert matrices and Fox calculus, leaving a discussion of Reidemeister torsion to Chapter 10. Indeed, in the current chapter, we chose to rely on covering spaces and elementary topological considerations, delaying the use of twisted homology to further chapters.

The study of Seifert surfaces will lead us to review knot signatures whose birth goes back to the sixties [66, 124, 128, 148]. As we shall see in Section 2.3, these signatures were generalized by Levine [108] and Tristram [147] to a function

$$\sigma_L \colon S^1 \to \mathbb{Z}.$$

Among other properties, this *Levine-Tristram signature* function provides lower bounds on the genus of the surfaces that L can bound in the 4-ball. Delaying precise definitions, this relation to the *4-genus* suggest that the Levine-Tristram signatures should admit a four dimensional interpretation. This step was taken in the early seventies by Viro [153] and Kauffman-Taylor [94] who reinterpreted $\sigma_L(\omega)$ in terms of branched covers over the 4-ball for $\omega \in S^1$ of *finite* order.

The Alexander module $H_1(\widehat{X}_L)$ also supports a pairing which was first discovered by Blanchfield [18]. Denoting by $Q := \mathbb{Q}(t)$ the field of fractions of $\Lambda = \mathbb{Z}[t^{\pm 1}]$, Section 2.4 will review this Q/Λ-valued pairing in the case of *knots*, the link case being treated in Chapter 6. Apart from being non-singular and Hermitian, this *Blanchfield pairing*

$$\mathrm{Bl}(K) \colon H_1(\widehat{X}_K) \times H_1(\widehat{X}_K) \to Q/\Lambda$$

can also be computed using Seifert matrices [76, 97, 110]. Arguably, the main application of the Blanchfield pairing lies in knot concordance [47], however it has also been used in the study of the unknotting number [20].

Finally, note that while several textbooks contain material on classical invariants [56, 95, 112, 140], the subjects of this chapter are not all treated equally often. Indeed, while the Alexander polynomial and the definition of the signature are almost systematically covered, the 4-dimensional aspects as well as the Blanchfield pairing are usually omitted. For this reason this chapter will provide slightly more details on these last two subjects.

2.2 The Alexander polynomial

In this section, we shall review the one variable Alexander polynomial $\Delta_L(t)$ of an oriented link L. Namely, Subsection 2.2.1 will recall the definition of $\Delta_L(t)$ by using orders of modules, Subsection 2.2.2 will deal with Fox calculus, while Subsection 2.2.3 is concerned with Seifert matrices. References include [95, 112, 150].

2.2.1 Definition of the Alexander polynomial

Given an oriented link $L = K_1 \cup \cdots \cup K_n$, the epimorphism $\psi\colon \pi_1(X_L) \to \mathbb{Z}$ given by $\gamma \mapsto \ell k(\gamma, K_1) + \ldots + \ell k(\gamma, K_n)$ induces the *total linking cover* $\widehat{X}_L \to X_L$. The homology of $\widehat{X}_L$ is naturally a module over $\Lambda = \mathbb{Z}[t^{\pm 1}]$, and the Λ-module $H_1(\widehat{X}_L)$ is called the *Alexander module* of the oriented link L. Since X_L is compact and Λ is Noetherian, the Alexander module is finitely generated over Λ [110, Proposition 1.1].

Remark 2.2.1. Note that in several references, the Alexander module refers to the relative homology group $H_1(\widehat{X}_L, \widehat{x})$, where $\widehat{x} \subset \widehat{X}_L$ is the fiber over a point $x \in X_L$ [57, 95]. For reasons which shall become apparent in Subsection 2.4.1, we shall sometimes denote the Alexander module by $H_1(X_L; \Lambda)$ instead of $H_1(\widehat{X}_L)$.

In order to extract invariants from the Alexander module, we briefly review orders and presentation matrices. Let A be an $m \times n$ matrix with coefficients in a Noetherian factorial domain R. Let us denote by $E_r(A)$ the ideal of R generated by all the $(m-r)\times(m-r)$ minors of A. By convention, $E_r(A) = (0)$ if $r < 0$ and $E_r(A) = R$ if $r \geq m$. Let $\Delta_r(A)$ denote the greatest common divisor of the elements of $E_r(A)$. Since R is a Noetherian factorial domain, $\Delta_r(A)$ exists and is well-defined up to multiplication by a unit of R. Given Δ and Δ' in R, let us note $\Delta \doteq \Delta'$ if $\Delta = u\Delta'$ for some unit u of R.

Now let M be a module over R. A *finite presentation* of M is an exact sequence $F \overset{\varphi}{\to} E \to M \to 0$, where E and F are free R-modules with finite basis. A matrix of φ is a *presentation matrix* of M. The r-th *elementary ideal* of M is the ideal of R given by $E_r(M) := E_r(A)$ where A is any presentation matrix of M. One can check that these ideals do not depend on the presentation of M. In particular the element $\Delta_r(M) := \Delta_r(A)$ of R is well defined up to multiplication by a unit of R. Finally $\Delta_0(M)$ is called the *order* of M.

Definition 1. The *Alexander polynomial* $\Delta_L(t)$ of an oriented link L is the order of $H_1(\widehat{X}_L)$.

The Alexander polynomial is only well defined up to multiplication by units of Λ, that is up to multiplication by $\pm t^n$ for $n \in \mathbb{Z}$. More generally, $\Delta^r_L(t) := \Delta_r(H_1(\widehat{X}_L))$ is called the

r-th *Alexander polynomial* of L and $\Delta_L^{\mathrm{tor}}(t)$ shall denote the first non-vanishing Alexander polynomial of L.

Remark 2.2.2. If K is a knot, then multiplication by $t-1$ induces an isomorphism on the Alexander module [110, Proposition 1.2]. It follows that $H_1(\widehat{X}_K)$ is a Λ-torsion module and consequently $\Delta_K(t)$ is not identically zero [110, Corollary 1.3]. This situation is in stark contrast with the case of links, where the Alexander polynomial may very well vanish identically.

The behavior of the Alexander polynomial under mirror image, orientation reversal, band sums, cabling and various other satellite operations is well understood [112, Chapter 6].

2.2.2 Fox calculus

Consider the free group F_n on $x_1, \dots, x_n$ and let $\mathrm{aug}\colon \mathbb{Z}[F_n] \to \mathbb{Z}, \sum n_f f \mapsto \sum n_f$ be the augmentation map. An additive homomorphism $D\colon \mathbb{Z}[F_n] \to \mathbb{Z}[F_n]$ is called a *derivative* if $D(ab) = D(a)\mathrm{aug}(b) + aD(b)$ for all $a, b \in \mathbb{Z}[F_n]$. Since $D(k) = 0$ for any integer k and $D(a^{-1}) = -a^{-1}D(a)$ for any $a \in F_n$, D is uniquely determined by its values on the generators of F_n. It then turns out that for all $j = 1, \dots, n$ there exists a unique *Fox derivative*

$$\frac{\partial}{\partial x_j}\colon \mathbb{Z}[F_n] \to \mathbb{Z}[F_n]$$

satisfying $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$ [58, Chapter VII]. Given a link L and a presentation $\langle x_1, \dots, x_n \mid r_1, \dots, r_m\rangle$ of $\pi_1(X_L)$, we denote by $\mathrm{pr} : \mathbb{Z}[F_n] \to \mathbb{Z}[\pi_1(X_L)]$ the ring homomorphism induced by the projection. The *Fox matrix* A whose (i,j) coefficient is $\psi(\mathrm{pr}(\frac{\partial r_i}{\partial x_j}))$ provides a presentation matrix for the module $H_1(\widehat{X}_L, \widehat{x})$ which we encountered in Remark 2.2.1, see [95, Theorem 7.1.5 and Exercise 7.3.11].

Remark 2.2.3. Let $\langle x_1, \dots, x_n | r_1, \dots, r_{n-1}\rangle$ be a Wirtinger presentation for $\pi_1(X_L)$ [112, Chapter 11]. We denote by A_i the square matrix obtained from the Fox matrix A by deleting its i-th column. Since the presentation has deficiency one, it can be checked that for each i, one has $\Delta_L(t) \doteq \det(A_i)$ [95, Lemma 7.3.2 and Exercise 7.3.11]. This fact will also be discussed in Subsection 3.3.2 and in Section 10.3.

Here is a practical application of Remark 2.2.3.

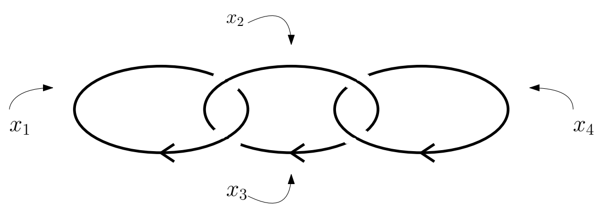

Figure 2.1: Performing Fox calculus.

Example 2.2.4. The Wirtinger presentation for the link depicted in Figure 2.1 has generators x_1, x_2, x_3, x_4 and relators $x_2x_1x_2^{-1}x_1^{-1}, x_2x_4x_2^{-1}x_4^{-1}, x_4x_2x_4x_3^{-1}$. Performing Fox calculus yields the matrix

$$\begin{bmatrix} x_2(1-x_1x_2^{-1}x_1^{-1}) & 1-x_2x_1x_2^{-1} & 0 & 0 \\ 0 & 1-x_2x_4x_2^{-1} & 0 & x_2(1-x_4x_2^{-1}x_4^{-1}) \\ 0 & x_4 & -x_4x_2x_4^{-1}x_3^{-1} & 1-x_4x_2x_4^{-1} \end{bmatrix}.$$

Applying Remark 2.2.3 implies that the Alexander polynomial of L is $(t-1)^2$.

2.2.3 Computation via Seifert matrices

A *Seifert surface* for an oriented link L is a compact oriented surface F whose oriented boundary is L. While a Seifert surface may be disconnected, we require that it has no closed components. The existence of a Seifert surface can be proved constructively using *Seifert's algorithm* [112, Theorem 2.2]. Since F is orientable, it admits a regular neighborhood homeomorphic to $F \times [-1,1]$ in which F is identified with $F \times \{0\}$. For $\varepsilon = \pm 1$, the *push off maps* $i^\varepsilon \colon H_1(F) \to H_1(S^3 \setminus F)$ are defined by sending a (homology class of a) curve x to $i^\varepsilon(x) := x \times \{\varepsilon\}$. The *Seifert pairing* is the bilinear form

$$\begin{aligned} H_1(F) \times H_1(F) &\to \mathbb{Z} \\ (a,b) &\mapsto \ell k(i^-(a), b). \end{aligned}$$

A *Seifert matrix* for an oriented link L is a matrix for a Seifert pairing. Although Seifert matrices do not provide link invariants, their so-called *S-equivalence class* does [112, Chapter 8]. The following theorem is due to Seifert [141]; a proof can be found in [112, Chapter 6].

Theorem 2.2.5. *Let F be a connected Seifert surface for an oriented link L. If A is a Seifert matrix arising from F, then $tA - A^T$ presents the Alexander module and in particular*

$$\Delta_L(t) \doteq \det(tA - A^T).$$

The Seifert matrix approach has the advantage of providing immediate proofs of several elementary properties of the Alexander polynomial [112]. For instance, it immediately follows from Theorem 2.2.5 that $\Delta_L(t)$ is symmetric (i.e. $\Delta_L(t) \doteq \Delta_L(t^{-1})$) and if K is a knot, then $\Delta_K(1) \doteq 1$, since $A - A^T$ represents the nonsingular intersection pairing on the Seifert surface.

Example 2.2.6. Applying Seifert's algorithm to the link L depicted in Figure 2.1, we see that the 2×2 identity matrix is a Seifert matrix for L. It immediately follows from Theorem 2.2.5 that $\Delta_L(t) \doteq (t-1)^2$, as we already observed in Example 2.2.4.

2.3 Levine-Tristram signatures

In this second introductory section, we briefly review the Levine-Tristram signature and nullity of a link. Subsection 2.3.1 gives a definition using Seifert matrices, while Subsection 2.3.2 outlines a 4-dimensional interpretation which involves signatures of branched covers. References include [28, 93].

2.3.1 Definition and properties

Recall from Subsection 2.2.3 that a Seifert surface for an oriented link L is a compact oriented surface whose oriented boundary is L. Moreover, the first homology of F is endowed with a bilinear pairing for which a matrix is called a Seifert matrix of L. Given a Seifert matrix A, observe that the matrix $(1-\omega)A^T + (1-\overline{\omega})A$ is Hermitian for all ω lying in S^1.

Definition 2. Given $\omega \in S^1$, the *Levine-Tristram signature* of L at ω is defined as

$$\sigma_L(\omega) := \operatorname{sign}((1-\omega)A^T + (1-\overline{\omega})A),$$

where A is any Seifert matrix for L.

These signatures are well defined (i.e. independent of the choice of Seifert surface) [112], and varying ω along S^1, they give rise to a function $\sigma_L : S^1 \to \mathbb{Z}$. It is easily seen that $\sigma_L : S^1 \to \mathbb{Z}$ is piecewise constant. Moreover, its discontinuities only occur at zeroes of $(t-1)\Delta_L^{\text{tor}}(t)$ [81, Theorem 2.1].

Example 2.3.1. We saw in Example 2.2.4 that the link L depicted in Figure 2.1 admits the 2×2 identity matrix as a Seifert matrix. As an immediate consequence, the signature function of L is identically equal to 2 on $S^1 \setminus \{1\}$.

Let A be a Seifert matrix for an oriented link L resulting from the choice of a Seifert surface F with $\beta_0(F)$ components. Furthermore, let $H(\omega)$ denote the matrix $(1-\omega)A^T + (1-\overline{\omega})A$. The *nullity* of L is the function $\eta_L \colon S^1 \to \mathbb{Z}$ whose value at $\omega \in S^1$ is

$$\eta_L(\omega) = \operatorname{null}(H(\omega)) + \beta_0(F) - 1.$$

As for the signature function, η_L is a well defined invariant and Theorem 2.2.5 implies that given $\omega \in S^1 \setminus \{1\}$, the nullity $\eta_L(\omega)$ vanishes if and only if $\Delta_L(\omega) \neq 0$.

It is well known that the *Murasugi signature* $\sigma_K(-1)$ of a knot K is even [56]. More generally, given an n-component link L and $\omega \in S^1 \setminus \{1\}$, the integer $\sigma_L(\omega) + \eta_L(\omega) - n + 1$ is even [143], see also [41, Lemma 5.6]. In particular, if ω is not a root of $\Delta_L(t)$ and n is odd (resp. even), then $\sigma_L(\omega)$ is even (resp. odd). The behavior of the Levine-Tristram signature under mirror image, orientation reversal, band sums, cabling and various other satellite operations is also well understood [112, 114]. Moreover local relations for the signature are also known [41, Section 5], see also [79].

We now deal with properties related to 4-dimensional topology.

Definition 3. Two oriented m-component links L and J are smoothly (resp. topologically) *concordant* if there is a smooth (resp. locally flat) embedding into $S^3 \times I$ of a disjoint union of m annuli $A \hookrightarrow S^3 \times I$, such that the oriented boundary of A satisfies $\partial A = -L \sqcup J \subset -S^3 \sqcup S^3 = \partial(S^3 \times I)$.

The integers $\sigma_L(\omega)$ and $\eta_L(\omega)$ are known to be concordance invariants for any root of unity ω of prime power order [128, 147]. However it is only recently that Nagel and Powell gave a precise characterization of the $\omega \in S^1$ at which σ_L and η_L are concordance invariants [129]. In a similar spirit, the Levine-Tristram signatures satisfy the following *Murasugi-Tristram inequality* [128, 147].

Theorem 2.3.2. *If an oriented link L bounds an m-component smoothly properly embedded surface $F \subset D^4$, then for any root of unity ω of prime power order, the following inequality holds:*

$$|\sigma_L(\omega)| + |\eta_L(\omega) - m + 1| \leq \operatorname{rk} H_1(F).$$

Working with the "averaged Levine-Tristram signature" and in the topological locally flat category, Powell [134] recently gave a Murasugi-Tristram type inequality which holds for *each* $\omega \in S^1 \setminus \{1\}$. We also refer to Corollary 8.4.7 for a generalization of Theorem 2.3.2 which builds on Powell's approach.

2.3.2 Signatures and branched covers over the 4-ball

Given a smoothly properly embedded connected surface $F \subset D^4$, denote by W_F the complement of a tubular neighborhood of F. A short Mayer-Vietoris argument shows that $H_1(W_F)$ is infinite cyclic and one may consider the covering space $W_k \to W_F$ obtained by composing the abelianization homomorphism with the quotient map $H_1(W_F) \cong \mathbb{Z} \to \mathbb{Z}_k$. The restriction of this cover to $F \times S^1$ consists of a copy of $F \times p^{-1}(S^1)$, where the $p\colon S^1 \to S^1$ is the k-fold cover of the circle. Extending p to a cover $D^2 \to D^2$ branched along 0, and setting

$$\overline{W}_F := W_k \cup_{F \times S^1} (F \times D^2)$$

produces a cover $\overline{W}_F \to D^4$ branched along $F = F \times \{0\}$. Before moving on, let us recall some terminology. Given a compact orientable 4-manifold W, there is a symmetric bilinear pairing on $H^2(W, \partial W)$ obtained by evaluating the cup product of two cohomology classes on the fundamental class of W. Using duality, this gives rise to the *intersection form* on $H_2(W)$. Alternatively, this pairing can be computed by representing homology classes by embedded surfaces and computing their algebraic intersection number, see [23, Section VI.11] and Section 5.6 for more details.

Returning to the matter at hand, denote by t a generator of the finite cyclic group $\mathbb{Z}_k$. The $\mathbb{C}[\mathbb{Z}_k]$-module structure of $H_2(\overline{W}_F, \mathbb{C})$ gives rise to a complex vector space

$$H_2(\overline{W}_F, \mathbb{C})_\omega = \{x \in H_2(\overline{W}_F, \mathbb{C}) \mid tx = \omega x\}$$

for each root of unity ω of order k. Restricting the intersection form on $H_2(\overline{W}_F, \mathbb{C})$ to $H_2(\overline{W}_F, \mathbb{C})_\omega$ produces a Hermitian pairing whose signature we call the *ω-signature* of $\overline{W}_F$ and which we denote by $\sigma_\omega(\overline{W}_F)$. These signatures share most of the properties of the usual signature [93, Chapter XIII]. Although this fact is not altogether surprising (since we are merely restricting to summands of $H_2(\overline{W}_F; \mathbb{C})$), it can also be understood in the somewhat deeper context of G-signatures [28, 93].

The next result, originally due to Viro [153] and Kauffman-Taylor [94], provides a 4-dimensional interpretation of the Levine-Tristram signature.

Theorem 2.3.3. *Assume that an oriented link L bounds a smoothly properly embedded compact oriented surface $F \subset D^4$ and let $\overline{W}_F$ be the k-fold of D^4 branched along F. Then, for any root of unity ω of order k, the following equality holds:*

$$\sigma_L(\omega) = \sigma_\omega(\overline{W}_F).$$

A proof of Theorem 2.3.3 can be found in [93, Chapters XII and XIII], see also [41, Theorem 6.1] for a generalization. Note that there now are 4-dimensional interpretations of the signature which hold for all ω and rely on bordism arguments, see [53, 134, 153] and Chapter 8.

To conclude this section, let us outline how the Levine-Tristram signature arises as a particular case of the ρ-invariant of Atiyah-Patodi-Singer [7].

Remark 2.3.4. Let M be a closed 3-manifold and let $\alpha\colon \pi_1(M) \to U(k)$ be a unitary representation. Using differential geometry, Atiyah, Patodi and Singer associate a real number $\rho(M, \alpha)$ to each such pair (M, α). We shall not give the definition of $\rho(M, \alpha)$ but instead we focus on a simpler situation. Namely, assume that r disjoint copies of M bound a 4-manifold W and that there is a map $\beta\colon \pi_1(W) \to U(k)$ which restricts to α on each copy of M. In this setting, the difference

$$\frac{1}{r}(\operatorname{sign}_\beta(W) - k \operatorname{sign}(W))$$

between the "twisted signature" $\operatorname{sign}_\beta(W)$ (see Section 5.6) and the usual signature turns out to be independent of (W, β) and coincides with the ρ-invariant [7].

When $k = 1$ and $\alpha\colon \pi_1(W) \to U(1) = \mathbb{C}^*$ is a character of finite order, the situation simplifies drastically. For the sake of concreteness, assume that the image of α is the cyclic subgroup $\mathbb{Z}_q$ generated by $\omega = e^{2\pi i/q}$. Since the bordism group $\Omega_3(B\mathbb{Z}_q)$ is finite (see e.g.[49]), there is an r for which r copies of M bound a 4-manifold W, and the character α extends. Relying on Theorem 2.3.3, one can show that if M is obtained by 0-framed surgery along L and α sends each meridian of L to ω, then $\rho(M_L, \alpha) = \sigma_L(\omega)$ [28]. For further reading on this point of view, see [71, 72, 109], Remark 3.4.10 and Section 7.3.

2.4 The Blanchfield pairing of a knot

The aim of this section is to review the Blanchfield pairing $\operatorname{Bl}(K)$ of an oriented knot K. The organization is as follows: Subsection 2.4.1 begins with some preliminary remarks on twisted homology, while Subsection 2.4.2 deals with the definition of $\operatorname{Bl}(K)$. Note that a more thorough discussion of twisted homology can be found in Chapter 5 and that the Blanchfield pairing for links will be treated in Chapter 6. References for this section include [18, 76, 87, 133].

2.4.1 Twisted homology: a first take

Let X be a CW complex and let $Y \subset X$ be a possibly empty subcomplex. Given an epimorphism $\psi\colon \pi_1(X) \to \mathbb{Z}^\mu$, denote by $p\colon \widehat{X} \to X$ the regular cover of X corresponding to the kernel of ψ and set $\widehat{Y} := p^{-1}(Y)$. The left action of the deck transformation group $\mathbb{Z}^\mu$ on $\widehat{X}$ endows the chain complex $C_*(\widehat{X}, \widehat{Y})$ with the structure of a left $\Lambda_\mu := \mathbb{Z}[\mathbb{Z}^\mu]$-module. Moreover, let R be a commutative ring and let M be a (R, Λ_μ)-bimodule.

Definition 4. The chain complex $C_*(X, Y; M) = M \otimes_{\Lambda_\mu} C_*(\widehat{X}, \widehat{Y})$ of left R-modules will be called the *twisted chain complex* of the pair (X, Y) with coefficients in M. The corresponding homology left R-modules $H_*(X, Y; M)$ will be called the *twisted homology* modules of (X, Y) with coefficients in M.

If Y is empty, we write $H_*(X; M)$ instead of $H_*(X, \emptyset; M)$.

Example 2.4.1. If $R = \Lambda_\mu$ and $M = \Lambda_\mu$ is given its natural $(\Lambda_\mu, \Lambda_\mu)$-bimodule structure, then one can use the canonical Λ_μ-module isomorphism of $\Lambda_\mu \otimes_{\Lambda_\mu} C_*(\widehat{X}, \widehat{Y})$ with $C_*(\widehat{X}, \widehat{Y})$ to recover the homology of the covering space: $H_*(X; \Lambda_\mu) \cong H_*(\widehat{X})$. As alluded to in Remark 2.2.1, if $L = K_1 \cup \ldots \cup K_n$ is an oriented link and $\psi \colon \pi_1(X_L) \to \mathbb{Z}$ is the homomorphism $\gamma \mapsto \ell k(\gamma, K_1) + \ldots + \ell k(\gamma, K_n)$, then $H_1(X_L; \Lambda)$ is the Alexander module of L.

Denote by $x \mapsto \overline{x}$ the involution on Λ_μ induced by $\overline{t_i} = t_i^{-1}$ and let Q_μ be the field of fractions of Λ_μ. Given a ring $\Lambda_\mu \subset S \subset Q_\mu$ and a left (resp. right) S-module V, we denote by $\overline{V}$ the right (resp. left) S-module that has the same underlying additive group as V, but for which the action by S on V is precomposed with the involution on S. In our setting, $\overline{C_k(\widehat{X}, \widehat{Y})}$ and M are now both right Λ_μ-modules.

Definition 5. The cochain complex $C^*(X, Y; M) := \operatorname{Hom}_{\Lambda_\mu}(\overline{C_*(\widehat{X}, \widehat{Y})}, M)$ of left R-modules will be called the *twisted cochain complex* of the pair (X, Y) with coefficients in M. The corresponding cohomology left R-modules $H^*(X, Y; M)$ will be called the *twisted cohomology* modules of the pair (X, Y) with coefficients in M.

Contrasting with Example 2.4.1, when $M = \Lambda_\mu$, the group $H^*(X; \Lambda_\mu)$ coincides with the cohomology group $H^*_c(\widehat{X})$ *with compact support*, see [60, Section 5.2]. Despite this difference, one still has Poincaré duality isomorphisms [24, 136, 157], see Section 5.5 for a more thorough discussion.

Theorem 2.4.2. *Let M be a (R, Λ_μ)-bimodule and let N be a compact orientable n-manifold. There are Poincaré duality isomorphisms $H_k(N, \partial N; M) \cong H^{n-k}(N; M)$ and $H_k(N; M) \cong H^{n-k}(N, \partial N; M)$.*

The universal coefficient theorem also requires some additional work since R is not necessarily a principal ideal domain. While Section 5.4 will deal with the general setting, this section in concerned with a particular case, namely $M = Q_\mu/\Lambda_\mu$ with its natural $(\Lambda_\mu, \Lambda_\mu)$-bimodule structure. The next lemma will be proved in Section 5.4.

Lemma 2.4.3. *If $\psi \colon \pi_1(X) \to \mathbb{Z}^\mu$ induces an involution preserving ring homomorphism $\mathbb{Z}[\pi_1(X)] \to \Lambda_\mu$, then the following map is a well defined chain isomorphism of left Λ_μ-modules:*

$$\begin{aligned} \operatorname{Hom}_{\Lambda_\mu}(\overline{C_*(\widehat{X})}, Q_\mu/\Lambda_\mu) &\to \overline{\operatorname{Hom}_{\Lambda_\mu}(\Lambda_\mu \otimes_{\Lambda_\mu} C_*(\widehat{X}), Q_\mu/\Lambda_\mu)} \\ f &\mapsto \Big((n \otimes \sigma) \mapsto n\overline{f(\sigma)} \Big). \end{aligned}$$

Furthermore, there is a well defined evaluation homomomorphism of left Λ_μ-modules:

$$H_i(\overline{\operatorname{Hom}_{\Lambda_\mu}(\Lambda_\mu \otimes_{\Lambda_\mu} C_*(\widehat{X}), Q_\mu/\Lambda_\mu)}) \to \overline{\operatorname{Hom}_{\Lambda_\mu}(H_i(X; \Lambda_\mu), Q_\mu/\Lambda_\mu)}.$$

Combining the homomorphisms of Lemma 2.4.3, one obtains an *evaluation map*

$$\mathrm{ev} \colon H^i(X; Q_\mu/\Lambda_\mu) \to \overline{\operatorname{Hom}_{\Lambda}(H_i(X; \Lambda_\mu), Q_\mu/\Lambda_\mu)}$$

which has no reason of being an isomorphism. For the remainder of this section, we shall only concern ourselves with the case $\mu = 1$.

2.4.2 The Blanchfield pairing of a knot: definition and properties

Given an oriented knot K, we denote by X_K its exterior and by $\psi\colon \pi_1(X_K) \to \mathbb{Z}$ the abelianization homomorphism $\gamma \mapsto \ell k(\gamma, K)$. As we saw in Section 2.2, this data gives rise to the Alexander module $H_1(X_K;\Lambda) \cong H_1(\widehat{X}_K)$ of K. We start with a well known fact, see e.g. [83].

Lemma 2.4.4. *The inclusion induced map $i\colon H_1(X_K;\Lambda) \to H_1(X_K, \partial X_K;\Lambda)$ is an isomorphism.*

Proof. The restriction of the cover $\widehat{X}_K \to X_K$ to the torus ∂X_K consists of the infinite cylinder $S^1 \times \mathbb{R}$: indeed the longitude of K is mapped by ψ to zero while the meridian of K is mapped by ψ to one. It follows that the inclusion induced map $H_i(\partial X_K;\Lambda) \to H_i(X_K;\Lambda)$ is zero when $i = 1$ (since lifts of the longitude bound in $\widehat{X}_K$) and an isomorphism when $i = 0$ (since both spaces are connected). The long exact sequence of the pair $(X_K, \partial X_K)$ thus takes the form $H_1(\partial X_K;\Lambda) \xrightarrow{0} H_1(X_K;\Lambda) \xrightarrow{i} H_1(X_K, \partial X_K;\Lambda) \to H_0(\partial X_K;\Lambda) \xrightarrow{\cong} H_0(X_K;\Lambda)$, and the result now follows by exactness. □

Observe that a homomorphism $f\colon M \to N$ of (Λ, Λ)-bimodules induces both maps $H_*(X;M) \to H_*(X;N)$ and $H^*(X;M) \to H^*(X;N)$ via the respective chain maps $m \otimes x \mapsto f(m) \otimes x$ and $\varphi \mapsto f \circ \varphi$. Moreover, short exact sequences of coefficients induce long exact sequences on twisted homology and cohomology. The resulting boundary homomorphisms are often called *Bockstein homomorphisms.*

Lemma 2.4.5. *Consider the short exact sequence $0 \to \Lambda \to Q \to Q/\Lambda \to 0$ of (Λ, Λ)-bimodules. The Bockstein homomorphism $BS\colon H^1(X_K;\Lambda) \to H^2(X_K;Q/\Lambda)$ is an isomorphism.*

Proof. Recall from Remark 2.2.2 that the Alexander module of a knot is Λ-torsion. Since Q is the field of fractions of Λ, one has $H_1(X_K;Q) \cong Q \otimes_\Lambda H_1(X_K;\Lambda) = 0$, see Section 2.5 for details. Using Lemma 2.4.4 and the Poincaré duality isomorphisms of Theorem 2.4.2, it follows that $H^2(X_K;Q)$ vanishes. Similarly, the universal coefficient theorem implies that $H^1(X_K;Q) = 0$. The result now follows from the long exact sequence in cohomology which arises from the short exact sequence $0 \to \Lambda \to Q \to Q/\Lambda \to 0$ of coefficients. □

We delay the proof of the following lemma to Section 2.5.

Lemma 2.4.6. *The evaluation map $\mathrm{ev}\colon H^1(X_K;Q/\Lambda) \to \overline{\mathrm{Hom}_\Lambda(H_1(X_K;\Lambda), Q/\Lambda)}$ described at the end of Subsection 2.4.1 is an isomorphism.*

Denote by Ω the following composition

$$H_1(X_K;\Lambda) \xrightarrow{i} H_1(X_K, \partial X_K;\Lambda) \xrightarrow{\mathrm{PD}} H^2(X_K;\Lambda) \xrightarrow{\mathrm{BS}^{-1}} H^1(X_K;Q/\Lambda) \xrightarrow{\mathrm{ev}} \overline{\mathrm{Hom}_\Lambda(H_1(X_K;\Lambda), Q/\Lambda)}$$

of the four left Λ-linear homomorphisms defined as follows. The first homomorphism is given by Lemma 2.4.4, the second homomorphism is Poincaré duality, the third homomorphism is the inverse of the Bockstein isomorphism described in Lemma 2.4.5 and the fourth homomorphism is described in Lemma 2.4.6.

The main definition of this section is the following.

Definition 6. The *Blanchfield pairing* of an oriented knot K is the map

$$\mathrm{Bl}(K)\colon H_1(X_K;\Lambda) \times H_1(X_K;\Lambda) \to Q/\Lambda$$

defined by $\mathrm{Bl}(K)(x,y) = \Omega(y)(x)$.

To describe some elementary properties of the Blanchfield pairing, we briefly introduce some terminology. Given a torsion Λ-module H, a *linking pairing* is a pairing $b\colon H \times H \to Q/\Lambda$. Such a pairing is *sesquilinear* if $b(px,qy) = pb(x,y)\overline{q}$ and *nonsingular* if its *adjoint* (i.e. the Λ-linear map $H \to \overline{\mathrm{Hom}_\Lambda(H,Q/\Lambda)}$) is an isomorphism. A sesquilinear pairing is *Hermitian* if $b(y,x) = \overline{b(x,y)}$ for all $x,y \in H$. Two linking pairings b,b' are *isometric* if there is a Λ-linear homomorphism $\psi\colon H \to H'$ satisfying $b'(\psi(x),\psi(y)) = b(x,y)$. The following proposition is well known [18, 76, 133].

Proposition 2.4.7. *The Blanchfield pairing of a knot is a nonsingular and Hermitian linking pairing.*

Proof. Combining Lemma 2.4.4, Theorem 5.5.1, Lemma 2.4.5 and Lemma 2.4.6, one observes that Ω is an isomorphism of left Λ-modules. Consequently, the pairing is nonsingular. Given x,y in $H_1(X_K;\Lambda)$ and p,q in Λ, since $\Omega(y)$ is left Λ-linear, one certainly has $\mathrm{Bl}(K)(px,y) = \Omega(y)(px) = p\Omega(y)(x) = p\,\mathrm{Bl}(K)(x,y)$. Since Ω is left Λ-linear and using the involuted module structure on $\overline{\mathrm{Hom}_\Lambda(H_1(X_K;\Lambda),Q/\Lambda)}$, one obtains $\mathrm{Bl}(K)(x,qy) = \Omega(qy)(x) = (q\cdot\Omega)(x) = \Omega(y)(x)\overline{q} = \mathrm{Bl}(K)(x,y)\overline{q}$. The proof that $\mathrm{Bl}(K)$ is Hermitian can be found in [133, Proposition 3.8]. □

The following result was originally observed by Kearton [96] (see also [110]) before being proved by Friedl-Powell [76]. We will also give a different proof of this fact in Chapter 6.

Theorem 2.4.8. *If K is a knot and if A is a Seifert matrix for K of size $2g$, then the Blanchfield pairing of K is isometric to the linking pairing*

$$\begin{array}{rcl} \Lambda^{2g}/(tA-A^T)\Lambda^{2g} \times \Lambda^{2g}/(tA-A^T)\Lambda^{2g} & \to & Q/\Lambda \\ (a,b) & \mapsto & a^T(t-1)(A-tA^T)^{-1}\overline{b}. \end{array}$$

In particular, Theorem 2.4.8 gives another proof that the Blanchfield pairing is Hermitian, see [76, Corollary 1.6]. Note that conflicting statements of Theorem 2.4.8 can be found in the literature. For instance, Kearton [96] writes $(t-1)(tA-A^T)^{-1}$ instead of $(t-1)(A-tA^T)^{-1}$ which leads to some trouble, as explained in [76, Section 1.2].

2.5 Reviewing some homological algebra

In this section, we review some homological algebra and prove Lemma 2.4.6. A standard reference is [158].

Let R be a ring and let **RMod** denote the category of (left) R-modules and R-linear maps. An R-module is *projective* if the covariant functor $\mathrm{Hom}_R(P,-)\colon \mathbf{RMod} \to \mathbf{Ab}$ is exact. Here **Ab** denotes the category of abelian groups and group homomorphisms. A *projective resolution* for an R-module A is an exact sequence $\ldots \to P_1 \to P_0 \to A \to 0$ of R-modules in which each P_i is projective. An R-module always possesses a projective resolution [158, Lemma 2.2.5]. Given two R-modules A and B, the *Ext group* $\mathrm{Ext}^n_R(A,B)$ is defined as follows. Pick a projective resolution $\ldots \to P_1 \to P_0 \to A \to 0$ for A and define the abelian group $\mathrm{Ext}^n_R(A,B)$ as the n-th homology group of the chain complex

$$\ldots \leftarrow \mathrm{Hom}_R(P_1,B) \leftarrow \mathrm{Hom}_R(P_0,B) \leftarrow 0$$

of abelian groups. It turns out that $\mathrm{Ext}^n_R(A,B)$ is independent of the chosen resolution. Furthermore, $\mathrm{Ext}^n_R(-,B)\colon \mathbf{RMod} \to \mathbf{Ab}$ is a contravariant functor while $\mathrm{Ext}^n_R(A,-)\colon \mathbf{RMod} \to \mathbf{Ab}$ is a covariant functor [158, Chapter 3].

Example 2.5.1. For any projective R-module A and any $n > 0$, one has $\mathrm{Ext}^n_R(A,B) = 0$ for all B: one can use the projective resolution $0 \to A \xrightarrow{\mathrm{id}} A \to 0$. If R is field, the same proof shows that $\mathrm{Ext}^n_R(A,B)$ vanishes for all $k > 0$, indeed free modules are projective. If R is principal, then $\mathrm{Ext}^n_R(A,B)$ vanishes for all $n > 1$ [158, Theorem 4.2.11]. Passing to a slightly more involved example, take $R = \Lambda$, $B = \Lambda$ with the Λ-module structure induced by multiplication and $A = \mathbb{Z}$ with the Λ-module structure $\lambda \cdot n = \varepsilon(\lambda)n$, where $\varepsilon\colon \Lambda \to \mathbb{Z}$ is the augmentation map. Since $\mathbb{Z}$ admits the free resolution $\Lambda \xrightarrow{t-1} \Lambda \to \mathbb{Z} \to 0$, it follows that $\mathrm{Ext}^n_\Lambda(\mathbb{Z};\Lambda) = 0$ for $n \geq 2$.

The abelian groups $\mathrm{Ext}^n_R(A,B)$ can also be computed using resolutions of B. An R-module I is *injective* if the contravariant functor $\mathrm{Hom}_R(-,I)$ is exact. An *injective resolution* for an R-module B is an exact sequence $0 \to B \to I_0 \to I_1 \to \ldots$ for which each I_i is injective. An R-module always possesses a injective resolution [158, Exercise 2.3.5]. Given two R-modules A and B, the Ext functor $\mathrm{Ext}^n_R(A,B)$ can be computed by picking an injective resolution $0 \to B \to I_0 \to I_1 \to \ldots$ for B and computing the n-th homology group of the chain complex $0 \to \mathrm{Hom}_R(A,I_0) \to \mathrm{Hom}_R(A,I_1) \to \ldots$, see [158, Theorem 2.7.6].

Example 2.5.2. Reasoning as in Example 2.5.1, $\mathrm{Ext}^n_R(-,B)$ vanishes whenever B is injective and $n > 0$. In particular if $Q(R)$ is the field of fractions of an integral domain R, then $Q(R)$ is injective over R and thus $\mathrm{Ext}^n_R(-,Q(R))$ vanishes for all $n > 0$. Note also that if R is a principal ideal domain, then $Q(R)/R$ is injective [158, Section 2.3]. For instance if $R = \mathbb{Q}[t^{\pm 1}]$, one has $\mathrm{Ext}^n_R(A,Q(R)/R) = 0$ for all A and for all $n > 0$. On the other hand, this is not the case for $R = \Lambda$.

Given a right R-module A and a left R-module B, define the *Tor group* $\mathrm{Tor}^R_n(A,B)$ by considering a right R-module projective resolution $\ldots \to P_1 \to P_0 \to A \to 0$ for A, applying the functor $- \otimes_R B$ and taking the n-th homology group of the chain complex $\ldots \to P_1 \otimes_R B \to P_0 \otimes_R B \to 0$ of abelian groups. Once again, $\mathrm{Tor}^R_n(A,B)$ is independent of the chosen projective resolution and yields a bifunctor $\mathrm{Tor}^R_n\colon \mathbf{ModR} \times \mathbf{Rmod} \to \mathbf{Ab}$.

Remark 2.5.3. Note that $\mathrm{Tor}^R_n(A,B)$ can also be computed by taking a projective resolution for B instead of a projective resolution for A [158, Theorem 2.7.2]. In particular, arguing as in Example 2.5.1, $\mathrm{Tor}^R_n(A,B)$ vanishes if either A or B is projective or if R is a field. More generally, a left R-module B is *flat* over R if $- \otimes_R B$ is exact, while a right R-module A is flat over R if $A \otimes_R -$ is exact. Projective modules are flat but the converse is not true. If either A or B is flat over R, then $\mathrm{Tor}^R_n(A,B) = 0$ for all $n \geq 1$. In particular this conclusion holds if A or B is $Q(R)$, the field of fractions of an integral domain R, or more generally, for any localization of R.

Up to now, the Tor and Ext functors took values in the category of abelian groups. However, if we work with bimodules, the Tor and Ext groups are naturally endowed with module structures.

Remark 2.5.4. If B is a (R,T)-bimodule, then $\mathrm{Hom}_{\textit{left-}R}(-,B)$ defines a functor from the category **RMod** of left R-modules to the category **ModT** of right T-modules. We shall argue that the Ext functors then define bifunctors $\mathbf{RMod}^{op} \times \mathbf{RModT} \to \mathbf{ModT}$. This becomes apparent by following the construction of Ext step by step. First, one takes a resolution $0 \to B \to I_0 \to I_1 \to \ldots$, where each I_k is a (R,T)-bimodule which is injective as an R-module. Given a left R-module A, it then follows that each $\mathrm{Hom}_{\textit{left-}R}(A, I_k)$ is a right T-module and the same goes for $\mathrm{Ext}^n_R(A,B)$. Naturally, the underlying abelian group can also be computed by taking a projective resolution of A, as previously described. The same remark applies to the Tor functors which define bifunctors $\mathbf{TModR} \times \mathbf{RMod} \to \mathbf{TMod}$.

Next, we collect two *universal coefficient spectral sequences* (UCSS). We refer to Weibel's monograph [158, Chapter 5] regarding background material on spectral sequences. However, we caution the reader that Weibel's conventions regarding the degrees of the differentials in cohomological spectral sequences differ from those used here.

Let R and T be rings, let C_* be a chain complex of free left R-modules and let S be a (R,T)-bimodule. This way $H_*(\mathrm{Hom}_{\text{left-}R}(C_*, S))$ is a right T-module, as are $\mathrm{Hom}_{\text{left-}R}(H_*(C_*), S)$ and more generally $\mathrm{Ext}^q_R(H_*(C_*), S)$. We refer to [110, Theorem 2.3] for the proof of the following theorem.

Theorem 2.5.5. *Let R, T be rings, let C_* be a chain complex of free left R-modules and let S be a (R,T)-bimodule. There exists a spectral sequence*

1. *converging to $H_*(\mathrm{Hom}_{\text{left-}R}(C_*, S))$,*
2. *with $E_2^{p,q} \cong \mathrm{Ext}^q_R(H_p(C_*), S)$,*
3. *with differentials $d_r^{p,q}$ of degree $(1-r, r)$.*

More specifically, there is a filtration

$$0 \subset F_0^n \subset F_1^n \subset \cdots \subset F_n^n = H_n(\mathrm{Hom}_{\text{left-}R}(C_*, S))$$

with $F_p^n / F_{p-1}^n \cong E_\infty^{p,n-p}$. All objects and isomorphisms are as right *T-modules. Furthermore, the edge homomorphism $H_*(\mathrm{Hom}_{\text{left-}R}(C_*, S)) \to \mathrm{Hom}_{\text{left-}R}(H_*(C_*), S)$ is the usual evaluation.*

Because of the involuted structures we are dealing with, we need to slightly modify Theorem 2.5.5. Indeed, we wish to apply this spectral sequence to the case where N is a (R,Λ)-bimodule in order to study the homomorphism of *left* R-modules

$$H_i(\overline{\mathrm{Hom}_{\text{left-}R}(N \otimes_\Lambda C_*(\widehat{X}), S)}) \to \overline{\mathrm{Hom}_{\text{left-}R}(H_*(X;N), S)}.$$

Remark 2.5.6. Let A be a left R-module and let B be a (R,T)-bimodule. Because of the involuted module structures, we need to modify the module structure on $\mathrm{Ext}^n_R(A,B)$. Let us follow the construction of $\mathrm{Ext}^n_R(A,B)$ step by step, starting from an injective resolution $0 \to B \to I_0 \to I_1 \to \ldots$ of (R,T)-bimodules for B. We wish to define $\mathrm{Ext}^n_R(A,B)$ as the left T-module obtained by taking the homology of the chain complex $0 \to \overline{\mathrm{Hom}_{\text{left-R}}(A, I_0)} \to \overline{\mathrm{Hom}_{\text{left-R}}(A, I_1)} \to \ldots$ of left T-modules. In other words, we want to convert $\mathrm{Ext}(-, B)\colon \mathbf{RMod} \to \mathbf{ModT}$ to a functor $\mathbf{RMod} \to \mathbf{TMod}$. Following the construction, one observes that the result is nothing but $\overline{\mathrm{Ext}_R(A, B)}$.

From now on, we will keep Remark 2.5.6 in mind but we will often slightly abuse notations by simply writing $\mathrm{Ext}_R(A, B)$, omitting the subtleties involving involuted module structures. Nevertheless, here is the version of Theorem 2.5.5 which fits our set up.

Theorem 2.5.7. *Let R, T be rings, let C_* be a chain complex of free left R-modules and let S be a (R, T)-bimodule. There exists a spectral sequence*

1. *converging to $H_*(\overline{\mathrm{Hom}_{\text{left-}R}(C_*, S)})$,*
2. *with $E_2^{p,q} \cong \mathrm{Ext}^q_R(H_p(C_*), S)$,*
3. *with differentials d^r of degree $(1 - r, r)$.*

More specifically, there is a filtration

$$0 \subset F_0^n \subset F_1^n \subset \cdots \subset F_n^n = H_n(\overline{\mathrm{Hom}_{\text{left-}R}(C_*, S)})$$

with $F_p^n / F_{p-1}^n \cong E_\infty^{p,n-p}$. All objects and isomorphisms are as left T*-modules. Furthermore, the edge homomorphism $H_*(\overline{\mathrm{Hom}_{\text{left-}R}(C_*, S)}) \to \overline{\mathrm{Hom}_{\text{left-}R}(H_*(C_*), S)}$ is the usual evaluation.*

Using Theorem 2.5.7, we can now prove Lemma 2.4.6, whose statement we recall for the reader's convenience.

Lemma 2.4.6. *The map $\mathrm{ev}\colon H^1(X_K; Q/\Lambda) \to \overline{\mathrm{Hom}_\Lambda(H_1(X_K; \Lambda), Q/\Lambda)}$ is an isomorphism.*

Proof. Recall from Lemma 2.4.3 that ev was defined as the composition of the isomorphism $H^1(X_K; Q/\Lambda) \to H_1(\overline{\mathrm{Hom}_{\text{left-}\Lambda}(C_*(\widehat{X}_K), Q/\Lambda)})$ with the map $H_1(\overline{\mathrm{Hom}_{\text{left-}\Lambda}(C_*(\widehat{X}_K), Q/\Lambda)}) \to \overline{\mathrm{Hom}_{\text{left-}\Lambda}(H_1(X_K; \Lambda), Q/\Lambda)}$. Consequently, in order to show that ev is an isomorphism, we shall use the UCSS of Theorem 2.5.7 and argue that this latter map is an isomorphism.

Since $H_p(X_K; \Lambda) = 0$ for $p \geq 2$ [112, Chapter 11], all the differentials on the second page will vanish if we manage to show that $E_{0,q} = \mathrm{Ext}^q_\Lambda(H_0(X_K; \Lambda), Q/\Lambda) = 0$ for all $q \geq 2$. Temporarily assuming this fact, the spectral sequence collapses at the second page. Using the notations of Theorem 2.5.7, we get a filtration of $H_1(\overline{\mathrm{Hom}_{\text{left-}\Lambda}(C_*(\widehat{X}_K), Q/\Lambda)}) = F_1^1 \supset F_0^1 \supset 0$ with $F_0^1 = E_\infty^{0,1} = E_2^{0,1} = \mathrm{Ext}^1_\Lambda(H_0(X_K; \Lambda), Q/\Lambda)$ and $F_1^1/F_0^1 = E_\infty^{1,0} = \overline{\mathrm{Hom}_{\text{left-}\Lambda}(H_1(X_K; \Lambda), Q/\Lambda)}$. Consequently, if we prove that $\mathrm{Ext}^q_\Lambda(H_0(X_K; \Lambda), Q/\Lambda)$ vanishes for $q \geq 1$, then we are done.

The trivial Λ-module $H_0(X_K; \Lambda) = \mathbb{Z}$ admits the length one free resolution $\Lambda \xrightarrow{t-1} \Lambda \to \mathbb{Z} \to 0$. One deduces that $\mathrm{Ext}^q_\Lambda(\mathbb{Z}; \Lambda) = 0$ for $q \geq 2$, see Example 2.5.1. The short exact sequence of coefficients $0 \to \Lambda \to Q \to Q/\Lambda \to 0$ induces a long exact sequence of Ext groups, with $\ldots \to \mathrm{Ext}^q_\Lambda(\mathbb{Z}; Q) \to \mathrm{Ext}^q_\Lambda(\mathbb{Z}; Q/\Lambda) \to \mathrm{Ext}^{q+1}_\Lambda(\mathbb{Z}; \Lambda) \to \mathrm{Ext}^{q+1}_\Lambda(\mathbb{Z}; Q) \ldots$ being the portion of interest. Recall from Example 2.5.2 that Q is injective over Λ. Thus, assuming that $q \geq 1$ and using the above discussion, the two outer terms of the exact sequence vanish and the lemma follows. $\square$

We now recall the corresponding UCSS in homology, see e.g. [87, Chapter 2].

Theorem 2.5.8. *Let R, Z be rings, let C_* be a chain complex of free left Z-modules and let S be a (R, Z)-bimodule. There exists a spectral sequence*

1. *converging to $H_*(S \otimes_Z C_*)$,*
2. *with $E^2_{p,q} \cong Tor^Z_p(H_q(C_*), S)$,*
3. *with differentials d_r of degree $(-r, r-1)$.*

More specifically, there is a filtration

$$0 \subset F^0_n \subset F^1_n \subset \cdots \subset F^n_n = H_n(R \otimes_Z C_*)$$

with $F^p_n / F^{p-1}_n \cong E^\infty_{p,n-p}$. All objects and isomorphisms are as left R-modules.

We conclude with a well known fact whose proof uses the terminology of this section.

Remark 2.5.9. Let R be a Noetherian integral domain, let $Q(R)$ be the field of fractions of R and let C_* be a chain complex of finitely generated free right R-modules. We claim that $H_*(\mathrm{Hom}_{\text{right-}R}(C_*, Q(R)))$ is canonically isomorphic to $Q(R) \otimes_R H_*(\mathrm{Hom}_{\text{right-}R}(C_*, R))$. Since $Q(R)$ is flat over R (recall Remark 2.5.3), our assumptions ensure that the chain complexes $\mathrm{Hom}_{\text{right-}R}(C_*, Q(R)) \cong \mathrm{Hom}_{\text{right-}R}(C_*, Q(R) \otimes_R R)$ and $Q(R) \otimes_R \mathrm{Hom}_R(C_*, R)$ are canonically chain isomorphic. The claim now follows from the universal coefficient theorem in homology: indeed, as we mentioned in Example 2.5.2, $Q(R)$ is injective over R.

Chapter 3

Invariants of colored links

3.1 Introduction

Given an oriented link L, Chapter 2 dealt with invariants which were extracted from the first homology of the total linking cover of the link exterior X_L. If L is a knot, this cover corresponds to the kernel of the abelianization homomorphism. On the other hand, for a link of more than one component, this homomorphism is obtained by first abelianizing and then mapping each meridian to one. In particular, there are various other abelian covers one could consider instead of the total linking cover. Of particular interest are the multivariable invariants which are extracted from the cover corresponding to the kernel of the abelianization homomorphism $\pi_1(X_L) \to H_1(X_L)$. However, in order to obtain results which are valid both in the one variable and in the multivariable case, we shall use *colored links.*

Definition 7. A *μ-colored link* is an oriented link L in S^3 whose components are partitioned into μ sublinks $L_1 \cup \cdots \cup L_\mu$.

Thus, in the case $\mu = 1$, a colored link is just an oriented link, while if L has n components and $\mu = n$, then L is an ordered link. Mimicking the one variable case, the *Alexander polynomial* $\Delta_L(t_1, \ldots, t_\mu)$ of a μ-colored link L can be extracted from the (colored) *Alexander module* of L. Thus, when $\mu = 1$, one recovers the one variable Alexander polynomial, while the case $\mu = n$ is precisely the multivariable Alexander polynomial defined by Alexander [1], see also [69]. Pursuing the analogy, $\Delta_L(t_1, \ldots, t_\mu)$ can be computed both with Fox calculus and by using a generalization of Seifert surfaces.

We focus on this latter approach and therefore begin by reviewing C-complexes in Section 3.2. These objects which were first introduced by Cooper [54] and further studied by Cimasoni [36] lead to 2^μ *generalized Seifert matrices* which generalize the usual Seifert matrix. After giving the definition of the Alexander polynomial in Section 3.3, we shall see how generalized Seifert matrices yield presentation matrices for the Alexander module of the colored link L.

Since Seifert matrices also lead to the Levine-Tristram signature, one might expect the generalized Seifert matries to produce a *multivariable signature.* This is indeed the case and the construction, which is due to Cooper [54] for links of two component and to Cimasoni-Florens [41] in the general case, will be discussed in Section 3.4.

3.2 C-complexes and generalized Seifert matrices

As we mentioned above, the theory of C-complexes and generalized Seifert matrices should be viewed as a generalization of the theory of Seifert surfaces which was reviewed in Subsection 2.2.3. References for this short section include [36, 41, 54].

Definition 8. A *C-complex* for a μ-colored link $L = L_1 \cup \ldots \cup L_\mu$ is a union $F = F_1 \cup \cdots \cup F_\mu$ of surfaces in S^3 such that:

1. for all i, F_i is a Seifert surface for L_i (possibly disconnected, but with no closed components);
2. for all $i \neq j$, $F_i \cap F_j$ is either empty or a union of clasps (see Figure 3.1);
3. for all i, j, k pairwise distinct, $F_i \cap F_j \cap F_k$ is empty.

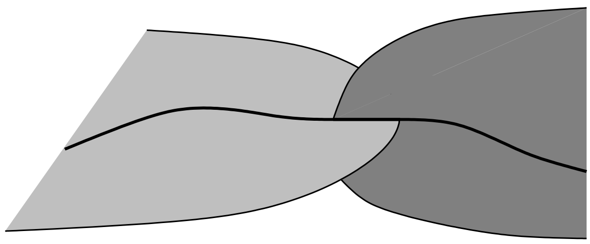

Figure 3.1: A clasp intersection.

The existence of a C-complex for an arbitrary colored link was established by Cimasoni [36, Lemma 1]. Note that in the case $\mu = 1$, a C-complex for L is nothing but a Seifert surface for the oriented link L. Moreover, as in the one variable case, there is a sequence of moves relating any two C-complexes of a given colored link [36, Lemma 3], see also [41, Lemma 2.2].

Given a sequence $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\mu)$ of ± 1's, let $i^\varepsilon \colon H_1(F) \to H_1(S^3 \setminus F)$ be defined as follows. Any homology class in $H_1(F)$ can be represented by an oriented cycle x which behaves as illustrated in Figure 3.1 whenever crossing a clasp. Then, define $i^\varepsilon([x])$ as the class of the 1-cycle obtained by pushing x in the ε_i-normal direction off F_i for $i = 1, \ldots, \mu$. Finally, consider the bilinear form

$$\begin{aligned} \alpha^\varepsilon \colon H_1(F) \times H_1(F) &\to \mathbb{Z} \\ (x, y) &\mapsto \ell k(i^\varepsilon(x), y)\,, \end{aligned}$$

where ℓk denotes the linking number. Fix a basis of $H_1(F)$ and denote by A^ε the resulting matrix of α^ε. This produces 2^μ matrices which are called *generalized Seifert matrices* for the colored link L. In fact, since for all ε, $A^{-\varepsilon}$ is equal to $(A^\varepsilon)^T$, we only need to compute $2^{\mu-1}$ of these matrices. For instance in the case $\mu = 1$, the matrix A^- is nothing but a Seifert matrix for the oriented link, while A^+ is its transpose.

Example 3.2.1. Consider the 2-colored link L depicted in the left-hand side of Figure 3.2. A computation involving the C-complex depicted on the right-hand side of Figure 3.2 gives $A^{++} = A^{--} = (-1)$ and $A^{+-} = A^{-+} = (0)$.

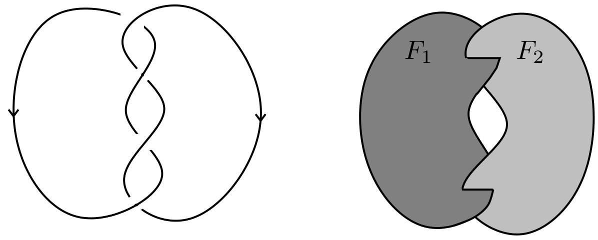

Figure 3.2: The colored link of Example 3.4.1 and a natural C-complex for it.

3.3 The multivariable Alexander polynomial

In this section, we review the Alexander polynomial of a colored link. The organization is the following: Subsection 3.3.1 deals with the definition of the Alexander polynomial of a colored link. In Subsection 3.3.2, we shall see that the Alexander polynomial can be computed using Fox calculus, while Subsection 3.3.3 will explain its relation with generalized Seifert matrices. References for this section include [36, 41, 95, 150].

3.3.1 Definition and properties

Let $L = L_1 \cup \cdots \cup L_\mu$ be a colored link and let X_L denote its exterior. The epimorphism $\psi\colon \pi_1(X_L) \to \mathbb{Z}^\mu$ given by $\gamma \mapsto (\ell k(\gamma, L_1), \ldots, \ell k(\gamma, L_\mu))$ induces a regular $\mathbb{Z}^\mu$-covering $\widehat{X}_L \to X_L$. The homology of $\widehat{X}_L$ is naturally a module over $\Lambda_\mu = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$. The Λ_μ-module $H_1(\widehat{X}_L)$ is called the *Alexander module* of the colored link L.

Remark 3.3.1. In several references, the Alexander module refers to the relative homology group $H_1(\widehat{X}_L, \widehat{x})$, where $\widehat{x} \subset \widehat{X}_L$ is the fiber over a point $x \in X_L$ [57, 95]. As in Chapter 2, we shall sometimes denote the Alexander module by $H_1(X_L; \Lambda_\mu)$ instead of $H_1(\widehat{X}_L)$. Chapter 5.2 will shed more light on this notation although Subsection 2.4.1 already gave several hints.

Since Λ_μ is a Noetherian factorial domain, the Alexander module can once again be studied via its elementary ideals $E_r(L) = E_r(H_1(\widehat{X}_L))$ and in particular via its order, see Subsection 2.2.1 for the definition of these notions.

Definition 9. The *Alexander polynomial* $\Delta_L(t_1, \ldots, t_\mu)$ of a μ-colored link L is the order of its Alexander module.

As in the one variable case, the Alexander polynomial is only well-defined up to units of Λ_μ, that is, up to multiplication by powers of $\pm t_i$. More generally, the polynomial $\Delta_L^r(t_1, \ldots, t_\mu) := \Delta_r(H_1(\widehat{X}_L))$ is called the *r-th Alexander polynomial* of the colored link L and we shall denote by Δ_L^{tor} the first non-vanishing Alexander polynomial of L.

Remark 3.3.2. Note that Δ_L^{tor} can also be defined as the order of $TH_1(\widehat{X}_L)$, the torsion submodule of the Alexander module. This observation is a consequence of the following algebraic fact. Given an R-module M, for $k \geq 0$, let $\Delta^{(k)}(M)$ denote the greatest common divisor of all $(m-k) \times (m-k)$ minors of an $m \times n$ presentation matrix of M. Denoting by r the rank of M and by $\text{Tor}_R(M)$ its torsion submodule, it is known that the order of $\text{Tor}_R(M)$ is equal to $\Delta^{(r)}(M)$, see [150, Lemma 4.9].

When $\mu = 1$, a colored link is nothing but an oriented link, and Definition 9 recovers the one variable Alexander polynomial discussed in Section 2.2. On the other hand, when $\mu = n$ and ψ is the abelianization map, then Δ_L is the multivariable Alexander polynomial of the ordered link L, as defined by Alexander [1], see also [69]. The relationship between these different polynomials is expressed in the following proposition, see [95, Proposition 7.3.10] for a proof.

Proposition 3.3.3. *Given a $(\mu+1)$-colored link $L = L_1 \cup \cdots \cup L_\mu \cup L_{\mu+1}$, consider the μ-colored link $L' = L'_1 \cup \cdots \cup L'_\mu$ obtained by setting $L'_i = L_i$ for $i < \mu$ and $L'_\mu = L_\mu \cup L_{\mu+1}$, then*

$$\Delta_{L'}(t_1, \dots, t_\mu) \doteq \begin{cases} (t_1 - 1)\Delta_L(t_1, \dots, t_1, t_1) & \text{if } \mu = 1, \\ \Delta_L(t_1, \dots, t_\mu, t_\mu). & \text{if } \mu > 1. \end{cases}$$

As in the one variable case, the Alexander polynomial is known to be symmetric in the sense that $\Delta_L(t_1^{-1}, \dots, t_\mu^{-1}) \doteq \Delta_L(t_1, \dots, t_\mu)$. On the other hand, the Alexander polynomial of an n-component ordered link $L = K_1 \cup \ldots \cup K_n$ satisfies the additional *Torres formula*

$$\Delta_L(t_1, \dots, t_{n-1}, 1) \doteq \begin{cases} \frac{t_1^{l_{1,2}} - 1}{t_1 - 1}\Delta_{L'}(t_1) & \text{if } n = 2 \\ (t_1^{l_{1,\mu}} \cdots t_{n-1}^{l_{n-1,n}} - 1)\Delta_{L'}(t_1, \dots, t_{n-1}) & \text{if } n > 2, \end{cases} \tag{3.1}$$

where L' is the sublink $K_1 \cup \cdots \cup K_{n-1}$ and $l_{i,j}$ is a shorthand for $\ell k(K_i, K_j)$, see [146]. Finally, the behavior of the multivariable Alexander polynomial under mirror image, orientation reversal, band sums, satellites and various other splices is well understood [36, 37, 65, 95, 149].

3.3.2 Computation via Fox calculus

Fox calculus provides a straightforward algorithm to compute the Alexander polynomial of a colored link. Given a presentation $\langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ of the group $\pi_1(X_L)$, denote by $\mathrm{pr} : \mathbb{Z}[F_n] \to \mathbb{Z}[\pi_1(X_L)]$ the ring homomorphism induced by the projection. The *Fox matrix* A whose (i,j) coefficient is $\psi(\mathrm{pr}(\frac{\partial r_i}{\partial x_j}))$ provides a presentation matrix for the module $H_1(\widehat{X}_L, \widehat{x})$ which we encountered in Remark 3.3.1, see [95, Theorem 7.1.5 and Exercise 7.3.11].

Remark 3.3.4. If $\langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle$ is a Wirtinger presentation for $\pi_1(X_L)$, we denote by A_i the matrix obtained from the Fox matrix A by deleting its i-th column. Since the presentation has deficiency one, it can be checked that for each i and each $\mu \geq 2$, one has $(\psi(x_i) - 1)\Delta_L(t_1, \dots, t_\mu) \doteq \det(A_i)$ [95, Lemma 7.3.2 and Exercise 7.3.11]. This fact will also be discussed in Chapter 10, see in particular Proposition 10.3.2. We also refer to Remark 2.2.3 for the case $\mu = 1$.

Here is a practical application of Remark 3.3.4.

Example 3.3.5. A Wirtinger presentation for the link depicted in Figure 2.1 has generators x_1, x_2, x_3, x_4 and relators $x_2 x_1 x_2^{-1} x_1^{-1}, x_2 x_4 x_2^{-1} x_4^{-1}, x_4 x_2 x_4 x_3^{-1}$. Performing Fox calculus and applying ψ yields the matrix

$$\begin{bmatrix} t_2 - 1 & 1 - t_1 & 0 & 0 \\ 0 & 1 - t_3 & 0 & t_2 - 1 \\ 0 & t_2 & -1 & 1 - t_2 \end{bmatrix},$$

and consequently Remark 3.3.4 implies that the multivariable Alexander polynomial of L is $\Delta_L(t_1, t_2, t_3) \doteq t_2 - 1$. Applying Proposition 3.3.3, one recovers the results of Examples 2.2.4 and 2.2.6, namely that the one variable Alexander polynomial of L is given by $\Delta_L(t) \doteq (t-1)^2$.

3.3.3 Computation via C-complexes

Recall from Theorem 2.2.5 that a presentation matrix for the Alexander module of an oriented link can be obtained by means of *connected* Seifert surfaces. Following [41, Section 3], we recall the corresponding statement in the multivariable case.

Definition 10. A C-complex $F = F_1 \cup \cdots \cup F_\mu$ is *totally connected* if each F_i is connected and $F_i \cap F_j \neq \emptyset$ for all $i \neq j$.

Using the moves described in [36, Lemma 1], any link admits a totally connected C-complex. In order to give a presentation matrix of the Alexander module in terms of generalized Seifert matrices, we begin by describing the first homology of a totally connected C-complex F. For $i = 1, \dots, \mu$ choose some interior point v_i of $F_i \setminus \bigcup_{j \neq i} F_i \cap F_j$. Given a clasp in $F_i \cap F_j$ with $i < j$, consider an oriented edge in $F_i \cup F_j$ joining v_i and v_j and passing through this single clasp. This leads to a collection of oriented edges $\{e_{ij}^1, \dots, e_{ij}^{c(i,j)}\}$, where $c(i,j)$ denotes the number of clasps in $F_i \cap F_j$. Let $K_{ij} \subset F_i \cup F_j$ denote the graph given by the union of these edges. Finally, let K_μ be the complete graph with vertices $\{v_i\}_{1 \leq i \leq \mu}$ and edges $\{e_{ij}^1\}_{1 \leq i < j \leq \mu}$.

The proof of the following lemma can be found in [41, Lemma 3.1].

Lemma 3.3.6. *The first homology group of a totally connected C-complex $F = F_1 \cup \cdots \cup F_\mu$ can be described as*

$$H_1(F) \cong \bigoplus_{1 \leq i \leq \mu} H_1(F_i) \oplus \bigoplus_{1 \leq i < j \leq \mu} H_1(K_{ij}) \oplus H_1(K_\mu).$$

Furthermore, a basis of $H_1(K_{ij})$ is given by $(\beta_{ij}^\ell)_{1 \leq \ell \leq c(i,j)-1}$, where $\beta_{ij}^\ell = e_{ij}^\ell - e_{ij}^{\ell+1}$. Finally, a basis of $H_1(K_\mu)$ is given by $(\gamma_{1ij})_{2 \leq i < j \leq \mu}$, where $\gamma_{ijk} = e_{ij}^1 - e_{ki}^1 + e_{jk}^1$.

A presentation matrix for the Alexander module of a colored link can be obtained by means of totally connected C-complexes [41, Theorem 3.2].

Theorem 3.3.7. *Let $L = L_1 \cup \cdots \cup L_\mu$ be a colored link, and consider a totally connected C-complex $F_1 \cup \cdots \cup F_\mu$ for L. Let $\alpha \colon H_1(F) \otimes_{\mathbb{Z}} \Lambda_\mu \to H_1(S^3 \setminus F) \otimes_{\mathbb{Z}} \Lambda_\mu$ be the homomorphism of Λ_μ-modules given by*

$$\alpha = \sum_{\varepsilon} \varepsilon_1 \dots \varepsilon_\mu t_1^{\frac{\varepsilon_1 + 1}{2}} \cdots t_\mu^{\frac{\varepsilon_\mu + 1}{2}} i^\varepsilon,$$

where the sum is on all sequences $\varepsilon = (\varepsilon_1, \dots \varepsilon_\mu)$ of ± 1's. Now set

$$\widehat{H} = \bigoplus_{i \leq i \leq \mu} H_1(F_i) \oplus \bigoplus_{1 \leq i < j \leq \mu} H_1(K_{ij}) \oplus \bigoplus_{1 \leq i < j < k \leq \mu} \mathbb{Z}\gamma_{ijk}$$

and let $\widehat{\alpha}$ be given by

1. *$\prod_{n \neq i} (t_n - 1)^{-1} \alpha$ on $H_1(F_i)$ for $1 \leq i \leq \mu$,*

2. $\prod_{n\neq i,j}(t_n-1)^{-1}\alpha$ *on* $H_1(K_{ij})$ *for* $1\leq i<j\leq\mu$*,*

3. $\prod_{n\neq i,j,k}(t_n-1)^{-1}\alpha(\gamma_{ijk})$ *for* $1\leq i<j<k\leq\mu$*.*

Then the Alexander module $H_1(\widehat{X}_L)$ *admits the finite presentation*

$$\widehat{H}\otimes_{\mathbb{Z}}\Lambda_\mu\overset{\widehat{\alpha}}{\to}H_1(S^3\setminus F)\otimes_{\mathbb{Z}}\Lambda_\mu\to H_1(\widehat{X}_L)\to 0.$$

For $\mu=1$, Theorem 3.3.7 recovers Theorem 2.2.5, while the case $\mu=2$ was first obtained by Cooper [54]. As mentioned in [41, Corollary 3.3], it also follows that the Alexander polynomial of a 2-colored link L with a (totally) connected C-complex $F=F_1\cup F_2$ is given by

$$\Delta_L(t_1,t_2)\doteq(t_1-1)^{-2g_2}(t_2-1)^{-2g_1}\det(t_1t_2A^{--}-t_1A^{-+}-t_2A^{+-}+A^{++}),$$

where g_i denotes the genus of F_i for $i=1,2$.

Remark 3.3.8. Observe that the statement given in Theorem 3.3.7 involves the linear form $i^\varepsilon\colon H_1(F)\to H_1(S^3\setminus F)$ and not the bilinear form $\alpha^\varepsilon\colon H_1(F)\times H_1(F)\to\mathbb{Z}$. Fixing Alexander-dual bases for $H_1(F)$ and $H_1(S^3\setminus F)$ and recalling that $\alpha^\varepsilon(x,y)=\ell k(i^\varepsilon(x),y)$, the generalized Seifert matrix A^ε is the *transpose* of the matrix representing i^ε.

Next, we show that the condition on the C-complex F is necessary for Theorem 3.3.7 to hold.

Example 3.3.9. The link L depicted on the left hand side of Figure 3.3 admits a contractible C-complex depicted on the right hand side of Figure 3.3. If this C-complex could be used to compute the Alexander module, then $\Delta_L(t_1,t_2,t_3)$ would identically be equal to 1 which is not the correct result, see Example 3.3.5.

Figure 3.3: On the left-hand side: the link L which appears in Example 3.3.5. On the right-hand side: a connected C-complex for L which is not totally connected.

In order to produce a totally connected C-complex, we artificially add a pair of canceling clasps as shown in Figure 3.4. A computation now shows that generalized Seifert matrices are given by $A^{+++}=A^{+-+}=\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$, $A^{++-}=\left[\begin{smallmatrix}0&0\\-1&1\end{smallmatrix}\right]$ and $A^{-++}=\left[\begin{smallmatrix}0&-1\\0&0\end{smallmatrix}\right]$. It follows from Theorem 3.3.7 that a presentation matrix for the Alexander module is given by $\left[\begin{smallmatrix}0&t_1(1-t_2)\\t_3(1-t_2)&t_3-t_1t_2\end{smallmatrix}\right]\left[\begin{smallmatrix}(t_2-1)^{-1}&0\\0&1\end{smallmatrix}\right]=\left[\begin{smallmatrix}0&t_1(1-t_2)\\t_3&t_3-t_1t_2\end{smallmatrix}\right]$ and consequently $\Delta_L(t_1,t_2,t_3)\doteq t_1t_3(1-t_2)$, which agrees with Example 3.3.5.

In general, if one only desires to compute the multivariable Alexander polynomial of a link without obtaining a presentation matrix for the Alexander module (but assuming the C-complex $F=F_1\cup\ldots\cup F_\mu$ to be connected), Cimasoni [36] provides the formula

$$\Delta_L(t_1,\ldots,t_\mu)\doteq\prod_{i=1}^{\mu}(1-t_i)^{\chi(F\setminus F_i)-1}\det\left(\sum_\varepsilon\varepsilon_1\cdots\varepsilon_\mu\,t_1^{\frac{1-\varepsilon_1}{2}}\cdots t_\mu^{\frac{1-\varepsilon_\mu}{2}}A^\varepsilon\right),\tag{3.2}$$

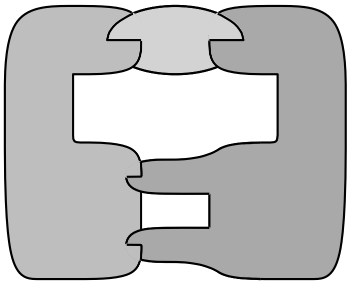

Figure 3.4: A totally connected C-complex for the link depicted in Figure 3.3.

where $\chi(F \setminus F_i)$ denotes the Euler characteristic of $F \setminus F_i$. In the case at hand, one can now use the contractible C-complex depicted in Figure 3.4: indeed (3.2) immediately gives $\Delta_L(t_1, t_2, t_3) \doteq 1 - t_2$ as desired.

Summarizing, Theorem 3.3.7 implies that for μ-colored links with $\mu \leq 3$ colors, C-complexes lead to *square* presentation matrices for the Alexander module. On the other hand, observe that for $\mu \geq 4$, Theorem 3.3.7 does not produce a square presentation matrix. In fact, this is not a surprise:

Remark 3.3.10. Crowell and Strauss proved that if an ordered link has $n \geq 4$ components and if $\Delta_L \neq 0$, then its Alexander module does not admit any square presentation matrix [57], see also [88, Lemma 2.2] and [95, Proposition 7.3.9]. This result applies to colored links as well.

Despite Remark 3.3.10, it is possible to use square matrices to compute the Alexander invariants up to some indeterminacy. More precisely let Λ_S denote the localization of the ring Λ_μ with respect to the multiplicative system generated by $\{t_i - 1\}_{1 \leq i \leq \mu}$. The following result is also due to Cimasoni-Florens [41, Corollary 3.6].

Corollary 3.3.11. *Let L be a μ-colored link. Consider a totally connected C-complex F for L. Then the matrix*

$$A(t_1, \ldots, t_\mu) = \sum_{\varepsilon} \varepsilon_1 \cdots \varepsilon_\mu t_1^{\frac{1-\varepsilon_1}{2}} \cdots t_\mu^{\frac{1-\varepsilon_\mu}{2}} A^{\varepsilon}$$

is a presentation matrix for the Λ_S-module $\Lambda_S \otimes_{\Lambda_\mu} H_1(\widehat{X}_L)$. In particular, for all r, there are non-negative integers m_i such that the following equality holds in Λ_μ :

$$\prod_{i=1}^{\mu} (1 - t_i)^{m_i} \Delta_r(L) \doteq \Delta_r(A(t_1, \ldots, t_\mu)).$$

In fact, Corollary 3.3.11 can even be improved upon, but for that, we first need to introduce some terminology.

Definition 11. Given a C-complex, the associated *C-complex matrix* is the Λ_μ-valued square matrix

$$H \ := \sum_{\varepsilon} \prod_{i=1}^{\mu} (1 - t_i^{\varepsilon_i})\, A^{\varepsilon},$$

where the sum is on all sequences $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\mu)$ of ±1's.

For instance in the case $\mu = 1$, a C-complex is a Seifert surface and a C-complex matrix is given by $(1-t)A^T + (1-t^{-1})A$, where A is a Seifert matrix. In general, note that up to multiples of $(t_i - 1)$, the transpose H^T of the C-complex matrix H is equal to the matrix $A(t_1, \ldots, t_\mu)$ described in Corollary 3.3.11. We now refine this latter corollary.

Remark 3.3.12. We make two claims whose proofs will be similar: firstly, the torsion submodule $TH_1(X_L; \Lambda_S)$ of $H_1(X_L; \Lambda_S)$ is isomorphic to the torsion submodule of $\Lambda_S^n / H^T \Lambda_S^n$ for *any* C-complex matrix H of size n; secondly, Corollary 3.3.11 can be improved by replacing *totally connected C-complex* by *connected C-complex*.

We start by proving the first claim. As explained in [41, p. 1230] (see also [36]), if F and F' are two C-complexes for isotopic links, then the corresponding C-complex matrices H and H' are related by a finite number of the following two moves:

$$H \mapsto H \oplus (0) \quad \text{and} \quad H \mapsto \begin{pmatrix} H & \xi & 0 \\ \xi^* & \lambda & \alpha \\ 0 & \overline{\alpha} & 0 \end{pmatrix},$$

with α a unit of Λ_S. In the first case, the Λ_S-module $\Lambda_S^n / H^T \Lambda_S^n$ picks up a free rank 1 factor, so its torsion submodule is left unchanged. In the second case, since α is a unit in Λ_S, one can assume via the appropriate base change that H is transformed into $H \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and the module itself is left unchanged. To prove the second assertion, note that the transformation $H \mapsto H \oplus (0)$ only arises when one wishes to connect two disconnected components of a C-complex, see [41, page 1230]. Consequently, turning a connected C-complex into a totally connected one will not change the Λ_S-Alexander modules presented by the corresponding C-complex matrices.

We conclude with an application of Remark 3.3.12 which shall be used again in Chapter 6.

Remark 3.3.13. We claim that if a colored link L has non-zero Alexander polynomial Δ_L, then $\det(H)$ is non-zero for any C-complex matrix H. Arguing as in Remark 3.3.12, it can be shown that if $\Delta_L \neq 0$, then L cannot admit a disconnected C-complex. Thus, any C-complex matrix H must come from a connected C-complex. Remark 3.3.12 now implies that H^T presents $H_1(X_L; \Lambda_S)$. Since H^T and the matrix $A(t_1, \ldots, t_\mu)$ coincide up to $(t_i - 1)$ factors, Corollary 3.3.11 implies that $\det(H) = \det(H^T)$ is equal to Δ_L up to units of Λ_S. Since we assumed the latter polynomial to be non-zero, the claim follows.

3.4 Multivariable signatures

In this section, we review the multivariable signature and nullity developed by Cooper [54, 55] and Cimasoni-Florens [41]. In Subsection 3.4.1, we shall recall the definition of the multivariable signature and its various properties before moving on to its 4-dimensional interpretation via branched covers in Subsection 3.4.2. A different 4-dimensional set-up will be studied in Chapter 8. The main reference for this section is [41].

3.4.1 Definition and properties

Let F be a C-complex for a colored link L. Denote by $\beta_0(F)$ the number of components of F, fix a basis of $H_1(F)$ and denote by A^ε the resulting generalized Seifert matrices. Since for all ε, $A^{-\varepsilon}$ is equal to $(A^{\varepsilon})^T$, one can check that for any $\omega = (\omega_1, \ldots, \omega_\mu)$ in the μ-dimensional torus $\mathbb{T}^\mu$, the matrix

$$H(\omega) = \sum_{\varepsilon} \prod_{i=1}^{\mu} (1 - \overline{\omega}_i^{\varepsilon_i})\, A^{\varepsilon}$$

is Hermitian. Observe that when $\mu = 1$, $H(\omega)$ is nothing but the matrix $(1-\omega)A^T + (1-\overline{\omega})A$ which we encountered in Section 2.3. Furthermore, note that $H(\omega)$ is the evaluation of the C-complex matrix H of Definition 11 at $(t_1, \ldots, t_\mu) = (\omega_1, \ldots, \omega_\mu)$.

Definition 12. The *multivariable signature and nullity* of the μ-colored link L are the functions

$$\sigma_L, \eta_L \colon \mathbb{T}^\mu \to \mathbb{Z}\,,$$

where $\sigma_L(\omega)$ is the signature of the Hermitian matrix $H(\omega)$, and $\eta_L(\omega) := \operatorname{nullity}(H(\omega)) + \beta_0(F) - 1$.

Note that in the case $\mu = 1$, one recovers the Levine-Tristram signature and nullity described in Section 2.3. For arbitrary μ, these multivariable generalizations are well-defined (i.e. independent of the choice of the C-complex) [41, Theorem 2.1], and satisfy all the properties of the Levine-Tristram signature and nullity, generalized from oriented links to colored links [41, Section 2.3].

We start with an example involving the now familiar link depicted in Figure 3.2.

Example 3.4.1. Recall from Example 3.2.1 that for the link L and the C-complex depicted in Figure 3.2, generalized Seifert matrices are given by $A^{++} = A^{--} = (-1)$ and $A^{+-} = A^{-+} = (0)$. This leads to $\sigma_L(\omega_1, \omega_2) = -\operatorname{sgn}(\operatorname{Re}[(1-\omega_1)(1-\omega_2)])$, see Figure 3.5. On the other hand, the nullity $\eta_L(\omega_1, \omega_2)$ is equal to one or zero, according to whether $\Delta_L(\omega_1, \omega_2) = \omega_1\omega_2 + 1$ vanishes or not.

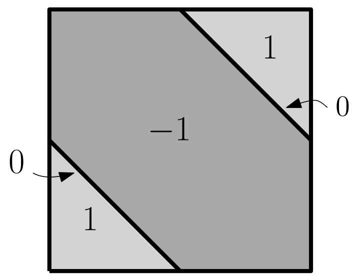

Figure 3.5: The values of σ_L for the link L of Example 3.4.1.

As an easier example, we consider the positive Hopf link.

Example 3.4.2. View the positive Hopf link L as a 2-colored link. Since L admits a contractible C-complex, its multivariable signature vanishes identically. On the other hand, using Seifert matrices, the Levine-Tristram signature of L, viewed as an oriented link, is -1 at $\omega \neq 1$ and vanishes at $\omega = 1$.

The behavior of the multivariable signature and nullity under mirror image, orientation reversal and band sums is well understood [41, Section 2.3]. We refer to [61] for the behavior of σ_L under various satellite operations. The multivariable signature also satisfies local relations which are described in [41, Section 5].

The following proposition collects some other properties of the multivariable signature.

Proposition 3.4.3. *Let L be a μ-colored link.*

1. *For all $\omega \in S^1 \setminus \{1\}$, the Levine-Tristram signature and nullity of the underlying oriented link can be recovered from their multivariable analogue as follows:*

$$\sigma_L(\omega) = \sigma_L(\omega, \dots, \omega) - \sum_{i<j} \ell k(L_i, L_j) \quad \textit{and} \quad \eta_L(\omega) = \eta_L(\omega, \dots, \omega)\,.$$

2. *The multivariable signature is constant on the connected components of the complement in $\mathbb{T}^\mu_* := (S^1 \setminus \{1\})^\mu$ of the zeroes of $\Delta_L(t_1, \dots, t_\mu)$.*

Proof. The first statement is a direct corollary of [41, Proposition 2.5], while the proof of the second statement can be found in [41, Corollary 4.2]. □

Note that in both assertions of Proposition 3.4.3, the assumption on ω is necessary: this is illustrated by Examples 3.4.2 and 3.4.1 respectively. However, the upshot of Proposition 3.4.3 is that many values of the multivariable signature can be recovered from the knowledge of the Levine-Tristram signature, the zero locus of the multivariable Alexander polynomial and the linking matrix. If needed, the remaining unknown values may be computed using C-complexes.

Example 3.4.4. We return to the oriented link L of Example 3.4.1. Using Section 2.3, it can be checked that the Levine-Tristram signature of L is given by $\sigma_L(\omega) = -1$ for $|\mathrm{Arg}(\omega)| < \pi/2$ and $\sigma_L(\omega) = -3$ for $|\mathrm{Arg}(\omega)| > \pi/2$. We now view L as a 2-colored link and use Proposition 3.4.3 to compute several values of $\sigma_L(\omega_1, \omega_2)$. Since the linking number of the components of L is equal to 2, the first statement of Proposition 3.4.3 implies that the values of the multivariable signature on the diagonal are $\sigma_L(\omega, \omega) = 1$ for $|\mathrm{Arg}(\omega)| < \pi/2$ and $\sigma_L(\omega, \omega) = -1$ for $|\mathrm{Arg}(\omega)| > \pi/2$. Since the multivariable Alexander polynomial of L is equal to $\Delta_L(t_1, t_2) = 1 + t_1 t_2$, the second statement of Proposition 3.4.3 implies that these values actually determine the multivariable signature outside of the intersection of the zero locus of $\Delta_L(t_1, t_2)$ with $\mathbb{T}^2_*$, see Figure 3.5.

Conversely, the first point of Proposition 3.4.3 can also be used to efficiently compute Levine-Tristram signatures.

Example 3.4.5. Since the C-complex for the 3-component link L depicted in Figure 3.3 is contractible, $\sigma_L(\omega_1, \omega_2, \omega_3)$ vanishes identically. As the sum of the linking numbers is -2, it follows from Proposition 3.4.3 that the Levine-Tristram signature of L is identically equal to 2, which is the result obtained in Example 2.3.1.

As in the one variable case, for appropriate $\omega \in \mathbb{T}^\mu$, the integers $\sigma_L(\omega)$ and $\eta_L(\omega)$ are related to 4-dimensional topology. In order to give precise statements, we introduce some terminology and notation.

Definition 13. Two colored links L and J with m components are said to be smoothly (resp. topologically) *concordant* if there exists a smooth (resp. locally flat) proper embedding of a collection of disjoint annuli $A_1, \dots, A_m$ in $S^3 \times [0, 1]$, such that for all i, A_i is a concordance between the components of L and J of the same color.

As in the one variable case, the multivariable signatures are not concordance invariants at every point of $\mathbb{T}^\mu$. Denote by $\mathbb{T}^\mu_P$ the dense subset of $\mathbb{T}^\mu$ constituted by the elements $\omega = (\omega_1, \dots, \omega_\mu)$ which satisfy the following condition: there exists a prime p such that for all i, the order of ω_i is a power of p. The following theorem is due to Cimasoni-Florens [41, Theorem 7.1].

Theorem 3.4.6. *$\sigma_L(\omega)$ and $\eta_L(\omega)$ are smooth concordance invariants for all $\omega \in \mathbb{T}^\mu_P$.*

Theorem 3.4.6 will be generalized in Chapter 8, see Corollary 8.4.6. Next, in order to state the generalized Murasugi-Tristram inequality (recall Theorem 2.3.2), we make the following definition.

Definition 14. A *colored bounding surface* F for a μ-colored link L consists of a union $F_1 \cup \dots \cup F_\mu \subset D^4$ of properly embedded, locally flat, oriented surfaces which only intersect each other transversally in double points and whose boundary is L. A colored bounding surface is *smooth* (respectively *well-connected*) if each surface F_i is smoothly embedded (respectively connected). It is also understood that a colored bounding surface has no closed components.

We wish to emphasize that the following theorem, which is due to Cimasoni-Florens [41, Theorem 7.2], deals with colored bounding surfaces which are both smooth and well-connected. In Chapter 8, we shall drop these extra assumptions.

Theorem 3.4.7. *Suppose that a μ-colored link L admits a smooth well-connected colored bounding surface F. Set $\beta_1 = \sum_{i=1}^{\mu} \operatorname{rk} H_1(F_i)$, and let c be the number of double points of F. Then for all $\omega \in \mathbb{T}^\mu_P$, the following inequality holds:*

$$|\sigma_L(\omega)| + |\eta_L(\omega) - \mu + 1| \leq \beta_1 + c.$$

Setting $\mu = 1$, Theorem 3.4.7 resembles the classical Murasugi-Tristram inequality from Theorem 2.3.2; however the surfaces are now required to be connected. On the other hand, for links with pairwise vanishing linking numbers, Theorem 3.4.7 recovers a theorem proved by Florens [68]. Note that Theorem 3.4.7 will be generalized in Chapter 8, see Corollary 8.4.7.

3.4.2 Multivariable signatures and branched covers of the 4-ball

Just as the Levine-Tristram signature, the multivariable signature also admits a 4-dimensional interpretation, recall Subsection 2.3.2. Following [41, Section 6.1], we outline this construction which relies on branched covers. We refer to Chapter 8 for a set-up involving twisted homology.

Recall from Definition 14 that a colored bounding surface F consists of a union $F_1 \cup \dots \cup F_\mu \subset D^4$ of properly embedded surfaces which only intersect each other transversally in double points. Writing νF for the union of some choice of tubular neighborhoods of the F_i, we refer to $W_F := D^4 \setminus \nu F$ as the *exterior* of F in D^4.

Although the 4-dimensional interpretation of the multivariable signature involves colored bounding surfaces, we start out with a well known result in which the coloring is irrelevant.

Lemma 3.4.8. *Let $F = F_1 \cup \ldots \cup F_m$ be a union of properly embedded, locally flat, compact, connected and oriented surfaces $F_i \subset D^4$ which only intersect each other transversally in double points. Then the abelian group $H_1(W_F)$ is freely generated by the meridians of the components F_i.*

Proof. Pick a small ball B_x around each intersection point x of F. Note that $W_F = D^4 \setminus (\bigcup_x B_x \cup \bigcup_i \nu F_i^\circ)$, where the surface F_i° is F_i with little discs removed around the intersection points. Consider the Mayer-Vietoris sequence of $D^4 \setminus \bigcup_x B_x = W_F \cup \bigcup_i \nu F_i^\circ$ with $\mathbb{Z}$-coefficients:

$$\dots \to H_2\Big(D^4 \setminus \bigcup_x B_x\Big) \to H_1\Big(\bigcup_i F_i^\circ \times S^1\Big) \to H_1\Big(\bigcup_i F_i^\circ \times D^2\Big) \oplus H_1(W_F) \to H_1\Big(D^4 \setminus \bigcup_x B_x\Big) \to \dots .$$

This simplifies via the Künneth theorem to $0 \to H_1\big(\bigcup_i \{pt_i\} \times S^1; \mathbb{Z}\big) \to H_1(W_F) \to 0$, where $pt_i \in F_i$. $\square$

Let F be a smooth well-connected colored bounding surface for a μ-colored link L. Using Lemma 3.4.8, given positive integers $k_1, \dots, k_\mu$, the canonical projection $H_1(W_F) \cong \mathbb{Z}^\mu \longrightarrow G := C_{k_1} \times \dots \times C_{k_\mu}$ induces a finite abelian covering of W_F. Using this G-cover, one can then form a G-cover $\overline{W}_F$ of D^4 branched along F, see [41, page 1253] for details. For roots of unity $\omega_1, \dots, \omega_\mu$ of respective orders $k_1, \dots, k_\mu$, denote by $\chi : G \longrightarrow \mathbb{C}^*$ the character sending each generator t_i of C_{k_i} to ω_i. Since $H_2(\overline{W}_F; \mathbb{C})$ is endowed with the structure of a $\mathbb{C}[G]$-module, we can then form the complex vector space

$$H_2(\overline{W}_F)_\chi = \{x \in H_2(\overline{W}_F; \mathbb{C}) \mid gx = \chi(g)x \text{ for all } g \in G\}.$$

Restricting the intersection form on $H_2(\overline{W}_F; \mathbb{C})$ to $H_2(\overline{W}_F, \mathbb{C})_\chi$ produces a Hermitian pairing whose signature we call the *ω-signature* of $\overline{W}_F$ and which we denote by $\sigma_\omega(\overline{W}_F)$, see Subsections 17.1.1 and 17.1.2 for details. Cimasoni and Florens [41, Theorem 6.1] give the following 4-dimensional interpretation of the multivariable signature.

Theorem 3.4.9. *Let $k_1, \dots, k_\mu$ be positive integers and set $G := C_{k_1} \times \dots \times C_{k_\mu}$. Assume that a μ-colored link L bounds a smooth well-connected colored bounding surface $F \subset D^4$ and let $\overline{W}_F$ be the G-fold cover of D^4 branched along F. Then, for any $\omega = (\omega_1, \dots, \omega_\mu) \in \mathbb{T}^\mu_*$ of orders $k_1, \dots, k_\mu$, the following equality holds:*

$$\sigma_L(\omega) = \sigma_\omega(\overline{W}_F).$$

Setting $\mu = 1$, Theorem 3.4.9 reduces to the corresponding statement for the Levine-Tristram signature which was stated in Theorem 2.3.3. This 4-dimensional interpretation of the signature will be generalized in Chapter 8, see in particular Theorem 8.1.1.

Remark 3.4.10. Looking back at Remark 2.3.4, it is tempting to view the multivariable signature as a rho invariant. Given an m-component algebraically split link L (i.e. with pairwise vanishing linking numbers) and $\alpha := e^{2\pi i/q}$, consider the character $\chi \colon H_1(M_L) \to \mathbb{C}$ which maps the meridian of the i-th component of L to α^{n_i} for some integer n_i which is coprime to q. Viewing L as an m-colored link and setting $\omega = (\alpha_1^{n_1}, \dots, \alpha_m^{n_m})$, [41, Theorem 6.7] implies that $\sigma_L(\omega) = \rho(M_L, \chi)$. For arbitrary links, Enrico Toffoli plans to relate the multivariable signature to an invariant defined by Kirk and Lesch [100, 101].

3.5 Further remarks

This section is devoted to collecting some results which shall be used in Chapters 4, 6 and 8; it is organized as follows. Subsection 3.5.1 relates the multivariable nullity to the rank of the Alexander module, while Subsection 3.5.2 applies the theory of C-complexes to boundary links.

3.5.1 Nullities and the rank of the Alexander module

The *Alexander nullity* $\beta(L)$ of a colored link L is defined as the rank of the Λ_μ-module $H_1(\widehat{X}_L)$. Clearly, $\beta(L)$ vanishes if and only if the Alexander polynomial of L is not identically zero. It is also known that the nullity of an n-component link is at most $n-1$ [95, Corollary 7.3.13]. Furthermore, $\beta(L)$ is a concordance invariant [95, Theorem 12.3.12]. The following proposition which appeared in [40] relates the Alexander nullity to the multivariable nullity, recall Definition 12.

Proposition 3.5.1. *The rank of the Alexander module of a colored link L is the minimal value of its nullity: $\beta(L) = \min\{\eta_L(\omega) \mid \omega \in \mathbb{T}^\mu_*\}$.*

Proof. Let $E_r(L)$ denote the ideal of Λ_μ generated by the $(m-r)\times(m-r)$ minors of an $m\times n$ presentation matrix of $H_1(\widehat{X}_L)$. Also, let Σ_r denote the set consisting of all $\omega \in \mathbb{T}^\mu_*$ such that $p(\omega) = 0$ for each $p \in E_{r-1}(L)$. Observe that the sets Σ_r form a decreasing sequence. By [41, Theorem 4.1], $\Sigma_r \setminus \Sigma_{r+1}$ consists of all ω such that $\eta_L(\omega) = r$. Therefore, if β denotes the minimal value of η_L, we have $\Sigma_r = \mathbb{T}^\mu_*$ for all $r \leq \beta$ and $\Sigma_{\beta+1} \neq \mathbb{T}^\mu_*$. Hence, β is equal to the maximal r such that $E_{r-1}(L) = 0$, which is nothing but the rank of the Alexander module. □

3.5.2 Boundary links

A *boundary link* is a link whose components bound disjoint Seifert surfaces. These links, which have their origin in high dimensional link theory, appear frequently in the literature since their behavior mimicks the one of knots, see for instance [45, 87, 137] and the references therein. Given an n-component boundary link $L = K_1 \cup \ldots \cup K_n$, let $F_1, \ldots, F_n$ be disjoint Seifert surfaces for the K_i and set $F = F_1 \sqcup \ldots \sqcup F_n$. Pushing curves off this *boundary Seifert surface* in the negative normal direction produces a homomorphism $i^-\colon H_1(F) \to H_1(S^3 \setminus F)$. The assignment $\theta(x,y) := \ell k(i^-(x), y)$ gives rise to a pairing on $H_1(F)$ and to a *boundary Seifert matrix for L*, see [105, p.670] for details. Since $H_1(F)$ decomposes as the direct sum of the $H_1(F_i)$, the restriction of θ to $H_1(F_i) \times H_1(F_j)$ produces matrices A_{ij}. For $i \neq j$, these matrices satisfy $A_{ij} = A_{ji}^T$, while A_{ii} is nothing but a Seifert matrix for the knot K_i.

We now argue that the generalized Seifert matrices for a boundary link L can be read off from the blocks of any boundary Seifert matrix for L.

Remark 3.5.2. Let F be a boundary Seifert surface which gives rise to a boundary Seifert matrix A for L. View F as a C-complex for L, and denote by A^ε_{ij} the restriction of the generalized Seifert matrix A^ε to $H_1(F_i) \times H_1(F_j)$. If $i \neq j$, since L is a boundary link, A^ε_{ij} is independent of ε and is equal to the block A_{ij} of the boundary Seifert matrix A. Similarly, for each ε with $\varepsilon_i = -1$, the restriction of A^ε to $H_1(F_i) \times H_1(F_i)$ is equal to the block A_{ii}.

Let $F = F_1 \sqcup \ldots \sqcup F_n$ be a boundary Seifert surface for a boundary link L. Let g_i denote the genus of F_i, let I_k be the $k \times k$ identity matrix, let τ be the block diagonal matrix whose diagonal blocks are $t_1 I_{2g_1}, t_2 I_{2g_2}, \ldots, t_n I_{2g_n}$ and set $g := g_1 + \ldots + g_n$. The following result appears to be known, see for instance [87, 88].

Proposition 3.5.3. *Let $L = K_1 \cup \ldots \cup K_n$ be a boundary link. Assume that A is a boundary Seifert matrix for L of size $2g$. Then $(A\tau - A^T)$ presents the torsion submodule $TH_1(X_L; \Lambda_S)$ of the Alexander module of L.*

Proof. Recall from Remark 3.3.12 that for any size m C-complex matrix H (see Definition 11), the torsion submodule of $\Lambda_S^m/H^T\Lambda_S^m$ is isomorphic to $TH_1(X_L;\Lambda_S)$. Consequently, we start by using the boundary Seifert matrices to compute such a C-complex matrix. Using Remark 3.5.2, , we observe that $H_i := (1-t_i)A_{ii}^T + (1-t_i^{-1})A_{ii}$ is a C-complex matrix for the knot K_i. Let u denote $\prod_{j=1}^n(1-t_j)$. Combining the definition of a C-complex matrix with Remark 3.5.2 shows that a C-complex matrix H for L is given by

$$H = \begin{bmatrix} u\overline{u}(1-t_1)^{-1}(1-t_1^{-1})^{-1}H_1 & u\overline{u}A_{12} & \dots & u\overline{u}A_{1n} \\ u\overline{u}A_{21} & u\overline{u}(1-t_2)^{-1}(1-t_2^{-1})^{-1}H_2 & \dots & u\overline{u}A_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ u\overline{u}A_{n1} & u\overline{u}A_{n2} & \dots & u\overline{u}(1-t_n)^{-1}(1-t_n^{-1})^{-1}H_n \end{bmatrix}.$$

Since $H_i = (1-t_i^{-1})(A_{ii} - t_iA_{ii}^T)$, the diagonal blocks of H can be rewritten as $u\overline{u}(1-t_i)^{-1}(A_{ii}-t_iA_{ii}^T)$. Using the equation $A_{ij} = A_{ji}^T$, we see that a C-complex matrix for L is given by

$$H = u\overline{u}(I_{2g}-\tau)^{-1}(A-\tau A^T). \tag{3.3}$$

It follows that $H^T = u\overline{u}(A^T - A\tau)(I_{2g}-\tau)^{-1}$. Since u is a unit of Λ_S and $(I_{2g}-\tau)^{-1}$ is an automorphism of Λ_S^{2g}, the module presented by H^T is canonically isomorphic to the module presented by $A\tau - A^T$.

Figure 3.6: Performing a trivial band clasping of the links L' and L''.

Since we now know that $TH_1(X_L;\Lambda_S)$ is isomorphic to the torsion submodule of $\Lambda_S^{2g}/H^T\Lambda_S^{2g}$, the proposition will follow once we prove that $\Lambda_S^{2g}/H^T\Lambda_S^{2g}$ is Λ_S-torsion. Band clasp trivially F_1 with F_2, F_2 with F_3, F_i with F_{i+1} and finally F_{n-1} with F_n (see Figure 3.6). The result is a link L' which bounds a connected C-complex F' for which the associated C-complex matrix is also H. Since L has pairwise vanishing linking numbers, L' does not. Consequently, using the Torres formula (3.1), the Alexander polynomial of L' is non-zero and thus its Alexander module is torsion. As we saw in Remark 3.3.12, if a C-complex matrix H arises from a *connected* C-complex, H^T presents the Λ_S-localized Alexander module. Thus H^T presents the torsion module $H_1(X_{L'};\Lambda_S)$ and the claim follows. $\square$

Note that Proposition 3.5.3 and its proof also imply that the first non-vanishing Alexander polynomial of L is equal to the determinant of $A\tau - A^T$ up to $t_i - 1$ factors; here we implicitly used Remark 3.3.2. We conclude this section with another known result which admits a quick proof using C-complexes.

Proposition 3.5.4. *The Alexander nullity of an n-component boundary link is equal to $n-1$.*

Proof. As we mentioned in Subsection 3.5.1 the Alexander nullity of any n-component link is at most $n-1$. Now, pick a boundary Seifert surface F for the boundary link L and let H denote the resulting C-complex matrix. Turn F into a connected C-complex F' for L by adding $n-1$ clasps using the move described in Figure 3.7. Using Remark 3.3.12, the Λ_S-Alexander module of L is presented by the resulting C-complex matrix H' for F'. Since each move adds a (0)-summand to H, the Λ_S-rank of H' is at least $n-1$. Since Q_μ is the field of fractions of both Λ_μ and Λ_S, we note that $\beta(L)$ is equal to the Λ_S-rank of $H_1(X_L;\Lambda_S)$. The conclusion follows by combining these observations. $\square$

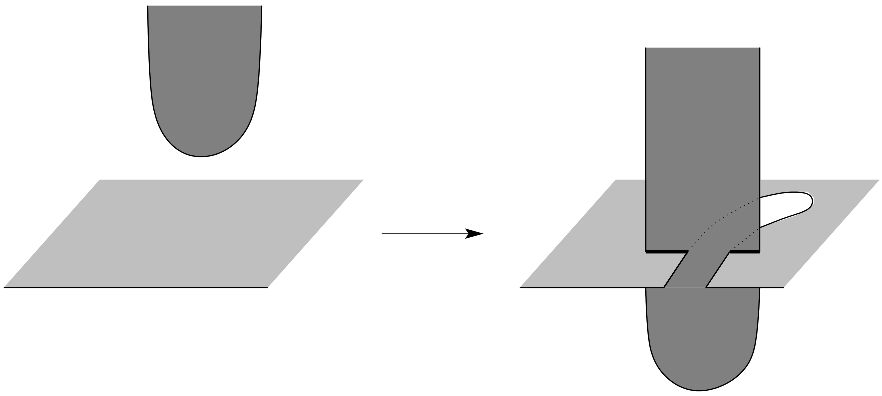

Figure 3.7: Adding a clasp between two surfaces of different colors.

Chapter 4

Splitting numbers and signatures

4.1 Introduction and statement of the results

Any link $L = K_1 \cup \cdots \cup K_n$ in S^3 can be turned into the split union of its components by a sequence of crossing changes between different components. Following Batson and Seed [10] and Cha, Friedl and Powell [32], we call *splitting number* of L the minimal number of crossing changes in such a sequence, and denote it by $\mathrm{sp}(L)$. Since upper bounds on $\mathrm{sp}(L)$ can be found by inspection of diagrams, the difficulty in computing it is to find lower bounds. The aim of this chapter (which is based on joint work with David Cimasoni and Kleopatra Zacharova [40]) is to provide such bounds using the multivariable signature and nullity. Let us first outline the previous work in the field.

As observed in [10], the linking numbers provide an elementary lower bound on the splitting number. First note that $\mathrm{sp}(L)$ has the same parity as the total linking number $\sum_{i<j} \ell k(K_i, K_j)$. Furthermore, given a two component link $K_i \cup K_j$, let $b_{\ell k}(K_i, K_j)$ be equal to 0 if $K_i \cup K_j$ is split, to 2 if it is non-split but $\ell k(K_i, K_j)$ vanishes, and to $|\ell k(K_i, K_j)|$ otherwise. Then, one may show that

$$\sum_{i<j} b_{\ell k}(K_i, K_j) \leq \mathrm{sp}(L)\,.$$

Since this *linking number bound* is not always sharp, Batson and Seed used Khovanov homology to obtain a new lower bound on $\mathrm{sp}(L)$ [10]. Testing it on links with up to 12 crossings, they found only 17 examples where this *Batson-Seed bound* is strictly stronger than the linking number bound. This enabled them to compute the splitting number of 7 of these links, while the remaining ones were left undetermined.

In [32], Cha, Friedl and Powell introduced two new techniques for computing splitting numbers. The first one is based on *covering link calculus*, while the second provides an obstruction in terms of the multivariable Alexander polynomial. This second result, originally stated as [32, Theorem 4.2], was then strengthened by Borodzik, Friedl and Powell in [21, Corollary 4.3]. It can be stated as follows: if the multivariable Alexander polynomial Δ_L of an n-component link does not vanish, then $\mathrm{sp}(L) \geq n-1$ and if $\mathrm{sp}(L) = n-1$, then

$$\Delta_L(t_1, \ldots, t_n) \doteq \prod_{i=1}^{n} \Delta_{K_i}(t_i) \cdot p(t_1, \ldots, t_n) \cdot p(t_1^{-1}, \ldots, t_n^{-1}) \cdot \prod_{i=1}^{n} (1-t_i)^{s_i}$$

for some $p \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ and $s_i \in \mathbb{Z}$. These two techniques together with the linking number bound allowed these authors to determine the splitting numbers of the 130 prime links with up to 9 crossings and to compute the splitting numbers of all of the 17 links in the Batson-Seed list.

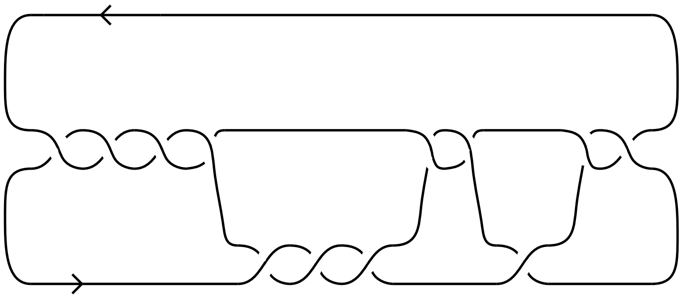

Figure 4.1: The 2-bridge link $C(4,3,2,1,2)$.

In a different direction, Borodzik and Gorsky found a Heegaard Floer theoretical criterion for bounding the splitting number [22]. As an application, they showed that for any positive a, the 2-bridge link with Conway normal form $C(2a,1,2a)$ has splitting number $2a$, even though the linking number of the two components vanishes (see Figure 4.1 for a more complicated 2-bridge link, namely $C(4,3,2,1,2)$).

The main result of this chapter is a new lower bound on the splitting number of a link in terms of its multivariable signature and nullity. Since these objects are defined for colored links, we shall say that the *splitting number* $\mathrm{sp}(L)$ of a colored link $L = L_1 \cup \cdots \cup L_\mu$ is the minimal number of crossing changes between sublinks of different colors required to turn L into the corresponding split colored link $L_1 \sqcup \cdots \sqcup L_\mu$. If μ is equal to the number of components, which is the case to keep in mind, one recovers the splitting number discussed above.

Our main result, which will be proved in Section 4.2, is the following inequality.

Theorem 4.1.1. *If $L = L_1 \cup \cdots \cup L_\mu$ is a colored link, then*

$$\Big|\sigma_L(\omega_1, \dots, \omega_\mu) - \sum_{i=1}^{\mu} \sigma_{L_i}(\omega_i)\Big| + \Big|\mu - 1 - \eta_L(\omega_1, \dots, \omega_\mu) + \sum_{i=1}^{\mu} \eta_{L_i}(\omega_i)\Big| \leq \mathrm{sp}(L)$$

for all $(\omega_1, \dots, \omega_\mu)$ in $\mathbb{T}_*^\mu := (S^1 \setminus \{1\})^\mu$.

As an immediately corollary of Theorem 4.1.1 and the first point of Proposition 3.4.3, we obtain the following lower bound for $\mathrm{sp}(L)$ in terms of the Levine-Tristram signature and nullity of L.

Corollary 4.1.2. *If $L = L_1 \cup \cdots \cup L_\mu$ is a colored link, then*

$$\Big|\sigma_L(\omega) + \sum_{i<j} \ell k(L_i, L_j) - \sum_{i=1}^{\mu} \sigma_{L_i}(\omega)\Big| + \Big|\mu - 1 - \eta_L(\omega) + \sum_{i=1}^{\mu} \eta_{L_i}(\omega)\Big| \leq \mathrm{sp}(L)$$

for all $\omega \in S^1 \setminus \{1\}$.

As we shall observe in Section 4.3, our bound is sharp for 127 out of the 130 prime links with up to 9 crossings, and two of the remaining splitting numbers can be determined with the linking number bound. Also, our method gives the splitting number of all but one of the 17 links in the Batson-Seed list. Our bound also implies the following generalization of [22, Theorem 7.12]: for any $n \geq 1$ and positive $a_1, \ldots, a_n, b_1, \ldots, b_{n-1}$, the splitting number of the 2-bridge link with Conway normal form $C(2a_1, b_1, 2a_2, b_2, \ldots, 2a_{n-1}, b_{n-1}, 2a_n)$ is equal to $a_1 + \cdots + a_n$, see Theorem 4.3.6.

Section 4.4 will deal with some further remarks which we briefly summarize. First, we show how Theorem 4.1.1 produces a signature obstruction in the spirit of the Alexander obstruction of Borodzik-Friedl-Powell. Next, we relate the splitting number to immersed concordances in $S^3 \times [0,1]$, leading to a 4-dimensional proof of Theorem 4.1.1. Finally, we shall see that our methods are sufficiently robust to provide lower bounds on variations of the splitting number as well as on the unlinking number. In particular, we will study the weak splitting number and explain how the existing methods generalize; this latter work is not contained in [40] and is currently unpublished.

4.2 Proof of Theorem 4.1.1

Let L' be a colored link obtained from L by a single crossing change involving sublinks of different colors. Then, a C-complex F' for L' can be obtained from a C-complex F for L by adding a clasp intersection, as illustrated in Figure 4.2.

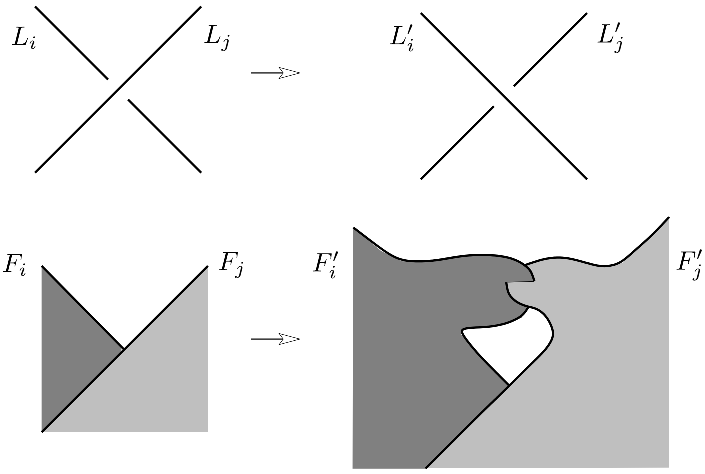

Figure 4.2: A crossing change resulting in the addition of a clasp intersection.

Since F may be assumed to be connected, it follows that $H_1(F') = H_1(F) \oplus \mathbb{Z}[\gamma]$ for some loop γ passing through the additional clasp. With respect to this choice of bases, the resulting Hermitian matrices can be written as $H'(\omega) = \begin{bmatrix} H(\omega) & z \\ \overline{z}^T & \lambda \end{bmatrix}$ for some vector z and real number λ. It follows that

$$|\sigma_L(\omega) - \sigma_{L'}(\omega)| + |\eta_L(\omega) - \eta_{L'}(\omega)| = 1$$

for all $\omega \in \mathbb{T}^\mu_*$. Consequently if $L = L^{(0)}, L^{(1)}, \ldots, L^{(s)} = L_1 \sqcup \cdots \sqcup L_\mu$ is a splitting sequence which realizes the splitting number, then

$$\begin{aligned}\mathrm{sp}(L) &= \sum_{i=1}^{s} \left(|\sigma_{L^{(i-1)}}(\omega) - \sigma_{L^{(i)}}(\omega)| + |\eta_{L^{(i-1)}}(\omega) - \eta_{L^{(i)}}(\omega)| \right) \\ &\geq |\sigma_L(\omega) - \sigma_{L^{(s)}}(\omega)| + |\eta_L(\omega) - \eta_{L^{(s)}}(\omega)| \,.\end{aligned}$$

The result now follows from the next claim whose proof can be found in [41, Proposition 2.13]. If $L \sqcup L'$ is a $(\mu + \mu')$-colored link given by the split union of the μ and μ'-colored links L and L', then, for all $\omega \in \mathbb{T}_*^{\mu}$ and $\omega' \in \mathbb{T}_*^{\mu'}$,

$$\sigma_{L \sqcup L'}(\omega, \omega') = \sigma_L(\omega) + \sigma_{L'}(\omega') \quad \text{and} \quad \eta_{L \sqcup L'}(\omega, \omega') = \eta_L(\omega) + \eta_{L'}(\omega') + 1\,.$$

This concludes the proof of Theorem 4.1.1.

4.3 Examples

In this section, we use our bounds on three types of examples: links with at most nine crossings, links of the Batson-Seed list [10], and a large class of 2-bridge links which generalize an example of Borodzik-Gorsky [22]. All the diagrams are taken from SnapPy [59].

As stated above, we tested Theorem 4.1.1 on all 130 prime links with fewer than ten crossings, using the notations and data from LinkInfo [34]. In 127 cases, the Levine-Tristram bound of Corollary 4.1.2 is enough to recover the splitting number. The three remaining links are $L9a47$ and $L9n27$ (for which the linking number bound is sharp) and $L8a9$ (whose splitting number can be recovered by the Alexander polynomial obstruction).

Before illustrating this with an example, note that there are 2^{μ} choices of orientations for the components of L, which give $2^{\mu-1}$ lower bounds on $\mathrm{sp}(L)$. Using Proposition [41, Proposition 2.8], these correspond to the value of the multivariable invariants on the $2^{\mu-1}$ diagonals of $\mathbb{T}_*^{\mu} \simeq (0, 2\pi)^{\mu}$.

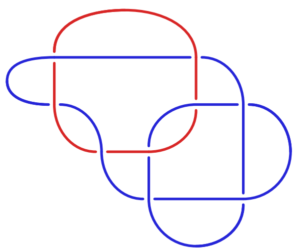

Figure 4.3: The link L9a29.

Example 4.3.1. The splitting number of the link $L = L9a29$ depicted in Figure 4.3 was shown to be 3 in [32, Section 4.2] by using the Alexander polynomial obstruction. Orienting L so that it consists of a right-handed trefoil L_1 and a trivial knot L_2 with linking number $\ell k(L_1, L_2) = -1$, we have

$$\sigma_L(-1) = 5, \quad \eta_L(-1) = 0, \quad \sigma_{L_1}(-1) = 2, \quad \eta_{L_1}(-1) = 0\,.$$

It follows that the bound

$$|\sigma_L(-1) + \ell k(L_1, L_2) - \sigma_{L_1}(-1)| + |2 - 1 - \eta_L(-1) - \eta_{L_1}(-1)| = 3$$

of Corollary 4.1.2 is sharp.

Next, we tested our bound on the 17 links of the Batson-Seed list, using Seifert matrix data kindly provided by J.C. Cha. Among these, there are seven 12-crossing links (namely, $L12n1342, L12n1350, L12n1357, L12n1363, L12n1367, L12n1274$ and $L12n1404$) for which both components are trefoils, and whose splitting number was shown to be equal to 3 by Batson and Seed. Cha, Friedl and Powell recovered these results via the Alexander obstruction. Using our techniques, these results are recovered most efficiently by using Corollary 4.4.1, whose statement can be found in Section 4.4. However, the splitting number of $L12n1342, L12n1350, L12n1367$ and $L12n1274$ can also be recovered by using the Levine-Tristram signature and Corollary 4.1.2 alone. Let us illustrate this with one example.

Example 4.3.2. Orient the link $L = L12n1367$ depicted in Figure 4.4 so that $\ell k(L_1, L_2) = 1$ and set $\omega = e^{\frac{i\pi}{3}}$. Using

$$\sigma_L(\omega) = 0, \quad \sigma_{L_1}(\omega) = 1, \quad \sigma_{L_2}(\omega) = -1, \quad \eta_L(\omega) = \eta_{L_1}(\omega) = \eta_{L_2}(\omega) = 1\,,$$

it follows that the bound

$$|\sigma_L(\omega) + \ell k(L_1, L_2) - \sigma_{L_1}(\omega) - \sigma_{L_2}(\omega)| + |2 - 1 - \eta_L(\omega) - \eta_{L_1}(\omega) - \eta_{L_2}(\omega)| = 3$$

of Corollary 4.1.2 is sharp.

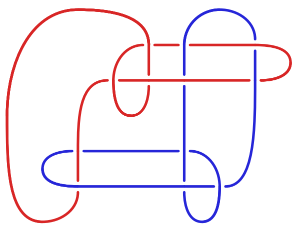

Figure 4.4: The link L12n1367.

For the remaining 10 links in this list, Batson and Seed could not determine whether the splitting number is 3 or 5. These links having two or three components, the Alexander polynomial obstruction cannot be applied. However, Cha, Friedl and Powell used various arguments based on covering link techniques to determine these values. As it turns out, our bound allows to easily determine these splitting number for all but one of them, namely $L12n1321$. Here are several of these examples.

Example 4.3.3. Consider the link $L = L11a372$ depicted in Figure 4.5, whose splitting number was shown to be 5 in [32, Section 5.2]. Orient L so that its trivial components L_1, L_2 satisfy $\ell k(L_1, L_2) = -1$. Since $\sigma_L(-1) = 5$ and $\eta_L(-1) = 0$, the bound

$$|\sigma_L(-1) + \ell k(L_1, L_2)| + |2 - 1 - \eta_L(-1)| = 5$$

given by the classical signature is enough to conclude.

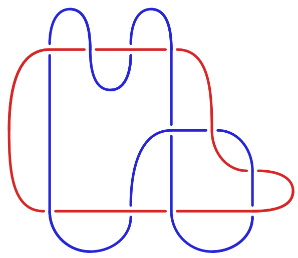

Figure 4.5: The link L11a372.

Example 4.3.4. It took the whole Section 5.3 of [32] to show that the splitting number of the 3-component link $L = L12a1622$ depicted in Figure 4.6 is equal to 5. Orienting L so that its trivial components L_1, L_2, L_3 satisfy $\ell k(L_1, L_2) = 0$, $\ell k(L_1, L_3) = 0$, $\ell k(L_2, L_3) = 1$, and picking $\omega = e^{\frac{3\pi i}{4}}$, we have $\sigma_L(\omega) = -4$ and $\eta_L(\omega) = 0$. Hence, the bound

$$|\sigma_L(\omega) + \ell k(L_2, L_3)| + |3 - 1 - \eta_L(\omega)| = 5$$

of Corollary 4.1.2 immediately provides the desired splitting number.

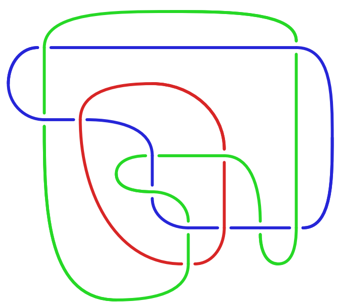

Figure 4.6: The link L12a1622.

Example 4.3.5. Consider the link $L = L12n1326$ depicted in Figure 4.7. Cha-Friedl-Powell [32, Section 5.2] used the twisted Alexander polynomial to compute the slice genus of a covering link, and concluded that $\mathrm{sp}(L) = 3$. Orienting L so that its trivial components L_1, L_2 satisfy $\ell k(L_1, L_2) = 1$, and picking $\omega = e^{\frac{\pi i}{5}}$ so that $\sigma_L(\omega) = 1$ and $\eta_L(\omega) = 0$, the bound

$$|\sigma_L(\omega) + \ell k(L_1, L_2)| + |2 - 1 - \eta_L(\omega)| = 3$$

of Corollary 4.1.2 immediately provides the desired splitting number.

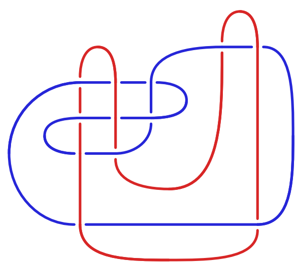

Figure 4.7: The link L12n1326.

In [22], Borodzik and Gorsky used Heegaard Floer techniques to compute the splitting number of the 2-bridge link with Conway normal form $C(2a, 1, 2a)$ (recall Figure 4.1 for an explanation of this notation). We conclude this section by showing how we can efficiently compute the splitting number of a wider class of 2-bridge links.

Theorem 4.3.6. *For any $n \geq 1$ and any positive integers $a_1, \dots, a_n, b_1, \dots, b_{n-1}$, the splitting number of the 2-bridge link $C(2a_1, b_1, 2a_2, b_2, \dots, 2a_{n-1}, b_{n-1}, 2a_n)$ is equal to $a_1 + \cdots + a_n$.*

Proof. Let us denote this link by L. Inspecting the standard diagram given by Conway's normal form, it is clear that $\mathrm{sp}(L) \leq a_1 + \cdots + a_n =: s$. Furthermore, as examplified in Figure 4.8, L admits a C-complex F made of two discs intersecting in s clasps, yielding a natural basis for $H_1(S)$ of cardinality $s - 1$. Computing the associated generalized Seifert matrices leads to

$$H(-1,-1) = 4 \begin{bmatrix} -2d_1 & 1 & & & \\ 1 & -2d_2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & & 1 \\ & & & 1 & -2d_{s-1} \end{bmatrix},$$

where the diagonal elements are positive integers whose precise value depends on the a_i's and b_i's, but will play no role here. Indeed, all the eigenvalues of a matrix of the form above are negative, so $\sigma_L(-1,-1) = 1 - s$ and $\eta_L(-1,-1) = 0$. Since L has $\mu = 2$ components, both unknotted, the result now follows from Theorem 4.1.1. $\square$

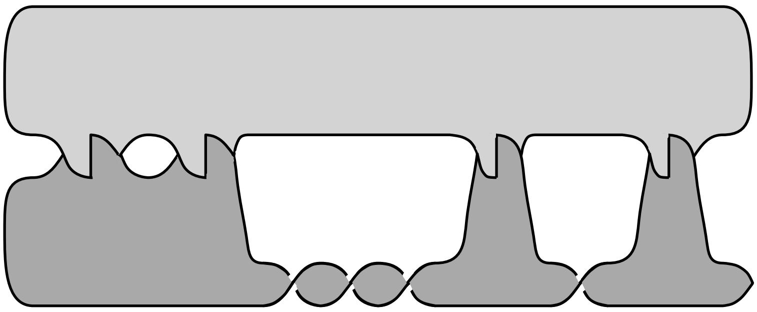

Figure 4.8: A C-complex for the 2-bridge link $C(4, 3, 2, 1, 2)$.

Note that this last result was obtained using the ordinary signature alone. Using the full power of Theorem 4.1.1, it is possible to determine the splitting number of many other 2-bridge links, such as all $C(a_1, \dots, a_n)$ with $n \leq 3$. The first example of a 2-bridge link whose splitting number is unknown to us is $L = C(4, 3, 1, 3)$: the linking number vanishes and our bound reads $\mathrm{sp}(L) \geq 2$, while the "obvious" splitting sequence has length 4.

4.4 Further remarks

This section is organized as follows. Subsection 4.4.1 shows how Theorem 4.1.1 produces a signature obstruction in the spirit of the Alexander obstruction of Borodzik-Friedl-Powell.

Subsection 4.4.2 provides a 4-dimensional proof of Theorem 4.1.1. Subsection 4.4.3 gives a bound on the unlinking number in terms of the multivariable signature. Finally, Subsection 4.4.4 is concerned with the so-called weak splitting number.

4.4.1 A signature obstruction

Let $L = K_1 \cup \cdots \cup K_n$ be an ordered link. Borodzik-Friedl-Powell [21, Corollary 4.3] prove that $n-1-\beta(L)$ is a lower bound for the splitting number, where $\beta(L)$ is the Alexander nullity of L discussed in Subsection 3.5.1. Furthermore, they show that if $\Delta_L \neq 0$ and $n-1 = \mathrm{sp}(L)$, then

$$\Delta_L(t_1, \ldots, t_n) \doteq \prod_{i=1}^{n} \Delta_{K_i}(t_i) \cdot p(t_1, \ldots, t_n) \cdot p(t_1^{-1}, \ldots, t_n^{-1}) \cdot \prod_{i=1}^{n} (1-t_i)^{s_i}$$

for some $p \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ and $s_i \in \mathbb{Z}$. The next proposition, which appeared in [40, Corollary 3.5] shows the analogue of this result in our setting. More precisely, we reprove the first inequality by using the multivariable nullity instead of the Alexander nullity. The case $\beta(L) = 0$ of our second statement can then be understood as the signature analog of the Alexander polynomial obstruction.

Proposition 4.4.1. *Let $L = L_1 \cup \cdots \cup L_\mu$ be a μ-colored link and let Δ_L^{tor} be its first non-vanishing Alexander polynomial. Then, one has*

$$\mu - 1 - \beta(L) \leq \mathrm{sp}(L)\,.$$

Furthermore, if $\mu - 1 - \beta(L) = \mathrm{sp}(L)$, then

$$\sigma_L(\omega_1, \ldots, \omega_\mu) = \sum_{i=1}^{\mu} \sigma_{L_i}(\omega_i) \quad \text{and} \quad \eta_{L_1}(\omega_1) = \cdots = \eta_{L_\mu}(\omega_\mu) = 0$$

for all $\omega = (\omega_1, \ldots, \omega_\mu) \in \mathbb{T}_*^\mu$ such that $\Delta_L^{\mathrm{tor}}(\omega) \neq 0$.

Proof. By Theorem 4.1.1 and Proposition 3.5.1, we have the inequalities

$$\begin{aligned} \mathrm{sp}(L) &\geq \left|\sigma_L(\omega) - \sum_{i=1}^{\mu} \sigma_{L_i}(\omega_i)\right| + \left|\mu - 1 - \eta_L(\omega) + \sum_{i=1}^{\mu} \eta_{L_i}(\omega_i)\right| \\ &\geq \mu - 1 - \eta_L(\omega) + \sum_{i=1}^{\mu} \eta_{L_i}(\omega_i) \\ &\geq \mu - 1 - \eta_L(\omega) \\ &\geq \mu - 1 - \beta(L)\,. \end{aligned}$$

Let us now assume that $\Delta_L^{\mathrm{tor}}(\omega) \neq 0$. Using the notations of the proof of Proposition 3.5.1, this implies that ω belongs to $\Sigma_{\beta(L)} \setminus \Sigma_{\beta(L)+1}$. By [41, Theorem 4.1], this means that $\eta_L(\omega) = \beta(L)$. The second statement now follows by setting equalities for each of the inequalities displayed above. □

Here is a concrete example of Proposition 4.4.1, see also [40, Example 4.2].

Example 4.4.2. The splitting number of the link $L = L9a24$ depicted on the left hand side of Figure 4.9 was shown to be 3 in [32, Section 6] by using the Alexander polynomial obstruction. Corollary 4.4.1 shows that the splitting number must be greater than one, and therefore equal to three due to the parity of the linking number. Indeed, one does not always have $\sigma_L(\omega_1, \omega_2) = \sigma_{L_1}(\omega_1) + \sigma_{L_2}(\omega_2)$, as shown by the zero locus of Δ_L illustrated on the right hand side of Figure 4.9.

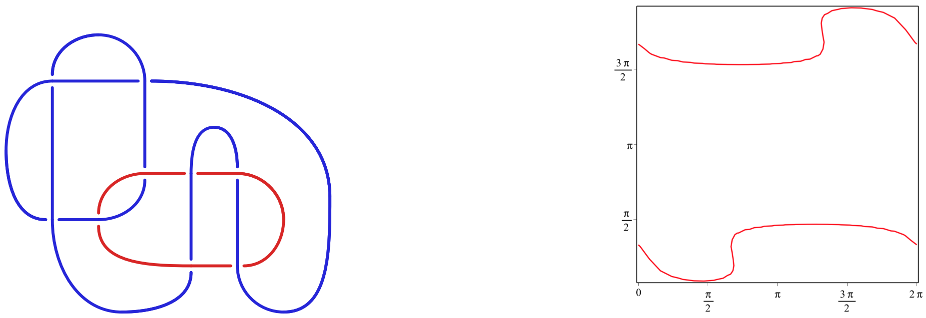

Figure 4.9: The link $L9a24$ and the intersection of the zero locus of its Alexander polynomial with $\mathbb{T}^2_*$.

4.4.2 Two 4-dimensional proofs of Theorem 4.1.1

In the case of an ordered link L and for ω in the set $\mathbb{T}^\mu_P$ described in Subsection 3.4.2, Theorem 4.1.1 can be proved by using 4-dimensional considerations. Indeed, a splitting sequence of length s provides an immersed concordance in $S^3 \times [0, 1]$ between L and the corresponding split link, with s transverse double points. Removing $B^3 \times [0, 1]$, where B^3 is a 3-ball in S^3 that meets each component of L in a small unknotted arc untouched by the crossing changes, we obtain a collection of discs in B^4 with s double points, bounding the link $L\#\overline{L_1}\# \ldots \#\overline{L_\mu}$. The result then follows from Theorem 3.4.7 together with the properties of the signature and nullity under mirror images and band sums, see [41, Proposition 2.10 and 2.12].

The result also follows in a more straightforward way from the following statement which is an immediate consequence of Theorem 8.1.2. If C is an immersed concordance in $S^3 \times I$ between L and L' with c double points, then

$$|\sigma_L(\omega) - \sigma_{L'}(\omega)| + |\eta_L(\omega) - \eta_{L'}(\omega)| \leq c$$

for all ω in a certain subset $\mathbb{T}^\mu_!$ of $\mathbb{T}^\mu_*$ which strictly contains $\mathbb{T}^\mu_P$, see Section 8.3. When L' is the split union of the components of L and C is the immersed concordance arising from a minimal splitting sequence, we recover Theorem 4.1.1 albeit for a smaller set of ω's.

4.4.3 Unlinking numbers

The techniques of Section 4.2 can also be used to obtain lower bounds on the unlinking number $u(L)$ of a link L. Namely, if $L = K_1 \cup \cdots \cup K_n$ is an n-component link, then one obtains

$$|\sigma_L(\omega_1, \ldots, \omega_n)| + |n - 1 - \eta_L(\omega_1, \ldots, \omega_n)| + \sum_{i<j} |\ell k(K_i, K_j)| \leq 2u(L)$$

for all $(\omega_1, \ldots, \omega_n) \in \mathbb{T}^n_*$. Unfortunately, this bound is not very powerful so we shall not discuss it any further.

4.4.4 Weak splitting numbers

This subsection is not contained in [40]: it involves subsequent unpublished work.

Following Borodzik-Friedl-Powell [21], the *weak splitting number* $\mathrm{wsp}(L)$ of an n-component link L is the minimal number of crossing changes needed to turn L into the split union of n knots. The weak splitting number must not be confused with the splitting number $\mathrm{sp}(L)$ which is the minimal number of crossing changes between *different* components needed to turn L into the split union of its components. To illustrate the difference between $\mathrm{sp}(L)$ and $\mathrm{wsp}(L)$, consider the Whitehead link $W = K_1 \cup K_2$ depicted in Figure 4.10. Since the splitting number must have the same parity as the linking number $\ell k(K_1, K_2)$ and L is non-trivial, inspection shows that $\mathrm{sp}(W) = 2$. On the other hand, changing the central crossing implies that $\mathrm{wsp}(W) = 1$. In general, one clearly has $\mathrm{wsp}(L) \leq \mathrm{sp}(L)$.

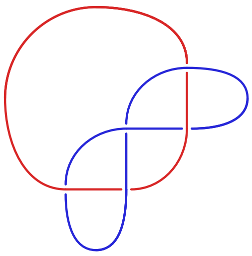

Figure 4.10: The Whitehead link has splitting number 2 but weak splitting number 1.

The difficulty in estimating $\mathrm{wsp}(L)$ lies in the following fact: the isotopy type of the components of the split link remains unknown. At the present time, to the best of our knowledge, the only known lower bound on wsp(L) is given by $n - 1 - \beta(L)$, where $\beta(L)$ is the Alexander nullity of L [21, Corollary 4.4]. Borodzik, Friedl and Powell also have an *Alexander polynomial obstruction* similar to the one described in Subsection 4.4.1. Namely, if $\Delta_L \neq 0$ and $n - 1 = \mathrm{wsp}(L)$, then they show that

$$\Delta_L(t_1, \ldots, t_n) \doteq f(t_1, \ldots, t_n) f(t_1^{-1}, \ldots, f_n^{-1}) \cdot \prod_{i=1}^{n} (t_i - 1)^{m_i} \cdot \prod_{i=1}^{n} p_i(t_i)$$

for some Laurent polynomials $p_i(t_i) \in \mathbb{Z}[t_i^{\pm 1}]$, $f \in \Lambda_n$, and some integers m_i.

In this subsection, we shall adapt the other various lower bounds and obstructions from the setting of splitting numbers to weak splitting numbers. These results can be used to compute the weak splitting number of links with up to 9 crossings, with 2 exceptions: $L9a29, L9a30$; some examples will be given.

We start with a *linking number bound* inspired by [32, Lemma 2.1] and [10]. Given an n-component link L, we claim that

$$\sum_{i<j} |\ell k(K_i, K_j)| \leq \mathrm{wsp}(L). \tag{4.1}$$

To prove (4.1), we introduce some terminology. A *self crossing change* (respectively a *mixed crossing change*) is a crossing change which occurs within a component of L (respectively involves two distinct components of L). To prove (4.1), set $N := \sum_{i<j} |\ell k(K_i, K_j)|$. A mixed crossing change alters the value of N by precisely one, while a self crossing change leaves it unaffected. If we perform a crossing change between two components with non-zero linking number, then N decreases by at most one. Since N vanishes on split links, the statement follows.

The following obstruction was suggested to us by Chuck Livingston.

Proposition 4.4.3. *Let $L = K_1 \cup K_2$ be a 2-component link with zero linking number. If L can be split using a single crossing change in K_2, then K_2 is nullhomotopic in the exterior X_{K_1} of K_1.*

Proof. Perform a crossing change within K_2 which yields a knot K_2' which is split from K_1. Since K_2' is split from K_1, it is nullhomotopic in X_{K_1}. Moreover since the crossing change in K_2 does not change its homotopy type, K_2 itself is nullhomotopic in X_{K_1}. □

Delaying examples, we move on to covering link calculus. Given a link $L = K_1 \cup \cdots \cup K_n$ where K_i is unknotted, one can form the 2-fold cover $p\colon S^3 \to S^3$ branched along K_i. The link $\widetilde{L} = p^{-1}(L \setminus K_i)$ is called the *covering link* of L with respect to K_i [32, 33, 106]. Note that a crossing change within K_j (with $j \neq i$) results in two crossing changes in the covering link, see for instance [32, Section 3]. Here is an immediate corollary of this observation.

Proposition 4.4.4. *Let $n \geq 2$ and let $L = K_1 \cup \cdots \cup K_n$ be an n-component link where K_i is unknotted. If L can be split via a single crossing change in K_j with $j \neq i$, then wsp$(\widetilde{L}) \leq 2$.*

Delaying examples for just a bit longer, we provide a lower bound on the weak splitting number which involves the multivariable signature and nullity.

Proposition 4.4.5. *If $L = K_1 \cup \cdots \cup K_n$ is an oriented n-component link, then*

$$\Big|\sigma_L(\omega) - \sum_{i=1}^{n} \sigma_{K_i}(\omega_i)\Big| + \Big|n - 1 - \eta_L(\omega) + \sum_{i=1}^{n} \eta_{K_i}(\omega_i)\Big| + 3\sum_{i<j} |\ell k(K_i, K_j)| \leq 4 \ \mathrm{wsp}(L).$$

for all $\omega \in \mathbb{T}^n_*$.

Proof. Assume an n-component link L can be split using $\mathrm{wsp}(L) = s + m$ crossing changes with s self crossing changes and m mixed crossing changes. We claim that L can be converted into the split union of its components in $2s + m$ crossing changes. To prove this claim, start by using $\mathrm{wsp}(L)$ crossing changes in order to convert $L = K_1 \cup \cdots \cup K_n$ into an n-component split link $K_1' \sqcup \cdots \sqcup K_n'$ for some knots $K_1', \ldots, K_n'$. Let s_i be the number of crossing changes needed to pass from K_i to K_i' while splitting L. To conclude the proof of the claim, perform $s = s_1 + \ldots + s_n$ additional crossing changes, converting the link $K_1' \sqcup \cdots \sqcup K_n'$ into $K_1 \sqcup \cdots \sqcup K_n$. Next, using the claim and proceeding as in Section 4.2, one obtains

$$\Big|\sigma_L(\omega) - \sum_{i=1}^{n} \sigma_{K_i}(\omega_i)\Big| + \Big|n - 1 - \eta_L(\omega) + \sum_{i=1}^{n} \eta_{K_i}(\omega_i)\Big| \leq 2(2s) + m = 4\mathrm{wsp}(L) - 3m.$$

The result now follows by observing that $\sum_{i<j} |\ell k(K_i, K_j)| \leq m$. □

Finally, we provide some examples of Propositions 4.4.3, 4.4.4 and 4.4.5

Example 4.4.6. Consider the 2-component link $L = L9a4 = K_1 \cup K_2$ pictured in Figure 4.11. Inspecting the diagram shows that the weak splitting number is at most 2. By means of contradiction, assume that L can be split using a single crossing change. Since $\ell k(K_1, K_2) = 0$, we must show that L cannot be split using a single self crossing change. First, assume that this undesirable crossing change takes place in the unknotted component K_2. Proposition 4.4.3 implies that K_2 is nullhomotopic in the exterior X_{K_1} of the trefoil. Under this assumption, a short computation involving the Wirtinger presentation shows that $\pi_1(X_{K_1})$ is infinite cyclic, contradicting the fact that K_1 is the trefoil.

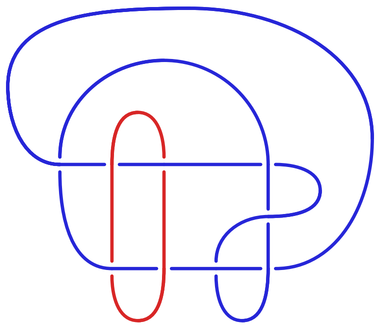

Figure 4.11: The link L9a4.

Next, assume that the self crossing change occurs in the knotted component K_1. Taking the 2-fold covering branched along the unknotted component K_2 produces a 2-component link $\widetilde{L} = J_1 \cup J_2$ with $\ell k(J_1, J_2) = 3$. Combining the linking number bound with Proposition 4.4.3 provides the desired contradiction.

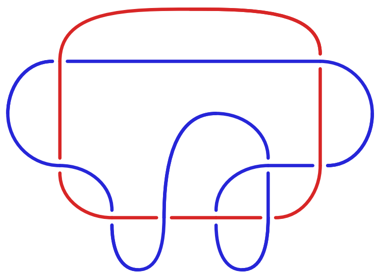

Figure 4.12: The link $L9a36$.

Example 4.4.7. Consider the 2-component link $L = L9a36$ depicted in Figure 4.12. Inspecting the diagram shows that the weak splitting number is at most 3. In particular, the Alexander polynomial obstruction cannot decide whether $\mathrm{wsp}(L)$ is 2 or 3. Orient L so that its (trivial) components K_1 and K_2 have linking number 2 (note that whatever the chosen orientation, the linking number bound is inconclusive). Using Linkinfo [34], $\sigma_L(-1) = -5$ and $\eta_L(-1) = 0$, and Proposition 4.4.5 now implies that

$$|-5+2| + 1 + 3 \cdot 2 = 10 \leq 4 \ \mathrm{wsp}(L).$$

Consequently, the weak splitting number must be at least 3. We conclude that $\mathrm{wsp}(L) = 3$.

Chapter 5

Twisted homology

5.1 Introduction

This chapter introduces a technical tool which shall be used throughout this thesis: twisted homology, also known as homology with local coefficients. Informally, twisted homology can be understood as a bridge between the theory of covering spaces and the formalism of homological algebra. Given a CW-complex X, a ring R and a $(R, \mathbb{Z}[\pi_1(X)])$-bimodule M, the main objects of study are the twisted homology left R-modules

$$H_*(X; M) = H_*(M \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X})),$$

where $\widetilde{X}$ denotes the universal cover of X. As we shall see in Section 5.2, varying M recovers both the cellular homology of X and the homology of the covering spaces of X.

The flexibility of this set-up, which was briefly introduced in Subsection 2.4.1, was already used in Section 2.4 to give a succint definition of the Blanchfield pairing of a knot. Although the use of twisted homology is not new, and is contained in some algebraic topology textbooks such as [60, 86], many different conventions seem to occur in the literature. For this reason, we shall take some time to set up the machinery carefully, especially regarding twisted cohomology in Section 5.3 and its relation to homology in Section 5.4. The same goes for our treatment of Poincaré duality in Section 5.5 and of twisted intersection forms in Section 5.6; note that these tools are used systematically in Chapters 6, 7 and 8.

Finally, in Section 5.7, we shall make several additional observations; here is a brief summary. While cellular homology is functorial, this is not the case for twisted homology, see also Chapter 11. We shall also discuss some technicalities related to long exact sequences in homology. Finally, we conclude with a few words on Λ_S-twisted homology, where Λ_S is the localization $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}, (1-t_1)^{-1}, \ldots, (1-t_\mu)^{-1}]$ of the Laurent polynomial ring Λ_μ.

5.2 Twisted homology

In this section, we review the twisted homology of a CW-complex X with coefficients in a $(R, \mathbb{Z}[\pi_1(X)])$-bimodule M. References include [60, 73, 75, 86, 102].

Let X be a CW complex and let $Y \subset X$ be a possibly empty subcomplex. Denote by $p \colon \widetilde{X} \to X$ the universal cover of X and set $\widetilde{Y} := p^{-1}(X)$. The left action of $\pi_1(X)$ on $\widetilde{X}$

endows the chain complex $C_*(\widetilde{X},\widetilde{Y})$ with the structure of a left $\mathbb{Z}[\pi_1(X)]$-module. Moreover, let R be a ring and let M be a $(R,\mathbb{Z}[\pi_1(X)])$-module.

Definition 15. The chain complex $C_*(X,Y;M) := M \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X},\widetilde{Y})$ of left R-modules will be called the *twisted chain complex* of (X,Y) with coefficients in M. The corresponding homology left R-modules $H_*(X,Y;M)$ will be called the *twisted homology* modules of (X,Y) with coefficients in M.

As usual, if Y is empty, we write $H_k(X;M)$ instead of $H_k(X,\emptyset;M)$. Note also that the isomorphism type of $H_*(X,Y;M)$ does not depend on the choice of a basepoint for the fundamental group; this is why we omitted it from the notation. The next lemma will show that the twisted homology of X encompasses the homology of its covering spaces. In particular, this will relate Definition 15 to the definition of twisted homology that we gave in Subsection 2.4.1. To make this precise, let $\psi\colon \pi_1(X) \to G$ be a surjective group homomorphism and let $\widehat{X}$ be the covering space associated to $\ker(\psi)$. Since ψ extends to a ring homomorphism $\psi\colon \mathbb{Z}[\pi_1(X)] \to \mathbb{Z}[G]$, any $(R,\mathbb{Z}[G])$-bimodule M can be also viewed as a $(R,\mathbb{Z}[\pi_1(X)])$-bimodule by setting $m\cdot\gamma := m\psi(\gamma)$ for m in M and γ in $\mathbb{Z}[\pi_1(X)]$.

Lemma 5.2.1. *Given a CW pair (X,Y), a surjective group homomorphism $\psi\colon \pi_1(X) \to G$ and a $(R,\mathbb{Z}[G])$-bimodule M, there is a canonical chain isomorphism of left R-modules*

$$M \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X},\widetilde{Y}) \cong M \otimes_{\mathbb{Z}[G]} C_*(\widehat{X},\widehat{Y}).$$

In particular $H_*(X,Y;M)$ can be computed from the chain complex $M \otimes_{\mathbb{Z}[G]} C_*(\widehat{X},\widehat{Y})$.

Proof. By associativity of the tensor product, it is enough to show that $\mathbb{Z}[G] \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X},\widetilde{Y})$ and $C_*(\widehat{X},\widehat{Y})$ are canonically chain isomorphic as left $\mathbb{Z}[G]$-modules. The isomorphism in question is the one induced by the projection $\widetilde{X} \to \widehat{X}$. □

Lemma 5.2.1 justifies why we often denote the Alexander module of a knot K by $H_1(X_K;\Lambda)$ or, more generally, the Alexander module of a μ-colored link L by $H_1(X_L;\Lambda_\mu)$. It also explains the definition of twisted homology we gave in Subsection 2.4.1.

Here are some additional examples of Lemma 5.2.1.

Example 5.2.2. If $R=\mathbb{Z}$ and $M=\mathbb{Z}$ is given the trivial right $\mathbb{Z}[\pi_1(X)]$-module structure $n\cdot\gamma = n$ and the left $\mathbb{Z}$-module structure induced by multiplication, then $H_*(X;\mathbb{Z})$ is nothing but the usual cellular homology of X. On the other hand, if $R=\mathbb{Z}[\pi_1(X)]$ and $M=\mathbb{Z}[\pi_1(X)]$ is given its natural $(\mathbb{Z},\mathbb{Z}[\pi_1(X)])$-bimodule structure, then $H_*(X;\mathbb{Z}[\pi_1(X)])$ is precisely the cellular homology of the universal cover of X.

We now review the relationship between the twisted homology of a space and the twisted homology of its fundamental group. Recall that the homology of a discrete group G with coefficients in a $(R,\mathbb{Z}[G])$-bimodule M is defined by $H_*(G;M) := \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z},M)$. Let BG denote the classifying space of G. Since its universal cover EG is contractible, the reduced cellular homology $\widetilde{H}_*(EG)$ vanishes identically, and thus its chain complex provides a free left $\mathbb{Z}[G]$-module resolution for $\mathbb{Z}$. It follows that $H_*(G;M) \cong H_*(BG;M)$. In particular, if a CW complex X is an Eilenberg-Maclane space $K(\pi_1(X),1)$, it immediately follows that $H_*(X;M) \cong H_*(\pi_1(X);M)$. Indeed, since $\pi_1(X)$ is discrete, $B\pi_1(X)$ is a $K(\pi_1(X),1)$. If X is not a $K(\pi_1(X),1)$, the situation remains salvageable:

Remark 5.2.3. Given a CW complex X and a $(R, \mathbb{Z}[\pi_1(X)])$-bimodule M, it is well known that $H_i(X; M) \cong H_i(\pi_1(X); M)$ for $i = 0, 1$. Here is an outline of the argument. Attach cells of dimension $k \geq 3$ to X in order to kill its higher homotopy groups. The result is a $K(\pi_1(X), 1)$ space Z. Using the previous discussion $H_i(Z; M) \cong H_i(\pi_1(X); M)$ for all i, and the claim now follows from the long exact sequence of the pair (Z, X).

Using the sphere theorem, the exterior X_K of a knot K is a $K(\pi_1(X_K), 1)$ [112, Chapter 11], so Remark 5.2.3 is not needed to conclude that $H_*(X_K; \Lambda) = H_*(\pi_1(X_K); \Lambda)$. On the other hand, while the exterior X_L of a μ-colored link L might not be aspherical, Remark 5.2.3 nevertheless implies that $H_i(X_L; \Lambda_\mu) = H_i(\pi_1(X_L); \Lambda_\mu)$ for $i = 0, 1$. Consequently $\Delta_L(t_1, \ldots, t_\mu)$ can be computed by using any CW-complex whose fundamental group coincides with $\pi_1(X_L)$.

Next, we give an example which shall be used systematically in Chapters 7 and 8.

Example 5.2.4. Let $\psi\colon \pi_1(X) \to \mathbb{Z}^\mu = \langle t_1, \ldots, t_\mu \rangle$ be a homomorphism and let $\omega = (\omega_1, \ldots, \omega_\mu)$ lie in $\mathbb{T}^\mu$. Composing the induced ring homomorphism $\mathbb{Z}[\pi_1(X)] \to \mathbb{Z}[\mathbb{Z}^\mu]$ with the ring homomorphism $\mathbb{Z}[\mathbb{Z}^\mu] \to \mathbb{C}$ which evaluates t_i at ω_i, produces a ring homomorphism $\phi\colon \mathbb{Z}[\pi_1(W)] \to \mathbb{C}$. In turn, ϕ endows $\mathbb{C}$ with a $(\mathbb{C}, \mathbb{Z}[\pi_1(X)])$-bimodule structure. To emphasize the choice of ω, we shall write $\mathbb{C}^\omega$ instead of $\mathbb{C}$. Since $\mathbb{C}^\omega$ is a $(\mathbb{C}, \mathbb{Z}[\pi_1(W)])$-bimodule, we may consider the complex vector space $H_k(X; \mathbb{C}^\omega)$. For instance, in Chapter 8, we shall reinterpret the multivariable nullity as the dimension of such a complex vector space, see Theorem 8.1.1.

We conclude with a well-known remark on the Euler characteristic.

Remark 5.2.5. Let X be a finite CW-complex, let R be a commutative ring and let M be a $(R, \mathbb{Z}[\pi_1(X)])$-bimodule which is free of rank d as a left R-module. We claim that the Euler characteristic of the twisted homology R-modules $H_*(X; M)$ equals d times the Euler characteristic of X. This is proved promptly since $C_k(\widetilde{X})$ is a free left $\mathbb{Z}[\pi_1(X)]$ module whose rank is equal to the number n_k of k-cells of X: indeed $C_k(X; M) = M \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X})$ must now be a free R-module of rank $n_k d$, and the claim follows.

5.3 Twisted cohomology

In this section, we review the twisted cohomology of a CW-complex X with coefficients in a $(R, \mathbb{Z}[\pi_1(X)])$-bimodule M. References include [60, 73, 75, 86, 102].

Let S be a ring endowed with an involution $s \mapsto \overline{s}$ satisfying $\overline{rs} = \overline{s}\,\overline{r}, \overline{r+s} = \overline{r} + \overline{s}$ and $\overline{1} = 1$; an example to keep in mind is the group ring $\mathbb{Z}[G]$ with the involution induced by $g \mapsto g^{-1}$. Given a left (resp. right) S-module N, we denote by $\overline{N}$ the right (resp. left) S-module that has the same underlying additive group as N, but for which the action by S on N is precomposed with the involution on S. This way, if M is a $(R, \mathbb{Z}[\pi_1(X)])$-bimodule, then both $\overline{C_*(\widetilde{X}, \widetilde{Y})}$ and M are right $\mathbb{Z}[\pi_1(X)]$-modules.

Definition 16. The cochain complex $C^*(X, Y; M) = \mathrm{Hom}_{\text{right-}\mathbb{Z}[\pi_1(X)]}(\overline{C_*(\widetilde{X}, \widetilde{Y})}, M)$ of left R-modules will be called the *twisted cochain complex* of (X, Y) with coefficients in M. The corresponding cohomology left R-modules $H^*(X, Y; M)$ will be called the *twisted cohomology* modules of (X, Y) with coefficients in M.

The content of Lemma 5.2.1 also holds for cohomology. Namely, given a surjective group homomorphism $\psi\colon \pi_1(X) \to G$ and a $(R, \mathbb{Z}[G])$-bimodule M, there are canonical chain isomorphisms of left R-modules

$$\mathrm{Hom}_{\text{right-}\mathbb{Z}[\pi_1(X)]}(\overline{C_*(\widetilde{X}, \widetilde{Y})}, M) \cong \mathrm{Hom}_{\text{right-}\mathbb{Z}[\pi_1(G)]}(\overline{C_*(\widehat{X}, \widehat{Y})}, M),$$

where $\widehat{X}$ is the cover of X corresponding to $\ker(\psi)$. In particular, $H^*(X, Y; M)$ can be computed from the cochain complex $\mathrm{Hom}_{\text{right-}\mathbb{Z}[G]}(\overline{C(\widehat{X}, \widehat{Y})}, M)$. However, the analogy with twisted homology cannot be pursued entirely. Indeed, taking $R = \mathbb{Z}[G]$ and endowing $M = \mathbb{Z}[G]$ with its natural $(\mathbb{Z}, \mathbb{Z}[G])$-bimodule structure, the cohomology groups $H^*(X; \mathbb{Z}[G])$ do *not* recover the cohomology of $\widehat{X}$, but only its cohomology with compact support $H^*_c(\widehat{X})$, see [60, Section 5.2].

5.4 The evaluation map

In this section, we discuss evaluation maps from twisted cohomology to twisted homology. Much of this section owes to discussions with Stefan Friedl, especially the statement of Lemma 5.4.2. References include [73, 102].

Let R be a ring with involution, let M, N be $(R, \mathbb{Z}[\pi_1(X)])$-bimodules and let S be a (R, R)-bimodule. Furthermore, let $\langle -, - \rangle\colon M \times N \to S$ be a nonsingular $\pi_1(X)$-invariant sesquilinear pairing, in the sense that $\langle m\gamma, n\gamma \rangle = \langle m, n \rangle$ and $\langle rm, sn \rangle = r\langle m, n \rangle \overline{s}$ for all $\gamma \in \pi_1(X)$, all $r, s \in R$ and all $m \in M, n \in N$. Here is an example we wish to keep in mind.

Example 5.4.1. Let $\pi_1(X) \to \mathbb{Z}^\mu$ be a group homomorphism which induces an involution preserving ring homomorphism $\psi\colon \mathbb{Z}[\pi_1(X)] \to \Lambda_\mu$. Consider the case where $R = \Lambda_\mu$ and use ψ to endow both $M = Q_\mu/\Lambda_\mu$ and $N = \Lambda_\mu$ with a $(\Lambda_\mu, \mathbb{Z}[\pi_1(X)])$-bimodule structure. Finally, endow $S = Q_\mu/\Lambda_\mu$ with its natural $(\Lambda_\mu, \Lambda_\mu)$-bimodule structure and consider the nonsingular sesquilinear $\pi_1(X)$-invariant pairing $Q_\mu/\Lambda_\mu \times \Lambda_\mu \to Q_\mu/\Lambda_\mu, (p, q) \mapsto p\overline{q}$. For $\mu = 1$, we already encountered this pairing in Subsection 2.4.1; we will use the multivariable case in Chapter 6.

Let us keep in mind that the left R-module structure on $\mathrm{Hom}_{\text{right-}\mathbb{Z}[\pi_1(X)]}(\overline{C_*(\widetilde{X}, \widetilde{Y})}, M)$ is given by $(r \cdot \varphi)(\sigma) = r\varphi(\sigma)$, while the left R-module structure on $\overline{\mathrm{Hom}_{\text{left-}R}(N \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X}, \widetilde{Y}), S)}$ is given by $(r \cdot \varphi)(n \otimes \sigma) = \varphi(n \otimes \sigma)\overline{r}$. The point of this set-up is to relate twisted homology and cohomology via an evaluation map.

Lemma 5.4.2. *Let (X, Y) be a CW-pair. Set $\pi := \pi_1(X)$. The following map is chain isomorphism of left R-modules:*

$$\begin{aligned}\kappa\colon \mathrm{Hom}_{\text{right-}\mathbb{Z}[\pi]}(\overline{C_*(\widetilde{X}, \widetilde{Y})}, M) &\to \overline{\mathrm{Hom}_{\text{left-}R}(N \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}, \widetilde{Y}), S)} \\ f &\mapsto ((n \otimes \sigma) \mapsto \langle f(\sigma), n \rangle)\,.\end{aligned}$$

Furthermore, there is a well-defined evaluation homomomorphism of left R-modules:

$$H_i(\overline{\mathrm{Hom}_{\text{left-}R}(N \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}, \widetilde{Y}), S)}) \to \overline{\mathrm{Hom}_{\text{left-}R}(H_i(X, Y; N), S)}$$

Proof. We begin by showing that the first map is well-defined. Given f in $\mathrm{Hom}_{\text{right-}\mathbb{Z}[\pi]}(\overline{C_*(\widetilde{X},\widetilde{Y})},M)$, we check that $\kappa(f)$ is well-defined. Given $\gamma \in \pi$, $n \in N$ and $\sigma \in \overline{C(\widetilde{X},\widetilde{Y})}$, we need to check that $\kappa(f)(n \otimes \gamma\sigma) = \kappa(f)(n\gamma \otimes \sigma)$. Using successively the definition of $\kappa(f)$, the definition of the right action on $\overline{C_*(\widetilde{X},\widetilde{Y})}$, the π-invariance of the inner product and the definition of $\kappa(f)$, we get

$$\kappa(f)(n \otimes \gamma\sigma) = \langle f(\gamma\sigma), n\rangle = \langle f(\sigma)\gamma^{-1}, n\rangle = \langle f(\sigma), n\gamma\rangle = \kappa(f)(n\gamma \otimes \sigma),$$

as desired. Next, we check that κ is left R-linear i.e that $r \cdot \kappa(f) = \kappa(r \cdot f)$ for r in R. Using successively the definition of the module structures, the definition of κ, the anti-linearity of $\langle -,-\rangle$, the definition of the module structures and the definition of $\kappa(f)$, we get

$$(r \cdot \kappa(f))(n \otimes \sigma) = \kappa(f)(n \otimes \sigma)\overline{r} = \langle f(\sigma), n\rangle\overline{r} = \langle rf(\sigma), n\rangle = \langle (r \cdot f)(\sigma), n\rangle = \kappa(r \cdot f)(n \otimes \sigma).$$

Finally, the fact that $\kappa(f)$ is indeed left R-linear follows from the linearity of $\langle -,-\rangle$ in its first variable. The second assertion is left to the reader and so our last task is to show that κ is indeed an isomorphism.

Injectivity immediately follows from the non-degeneracy of $\langle -,-\rangle$. To prove nonsingularity, assume we are given a left R-linear map $\Phi\colon N \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X},\widetilde{Y}) \to S$; our goal is to produce a right $\mathbb{Z}[\pi]$-linear map $f\colon \overline{C_*(\widetilde{X},\widetilde{Y})} \to M$ such that $\Phi(n \otimes \sigma) = \langle f(\sigma), n\rangle$ for all n in N and all σ in $C_*(\widetilde{X},\widetilde{Y})$. Since $\langle -,-\rangle$ is nonsingular, M and $\overline{\mathrm{Hom}_{\text{left-}R}(N,S)}$ are isomorphic as $(R,\mathbb{Z}[\pi_1(X)])$-bimodules. Given σ in $\overline{C_*(\widetilde{X},\widetilde{Y})}$, define $f(\sigma)$ as follows: observe that the map Φ determines an element $\Phi(- \otimes \sigma)$ of $\overline{\mathrm{Hom}_{\text{left-}R}(N,S)}$, and define $f(\sigma)$ as the corresponding element in M. $\square$

Summarizing, if we are given a ring R with involution, $(R,\mathbb{Z}[\pi_1(X)])$-bimodules M,N, a (R,R)-bimodule S, and a nonsingular $\pi_1(X)$-invariant sesquilinear pairing $\langle -,-\rangle\colon M \times N \to S$, Lemma 5.4.2 produces an *evaluation map*

$$\mathrm{ev}\colon H^i(X,Y;M) \to \overline{\mathrm{Hom}_{\text{left-}R}(H_i(X,Y;N),S)}.$$

The following example is important in the study of the Blanchfield pairing, see Chapter 6.

Example 5.4.3. Returning to the case described in Example 5.4.1, the map $Q_\mu/\Lambda_\mu \times \Lambda_\mu \to Q_\mu/\Lambda_\mu, (p,q) \mapsto p\overline{q}$ induces an evaluation map $H^k(X;Q_\mu/\Lambda_\mu) \to \overline{\mathrm{Hom}_{\Lambda_\mu}(H_i(X;\Lambda_\mu),Q_\mu/\Lambda_\mu)}$ which we encountered in Section 2.4 for $\mu = 1$.

The evaluation map unfortunately has no reason to be an isomorphism. If R is a principal ideal domain, then the universal coefficient theorem implies that $H^i(X,Y;N)$ decomposes as a direct sum of $\overline{\mathrm{Hom}_{\text{left-}R}(H_i(X,Y;N),S)}$ and an Ext term. In general, the evaluation map can be studied using the universal coefficient spectral sequence of Theorem 2.5.7, as follows.

Theorem 5.4.4. *Let R be a ring with involution, let M,N be $(R,\mathbb{Z}[\pi_1(X)])$-bimodules and let S be a (R,R)-bimodule. Furthermore, let $\langle -,-\rangle\colon M \times N \to S$ be a nonsingular $\pi_1(X)$-invariant sesquilinear pairing. Then there exists a spectral sequence*

1. *converging to $H^*(X,Y;M)$,*
2. *with $E_2^{p,q} \cong \mathrm{Ext}^q_R(H_p(X,Y;N),S)$,*

3. *with differentials d^r of degree $(1-r, r)$.*

More specifically, there is a filtration

$$0 \subset F_0^n \subset F_1^n \subset \cdots \subset F_n^n = H^n(X, Y; M)$$

with $F_p^n / F_{p-1}^n \cong E_\infty^{p,n-p}$. All objects and isomorphisms are as left *R-modules.*

Proof. Use the first assertion of Lemma 5.4.2 to identify $H^*(X, Y; M)$ with the homology of the cochain complex $\overline{\mathrm{Hom}_{\text{left-}R}(N \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}, \widetilde{Y}), S)}$. The result now follows from the spectral sequence of Theorem 2.5.7. □

5.5 Poincaré duality

In this section, we review Poincaré duality with twisted coefficients. This material goes at the very least back to Wall [157], but we take the time to give some details since there are many conflicting conventions in the literature. References include [73, 74, 102, 123, 136].

Let N be a compact oriented n-manifold. Endow $\mathbb{Z}[\pi_1(N)]$ with its natural $(\mathbb{Z}[\pi_1(N)], \mathbb{Z}[\pi_1(N)])$-bimodule structure. Assume N can be triangulated, pick a triangulation for N (which we also denote by N) and let N' denote the manifold N equipped with the dual triangulation, see [150, Section 14] or [74, Section 4.1] for details. By [123, Lemma 1], the left $\mathbb{Z}[\pi_1(N)]$-module $C_{n-k}(\widetilde{N})$ is canonically isomorphic to the $\mathbb{Z}[\pi_1(N)]$-module $\overline{\mathrm{Hom}_{\text{left-}\mathbb{Z}[\pi_1(N)]}(C_k(\widetilde{N'}, \widetilde{\partial N'}), \mathbb{Z}[\pi_1(N)])}$ and the resulting chain isomorphisms are sometimes called *universal Poincaré duality*. The next theorem (which is well-known) shows that Poincaré duality holds with *any* coefficients [157].

Theorem 5.5.1. *Let N be a compact oriented n-manifold and let M be a $(R, \mathbb{Z}[\pi_1(N)])$-bimodule. There are Poincaré duality isomorphisms $H_k(N, \partial N; M) \cong H^{n-k}(N; M)$ and $H_k(N; M) \cong H^{n-k}(N, \partial N; M)$.*

Proof. Set $\pi := \pi_1(N)$ and let M be a $(R, \mathbb{Z}[\pi])$-bimodule. If one tensors the universal Poincaré dualité isomorphisms by M, then one deduces that $M \otimes_{\mathbb{Z}[\pi]} C_{n-*}(\widetilde{N})$ is chain isomorphic to $M \otimes_{\mathbb{Z}[\pi]} \overline{\mathrm{Hom}_{\text{left-}\mathbb{Z}[\pi]}(C_*(\widetilde{N'}, \widetilde{\partial N'}), \mathbb{Z}[\pi])}$. Consequently, to prove the theorem, it is enough to show that there is a left $\mathbb{Z}[\pi]$-module chain isomorphism

$$M \otimes_{\mathbb{Z}[\pi]} \overline{\mathrm{Hom}_{\text{left-}\mathbb{Z}[\pi]}(C_*(\widetilde{N'}, \widetilde{\partial N'}), \mathbb{Z}[\pi])} \cong \mathrm{Hom}_{\text{right-}\mathbb{Z}[\pi]}(\overline{C_*(\widetilde{N'}, \widetilde{\partial N'})}, M).$$

This can be proved in two steps. Firstly, note that there is a canonical isomorphism of left $\mathbb{Z}[\pi]$-modules between $\overline{\mathrm{Hom}_{\text{left-}\mathbb{Z}[\pi]}(C_*(\widetilde{N'}, \widetilde{\partial N'}), \mathbb{Z}[\pi])}$ and $\mathrm{Hom}_{\text{right-}\mathbb{Z}[\pi]}(\overline{C_*(\widetilde{N'}, \widetilde{\partial N'})}, \mathbb{Z}[\pi])$, for instance by applying Lemma 5.4.2 with $R = N = M = S = \mathbb{Z}[\pi]$ and the inner product $M \otimes N \to S$ given by $(p, q) \mapsto p\overline{q}$. Secondly since $\overline{C_*(\widetilde{N'}, \widetilde{\partial N'})}$ is finitely generated and free as a right $\mathbb{Z}[\pi]$-module, we deduce that $M \otimes_{\mathbb{Z}[\pi]} \mathrm{Hom}_{\text{right-}\mathbb{Z}[\pi]}(\overline{C_*(\widetilde{N'}, \widetilde{\partial N'})}, \mathbb{Z}[\pi])$ is canonically left $\mathbb{Z}[\pi]$-isomorphic to $\mathrm{Hom}_{\text{right-}\mathbb{Z}[\pi]}(\overline{C_*(\widetilde{N'}, \widetilde{\partial N'})}, M \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi])$. Combining these isomorphisms concludes the proof. □

5.6 Twisted intersection pairings

In this section, we review twisted intersection forms. Once again, although there are no new results, we give some details since many conflicting conventions appear in the literature. References include [31, 102, 157].

Let W be a compact oriented $2k$-manifold. Let R be a ring with involution, let M be a $(R, \mathbb{Z}[\pi_1(W)])$-bimodule and let S be a (R,R)-bimodule. Furthermore, pick a nonsingular $\pi_1(W)$-invariant sesquilinear pairing $M \times M \to S$. Compose the homomorphism induced by the inclusion $(W, \emptyset) \to (W, \partial W)$ with the Poincaré duality isomorphism of Theorem 5.5.1 and the evaluation homomorphism described in Section 5.4. The result is the following homomorphism of left R-modules.

$$\Phi\colon H_k(W;M) \to H_k(W,\partial W;M) \xrightarrow{\text{PD}} H^k(W;M) \xrightarrow{\text{ev}} \overline{\text{Hom}_{\text{left-}R}(H_k(W;M),S)}.$$

The main definition of this section is the following.

Definition 17. The *twisted intersection pairing*

$$\lambda_M(W)\colon H_k(W;M) \times H_k(W;M) \to S$$

is defined by $\lambda_M(W)(x,y) = \Phi(y)(x)$.

Note that while $\lambda_M(W)$ is $(-1)^k$-hermitian, it has no reason to be nonsingular. In particular, the space $\text{im}(H_1(\partial W;M) \to H_1(W;M))$ is annihilated by $\lambda_M(W)$. We start by unravelling Definition 17 in the untwisted case.

Example 5.6.1. As we saw in Example 5.2.2, if $R = \mathbb{Z}$ and $M = \mathbb{Z}$ is given the trivial right $\mathbb{Z}[\pi_1(X)]$-module structure $n \cdot \gamma = n$ with the left $\mathbb{Z}$-module structure induced by multiplication, then $H_*(W;\mathbb{Z})$ is nothing but the cellular homology of W. In this case, assuming W to be connected, we claim that $\lambda_{\mathbb{Z}}(W)(a,b) = \langle PD(a), b\rangle$ coincides with the usual definition $\langle \text{PD}(a) \cup \text{PD}(b), [W]\rangle$ of the intersection form, where $\cup$ denotes the cup product, $[W]$ is the fundamental class and $\langle -,-\rangle$ denotes the evaluation of cohomology classes on homology classes.

To see this, we use three standard facts of algebraic topology. First, under the identification of $H_0(W)$ with $\mathbb{Z}$, looking at the definition of the cap product $\cap$, we see that capping classes of complementary dimensions corresponds to evaluation. Secondly, the relation $(\varphi \cup \psi) \cap \sigma = \varphi \cap (\psi \cap \sigma)$ holds, where φ, ψ and σ are (co)homology classes of the approriate dimension [23, Proposition 5.1 (iii)]. Thirdly, the Poincaré duality isomorphism described in Section 5.5 satisfies the relation $PD^{-1}(\varphi) = \varphi \cap [W]$. Using successively these facts, the claim follows:

$$\langle \text{PD}(a) \cup \text{PD}(b), [W]\rangle = (PD(a) \cup PD(b)) \cap [W] = PD(a) \cap (PD(b) \cap [W]) = PD(a) \cap b = \langle PD(a), b\rangle.$$

In particular, as we briefly mentioned in Section 2.3.2, in dimension four, $\lambda_{\mathbb{Z}}(W)$ can be computed by counting algebraic intersections of embedded surfaces [23, Section 6.11]. Finally, note that in the untwisted case with rational coefficients, the *signature* of W is defined as $\text{sign}(W) := \text{sign}(\lambda_{\mathbb{Q}}(W))$.

Most often, we shall be dealing with twisted homology Λ_μ-modules. In this case, the twisted intersection pairing can also be understood geometrically. Before spelling this out, let us start out in a slightly greater generality which will be useful in Section 8.5.

Example 5.6.2. Let W be a compact connected oriented 4-manifold. Set $\pi = \pi_1(W)$ and let $\pi^{(n)} = [\pi^{(n-1)}, \pi^{(n-1)}]$ denote its derived series starting at $\pi^{(0)} = \pi$. The projection $\pi \to \pi/\pi^{(n)}$ gives rise to the left $\mathbb{Z}[\pi/\pi^{(n)}]$-modules $H_k(W; \mathbb{Z}[\pi/\pi^{(n)}])$ and we may consider the $\mathbb{Z}[\pi/\pi^{(n)}]$-twisted intersection pairing

$$\lambda_n(W) := \lambda_{\mathbb{Z}[\pi/\pi^{(n)}]}(W)\colon H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) \times H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) \to \mathbb{Z}[\pi/\pi^{(n)}],$$

as in Definition 17. As we mentioned above, of particular interest to us is the case where $n = 1$ and $\pi/\pi^{(1)} = H_1(W)$ is free abelian of rank μ. In this situation, $\mathbb{Z}[\pi/\pi^{(1)}]$ is nothing but the commutative ring Λ_μ of Laurent polynomials. We can now be even more explicit: if a and b belong to $H_2(W; \Lambda_\mu)$, then the twisted intersection form is given by

$$\lambda_1(W)(a, b) = \sum_{g \in \mathbb{Z}^\mu} \langle a, gb \rangle g \in \Lambda_\mu,$$

where $\langle \ , \ \rangle$ denotes the algebraic intersection number on the $\mathbb{Z}^\mu$-cover $\widehat{W}$ of W (and not the evaluation homomorphism) [157]. The same remark holds if we replace Λ_μ with the localized ring $\Lambda_S = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}, (1-t_1)^{-1}, \ldots, (1-t_\mu)^{-1}]$.

We conclude with an example which shall be used several times both in Chapter 7 and in Chapter 8.

Example 5.6.3. Let W be a 4-dimensional manifold with (possibly empty) boundary. Let $\psi\colon \pi_1(W) \to \mathbb{Z}^\mu = \langle t_1, \ldots, t_\mu \rangle$ be a homomorphism and let $\omega = (\omega_1, \ldots, \omega_\mu)$ lie in $\mathbb{T}^\mu$. Composing the induced map $\mathbb{Z}[\pi_1(W)] \to \mathbb{Z}[\mathbb{Z}^\mu]$ with the map $\mathbb{Z}[\mathbb{Z}^\mu] \to \mathbb{C}$ which evaluates t_i at ω_i, produces a morphism $\phi\colon \mathbb{Z}[\pi_1(W)] \to \mathbb{C}$ of rings with involutions. In turn, ϕ endows $\mathbb{C}$ with a $(\mathbb{C}, \mathbb{Z}[\pi_1(X)])$-bimodule structure. To emphasize the choice of ω, we shall write $\mathbb{C}^\omega$ instead of $\mathbb{C}$. Since $\mathbb{C}^\omega$ is a $(\mathbb{C}, \mathbb{Z}[\pi_1(W)])$-bimodule, we consider the complex vector spaces $H_k(W; \mathbb{C}^\omega)$ which we already encountered in Example 5.2.4. Using Definition 17, we obtain the twisted intersection form $\lambda_{\mathbb{C}^\omega}(W)\colon H_2(W; \mathbb{C}^\omega) \times H_2(W; \mathbb{C}^\omega) \to \mathbb{C}$. We also set $\operatorname{sign}_\omega(W) := \operatorname{sign}(\lambda_{\mathbb{C}^\omega}(W))$.

5.7 Further remarks

This section is devoted to collecting some results which shall be used in Chapters 6, 8 9 and 11; it is organized as follows. Subsection 5.7.1 discusses induced maps in twisted homology. Subsection 5.7.2 deals with some technicalities related to the long exact sequence of the pair. Finally, Subsection 5.7.3 is concerned with Λ_S-twisted homology, where Λ_S denotes the localization $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}, (1-t_1)^{-1}, \ldots, (1-t_\mu)^{-1}]$ of the Laurent polynomial ring Λ_μ.

5.7.1 Induced maps in twisted homology

Let T and S be rings and let M be a right S-module. Given a ring homomorphism $f\colon T \to S$, the right T module f^*M obtained by *restriction of scalars* has the same underlying abelian group as M but is endowed with the action $m \cdot t = mf(t)$. Note that restriction of scalars produces a functor $\mathbf{ModS} \to \mathbf{ModT}$. Let X, Y be CW-complexes and let M be a $(R, \mathbb{Z}[\pi_1(Y)])$-bimodule. Any continuous map $f\colon X \to Y$ induces a group homomorphism $\pi_1(X) \to \pi_1(Y)$ and thus a ring homomorphism $\mathbb{Z}[\pi_1(X)] \to \mathbb{Z}[\pi_1(Y)]$. Consequently, by restriction of scalars, M naturally becomes a $(R, \mathbb{Z}[\pi_1(X)])$-bimodule which we denote by $f^*(M)$.

Lemma 5.7.1. *Given a $(R, \mathbb{Z}[\pi_1(Y)])$-bimodule M, any continuous map $f\colon X \to Y$ between CW complexes induces a left R-module homomorphism $f_*^M\colon H_*(X; f^*M) \to H_*(Y; M)$.*

Proof. Fix basepoints for X and Y. The map f induces a group homomorphism $f_*\colon \pi_1(X) \to \pi_1(Y)$ and a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ on the universal cover. Consequently $\widetilde{f}$ gives rise to a $\mathbb{Z}$-linear map $C_*(\widetilde{X}) \to C_*(\widetilde{Y})$ which we also denote by $\widetilde{f}$ and which satisfies $\widetilde{f}(\gamma\sigma) = f_*(\gamma)\widetilde{f}(\sigma)$ for γ in $\pi_1(X)$ and σ in $C_*(\widetilde{X})$. We define $f_\#\colon f^*(M) \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X}) \to M \otimes_{\mathbb{Z}[\pi_1(Y)]} C_*(\widetilde{Y})$ by $m \otimes \sigma \mapsto m \otimes \widetilde{f}(\sigma)$. Let us check that $f_\#$ is well defined. Given γ in $\pi_1(X)$ and σ in $C_*(\widetilde{X})$, we must check that $m \otimes \gamma\sigma$ and $mf_*(\gamma) \otimes \sigma$ have the same image under $f_\#$. Since these elements are respectively sent to $m \otimes \widetilde{f}(\gamma\sigma)$ and $mf_*(\gamma) \otimes \widetilde{f}(\sigma)$, the claim now follows from the equality $\widetilde{f}(\gamma\sigma) = f_*(\gamma)\widetilde{f}(\sigma)$ which was mentioned above. It can then be checked that $f_\#$ is a R-linear chain map and thus induces the desired map in homology. □

Let $f\colon R \to S$ and $g\colon S \to T$ be ring homomorphisms and let M be a right T-module. Observe that $(gf)^*M$ and $f^*(g^*M)$ are equal as right R-modules. To see this, first note that they are equal as abelian groups. Then, notice that the right R-action on $(gf)^*M$ is given by $m \cdot r = mg(f(r))$ while the right R-action on $f^*(g^*(M))$ is $m \cdot r = m \cdot_{g^*(M)} f(r) = mg(f(r))$. Somewhat pedantically, we are considering the contravariant functor $\mathbf{Rng} \to \mathbf{Cat}$ which associates to a ring R the category $\mathbf{ModR}$ of right R-modules, and to a ring homomorphism $f\colon R \to S$ the restriction of scalars functor $\mathbf{ModS} \to \mathbf{ModR}$.

Corollary 5.7.2. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps, and M is a $(R, \mathbb{Z}[\pi_1(Z)])$-bimodule, then the left R-linear maps $H_*(X; (gf)^*M) \overset{(gf)_*^M}{\to} H_*(Z; M)$ and $H_*(X; f^*(g^*M)) \overset{f_*^{g^*M}}{\to} H_*(Y; g^*M) \overset{g_*^M}{\to} H_*(Z; M)$ agree.*

Proof. We argued in the discussion preceding the lemma that $(gf)^*M$ is equal to $f^*(g^*M)$, and we know from Lemma 5.7.1 that there are corresponding well defined maps on homology. Consequently, we just need to check that these two maps agree. The fist map sends $m \otimes \sigma$ to $m \otimes \widetilde{gf}(\sigma)$, while the second sends it to $m \otimes \widetilde{g}\widetilde{f}(\sigma)$. By covering space theory (and fixing basepoints), these two twisted chains are equal. □

5.7.2 The long exact sequence of the pair

Let X be a CW complex and let $Y \subset X$ be a possibly empty subcomplex. Denote by $p\colon \widetilde{X} \to X$ the universal cover of X and set $\widetilde{Y} := p^{-1}(Y)$. The left action of $\pi_1(X)$ on $\widetilde{X}$ endows the chain subcomplex $C_*(\widetilde{Y})$ of $C_*(\widetilde{X})$ with the structure of a left $\mathbb{Z}[\pi_1(X)]$-module. Given a ring R and a $(R, \mathbb{Z}[\pi_1(X)])$-bimodule M, we define the following left R-module:

$$H_*(Y \subset X; M) := H_*(M \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{Y})).$$

Since short exact sequences of chain complexes give rise to long exact sequences in homology, there is a long exact sequence $\ldots \to H_k(Y \subset X; M) \to H_k(X; M) \to H_k(X, Y; M) \to \ldots$

of left R-modules. From the point of view of "classical" cellular homology, this is slightly unsatisfactory since one might wish to use $H_*(Y;M)$ instead of $H_*(Y \subset X;M)$.

To achieve this, first note that a $(R,\mathbb{Z}[\pi_1(X)])$-bimodule M can also be viewed as a $(R,\mathbb{Z}[\pi_1(Y)])$-bimodule thanks to the inclusion induced map $\pi_1(X) \to \pi_1(Y)$. Next, assume that Y is connected, let $\widehat{Y}$ be the universal cover of Y and let $\widetilde{Y}$ be the inverse image of Y under the universal covering map $p\colon \widetilde{X} \to X$. It is proved in [73, Lemma 2.1] that there is a canonical chain isomorphism

$$M \otimes_{\mathbb{Z}[\pi_1(Y)]} C_*(\widehat{Y}) \to M \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{Y}) \tag{5.1}$$

of left R-modules and in particular $H_*(Y \subset X;M)$ is canonically isomorphic to $H_*(Y;M)$.

Note that [73, Lemma 2.1] actually deals with disconnected subcomplexes. The subtlety lies in the choice of the basepoint: for instance, note that in (5.7.2), the basepoint for $\pi_1(X)$ must be taken in Y for the notation $H_*(Y;M)$ to make sense. Nevertheless, there is a long exact sequence

$$\ldots \to H_n(Y;M) \to H_n(X,Y;M) \to H_n(X,Y;M) \to H_{n-1}(Y;M) \to \ldots$$

in homology but with the following caveat: if $Y = \bigsqcup_{i \in I} Y_i$ is disconnected, $H_*(Y;M)$ is defined as $\bigoplus_{i \in I} H_i(Y_i;M)$, see [73, Section 2.1] for details.

5.7.3 Λ_S-coefficients

We now restrict to a situation which we have already encountered in Section 3.3 and that will figure prominently in Chapter 6. Namely, let $\Lambda_\mu := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$ denote the ring of Laurent polynomials, let S be the multiplicative subset of Λ_μ generated by $(1-t_1), \ldots, (1-t_\mu)$ and let Λ_S be the localization of Λ_μ with respect to S. Since localizations are flat, note that $H_*(X,Y;\Lambda_S)$ is isomorphic to $\Lambda_S \otimes_{\Lambda_\mu} H_*(X,Y;\Lambda_\mu)$, see Section 2.5.

Although the usefulness of Λ_S coefficients should already be apparent from Corollary 3.3.11 and Remark 3.3.12, another reason to use them can be stated informally as follows: "Λ_S coefficients kill the twisted H_0 module". The next lemma contains a more precise statement.

Lemma 5.7.3. *Let X be a CW-complex.*

1. *If the composition $\pi_1(S^1) \to \pi_1(X \times S^1) \xrightarrow{\varphi} \mathbb{Z}^\mu$ sends a generator of $\pi_1(S^1)$ to a non-trivial element z of $\mathbb{Z}^\mu$, then the chain complex $C_*(X \times S^1; \mathbb{Z}[\mathbb{Z}^\mu][(z-1)^{-1}])$ is acyclic.*
2. *If z is a non-trivial element in the image of $\psi\colon \pi_1(X) \to \mathbb{Z}^\mu$, then $H_0(X;\mathbb{Z}[\mathbb{Z}^\mu][(z-1)^{-1}])$ vanishes.*

Proof. A proof of the first statement can be found in [132, Example 2.7]. To prove the second statement, we first describe $H_0(X;M)$ for an arbitrary $(R,\mathbb{Z}[\pi_1(X)])$-bimodule M: it is isomorphic to the quotient of M by the submodule generated by the $m\gamma - m$, with m in M an γ in $\pi_1(X)$. This identification of $H_0(X;M)$ with the *coinvariants* of M can for instance be seen using the relationship with group homology, see Remark 5.2.3. The second statement now follows by taking $M = \mathbb{Z}[\mathbb{Z}^\mu]$ and using the fact that $\mathbb{Z}[\mathbb{Z}^\mu][(z-1)^{-1}]$ is flat over $\mathbb{Z}[\mathbb{Z}^\mu]$. $\square$

Chapter 6

Blanchfield pairing of colored links

6.1 Introduction and statement of the results

In Section 2.4, we saw that the Alexander module $H_1(X_K;\mathbb{Z}[t^{\pm 1}])$ of a knot K supports a nonsingular Hermitian $\mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$-valued pairing $\mathrm{Bl}(K)$ called the Blanchfield pairing. Furthermore, recall that we denote by $\Lambda_S := \mathbb{Z}[t_1^{\pm 1},\dots,t_\mu^{\pm 1},(1-t_1)^{-1},\dots,(1-t_\mu)^{-1}]$ the localization of the ring $\Lambda_\mu = \mathbb{Z}[t_1^{\pm 1},\dots,t_\mu^{\pm 1}]$. As we shall see in Section 6.2, in the case of links, the Blanchfield pairing generalizes to a sesquilinear pairing

$$\mathrm{Bl}(L)\colon TH_1(X_L;\Lambda_S)\times TH_1(X_L;\Lambda_S)\to Q_\mu/\Lambda_S$$

on the torsion submodule of the Λ_S coefficient Alexander module $H_1(X_L;\Lambda_S)$ [87, 88]. There are two main reasons for which Λ_S coefficients are used instead of the more conventional Λ_μ coefficients. The first is to ensure that the Alexander module $H_1(X_L;\Lambda_S)$ admits a square presentation matrix: the corresponding statement is false over Λ_μ, see Remark 3.3.10. The second is to guarantee that the Blanchfield pairing is non-degenerate after quotienting $TH_1(X_L;\Lambda_S)$ by the so-called *maximal pseudonull submodule*, see [87]. Note that for knots, the Alexander module over Λ_S is the same as the Alexander module over $\Lambda=\mathbb{Z}[t^{\pm 1}]$: indeed multiplication by $t-1$ induces an isomorphism on $H_1(X_K;\Lambda)$, see Remark 2.2.2.

In the case of links, although $\mathrm{Bl}(L)$ is used to investigate concordance [30, 44, 45, 99], unlinking numbers and splitting numbers [21], there is no general formula to compute it. However, since we saw in Theorem 2.4.8 that the Blanchfield pairing of a knot K can be computed using Seifert matrices, one could expect generalized Seifert matrices to play a role. More precisely, given a Seifert matrix A for K of size $2g$, Theorem 2.4.8 states that the Blanchfield pairing of K is isometric to the linking pairing

$$\begin{aligned}\Lambda^{2g}/(tA-A^T)\Lambda^{2g}\times\Lambda^{2g}/(tA-A^T)\Lambda^{2g} &\to Q/\Lambda \\ (a,b) &\mapsto a^T(t-1)(A-tA^T)^{-1}\bar{b}.\end{aligned}\tag{6.1}$$

Implicit in this statement is the fact that the Alexander module of a knot admits a square presentation matrix. In general however, $\mathrm{Bl}(L)$ is defined on $TH_1(X_L;\Lambda_S)$ and to the best of our knowledge, the latter Λ_S-module has no reason of admitting a square presentation matrix. Thus a direct generalization of (6.1) seems out of reach. On the other hand, if we restrict to links whose Alexander polynomial is nonzero, then $TH_1(X_L;\Lambda_S)$ is equal to $H_1(X_L;\Lambda_S)$ and consequently admits a square presentation matrix, see Corollary 3.3.11. Restricting to this setting, a computation of $\mathrm{Bl}(L)$ in the spirit of Theorem 2.4.8 once again seems within reach.

In fact, when the Alexander polynomial of L is nonzero, we obtained such a result together with Stefan Friedl and Enrico Toffoli [52]. In order to give a precise statement, recall from Section 3.2 that a C-complex for a μ-colored link L consists of a collection of Seifert surfaces $F_1, \dots, F_\mu$ for the sublinks $L_1, \dots, L_\mu$ that intersect only pairwise along clasps. Given such a C-complex and a sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_\mu)$ of ± 1's, we also saw in Section 3.2 that there are 2^μ generalized Seifert matrices A^ε which extend the usual Seifert matrix. Furthermore, recall from Definition 11 that the associated *C-complex matrix* is the Λ_μ-valued square matrix

$$H \;:=\; \sum_{\varepsilon} \prod_{i=1}^{\mu} (1 - t_i^{\varepsilon_i})\, A^\varepsilon,$$

where the sum is on all sequences $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_\mu)$ of ± 1's. Finally, recall from Remark 3.3.13 that if a colored link L has non-zero Alexander polynomial, then $\det(H)$ is non-zero for any C-complex matrix H. In this situation, H^{-1} shall denote the inverse of H over Q_μ.

In [52, Theorem 1.1], we proved the following result.

Theorem 6.1.1. *Let L be a μ-colored link and let H be a C-complex matrix for L of size n. If $H_1(X_L; \Lambda_S)$ is Λ_S-torsion, then the Blanchfield pairing* $\mathrm{Bl}(L)$ *is isometric to the pairing*

$$\begin{aligned} \Lambda_S^n / H^T \Lambda_S^n \times \Lambda_S^n / H^T \Lambda_S^n \;&\to\; Q_\mu / \Lambda_S \\ (a, b) \;&\mapsto\; -a^T H^{-1} \overline{b}. \end{aligned} \tag{6.2}$$

Let us briefly discuss some features of Theorem 6.1.1 even though it will be generalized in Theorem 6.1.2 below. Firstly, in [52, Theorem 1.1] we actually required the C-complex matrix to arise from a totally connected C-complex. However, this assumption later turned out not to be necessary, see Theorem 6.1.2. Secondly, note that Theorem 6.1.1 recovers Theorem 2.4.8 in the knot case, see Theorem 6.6.7 and [52, Section 5]. Finally, as we shall see in Section 6.4, although Theorem 6.1.1 only holds for links with non-zero Alexander polynomial, its proof in fact recovers Corollary 3.3.11.

Returning to arbitrary links, since the torsion submodule $TH_1(X_L; \Lambda_S)$ of $H_1(X_L; \Lambda_S)$ has no reason of admitting a square presentation matrix, a direct generalization of (6.2) once again seems difficult. In order to circumvent this issue, we adapt the definition of the pairing described in (6.2) as follows. Let Δ denote the order of $\mathrm{Tor}_{\Lambda_S}(\Lambda_S^n / H^T \Lambda_S^n)$, the Λ_S-torsion submodule of $\Lambda_S^n / H^T \Lambda_S^n$. Note that for any class $[x]$ in $\mathrm{Tor}_{\Lambda_S}(\Lambda_S^n / H^T \Lambda_S^n)$, there exists an x_0 in Λ_S^n such that $\Delta x = H^T x_0$. As we shall see in Proposition 6.6.2, the assignment $(v, w) \mapsto \frac{1}{\Delta^2} v_0^T H \overline{w_0}$ induces a well-defined pairing

$$\lambda_H \colon \mathrm{Tor}_{\Lambda_S}\big(\Lambda_S^n / H^T \Lambda_S^n\big) \times \mathrm{Tor}_{\Lambda_S}\big(\Lambda_S^n / H^T \Lambda_S^n\big) \to Q_\mu / \Lambda_S,$$

which recovers the pairing described in (6.2) when $\det(H) \neq 0$. The main theorem of this chapter, which appeared in [51], reads as follows.

Theorem 6.1.2. *The Blanchfield pairing of a colored link L is isometric to the pairing $-\lambda_H$, where H is any C-complex matrix for L.*

Theorem 6.1.2 generalizes Theorem 6.1.1 to links whose Alexander module $H_1(X_L;\Lambda_S)$ is not torsion and recovers it if $H_1(X_L;\Lambda_S)$ is torsion. As we mentioned above, the statement of Theorem 6.1.1 which appeared in [52, Theorem 1.1] required H to arise from a *totally connected* C-complex, whereas Theorem 6.1.2 gets rid of this extraneous assumption. Furthermore, Theorem 6.1.2 also recovers previous computations (see [87, 88]) of $\mathrm{Bl}(L)$ when L is a boundary link, see Theorem 6.6.7. Finally, note that to the best of our knowledge, Theorem 6.1.2 was not even known in the case of oriented links (i.e. $\mu = 1$) and the result might be of independent interest.

While the Blanchfield pairing of a knot is known to be Hermitian and nonsingular, see Proposition 2.4.7, the corresponding statements for links require some more work. The Hermitian property of $\mathrm{Bl}(L)$ was taken care of by Powell [133], whereas Hillman [87] quotients $TH_1(X_L;\Lambda_S)$ by its *maximal pseudo null submodule* in order to turn $\mathrm{Bl}(L)$ into a non-degenerate pairing, see also [21, Section 2.5]. Even though we avoid discussing the non-degeneracy of the Blanchfield pairing, we observe that Theorem 6.1.2 provides a quick proof that $\mathrm{Bl}(L)$ is hermitian. Namely, using Δ_L^{tor} to denote the first non-vanishing Alexander polynomial of L over Λ_S, we obtain the following corollary.

Corollary 6.1.3. *The Blanchfield pairing of a link L is Hermitian and takes values in $\Delta_L^{\mathrm{tor}\,-1}\Lambda_S/\Lambda_S$.*

Since the definition of the pairing λ_H is quite manageable, we also use Theorem 6.1.2 to obtain quick proofs regarding the behavior of $\mathrm{Bl}(L)$ under connected sums, disjoint unions, band claspings, mirror images and orientation reversals, see Proposition 6.6.4, Proposition 6.6.5 and Proposition 6.6.6.

In order to prove Theorem 6.1.2, we shall push a C-complex for L in the 4-ball and study the algebraic topology of the exterior. Notably, we will describe the fundamental group of this space in Proposition 6.3.3 and its twisted homology Λ_S-modules in Corollary 6.3.4 and Proposition 6.4.2. Our main technical result however will be the computation of its twisted intersection form. Since this result plays an important role in Chapter 8, we state it as follows:

Theorem 6.1.4. *Let L be a colored link, let F be a C-complex for L and let W be the exterior of a push-in of F into the 4-ball D^4. Then the intersection pairing on $H_2(W;\Lambda_S)$ is represented by a C-complex matrix H associated to F.*

6.2 The Blanchfield pairing: definition and properties

This section deals with the definition of the Blanchfield pairing of a link. References include [87, 88, 107, 133].

Let $L = L_1 \cup \cdots \cup L_\mu$ be a colored link and denote its exterior by X_L. Recall from Section 3.3 that the epimorphism $\pi_1(X_L) \to \mathbb{Z}^\mu$ defined by $\gamma \mapsto (\ell k(\gamma, L_1), \ldots, \ell k(\gamma, L_\mu))$ gives rise to the *Alexander module* $H_1(X_L;\Lambda_\mu)$ of L. Since Λ_S is flat over Λ_μ, the Λ_S-Alexander module $H_1(X_L;\Lambda_S)$ is canonically isomorphic to $\Lambda_S \otimes_{\Lambda_\mu} H_1(X_L;\Lambda_\mu)$. Furthermore, the short exact sequence $0 \to \Lambda_S \to Q_\mu \to Q_\mu/\Lambda_S \to 0$ of coefficients gives rise to the long exact sequence

$$\ldots \to H^k(X,Y;Q_\mu) \to H^k(X,Y;Q_\mu/\Lambda_S) \to H^{k+1}(X,Y;\Lambda_S) \to H^{k+1}(X,Y;Q_\mu) \to \ldots \tag{6.3}$$

in cohomology. The boundary homomorphism $H^k(X,Y;Q_\mu/\Lambda_S) \to H^{k+1}(X,Y;\Lambda_S)$ is referred to as the *Bockstein homomorphism* and will be denoted by BS. As usual, we denote by ev the evaluation map from $H^1(X_L;Q_\mu/\Lambda_S)$ to $\overline{\operatorname{Hom}_{\Lambda_S}(H_1(X_L;\Lambda_S),Q_\mu/\Lambda_S)}$. The following lemma replaces Lemma 2.4.5 in the case of links:

Lemma 6.2.1. *Let L be a link in S^3.*

1. *Poincaré duality restricts to a well-defined isomorphism*

$$TH_1(X_L;\Lambda_S) \to \ker(H^2(X_L;\Lambda_S) \to H^2(X_L;Q_\mu)).$$

2. *The Bockstein homomorphism induces a well-defined isomorphism*

$$\ker(H^2(X_L;\Lambda_S) \to H^2(X_L;Q_\mu)) \to \frac{H^1(X_L;Q_\mu/\Lambda_S)}{\ker(H^1(X_L;Q_\mu/\Lambda_S) \xrightarrow{BS} H^2(X_L;\Lambda_S))}.$$

3. *The evaluation map induces a well-defined homomorphism*

$$\frac{H^1(X_L;Q_\mu/\Lambda_S)}{\ker(H^1(X_L;Q_\mu/\Lambda_S) \xrightarrow{BS} H^2(X_L;\Lambda_S))} \to \overline{\operatorname{Hom}_{\Lambda_S}(TH_1(X_L;\Lambda_S),Q_\mu/\Lambda_S)}.$$

Proof. To prove the first assertion, we first show that the Poincaré duality isomorphism PD restricts as claimed. Namely, we claim that torsion elements in $H^2(X_L;\Lambda_S)$ are mapped to zero in $H^2(X_L;Q_\mu)$. Given x in $H^2(X_L;\Lambda_S)$, assume that there is a non-zero λ in Λ_S for which $\lambda x = 0$. Since $H^2(X_L;Q_\mu)$ is a Q_μ-vector space and λ is non-zero, this implies that x must be zero, proving the claim. Since PD is an isomorphism, the first statement will immediately follow if we manage to show $\ker(H^2(X_L;\Lambda_S) \to H^2(X_L;Q_\mu))$ is isomorphic to $TH^2(X_L;\Lambda_S)$. This is a consequence of the next claim, which was proved in Remark 2.5.9: $H^2(X_L;Q_\mu)$ is isomorphic to $Q_\mu \otimes_{\Lambda_S} H^2(X_L;\Lambda_S)$.

The second statement immediately follows from the long exact sequence displayed in (6.3): indeed, by exactness $\ker(H^2(X_L;\Lambda_S) \to H^2(X_L;Q_\mu))$ is equal to $\operatorname{im}(BS) \cong \frac{H^1(X_L;Q_\mu/\Lambda_S)}{\ker(\mathrm{BS})}$, where the isomorphism is induced by the Bockstein homomorphism. To prove the third statement, we must show that elements of $\ker(BS)$ evaluate to zero on elements of $TH_1(X_L;\Lambda_S)$. Since $\ker(BS) = \operatorname{im}(H^1(X_L;Q_\mu) \to H^1(X_L;Q_\mu/\Lambda_S))$, elements of $\ker(BS)$ are represented by cocycles which factor through Q_μ-valued homomorphisms. Since Q_μ is a field, these latter cocycles vanish on torsion elements, and thus so do the elements of $\ker(BS)$. □

We shall now define the adjoint of the Blanchfield pairing by following the same steps as in Section 2.4. Namely, we denote by Ω the composition

$$\begin{aligned} TH_1(X_L;\Lambda_S) &\xrightarrow{(i)} TH_1(X_L,\partial X_L;\Lambda_S) \\ &\xrightarrow{(ii)} \ker(H^2(X_L;\Lambda_S) \to H^2(X_L;Q_\mu)) \\ &\xrightarrow{(iii)} \frac{H^1(X_L;Q_\mu/\Lambda_S)}{\ker(H^1(X_L;Q_\mu/\Lambda_S) \xrightarrow{\mathrm{BS}} H^2(X_L;\Lambda_S))} \\ &\xrightarrow{(iv)} \overline{\operatorname{Hom}_{\Lambda_S}(TH_1(X_L;\Lambda_S),Q_\mu/\Lambda_S)} \end{aligned}$$

of the four Λ_S-homomorphisms defined as follows. The inclusion induced homomorphism $H_1(X_L;\Lambda_S) \to H_1(X_L,\partial X_L;\Lambda_S)$ gives rise to the first homomorphism. The next three homomorphisms are defined in Lemma 6.2.1, namely using Poincaré duality, the Bockstein homomorphism and the evaluation homomorphism. The main definition of this chapter is the following.

Definition 18. The *Blanchfield pairing* of a μ-colored link L is the pairing

$$\mathrm{Bl}(L)\colon TH_1(X_L;\Lambda_S)\times TH_1(X_L;\Lambda_S)\to Q_\mu/\Lambda_S$$

defined by $\mathrm{Bl}(L)(a,b)=\Omega(b)(a)$.

It follows from Definition 18 that the Blanchfield pairing is sesquilinear over Λ_S, in the sense that $\mathrm{Bl}(L)(pa,qb)=p\,\mathrm{Bl}(L)(a,b)\overline{q}$ for any a,b in $TH_1(X_L;\Lambda_S)$ and any p,q in Λ_S, see the proof of Proposition 2.4.7.

Remark 6.2.2. The homomorphisms $(i),(ii)$ and (iii) are in fact isomorphisms: for (ii) and (iii), this is contained in Lemma 6.2.1, while for (i), it follows from the long exact sequence of the pair $(X_L,\partial X_L)$ with Λ_S coefficients, see Lemma 5.7.3. Note that (i) is *not* an isomorphism with Λ_μ coefficients. This is a crucial difference with the knot case, see Lemma 2.4.4.

Despite the observations of Remark 6.2.2, it is *a priori* not clear from Definition 18 that $\mathrm{Bl}(L)$ is nonsingular. Indeed, while $(i),(ii)$ and (iii) are isomorphisms, it does not seem obvious to us that (iv) is an isomorphism. On the other hand, if $H_1(X_L;\Lambda_S)$ is Λ_S-torsion, then the argument showing that $\mathrm{Bl}(L)$ is nonsingular is the same as in the knot case, see the proof of Proposition 2.4.7. We conclude this subsection by observing that if L is a knot, then Definition 18 reduces to Definition 6.

Remark 6.2.3. If L is a knot, then the Bockstein homomorphism BS is invertible and the map described in the second point of Lemma 6.2.1 is precisely the inverse BS^{-1} of the Bockstein homomorphism. Since multiplication by $t-1$ induces an isomorphism on the Alexander module of a knot (recall Remark 2.2.2), Definition 18 does indeed reduce to Definition 6.

6.3 Proof of Theorem 6.1.2, part I: pushed-in C-complexes

In this section, we give more details regarding the "pushed-in C-complexes in the 4-ball D^4" which already appeared in Subsection 3.4.2. Namely, in Subsection 6.3.1, we study the exterior of such a pushed-in C-complex in order to compute its fundamental group in Subsection 6.3.2. Note that our approach differs slightly from the existing literature [41, 55]: instead of only pushing in the interiors of the Seifert surfaces, we also push in radially the corresponding sublinks. Moreover the different Seifert surfaces end up at different depths of the 4-ball.

Remark 6.3.1. Up to now, we have always used the letter "F" to denote C-complexes and their push-in. At this stage however, we wish to keep track of the difference between dimension 3 and dimension 4. Consequently, in this section and in the next, we adopt the following convention: we shall use the letter "S" for C-complexes and the letter "F" for their push-in.

Before diving into 4-dimensional considerations, we introduce some terminology and prove a short technical lemma which we shall need throughout the proof of Theorem 6.1.2.

Definition 19. A curve on a C-complex S is called *nice* if the following conditions are satisfied:

1. it has no self-intersections,
2. the restriction to each component S_i is an embedding,
3. it intersects each clasp at most once,
4. when it intersects a clasp, then it looks locally like in Figure 3.1.

The following lemma ensures that homology classes of curves on a C-complex can always be represented by nice curves.

Lemma 6.3.2. *Given a C-complex S, there exists a basis of $H_1(S)$ for which each element is represented by a nice curve.*

Proof. Up to homotopy equivalence, S can be constructed by taking the disjoint union of the surfaces S_i and adding an arc connecting S_i with S_j for each clasp. Contracting every surface to a point produces a graph Γ with μ vertices $V_1, \ldots, V_\mu$ and one edge for each clasp. This construction yields the short exact sequence

$$0 \to \bigoplus_{i=1}^{\mu} H_1(S_i) \to H_1(S) \xrightarrow{\pi} H_1(\Gamma) \to 0,$$

where the non-trivial maps are respectively induced by the inclusions of the disjoint F_i's into F, and the projection to the quotient. The surjectivity of π is immediate since any embedding of Γ into S produces a right inverse.

It is clear that each $H_1(S_i)$ admits a basis given by embedded curves that do not intersect any of the clasps. Thus it remains to find nice curves on S whose images under π form a basis for $H_1(\Gamma)$. Next, we say that a path on Γ is *simple* if it intersects each vertex and each edge at most once. Consequently, the lemma will follow from the following two assertions:

1. $H_1(\Gamma)$ admits a basis consisting of simple closed curves,
2. given any simple closed curve γ on Γ, there exists a nice curve s on S with $\pi(s) = \gamma$.

The first statement is of course well-known. For the reader's convenience we sketch the argument. Let T be a maximal tree in Γ and let $e_1, \ldots, e_k$ be the edges in $\Gamma \setminus T$. We can connect the end points of each e_i by a simple path p_i in the tree T. Now, for $i = 1, \ldots, k$, the curves $\gamma_i = e_i \cup p_i$ are simple and represent a basis for $H_1(\Gamma)$.

In order to prove the second assertion, observe that each vertex V_i crossed by a simple closed curve γ is both the initial point of a unique edge crossed by γ and the terminal point of a unique edge crossed by γ. Next, we pick an embedded curve γ_i on the corresponding surface S_i connecting the end points of the two clasps. Finally, we define s as the curve on S which is given by the union of the following paths:

1. for each edge crossed by γ, we take the corresponding clasp,
2. for each vertex V_i crossed by γ, we take the simple closed curve γ_i on S_i.

Since s is clearly nice and satisfies $\pi(s) = \gamma$, this concludes the proof. □

6.3.1 The exterior of a pushed-in C-complex

Let $S = S_1 \cup \cdots \cup S_\mu$ be a C-complex for a μ-colored link L and view S^3 as the boundary of D^4. For $i = 1, \ldots, \mu$, we pick a tubular neighborhood $S_i \times [-2, 2]$ of S_i in S^3. Furthermore for each i, we fix two surfaces with boundary S_i', S_i'' such that $S_i' \subset S_i \subset S_i''$, the complement $S_i'' \setminus \mathrm{int}(S_i')$ is a union of small annuli around $L_i = \partial S_i$, and the respective unions S' and S'' form C-complexes for links isotopic to L. Let us fix once and for all an embedding of $S^3 \times [0, \mu]$ in D^4 such that $S^3 \times \{0\}$ agrees with $S^3 = \partial D^4$. In order to prevent the different tubular neighborhoods from getting mixed up, we denote the image of (p, t) under this map by $p \star t$. For $i = 1, \ldots, \mu$, we write

$$F_i \;:=\; L_i \star \big[0, i - \frac{1}{4}\big] \cup_{L_i \star \{i - \frac{1}{4}\}} S_i \star \big\{i - \frac{1}{4}\big\}$$

and refer to $F := F_1 \cup \cdots \cup F_\mu$ as the *push-in* of S. In other words, F is obtained by pushing each sublink L_i radially (at a different depth) into the 4-ball and then capping it off with S_i. Observe that the F_i intersect pairwise in double points and consequently F is a colored bounding surface for L, see Definition 14. Since our goal is to study the exterior of F in D^4, we define

$$\nu F \;:=\; \bigcup_{i=1}^{\mu} \Big((\mathrm{int}(S_i'') \setminus S_i') \times (-1, 1) \star [0, i) \;\cup\; S_i \times (-1, 1) \star (i - \frac{1}{2}, i) \Big),$$

and $W_F := D^4 \setminus \nu F$. In order to compute the homology of W_F, we shall now decompose the latter space into more manageable pieces. First of all, denote by

$$B \;:=\; D^4 \setminus \left(\bigcup_{i=1}^{\mu} \mathrm{int}(S_i'') \times (-1, 1) \star [0, i) \right)$$

the complement of the whole trace of the push-in and observe that B is homeomorphic to a 4-ball. In order to recover W_F from B, we first set

$$Y_i \;:=\; S_i' \setminus \Big(\bigcup_{j \neq i} \mathrm{int}(S_j'') \times (-2, 2) \Big) \star [0, i-1] \cup S_i' \setminus \Big(\bigcup_{j=i+1}^{\mu} \mathrm{int}(S_j'') \times (-2, 2) \Big) \star [i-1, i - \frac{1}{2}]$$

for $i = 1, \ldots, \mu$. Observe that Y_i is a closed subset of $S_i' \star [0, i - \frac{1}{2}]$ which is homotopy equivalent to S_i. Moreover, making use of the neighborhoods of the S_i in S^3, it makes sense to consider the closed sets $Y_i \times [-1, 1] \subseteq D^4$. In the definition of the Y_i's, we removed large enough neighborhoods of the clasps in order to make these sets disjoint. It remains to add the clasp parts. For $i < j$, we define the space

$$X_{ij} \;:=\; (S_i' \cap S_j') \star [0, i - \frac{1}{2}]$$

which consists of a disjoint union $\bigsqcup X_{ij}^k$ of 2-disks, one for each clasp between S_i and S_j. Using the slightly larger neighborhoods of S_i and S_j in S^3, we consider the cross-shaped subset of $[-2, 2] \times [-2, 2]$ given by

$$K := [-1, 1] \times [-2, 2] \cup [-2, 2] \times [-1, 1].$$

This way, the space W_F decomposes as $B \cup Z$, where

$$Z := \bigcup_{i=1}^{\mu} Y_i \times [-1,1] \;\cup\; \bigcup_{i<j} X_{ij} \times K.$$

Observe that for each nice curve α in S (recall Definition 19), the push-offs $i^{\varepsilon}([\alpha])$ described in Subsection 3.2 can be represented by curves α^{ε} which are embedded in the intersection of S^3 with $\bigcup_{i=1}^{\mu} Y_i \times \{\pm 1\} \subseteq \partial B$.

6.3.2 The fundamental group and the boundary of W_F

Given a C-complex S, denote by J the subset of $\{1,\dots,\mu\}^2$ consisting of pairs (i,j) for which $i<j$ and there exists at least one clasp between the surfaces S_i and S_j.

Proposition 6.3.3. *The fundamental group of W_F admits the presentation*

$$\langle a_1,\dots,a_\mu \,|\, [a_i,a_j]=e \text{ for all } (i,j)\in J\rangle,$$

where the generators $a_1,\dots,a_\mu$ correspond to meridians for the surfaces $F_1,\dots,F_\mu$.

Proof. Recall from Subsection 6.3.1 that $W_F = B \cup Z$, where B is contractible and $Z = \bigsqcup_{i=1}^{\mu} Y_i \times [-1,1] \;\cup\; \bigsqcup_{i<j} X_{ij} \times K$. Observe that gluing $Y_1 \times [-1,1]$ to B is homotopically the same as identifying $Y_1 \times \{-1\}$ with $Y_1 \times \{1\}$ so that

$$\pi_1(B \cup (Y_1 \times [-1,1])) \cong \langle a_1 \,|\, a_1 \cdot 1 \cdot a_1^{-1} = 1\rangle = \langle a_1\rangle.$$

The generator a_1 is a meridian for the surface F_1. Gluing successively $Y_2 \times [-1,1],\dots,Y_\mu \times [-1,1]$ and repeating the argument adds one new generator a_i for each i, namely the meridian of the surface F_i. As each inclusion induced map $\pi_1(Y_i \times \{\pm 1\}) \to \pi_1(Y_i \times [-1,1])$ factors through the trivial group $\pi_1(B)$, no relations are added and thus

$$\pi_1\left(B \cup \bigcup_{i=1}^{\mu} Y_i \times [-1,1]\right) \;\cong\; \langle a_1,\dots,a_\mu\rangle.$$

In order to recover W_F, it remains to glue back in the contractible "clasp parts" $X_{ij}^k \times K$ (recall that $X_{ij} = \bigsqcup_k X_{ij}^k$), which are only non-empty when $(i,j) \in J$. Note that $X_{ij}^k \times K$ and $B \cup \bigsqcup_{i=1}^{\mu} Y_i \times [-1,1]$ intersect in $P := X_{ij}^k \times \partial K$ which is homotopy equivalent to a circle. Moreover, under the inclusion map of P into $B \cup \bigsqcup_{i=1}^{\mu} Y_i \times [-1,1]$, a generator of $\pi_1(P)$ is sent (up to inversion) to a commutator of the form $[a_i,a_j]$. Hence, by Van Kampen's theorem, one gets

$$\pi_1\left(B \cup \bigcup_{i=1}^{\mu} Y_i \times [-1,1] \cup (X_{ij}^k \times K)\right) \;\cong\; \langle a_1,\dots,a_\mu \,|\, [a_i,a_j]\rangle.$$

Repeating the process for each X_{ij}^k immediately yields the proposition. □

Let $p\colon \widehat{W}_F \to W_F$ be the cover of W_F corresponding to the kernel of the abelianization map $\pi_1(W_F) \to H_1(W_F)$, see Figure 6.2 for a schematic picture. Recall from Definition 10 that a C-complex S is totally connected if each S_i is connected and $S_i \cap S_j \neq \emptyset$ for all $i \neq j$. Proposition 6.3.3 implies both that the deck transformation group of $\widehat{W}_F$ is free abelian of rank μ and the following result.

Corollary 6.3.4. *If the C-complex S is totally connected, then $H_1(W_F; \Lambda_S)$ vanishes.*

Proof. If each pair of surfaces in S is joined by at least one clasp, then it follows from Proposition 6.3.3 that $\pi_1(W_F)$ is the free-abelian group on $t_1, \dots, t_\mu$. This implies that $H_1(W_F; \Lambda_\mu) = 0$, which by flatness of Λ_S also implies that $H_1(W_F; \Lambda_S) = 0$. □

We conclude this subsection by computing the Λ_S-twisted homology of the boundary of W_F. The upshot is that while the link exterior X_L is strictly contained in ∂W_F, both spaces look the same to Λ_S-twisted homology. In particular, we shall use this observation in order to reduce the computation of the Blanchfield pairing to the computation of a "Blanchfield-like" pairing on $H_1(\partial W_F; \Lambda_S)$.

Lemma 6.3.5. *$H_1(\partial W_F; \Lambda_S)$ is isomorphic to $H_1(X_L; \Lambda_S)$.*

Proof. Set $M_F := \overline{\nu F} \cap W_F$ so that the boundary of W_F decomposes as $\partial W_F = X_L \cup_{L \times S^1} M_F$; here M_F should be thought of as "the boundary of the tubular neighborhood of F, minus the link exterior". The resulting Mayer-Vietoris exact sequence for ∂W_F with Λ_S-coefficients is given by

$$H_1(L \times S^1; \Lambda_S) \to H_1(X_L; \Lambda_S) \oplus H_1(M_F; \Lambda_S) \to H_1(\partial W_F; \Lambda_S) \to H_0(L \times S^1; \Lambda_S).$$

As the restriction of $H_1(W_F) \to \mathbb{Z}^\mu$ to $H_1(L_i \times S^1; \mathbb{Z})$ sends each meridian to t_i, Lemma 5.7.3 ensures that $C_*(L \times S^1; \Lambda_S)$ is acyclic. Consequently, it only remains to show that $H_1(M_F; \Lambda_S)$ vanishes. Away from the c double points of F, the boundary of a tubular neighborhood of each F_i consists of $F_i \times S^1$ for $i = 1, \dots, \mu$. Given a double point $C_{ij}^k \in F_i \cap F_j$, remove the open disk D_{ij}^k which consists of the component of $\nu(F_j \cap F_i)$ containing C_{ij}^k. Repeating the process for each double point produces punctured surfaces $X_1, \dots, X_\mu$. The manifold M_F can now be recovered from the union of the $X_i \times S^1$ by gluing each $D_{ij}^k \times S^1$ along the tori $\partial D_{ij}^k \times S^1$. The corresponding Mayer-Vietoris exact sequence yields

$$\bigoplus_{i<j} \bigoplus_{k=1}^{c(i,j)} H_1\big(\partial D_{ij}^k \times S^1; \Lambda_S\big) \to \bigoplus_{i=1}^{\mu} H_1(X_i \times S^1; \Lambda_S) \to H_1(M_F; \Lambda_S) \to \bigoplus_{i<j} \bigoplus_{k=1}^{c(i,j)} H_0\big(\partial D_{ij}^k \times S^1; \Lambda_S\big).$$

As each S^1 factor arises as a meridian of the link L, one can apply Lemma 5.7.3 and the claim immediately follows. □

6.4 Proof of Theorem 6.1.2, part II: the intersection pairing

This section is organized as follows. In Subsection 6.4.1, we describe an explicit isomorphism $H_1(S) \otimes_{\mathbb{Z}} \Lambda_S \to H_2(W_F; \Lambda_S)$, while Subsection 6.4.2 computes the corresponding intersection form, thus proving Theorem 6.1.4.

Remark 6.4.1. Note that since we are dealing with *left* Λ_S-modules, we ought to write $\Lambda_S \otimes_{\mathbb{Z}} H_1(S)$ instead of $H_1(S) \otimes_{\mathbb{Z}} \Lambda_S$. However, we stick with former in order to align with the notations of [52]. Also, since Λ_S is commutative, this does not make a big difference.

6.4.1 A geometric basis for $H_2(W_F; \Lambda_S)$

The goal of this subsection is to provide a convenient basis for the Λ_S-module $H_2(W_F; \Lambda_S)$. In order to state the main result, we briefly introduce some notation. Given an arbitrary lift $\overline{B}$ of B to $\widehat{W}_F$ and a nice curve α on S, lift each push-off α^ε to $\overline{B}$, and call it $\overline{\alpha}^\varepsilon \subset \overline{B}$. Also, set

$$\operatorname{sgn}\varepsilon := \prod_{i=1}^{\mu} \varepsilon_i \in \{\pm 1\}, \qquad \mathbf{1} := (1, \dots, 1) \in \mathbb{Z}^\mu,$$

and notice that the following equality holds:

$$\sum_{\varepsilon} \operatorname{sgn}\varepsilon \, t^{\frac{\mathbf{1}+\varepsilon}{2}} = \prod_{i=1}^{\mu} (t_i - 1). \tag{6.4}$$

Next, using Lemma 6.3.2, we fix once and for all a basis $\mathscr{B}$ for $H_1(S)$ so that each element of $\mathscr{B}$ is represented by a nice curve, resulting in a set $\mathbf{B}$ of representatives. The remainder of this subsection will be devoted to the proof of the following result.

Proposition 6.4.2. *For each nice curve α on S, there is a closed surface Φ_α embedded in $\widehat{W}_F$, which intersects $p^{-1}(B \cap Z)$ in the curve $\sum_\varepsilon \operatorname{sgn}(\varepsilon) t^{\frac{\mathbf{1}+\varepsilon}{2}} \overline{\alpha}^\varepsilon$. The map*

$$\Phi \colon H_1(S) \otimes_{\mathbb{Z}} \Lambda_S \to H_2(W_F; \Lambda_S)$$

defined on the elements of $\mathbf{B}$ as $[\alpha] \otimes 1 \mapsto [\Phi_\alpha]$ is an isomorphism of Λ_S-modules.

The construction of the surfaces Φ_α

Given a nice cycle $\alpha \subset S$, there is a Seifert surface for each α^ε (viewed as an oriented knot) in $\partial B \cong S^3$. Pushing the interior of these surfaces inside B provides properly embedded surfaces S_α^ε in B whose boundary is α^ε. Lifting these surfaces to $\widehat{W}_F$ as subsets $\overline{S}_\alpha^\varepsilon$ of the fixed lift $\overline{B}$ of B, one has

$$\partial(t^{\frac{\mathbf{1}+\varepsilon}{2}} \overline{S}_\alpha^\varepsilon) = t^{\frac{\mathbf{1}+\varepsilon}{2}} \overline{\alpha}^\varepsilon.$$

In order to build a closed surface from all of these disjoint $t^{\frac{\mathbf{1}+\varepsilon}{2}} \overline{S}_\alpha^\varepsilon$, decompose α as

$$\alpha \;=\; \left(\bigcup_{i=1}^{\mu} \alpha_i \right) \;\cup\; \left(\bigcup_{i<j} \bigcup_{k=1}^{c(i,j)} \alpha_{ij}^k \right), \tag{6.5}$$

where α_i is the (possibly empty) subset of α which lies in $S_i \setminus \bigcup_{j \neq i} S_j$ and α_{ij}^k is the (possibly empty) subset of α corresponding to the clasp indexed by the triple (i, j, k). One can perform analogous decompositions for each push-off α^ε yielding segments α_i^ε and $\alpha_{ij}^{k,\varepsilon}$. Given two sequences $\varepsilon, \varepsilon'$ which differ only at the index j, say with $\varepsilon_j = -1$ and $\varepsilon'_j = +1$, connect the two surfaces $t^{\frac{\mathbf{1}+\varepsilon}{2}} \overline{S}_\alpha^\varepsilon$ and $-t^{\frac{\mathbf{1}+\varepsilon'}{2}} \overline{S}_\alpha^{\varepsilon'}$ by adding a cylinder $t^{\frac{\mathbf{1}+\varepsilon}{2}} \overline{\alpha}_j \times [-1, 1] \subseteq p^{-1}(Y_j \times [-1, 1])$. Repeating this process for all ε, ε' as above, we can now set

$$S_\alpha \;:=\; \bigcup_{\varepsilon} \left(\operatorname{sgn}(\varepsilon) t^{\frac{\mathbf{1}+\varepsilon}{2}} \overline{S}_\alpha^\varepsilon \;\cup\; \bigcup_{\{j \,|\, \varepsilon_j = -1\}} \operatorname{sgn}(\varepsilon) t^{\frac{\mathbf{1}+\varepsilon}{2}} \overline{\alpha}_j \times [-1, 1] \right),$$

where the sign $\mathrm{sgn}(\varepsilon) = \varepsilon_1 \cdots \varepsilon_\mu$ is added for the orientations to be consistent. We see from its construction that S_α is a surface whose boundary lies in the boundary of the union of (lifts of) the topological 4-balls $X_{ij}^k \times K$. Consequently, one can find a surface S^{clasp} whose components lie in those 4-balls and such that $\partial S_\alpha^{\mathrm{clasp}} = -\partial S_\alpha$. We can hence define a closed surface

$$\Phi_\alpha := S_\alpha \cup_\partial S_\alpha^{\mathrm{clasp}}.$$

Using our set $\mathbf{B}$ of nice representatives for the basis $\mathscr{B}$ of $H_1(S)$, we can now define a Λ_S-linear map

$$\Phi \colon H_1(S) \otimes_{\mathbb{Z}} \Lambda_S \to H_2(W_F; \Lambda_S)$$

by $[\alpha] \otimes 1 \mapsto [\Phi_\alpha]$ for $\alpha \in \mathbf{B}$. From the construction of Φ, we get the following formula.

Proposition 6.4.3. *Let $\partial \colon H_2(W_F; \Lambda_S) \to H_1(B \cap Z; \Lambda_S)$ be the boundary map in the Mayer-Vietoris sequence of $W_F = B \cup Z$. Then, for each nice cycle α, we have*

$$\partial\Phi([\alpha] \otimes 1) = \sum_{\varepsilon} \mathrm{sgn}(\varepsilon) t^{\frac{\mathbf{1}+\varepsilon}{2}} [\overline{\alpha}^{\varepsilon}]. \tag{6.6}$$

Our goal is now to prove that Φ is an isomorphism.

Reducing the problem to a commutativity statement

Recall from Subsection 6.3.1 that we decomposed W_F into $B \cup Z$, where $Z := \bigsqcup_{i=1}^{\mu} Y_i \times [-1,1] \cup \bigsqcup_{i<j} X_{ij} \times K$. We now set

$$Z_1 := \bigcup_{i=1}^{\mu} Y_i \times [-1,1], \qquad Z_2 = \bigcup_{i<j} X_{ij} \times K$$

so that three applications of the Mayer-Vietoris exact sequence (together with the fact that

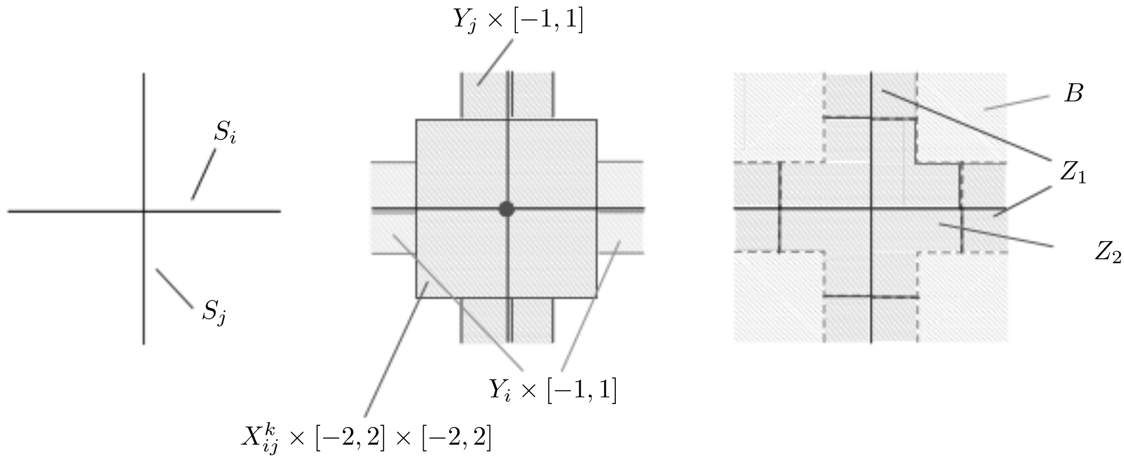

Figure 6.1: A dimensionally reduced sketch of Z_1 and Z_2 around a clasp.

Z_2 is made of contractible components) produce the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & H_1(B\cap Z_1;\Lambda_S) \arrow[r, "\psi"] \arrow[d, "a"] & H_1(Z_1;\Lambda_S) \arrow[d] \\
0 \arrow[r] H_2(W_F;\Lambda_S) \arrow[r, "\partial"] & H_1(B\cap Z;\Lambda_S) \arrow[r, "\varphi"] \arrow[d, "b"] & H_1(Z;\Lambda_S) \arrow[d] \\
 & H_0((B\cap Z_1)\cap(B\cap Z_2);\Lambda_S) \arrow[r, "L"] \arrow[d, "M"] & H_0(Z_1\cap Z_2;\Lambda_S) \arrow[d, "N"] \\
 & H_0(B\cap Z_1;\Lambda_S)\oplus H_0(B\cap Z_2;\Lambda_S) \arrow[r] & H_0(Z_1;\Lambda_S)\oplus H_0(Z_2;\Lambda_S).
\end{tikzcd} \tag{6.7}$$

The next lemma provides a first step towards the understanding of $\ker(\varphi)$.

Lemma 6.4.4. *The sequence*

$$0 \to \ker(\psi) \to \ker(\varphi) \to \ker(L)\cap\ker(M) \to 0 \tag{6.8}$$

of Λ_S-modules is exact.

Proof. The previous commutative diagram restricts to

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
H_1(B\cap Z_1;\Lambda_S) \arrow[r, "\psi"] \arrow[d] & H_1(Z_1;\Lambda_S) \arrow[d] \\
H_1(B\cap Z;\Lambda_S) \arrow[r, "\varphi"] \arrow[d, "b"] & H_1(Z;\Lambda_S) \arrow[d] \\
\ker(M) \arrow[r, "L"] \arrow[d, "M"] & \ker(N) \arrow[d, "N"] \\
0 & 0.
\end{tikzcd}$$

Applying the snake lemma produces the long exact sequence

$$0 \to \ker(\psi) \to \ker(\varphi) \to \ker(L)\cap\ker(M) \to \operatorname{coker}(\psi) \to \operatorname{coker}(\varphi) \to \ker(N)/\operatorname{im}(L|_{\ker M}) \to 0. \tag{6.9}$$

Since $Z_1 = \bigsqcup_{i=1}^{\mu} Y_i \times [-1,1]$ and $B\cap Z_1 = \bigsqcup_{i=1}^{\mu} Y_i \times \{\pm 1\}$, ψ is clearly surjective and the result follows. $\square$

Let us describe the strategy we shall use in order to show that the map $\Phi\colon H_1(S)\otimes_{\mathbb{Z}}\Lambda_S \to H_2(W_F;\Lambda_S)$ defined in Subsection 6.4.1 is an isomorphism. Using the short exact sequence

$$0 \to \bigoplus_{i=1}^{\mu} H_1(S_i) \xrightarrow{\iota} H_1(S) \xrightarrow{\pi} H_1(\Gamma) \to 0, \tag{6.10}$$

used in the proof of Lemma 6.3.2, we shall define isomorphisms σ and τ that fit into a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
\bigoplus_{i=1}^{\mu} H_1(S_i) \otimes_{\mathbb{Z}} \Lambda_S \arrow[rr, "\sigma"] \arrow[d, "\iota"] & & \ker(\psi) \arrow[d, "a"] \\
H_1(S) \otimes_{\mathbb{Z}} \Lambda_S \arrow[r, "\Phi"] \arrow[d, "\pi"] & H_2(W_F; \Lambda_S) \arrow[r, "\partial, \cong"] & \ker(\varphi) \arrow[d, "b"] \\
H_1(\Gamma) \otimes_{\mathbb{Z}} \Lambda_S \arrow[rr, "\tau"] \arrow[d] & & \ker(L) \cap \ker(M) \arrow[d] \\
0 & & 0.
\end{tikzcd} \tag{6.11}$$

The 5-lemma will then immediately imply that Φ is an isomorphism.

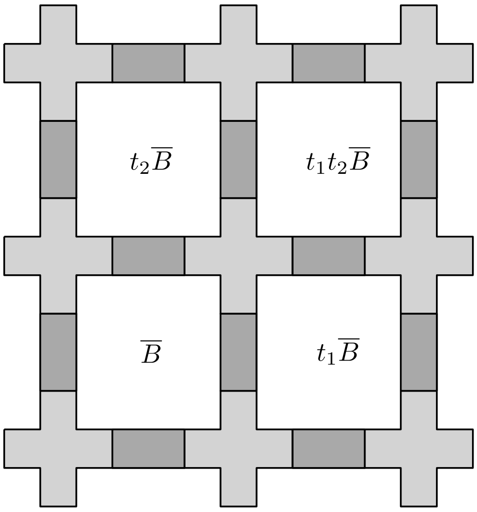

Figure 6.2: A schematic picture of a small portion of the cover $\widehat{W}_F$, in the simple case of only two surfaces and one clasp. We represented $p^{-1}(B)$ in white, $p^{-1}(Z_1)$ in dark gray and $p^{-1}(Z_2)$ in light gray.

The short exact sequence of $H_1(S)$

We fix some notation and recall the short exact sequence which was used in the proof of Lemma 6.3.2. Let $c(i,j)$ be the number of clasps between surfaces S_i and S_j in the C-complex S. The clasps will be denoted by C_{ij}^k for $k = 1, \ldots, c(i,j)$ resulting in a total number c of clasps. Up to homotopy equivalence, S can be constructed by taking the disjoint union of the surfaces S_i and adding an arc connecting S_i with S_j for each clasp C_{ij}^k. Contracting every surface to a point produces a graph Γ with μ vertices $\{V_k\}$ and c edges $\{E_{ij}^k\}$. We shall consider Γ as an oriented graph, where the edge E_{ij}^k travels from V_i to V_j if $i < j$. This construction yields the short exact sequence (6.10), where the non-trivial maps are respectively induced by the inclusions of the disjoint S_i's into S, and the projection to the quotient. To get the left vertical sequence of (6.11), we just tensor (6.10) with Λ_S, without changing the name of the maps.

Constructing the map σ

Recall from Subsection 6.4.1 that $W_F = B \cup Z$ was further decomposed by observing that $Z = Z_1 \cup Z_2$ with $Z_1 = \bigsqcup_{i=1}^{\mu} Y_i \times [-1,1]$ and $B \cap Z_1 = \bigsqcup_{i=1}^{\mu} Y_i \times \{\pm 1\}$. Consequently, if we lift $B \cap Z$ as a subspace of B, then the restriction of ψ to the k-th summand of $H_1(B \cap Z_1; \Lambda_S)$ is the map $(x, y) \mapsto x \otimes t_k + y \otimes 1$. Since S_k and Y_k are homotopy equivalent, it follows that

$$\ker(\psi) = \bigoplus_{i=1}^{\mu} \Lambda_S\{(-x \otimes 1, x \otimes t_i) \,|\, x \in H_1(S_i)\}.$$

One can now define the map $\sigma\colon \bigoplus_{i=1}^{\mu} H_1(S_i) \otimes_{\mathbb{Z}} \Lambda_S \to \ker(\psi)$ by

$$\sigma(x) \;=\; \Big(\prod_{j \neq k} (t_j - 1)\Big)(-x \otimes 1, x \otimes t_k)$$

for $x \in H_1(S_k)$. Using the description of $\ker(\psi)$, the map σ is well defined; it is an isomorphism since the $(t_j - 1)$'s are invertible in Λ_S. We conclude this paragraph by proving the commutativity of the top part of (6.10). Given a primitive element $x \in H_1(S_k)$, we represent $\iota x \in H_1(S)$ by a nice cycle α, which only belongs to S_k. Proposition 6.4.3 hence gives us

$$\partial\Phi\iota x \;=\; \sum_{\varepsilon} \operatorname{sgn}(\varepsilon) t^{\frac{\mathbf{1}+\varepsilon}{2}} [\overline{\alpha}^{\varepsilon}].$$

Notice that, since α is contained in S_k, the curve α^{ε} only depends on the value of ε_k. We thus only have two push-offs for it, which we denote by α^+ and α^-, as one does in the case when S consists of a single Seifert surface. We will also denote by $\hat{\varepsilon}$ the element in $\{\pm 1\}^{\mu-1}$ obtained from ε by ignoring ε_k. We can now rewrite the above formula as

$$\partial\Phi\iota x = \sum_{\varepsilon \in \{\pm 1\}^{\mu}} \operatorname{sgn}(\hat{\varepsilon}) \prod_{j \neq k} t_j^{\frac{1+\varepsilon_j}{2}} \varepsilon_k t_k^{\frac{1+\varepsilon_k}{2}} [\overline{\alpha}^{\varepsilon_k}] = \sum_{\varepsilon' \in \{\pm 1\}^{\mu-1}} \operatorname{sgn}(\varepsilon') \prod_{j \neq k} t_j^{\frac{1+\varepsilon_j}{2}} (-[\overline{\alpha}^-] + t_k[\overline{\alpha}^+]).$$

Applying (6.4) to the sum over the ε', we get

$$\partial\Phi\iota x \;=\; \Big(\prod_{j \neq k} (t_j - 1)\Big)(-[\overline{\alpha}^-] + t_k[\overline{\alpha}^+]).$$

Commutativity follows, because by definition of σ and a, we also have

$$a\sigma(x) \;=\; \Big(\prod_{j \neq k} (t_j - 1)\Big) a(-x \otimes 1, x \otimes t_k) \;=\; \Big(\prod_{j \neq k} (t_j - 1)\Big)(-[\overline{\alpha}^-] + t_k[\overline{\alpha}^+]).$$

Constructing the map τ

First, we describe the space $\ker(L) \cap \ker(M)$ by making use of the following portion of the Mayer-Vietoris diagram (6.7):

$$\begin{tikzcd} H_0((B \cap Z_1) \cap (B \cap Z_2); \Lambda_S) \arrow[r, "L"] \arrow[d, "M = M' \oplus M''"] & H_0(Z_1 \cap Z_2; \Lambda_S) \\ H_0(B \cap Z_1; \Lambda_S) \oplus H_0(B \cap Z_2; \Lambda_S). & \end{tikzcd}$$

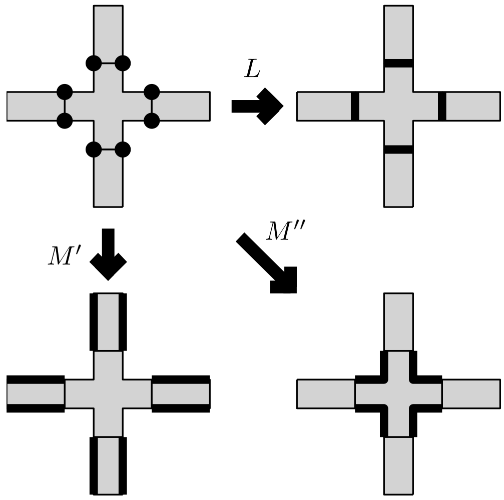

Figure 6.3: The inclusion induced maps L, M', M''.

The spaces $B\cap Z_2$, $Z_1\cap Z_2$ and $B\cap Z_1\cap Z_2$ are all of the form $\bigsqcup_{i<j} X_{ij}\times P$ for some contractible subset P of ∂K. Consequently the restriction of the cover to each of these subspaces is the trivial $\mathbb{Z}^\mu$-covering, and the inclusion induced maps L, M' and M'' can be understood using Figure 6.3. It follows that $\ker(L)\cap\ker(M'')$ is generated by c linearly independent elements v_{ij}^k, one for each clasp C_{ij}^k, as illustrated in Figure 6.4:

$$\ker(L)\cap\ker(M'') = \mathrm{Span}_{\Lambda_S}\{v_{ij}^k \,|\, i<j,\ 1\le k\le c(i,j)\}. \tag{6.12}$$

Since $\ker(L)\cap\ker(M)$ is the subspace of $\ker(L)\cap\ker(M'')$ which is annihilated by M', we now compute the image of v_{ij}^k under M'. Observing that $H_0(B\cap Z_1;\Lambda_S)\cong\bigoplus_{i=1}^{\mu} H_0(\overline{Y_i\times\{-1\}\sqcup Y_i\times\{1\}})\otimes_{\mathbb{Z}}\Lambda_S$ and denoting by $y_i^\pm$ a positive generator of $H_0(\overline{Y_i\times\{\pm 1\}})$, a short computation using Figure 6.3 shows that

$$M'v_{ij}^k = t_i(t_j-1)y_i^+ - (t_j-1)y_i^- - t_j(t_i-1)y_j^+ + (t_i-1)y_j^-. \tag{6.13}$$

Let $T\colon C_1(\Gamma)\to\mathrm{Span}_{\Lambda_S}\{v_{ij}^k\}_{ijk}$ be the $\mathbb{Z}$-linear map defined on generators by

$$T(E_{ij}^k) := \prod_{l\neq i,j}(t_l-1)\, v_{ij}^k.$$

Note that since Γ is a graph, its first homology group $H_1(\Gamma)$ is a subspace of $C_1(\Gamma)$.

Proposition 6.4.5. *The restriction of T to $H_1(\Gamma)$ takes values in* $\ker(L)\cap\ker(M)$.

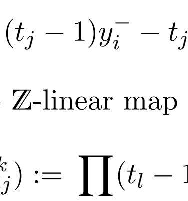

Figure 6.4: The elements v_{ij}^k that span $\ker(L)\cap\ker(M'')$.

Proof. Since this proof consists of a straightforward computation which combines (6.13) and the definition of T, we shall only give the details in a simple case. The general case follows the exact same steps. Suppose that in our C-complex there are clasps C^1_{ij}, C^1_{il}, C^1_{jk}, with $i<j<k$, then the element $\gamma := E^1_{ij} + E^1_{jk} - E^1_{ik}$ is in $H_1(\Gamma)$. Since

$$T\gamma = \prod_{l \neq i,j,k} (t_l - 1)\left((t_k-1)v^1_{ij} + (t_i - 1)v^1_{jk} - (t_j - 1)v^1_{ik}\right)$$

is contained in $\ker(L) \cap \ker(M'')$, it only remains to check that $M'T\gamma = 0$. The immediate computation

$$\begin{aligned} M'T\gamma = \prod_{l\neq i,j,k}(t_l-1)\cdot [&(t_k-1)(t_i(t_j-1)y_i^+ - (t_j-1)y_i^- - t_j(t_i-1)y_j^+ + (t_i-1)y_j^-) + \\ &(t_i-1)(t_j(t_k-1)y_j^+ - (t_k-1)y_j^- - t_k(t_j-1)y_k^+ + (t_j-1)y_k^-) \\ &-(t_j-1)(t_i(t_k-1)y_i^+ - (t_k-1)y_i^- - t_k(t_i-1)y_k^+ + (t_i-1)y_k^-)] = 0, \end{aligned}$$

which relies on (6.13) concludes the proof. □

We can now define the Λ_S-linear map

$$\tau\colon H_1(\Gamma)\otimes_{\mathbb{Z}}\Lambda_S \to \ker(L)\cap\ker(M)$$

as the extension to Λ_S-scalars of the $\mathbb{Z}$-linear map $T\colon H_1(\Gamma)\to \ker(L)\cap\ker(M)$.

Proposition 6.4.6. *The map τ is an isomorphism of Λ_S-modules.*

Proof. As earlier in this section, let c be the total number of clasps of the C-complex S. We will identify the codomain $\ker(L)\cap\ker(M)$ with the kernel of a $\mu\times c$ matrix with coefficients in Λ_S. During this proof, rows of $(\mu\times c)$-matrices will be indexed by integers $l = 1,\dots,\mu$ and columns by triples (i,j,k) with $1\le i<j\le\mu$ and $1\le k\le c(i,j)$. We start by noticing that, thanks to (6.12) and (6.13), we have

$$\ker(L)\cap\ker(M) \;=\; \left\{\sum_{i,j,k} a^k_{ij}v^k_{ij} \mid \forall\, l = 1,\dots,\mu,\ \sum_{j,k} a^k_{lj}(t_j-1) - \sum_{i,k} a^k_{il}(t_i-1) = 0\right\}.$$

It follows that $\ker(L)\cap\ker(M)$ is the subspace of $\mathrm{Span}_{\Lambda_S}\{v^k_{ij}\}_{ijk}$ given by the null space of the $\mu\times c$ matrix R with (l,ijk)-coefficient

$$(R^{ijk}_l) = ((t_j-1)\delta_{il} - (t_i-1)\delta_{jl}),$$

where δ_{ij} is the Kronecker delta function. For each l, multiplying the l-th row of R by (t_l-1) yields a $(\mu\times c)$-matrix Q whose kernel is still $\ker(L)\cap\ker(M)$ and whose (l,ijk)-coefficient is

$$(Q^{ijk}_l) = ((t_i-1)(t_j-1)(\delta_{il}-\delta_{jl})).$$

Next, multiplying each ijk-column of Q by $(t_i-1)^{-1}(t_j-1)^{-1}$ results in a matrix P whose (l,ijk)-coefficient is

$$(P^{ijk}_l) = (\delta_{il}-\delta_{jl}).$$

Since P represents the boundary operator $\partial\colon C_1(\Gamma) \to C_0(\Gamma)$, its kernel over Λ_S is isomorphic to $H_1(\Gamma) \otimes_{\mathbb{Z}} \Lambda_S$. Consequently, in order to conclude the proof, it only remains to show that $\ker(Q) \cong \ker(P)$. The operations we performed on the columns of Q give rise to a Λ_S-module isomorphism

$$v_{ij}^k \mapsto (t_i-1)^{-1}(t_j-1)^{-1}v_{ij}^k = \left(\prod_{l=1}\mu(t_l-1)\right)^{-1}\prod_{l\neq i,j}(t_l-1)v_{ij}^k = \left(\prod_{l=1}\mu(t_l-1)\right)^{-1}\tau(E_{ij}^k)$$

which restricts to an isomorphism $\ker(P) \to \ker(Q)$. Since the (t_i-1) are invertible in Λ_S, τ is an isomorphism. □

In order to conclude the proof of the reduction discussed in Subsection 6.4.1, it only remains to prove the next proposition.

Proposition 6.4.7. *The bottom square of (6.11) commutes.*

Proof. Given a nice cycle α in S and a clasp C_{ij}^k, define

$$n_{ij}^k = \begin{cases} 1 & \text{if } \alpha \text{ crosses } C_{ij}^k \text{ from } i \text{ to } j,\\ -1 & \text{if } \alpha \text{ crosses } C_{ij}^k \text{ from } j \text{ to } i,\\ 0 & \text{if } \alpha \text{ does not cross } C_{ij}^k \text{ at all.}\end{cases}$$

As we will see, the image of $[\alpha]\otimes 1$ under both $\tau\pi$ and $b\partial\Phi$ will only depend on this combinatorial data. We start with the computation of $\tau\pi([\alpha]\otimes 1)$. From the definitions of π and n_{ij}^k, it is clear that

$$\pi([\alpha]\otimes 1) = \sum_{i<j,\,k} n_{ij}^k E_{ij}^k,$$

and consequently the definition of τ yields

$$\tau\pi([\alpha]\otimes 1) = \sum_{i<j,\,k}\prod_{l\neq i,j}(t_l-1)\, n_{ij}^k v_{ij}^k.$$

To conclude, we compute $b\partial\Phi([\alpha]\otimes 1)$. Thanks to Proposition 6.4.3, we have $\partial\Phi([\alpha]\otimes 1) = \sum_\varepsilon \operatorname{sgn}(\varepsilon)t^{\frac{1+\varepsilon}{2}}[\overline{\alpha}^\varepsilon]$. Since the map b is the boundary homomorphism in the Mayer–Vietoris sequence for $B\cap Z = (B\cap Z_1)\cup(B\cap Z_2)$, it only depends on the behavior of α at the clasps. For $\varepsilon \in \{\pm1\}^\mu$, the part of $\overline{\alpha}^\varepsilon$ contained in $B\cap Z_2$ will be either empty or a path connecting $S_i\times\{\varepsilon_i\}$ to $S_j\times\{\varepsilon_j\}$, whose direction depends on the sign of n_{ij}^k. Since this data does not depend on the coordinates of ε different from i and j, we will denote such a strand by $\overline{\alpha}_{ij}^{k,\varepsilon_i\varepsilon_j}$. The part of $\partial\Phi([\alpha]\otimes 1)$ which is contained in $B\cap Z_2$ is hence

$$\sum_\varepsilon \operatorname{sgn}(\varepsilon)t^{\frac{1+\varepsilon}{2}}\sum_{i<j,\,k}\overline{\alpha}_{ij}^{k,\varepsilon_i\varepsilon_j} = \sum_{i<j,\,k}\sum_{\varepsilon'}\operatorname{sgn}(\varepsilon')\prod_{l\neq i,j}t^{\frac{1+\varepsilon_l}{2}}\left(t_it_j\overline{\alpha}_{ij}^{k,++} - t_i\overline{\alpha}_{ij}^{k,+-} - t_j\overline{\alpha}_{ij}^{k,-+} + \overline{\alpha}_{ij}^{k,--}\right),$$

where ε' now varies in $\{\pm1\}^{\mu-2}$. We observe now that the boundary of $(t_it_j\overline{\alpha}_{ij}^{k,++} - t_i\overline{\alpha}_{ij}^{k,+-} - t_j\overline{\alpha}_{ij}^{k,-+} + \overline{\alpha}_{ij}^{k,--})$ is given by $-n_{ij}^k v_{ij}^k$, see Figure 6.5. So, taking into account a minus sign coming from the Mayer-Vietoris sequence, we get

$$b\partial\Phi([\alpha]\otimes 1) = \sum_{i<j,\,k}\sum_{\varepsilon'}\operatorname{sgn}(\varepsilon')\prod_{l\neq i,j}t^{\frac{1+\varepsilon_l}{2}}\, n_{ij}^k v_{ij}^k.$$

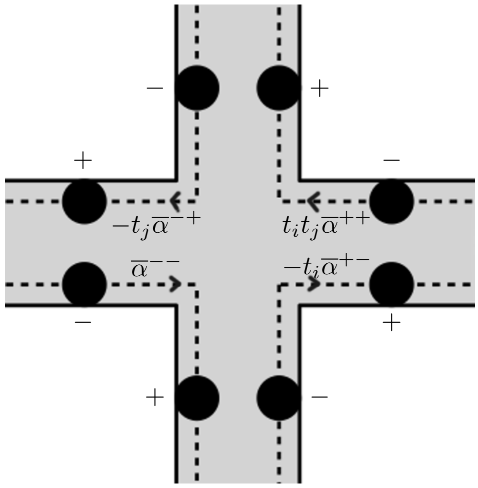

Figure 6.5: A representation of the curve $(t_it_j\overline{\alpha}_{ij}^{k,++} - t_i\overline{\alpha}_{ij}^{k,+-} - t_j\overline{\alpha}_{ij}^{k,-+} + \overline{\alpha}_{ij}^{k,--})$, in the case where $n_{ij}^k = 1$. Observe that its boundary gives the opposite of the element v_{ij}^k depicted in Figure 6.4.

As a last step, we apply (6.4) to the sum over the ε', obtaining

$$b\partial\Phi([\alpha] \otimes 1) = \sum_{i<j,\, k} \prod_{l \neq i,j} (t_l - 1)\, n_{ij}^k v_{i,j}^k.$$

This concludes the proof of the proposition.

□

Combining the reduction of Subsection 6.4.1 with the results of Subsection 6.4.1, we have now completed the proof of Proposition 6.4.2. Indeed, since we now know that the maps ∂, σ and τ are all isomorphisms, applying the 5-lemma to the diagram in Equation (6.11) implies that $\Phi\colon H_1(S) \otimes_{\mathbb{Z}} \Lambda_S \to H_2(W_F; \Lambda_S)$ is an isomorphism, as desired.

6.4.2 The twisted intersection pairing of W_F

In Subsection 6.3.1, we decomposed the exterior of the push-in as $W_F = B \cup Z$, where B was homeomorphic to the 4-ball. Then, in Subsection 6.4.1, we used this decomposition to build an isomorphism $\Phi\colon H_1(S) \otimes_{\mathbb{Z}} \Lambda_S \to H_2(W_F; \Lambda_S)$. In this subsection, we use the resulting basis of $H_2(W_F; \Lambda_S)$ (recall Proposition 6.4.2) to compute the twisted intersection form $\lambda_{\Lambda_S}(W_F)$ of W_F. More precisely, given $\alpha, \beta \in H_1(S)$, we relate $\lambda_{\Lambda_S}(W_F)([\Phi_\alpha], [\Phi_\beta])$ to the C-complex matrix H, recall Definition 11.

Recall from Subsection 5.6, that the formula for $\lambda_{\Lambda_S}(W_F)([\Phi_\alpha], [\Phi_\beta])$ involves the algebraic intersection of Φ_α with $t^g\Phi_\beta$ for each $g \in \mathbb{Z}^\mu$. In order to pinpoint where these intersections take place, we consider the space

$$T := \bigcup_{i=1}^{\mu} S_i \times [-1, 1] \subseteq S^3$$

so that $\Phi_\alpha \subseteq \bigsqcup_{\varepsilon\in\{\pm1\}^\mu} t^{\frac{\mathbf{1}+\varepsilon}{2}}\overline{B} \cup p^{-1}(T\star 0)$, where $p\colon \widehat{W}_F \to W_F$ denotes the covering corresponding to the kernel of the abelianization map $\pi_1(W_F) \to H_1(W_F)$ and $B \cap S^3 = S^3 \backslash \mathring{T}$. Moreover, smoothing the corners, T becomes a smooth submanifold of S^3 and, as such, its oriented boundary admits a neighborhood $\partial T \times [-\delta, \delta]$, where the positive part lives outside of T.

The next lemma (whose proof can be understood by looking at Figure 6.6) will make it possible to transfer information from ∂B to the standard S^3.

Lemma 6.4.8. *There exists an orientation preserving homeomorphism between ∂B and S^3, which brings $\partial T \star [0, \frac{1}{2}] \subseteq B$ to $\partial T \times [0, \delta] \subseteq S^3$. (Notice that $\partial T \star \{\frac{1}{2}\}$ is sent to $\partial T \times \{0\}$, while $\partial T \star \{0\}$ is sent to $\partial T \times \{\delta\}$).*

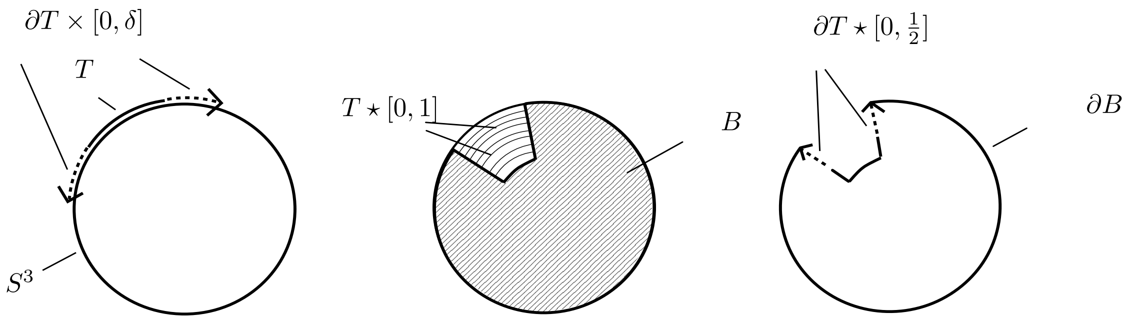

Figure 6.6: Schematic picture of the statement of Lemma 6.4.8.

We are now ready to prove the main result of this section. This result is precisely Theorem 6.1.4 from the introduction.

Theorem 6.4.9. *Let H be the C-complex matrix corresponding to the basis $\mathscr{B}$ of $H_1(S)$. With respect to the image of $\mathscr{B}$ by the isomorphism Φ of Proposition 6.4.2, the twisted intersection form $\lambda_{\Lambda_S}(W_F)\colon H_2(W_F; \Lambda_S) \times H_2(W_F; \Lambda_S) \to \Lambda_S$ is represented by H. More explicitly, if Φ_α and Φ_β are two of the surfaces constructed above, then we have the formula*

$$\lambda_{\Lambda_S}(W_F)([\Phi_\alpha], [\Phi_\beta]) = \sum_{\varepsilon} \prod_{i=1}^{\mu} (1 - t_i^{\varepsilon_i}) \ell k(\alpha^\varepsilon, \beta).$$

Proof. Fix $[\alpha], [\beta] \in \mathscr{B}$, such that α and β are the nice representatives in $\mathbf{B}$ we used to define the map Φ. We perform homotopies which push Φ_α and Φ_β inside the interior of W_F in such a way that

$$\Phi_\alpha \subseteq p^{-1}(B) \cup p^{-1}\big(T \star \frac{1}{4}\big) \quad , \quad \Phi_\beta \subseteq p^{-1}(B) \cup p^{-1}\big(T \star \frac{1}{2}\big),$$

and that Φ_α intersect $t^g \phi_\beta$ transversally for all $g \in \mathbb{Z}^\mu$. Recalling the construction of Φ_α and Φ_β, we see that the algebraic intersection of Φ_α with $t^g \Phi_\beta$ can now only happen in the disjoint union of the 4-balls $t^h \overline{B}$. It follows that

$$\lambda_{\Lambda_S}(W_F)([\Phi_\alpha], [\Phi_\beta]) \overset{\text{def}}{=} \sum_{g\in\mathbb{Z}^\mu} \langle \Phi_\alpha, t^g\, \Phi_\beta\rangle\, t^g = \sum_{g\in\mathbb{Z}^\mu} \sum_{\varepsilon,\varepsilon'\in\{\pm1\}^\mu} \operatorname{sgn}(\varepsilon) \operatorname{sgn}(\varepsilon') \langle t^{\frac{\mathbf{1}+\varepsilon}{2}} \overline{S}^\varepsilon_\alpha, t^g t^{\frac{\mathbf{1}+\varepsilon'}{2}} \overline{S}^{\varepsilon'}_\beta \rangle t^g.$$

Moreover, the two surfaces $t^{\frac{\mathbf{1}+\varepsilon}{2}} S^\varepsilon_\alpha$ and $t^g t^{\frac{\mathbf{1}+\varepsilon}{2}} S^{\varepsilon'}_\beta$ only intersect when they belong to the same lift of B, i.e. when $t^g t^{\frac{\mathbf{1}+\varepsilon'}{2}} = t^{\frac{\mathbf{1}+\varepsilon}{2}}$. This occurs precisely when $g = \frac{\varepsilon-\varepsilon'}{2}$ and, in this case, translation invariance gives us

$$\langle t^{\frac{1+\varepsilon}{2}}\overline{S}^{\varepsilon}_{\alpha}, t^g t^{\frac{1+\varepsilon'}{2}}\overline{S}^{\varepsilon'}_{\beta}\rangle = \langle S^{\varepsilon}_{\alpha}, S^{\varepsilon'}_{\beta}\rangle.$$

Homotope $\overline{S}^{\varepsilon}_{\alpha}, \overline{S}^{\varepsilon'}_{\beta} \subseteq \overline{B}$ so that their boundaries are respectively $\overline{\alpha}^{\varepsilon} \star \{\frac{1}{4}\}$ and $\overline{\beta}^{\varepsilon'} \star \{\frac{1}{2}\}$. Consequently, the algebraic intersection $\langle S^{\varepsilon}_{\alpha}, S^{\varepsilon'}_{\beta}\rangle$ coincides with the linking number in $\partial\overline{B}$ of $\overline{\alpha}^{\varepsilon}\star\{\frac{1}{4}\}$ and $\overline{\beta}^{\varepsilon'}\star\{\frac{1}{2}\}$, which in turn equals the linking number in ∂B of $\alpha^{\varepsilon}\star\{\frac{1}{4}\}$ and $\beta^{\varepsilon'}\star\{\frac{1}{2}\}$. Lemma 6.4.8 now provides the existence of an orientation preserving homeomorphism from ∂B to S^3 that brings $\alpha^{\varepsilon}\star\{\frac{1}{4}\}$ to $\alpha^{\varepsilon}\times\{\frac{\delta}{2}\}$, and $\beta^{\varepsilon'}\star\{\frac{1}{2}\}$ to $\beta^{\varepsilon'} = \beta^{\varepsilon'}\times\{0\}$. As a consequence, we obtain

$$\langle S^{\varepsilon}_{\alpha}, S^{\varepsilon'}_{\beta}\rangle = \ell k(\alpha^{\varepsilon}\times\{\delta/2\}, \beta^{\varepsilon'}) = \ell k(\alpha^{\varepsilon}, \beta).$$

Putting everything together, we get

$$\lambda_{\Lambda_S}(W_F)([\Phi_\alpha],[\Phi_\beta]) = \sum_{\varepsilon,\varepsilon'\in\{\pm 1\}^{\mu}} \operatorname{sgn}(\varepsilon)\operatorname{sgn}(\varepsilon')\ell k(\alpha^{\varepsilon},\beta) t^{\frac{\varepsilon-\varepsilon'}{2}}.$$

We will now algebraically manipulate this last expression in order to get the desired formula. Factoring out the terms involving ε' and using (6.4) (applied to the variables t_i^{-1}), we rewrite this as

$$\lambda_{\Lambda_S}(W_F)([\Phi_\alpha],[\Phi_\beta]) = \sum_{\varepsilon}\operatorname{sgn}(\varepsilon)\ell k(\alpha^{\varepsilon},\beta)\, t^{\frac{\varepsilon+\mathbf{1}}{2}}\sum_{\varepsilon'}\operatorname{sgn}(\varepsilon')\, t^{-\frac{\mathbf{1}+\varepsilon'}{2}} = \sum_{\varepsilon}\operatorname{sgn}(\varepsilon)\ell k(\alpha^{\varepsilon},\beta)\, t^{\frac{\varepsilon+\mathbf{1}}{2}}\prod_{i=1}^{\mu}(t_i^{-1}-1).$$

Then, from the identity $\varepsilon_i\, t_i^{\frac{\varepsilon_i+1}{2}}(t_i^{-1}-1) = 1-t_i^{\varepsilon_i}$, we get

$$\lambda_{\Lambda_S}(W_F)([\Phi_\alpha],[\Phi_\beta]) = \sum_{\varepsilon}\prod_{i=1}^{\mu}(1-t_i^{\varepsilon_i})\ell k(\alpha^{\varepsilon},\beta),$$

which concludes the proof of the theorem. □

This concludes the more geometrical part of the proof of Theorem 6.1.2. Next, we do some homological algebra.

6.5 Proof of Theorem 6.1.2, part III: homological algebra

We start by recalling the set-up and fixing some notation. Pushing a C-complex S into the 4-ball D^4 leads to properly embedded surfaces which only intersect transversally in double points. Let W be the exterior of such a pushed-in C-complex in D^4, i.e. W is the complement in D^4 of a tubular neighborhood of the pushed-in C-complex as described in Section 6.3. The aim is now to relate the Λ_S-twisted intersection form $\lambda_{\Lambda_S}(W)$ (which was computed in Theorem 6.4.9) to the Blanchfield form on $H_1(X_L;\Lambda_S)\cong H_1(\partial W;\Lambda_S)$. The first step is to study the cochain complexes of $\partial W, W$ and $(W,\partial W)$ with coefficients in Λ_S, Q_μ and Q_μ/Λ_S. These 9 cochain complexes fit in a commutative diagram whose columns and rows are exact.

Keeping this motivating example in mind, we make a short detour which shall only involve homological algebra. More precisely, given a commutative ring R, we shall consider the following commutative diagram of cochain complexes of R-modules whose columns and rows are assumed to be exact:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r, "h_B"] \arrow[d, "v_B"'] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D \arrow[r, "h_D"] \arrow[d, "v_D"'] & E \arrow[r] \arrow[d, "v_E"'] & F \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H \arrow[r, "h_H"] \arrow[d] & J \arrow[r, "h_J"] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0. & \end{tikzcd} \tag{6.14}$$

We shall write $H^*(D) \to H^*(J)$ for the homomorphism induced by any composition of the chain maps from D to J. Also, $H^*(J) \to H^{*+1}(C)$ will denote the composition of the boundary map from $H^*(J)$ to $H^{*+1}(B)$ with the homomorphism induced by the chain map from B to C. Alternatively, the latter map can also be described as the composition of the homomorphism induced by the chain map from J to K with the boundary homomorphism $\delta_K^v\colon H^*(K) \to H^{*+1}(C)$. Furthermore, δ_K^h will denote the boundary homomorphism from $H^*(K)$ to $H^{*+1}(H)$. Note that these boundary maps are of degree $+1$ since we are working with *co*chain complexes.

We now argue that there is a well-defined homomorphism from $v_D \ker(H^*(D) \to H^*(J))$ to $H^{*-1}(K)/\ker(\delta_K^h)$, which we shall denote by $(\delta_K^h)^{-1}$. Indeed, if x belongs to $\ker(H^*(D) \to H^*(J))$, the definition of the latter kernel implies that $(h_H \circ v_D)(x) = 0$. By exactness of the bottom row of (6.14), there is a k in $H^{*-1}(K)$ which satisfies $\delta_K^h(k) = v_D(x)$. Set $(\delta_K^h)^{-1}(v_D(x)) := k$. It can be checked that $(\delta_K^h)^{-1}$ is well-defined.

Similarly, we shall argue that there is a well-defined homomorphism from $h_D \ker(H^*(D) \to H^*(J))$ to $\frac{H^*(B)}{\ker(v_B)}$, which we shall denote by v_B^{-1}. Indeed, if x belongs to $\ker(H^*(D) \to H^*(J))$, the definition of the latter kernel implies that $(v_E \circ h_D)(x) = 0$. By exactness of the middle column of (6.14), there is a b in $H^*(B)$ which satisfies $v_B(b) = h_D(x)$. Set $v_B^{-1}(h_D(x)) := b$. It can be checked that v_B^{-1} is well-defined.

Finally, we claim that δ_K^v induces a well-defined map from $\frac{H^{*-1}(K)}{\ker(\delta_K^h)}$ to $\frac{H^*(C)}{\mathrm{im}(H^{*-1}(J) \to H^*(C))}$. To do this, we must show that if k lies in the kernel of δ_K^h, then $\delta_K^v(k)$ belongs to im $:= \mathrm{im}(H^{*-1}(J) \overset{h_J}{\to} H^{*-1}(K) \overset{\delta_K^v}{\to} H^*(C))$. By exactness of the bottom row of (6.14), we have $\ker(\delta_K^h) = \mathrm{im}(h_J)$. Consequently k lies in $\mathrm{im}(h_J)$ and thus $\delta_K^v(k)$ belongs to im, proving the claim.

We omit the proof of the following lemma which involves a lengthy but standard diagram chasing argument. Note that the statement of this lemma was inspired by [9, Lemma 4.4].

Lemma 6.5.1. *Given nine cochain complexes as in (6.14), the diagram below anticommutes:*

$$\begin{tikzcd}
\ker(H^*(D) \to H^*(J)) \arrow[r, "v_D"] \arrow[d, "h_D"] & v_D \ker(H^*(D) \to H^*(J)) \arrow[d, "(\delta_K^h)^{-1}"] \\
h_D \ker(H^*(D) \to H^*(J)) \arrow[d, "v_B^{-1}"] & \frac{H^{*-1}(K)}{\ker(\delta_K^h)} \arrow[d, "\delta_K^v"] \\
\frac{H^*(B)}{\ker(v_B)} \arrow[r, "h_B"] & \frac{H^*(C)}{\mathrm{im}(H^{*-1}(J) \to H^*(C))}.
\end{tikzcd} \tag{6.15}$$

This concludes our algebraic detour and we now return to topological matters, namely to the nine cochain complexes which arose when we considered the exterior W of a pushed-in C-complex in the 4-ball.

Denote by $i^W_{\Lambda_S, Q_\mu}$ the homomorphism from $H^2(W;\Lambda_S)$ to $H^2(W;Q_\mu)$ induced by the inclusion of Λ_S into Q_μ. We also denote by $i^{W,\partial W}_{\Lambda_S}$ the homomorphism from $H^2(W;\Lambda_S)$ to $H^2(\partial W;\Lambda_S)$. More generally, we will often implicitly follow this notational scheme: for instance $i^{(W,\partial W),W}_{Q_\mu/\Lambda_S}$ will denote the map from $H^2(W,\partial W;Q_\mu/\Lambda_S)$ to $H^2(W;Q_\mu/\Lambda_S)$.

Since BS plays the role of the boundary map δ^h_K in our algebraic detour, there is a well-defined map BS^{-1} from $i^{W,\partial W}_{\Lambda_S} \ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q_\mu))$ to $\frac{H^1(\partial W;Q_\mu/\Lambda_S)}{\ker(H^1(\partial W;Q_\mu/\Lambda_S)\xrightarrow{\mathrm{BS}}H^2(\partial W;\Lambda_S))}$. Similarly, translating the role of v_B into this setting, there is a well-defined map $(i^{(W,\partial W),W}_{Q_\mu})^{-1}$ from $i^W_{\Lambda_S,Q_\mu} \ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q_\mu))$ to $\frac{H^2(W,\partial W;Q_\mu)}{\ker(H^2(W,\partial W;Q_\mu)\to H^2(W;Q_\mu))}$. Furthermore, we shall denote by δ_{Q_μ/Λ_S} the boundary map which arises in the long sequence of the pair $(W,\partial W)$ with Q_μ/Λ_S coefficients.

Applying Lemma 6.5.1 to the cochain complexes of ∂W, of W and of $(W,\partial W)$ with coefficients in Λ_S, Q_μ and Q_μ/Λ_S immediately yields the following lemma.

Lemma 6.5.2. *Let W be the exterior of a pushed-in C-complex in D^4. The following diagram anticommutes:*

$$\begin{tikzcd}
\ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q_\mu)) \arrow[r, "i^{W,\partial W}_{\Lambda_S}"] \arrow[d, "i^W_{\Lambda_S,Q_\mu}"] & i^{W,\partial W}_{\Lambda_S} \ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q_\mu)) \arrow[d, "BS^{-1}"] \\
i^W_{\Lambda_S,Q_\mu} \ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q_\mu)) \arrow[d, "(i^{(W,\partial W),W}_{Q_\mu})^{-1}"] & \frac{H^1(\partial W;Q_\mu/\Lambda_S)}{\ker(H^1(\partial W;Q_\mu/\Lambda_S)\xrightarrow{BS}H^2(\partial W;\Lambda_S))} \arrow[d, "\delta_{Q_\mu/\Lambda_S}"] \\
\frac{H^2(W,\partial W;Q_\mu)}{\ker(H^2(W,\partial W;Q_\mu)\to H^2(W;Q_\mu))} \arrow[r, "i^{(W,\partial W)}_{Q_\mu,Q_\mu/\Lambda_S}"] & \frac{H^2(W,\partial W;Q_\mu/\Lambda_S)}{\mathrm{im}(H^1(\partial W;Q_\mu)\to H^2(W,\partial W;Q_\mu/\Lambda_S))}.
\end{tikzcd}$$

Recall from Section 5.5 that Poincaré duality provides isomorphisms from $H_1(\partial W;\Lambda_S)$ to $H^2(\partial W;\Lambda_S)$ and from $H_2(W,\partial W;\Lambda_S)$ to $H^2(W;\Lambda_S)$. Both these maps shall be denoted by PD. Furthermore, we denote by ∂ the map from $H_2(W,\partial W;\Lambda_S)$ to $H_1(\partial W;\Lambda_S)$ which arises in the long exact sequence of the pair $(W,\partial W)$. We shall abbreviate $TH_1(\partial W;\Lambda_S)$ by T. Finally, we recall that a C-complex $S = S_1 \cup \ldots \cup S_\mu$ is totally connected if each S_i is connected and $S_i \cap S_j \neq \emptyset$ for all $i \neq j$.

Lemma 6.5.3. *Let W be the exterior of a pushed-in C-complex in D^4.*

1. *Poincaré duality restricts to a well-defined map $\partial^{-1}(T) \to \ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q_\mu))$.*

2. *If the C-complex is totally connected, then Poincaré duality restricts to a well-defined map $T \to i^{W,\partial W}_{\Lambda_S} \ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q_\mu))$.*

Proof. In order to prove both statements, we shall consider the following commutative diagram:

$$\begin{tikzcd} H_2(W,\partial W;\Lambda_S) \arrow[r, "\mathrm{PD}"] \arrow[d, "\partial"] & H^2(W;\Lambda_S) \arrow[r, "i^W_{\Lambda_S,Q_\mu}"] \arrow[d, "i^{W,\partial W}_{\Lambda_S}"] & H^2(W;Q_\mu) \arrow[d, "i^{W,\partial W}_{Q_\mu}"] \\ H_1(\partial W;\Lambda_S) \arrow[r, "\mathrm{PD}"] & H^2(\partial W;\Lambda_S) \arrow[r, "i^{\partial W}_{\Lambda_S,Q_\mu}"] & H^2(\partial W;Q_\mu). \end{tikzcd} \tag{6.16}$$

We start with the first assertion. Given x in $\partial^{-1}(T)$, the goal is to show that $PD(x)$ lies in $\ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q_\mu))$ or in other words, we wish to show that $(i^{\partial W}_{\Lambda_S,Q_\mu} \circ i^{W,\partial W}_{\Lambda_S} \circ PD)(x)$ vanishes. Since $\partial(x)$ is a torsion element of $H_1(\partial W;\Lambda_S)$, there exists a non-zero λ in Λ_S for which $\lambda\partial(x) = 0$. The commutativity of (6.16) now implies that $\lambda(i^{\partial W}_{\Lambda_S,Q_\mu} \circ i^{W,\partial W}_{\Lambda_S} \circ PD)(x) = (i^{\partial W}_{\Lambda_S,Q_\mu} \circ PD)(\lambda\partial(x)) = 0$. Since $H^2(W;Q_\mu)$ is a vector space and λ is non-zero, the first claim is proved.

Next, we deal with the second claim. Given a in T, we must find a d in $\ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q_\mu))$ such that $i^{W,\partial W}_{\Lambda_S}(d) = PD(a)$. Since we now assume the C-complex to be totally connected, Corollary 6.3.4 implies that $H_1(W;\Lambda_S) = 0$ and thus ∂ is surjective. Consequently, there exists an x in $H_2(W,\partial W;\Lambda_S)$ for which $\partial(x) = a$. Since a is torsion, x is actually in $\partial^{-1}(T)$ and so the first claim implies that $PD(x)$ lies in $\ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q_\mu))$. Thus we set $d := PD(x)$ and observe that the commutativity of (6.16) implies $PD(a) = PD(\partial(x)) = i^{W,\partial W}_{\Lambda_S}(PD(x)) = i^{W,\partial W}_{\Lambda_S}(d)$, as desired. □

Next, we deal with the evaluation maps which were described in Section 5.4. More precisely we shall consider the map from $\overline{H^2(W,\partial W;Q_\mu)}$ to $\overline{\mathrm{Hom}_{\Lambda_S}(H_2(W,\partial W;\Lambda_S),Q_\mu)}$ and the map from $H^2(W,\partial W;Q_\mu/\Lambda_S)$ to $\overline{\mathrm{Hom}_{\Lambda_S}(H_2(W,\partial W;\Lambda_S),Q_\mu/\Lambda_S)}$.

Lemma 6.5.4. *Let W be the exterior of a pushed-in C-complex in D^4.*

1. *The evaluation map on $H^2(W,\partial W;Q_\mu)$ induces a well-defined map*

$$\mathrm{ev}\colon \frac{H^2(W,\partial W;Q_\mu)}{\ker(H^2(W,\partial W;Q_\mu) \to H^2(W;Q_\mu))} \to \overline{\mathrm{Hom}_{\Lambda_S}(\partial^{-1}(T),Q_\mu)}.$$

2. *The evaluation map on $H^2(W,\partial W;Q_\mu/\Lambda_S)$ induces a well-defined map*

$$\mathrm{ev}\colon \frac{H^2(W,\partial W;Q_\mu/\Lambda_S)}{\mathrm{im}(H^1(\partial W;Q_\mu) \to H^2(W,\partial W;Q_\mu/\Lambda_S))} \to \overline{\mathrm{Hom}_{\Lambda_S}(\partial^{-1}(T),Q_\mu/\Lambda_S)}.$$

Proof. From now on, we shall write $\langle \varphi, x\rangle$ instead of $(\mathrm{ev})(\varphi)(x)$. We start by proving the first assertion. First of all, by exactness we have $\ker(H^2(W,\partial W;Q_\mu) \to H^2(W;Q_\mu)) = \mathrm{im}(H^1(\partial W;Q_\mu) \xrightarrow{\delta_{Q_\mu}} H^2(W,\partial W;Q_\mu))$, where δ_{Q_μ} denotes the boundary map in the long exact sequence of the pair. Consequently, the goal is to show that for all φ in $H^1(\partial W;Q_\mu)$ and all x in $\partial^{-1}(T)$, one has $\langle \delta_{Q_\mu}\varphi, x\rangle = 0$. Consider the following commutative diagram:

$$
\begin{tikzcd}
H^1(\partial W;Q_\mu) \arrow[r, "\mathrm{ev}"] \arrow[d, "\delta_{Q_\mu}"] & \overline{\mathrm{Hom}_{\Lambda_S}(H_1(\partial W;\Lambda_S),Q_\mu)} \arrow[d, "\partial^*"] \\
H^2(W,\partial W;Q_\mu) \arrow[r, "\mathrm{ev}"] & \overline{\mathrm{Hom}_{\Lambda_S}(H_2(W,\partial W;\Lambda_S),Q_\mu)}.
\end{tikzcd}
\tag{6.17}
$$

Since ∂x is torsion, there exists a non-zero λ in Λ_S for which $\lambda\partial(x)$ vanishes. The diagram in (6.17) now gives $\lambda\langle\delta_{Q_\mu}\varphi,x\rangle = \lambda\langle\varphi,\partial x\rangle = \langle\varphi,\lambda\partial(x)\rangle = 0$. Since this equation takes place in the field Q_μ and λ is non-zero, we get $\langle\delta_{Q_\mu}\varphi,x\rangle = 0$, as desired.

To prove the second claim, start with φ in $H^1(\partial W;Q_\mu)$ and x in $\partial^{-1}(T)$. We denote by $i^{\partial W}_{Q_\mu,Q_\mu/\Lambda_S}$ the homomorphism from $H^1(\partial W;Q_\mu)$ to $H^1(\partial W;Q_\mu/\Lambda_S)$ and by δ_{Q_μ/Λ_S} the boundary homomorphism from $H^1(\partial W;Q_\mu/\Lambda_S)$ to $H^2(W,\partial W;Q_\mu/\Lambda_S)$. We wish to show that $\langle(\delta_{Q_\mu/\Lambda_S}\circ i^{\partial W}_{Q_\mu,Q_\mu/\Lambda_S})(\varphi),x\rangle = 0$. Since φ is Q_μ-valued and $\partial(x)$ is torsion, the result also follows from the commutativity of (6.17). $\square$

Recall that we denote by BS^{-1} the map from $i^{W,\partial W}_{\Lambda_S}\ker(H^2(W;\Lambda_S)\to H^2(\partial W;Q_\mu))$ to $\frac{H^1(\partial W;Q_\mu/\Lambda_S)}{\ker(H^1(\partial W;Q_\mu/\Lambda_S)\to H^2(\partial W;\Lambda_S))}$ which appeared in Lemma 6.5.2. Combining the previous results, we obtain the following lemma.

Lemma 6.5.5. *Let L be a μ-colored link and let W be the exterior of a pushed-in totally connected C-complex for L. The squares and triangle in the following diagram commute, while the top pentagon anticommutes. Furthermore, the map $\Gamma := \mathrm{ev}\circ BS^{-1}\circ PD$ coincides with the adjoint of the Blanchfield pairing* $\mathrm{Bl}(L)$.

$$
\begin{tikzcd}
\partial^{-1}(T) \arrow[r, "\partial"] \arrow[dd, "(i^{(W,\partial W),W}_{Q_\mu})^{-1}\circ i^{W}_{\Lambda_S,Q_\mu}\circ PD"'] & T \arrow[d, "BS^{-1}\circ PD"] \arrow[ddr, bend left, "\Gamma"] & \\
& \frac{H^1(\partial W;Q_\mu/\Lambda_S)}{\ker(BS)} \arrow[d, "\delta_{Q_\mu/\Lambda_S}"] \arrow[dr, "\mathrm{ev}"] & \\
\frac{H^2(W,\partial W;Q_\mu)}{\ker(H^2(W,\partial W;Q_\mu)\to H^2(W;Q_\mu))} \arrow[r, "i^{(W,\partial W)}_{Q_\mu,Q_\mu/\Lambda_S}"] \arrow[d, "\mathrm{ev}"] & \frac{H^2(W,\partial W;Q_\mu/\Lambda_S)}{\mathrm{im}(H^1(\partial W;Q_\mu)\to H^2(W,\partial W;Q_\mu/\Lambda_S))} \arrow[dr, "\mathrm{ev}"] & \overline{\mathrm{Hom}_{\Lambda_S}(T,Q_\mu/\Lambda_S)} \arrow[d, "\partial^*"] \\
\overline{\mathrm{Hom}_{\Lambda_S}(\partial^{-1}(T),Q_\mu)} \arrow[rr] & & \overline{\mathrm{Hom}_{\Lambda_S}(\partial^{-1}(T),Q_\mu/\Lambda_S)}.
\end{tikzcd}
\tag{6.18}
$$

Proof. We start by arguing that the maps in (6.18) are well-defined. For the upper right evaluation map, this follows from the same argument as the one which was used in Lemma 6.2.1. All the other maps are well-defined thanks to Lemma 6.5.2, Lemma 6.5.3 and Lemma 6.5.4. The top pentagon anticommutes thanks to Lemma 6.5.3 and Lemma 6.5.2. The remaining squares and triangle clearly commute. To prove the second assertion, we start by noting that Lemma 6.3.5 implies that the inclusion induced map $H_1(X_L;\Lambda_S)\to H_1(\partial W;\Lambda_S)$ is an isomorphism. Using this fact, we observe that Γ is defined exactly as the adjoint Ω of the Blanchfield pairing was, see Section 6.2. $\square$

Looking at the leftmost column of (6.18), we wish to define a pairing on $\partial^{-1}(T)$. To do this, we start by considering the composition

$$\begin{aligned}\Theta\colon \quad \partial^{-1}(T) &\xrightarrow{\mathrm{PD}} \ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q_\mu)) \\ &\xrightarrow{i^W_{\Lambda_S,Q_\mu}} i^W_{\Lambda_S,Q_\mu} \ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q_\mu)) \\ &\longrightarrow \frac{H^2(W,\partial W;Q_\mu)}{\ker(H^2(W,\partial W;Q_\mu) \to H^2(W;Q_\mu))} \\ &\xrightarrow{\mathrm{ev}} \overline{\mathrm{Hom}_{\Lambda_S}(\partial^{-1}(T),Q_\mu)}\end{aligned}$$

of Λ_S-linear homomorphisms, where the third arrow denotes the homomorphism $(i^{(W,\partial W),W}_{Q_\mu})^{-1}$ which was described in the discussion leading up to Lemma 6.5.2. Note that the first map is well-defined thanks to Lemma 6.5.3, the second map is obviously well-defined, the discussion prior to Lemma 6.5.2 ensures that the third map is well-defined, and the fourth map is well-defined thanks to Lemma 6.5.4. We define the desired pairing on $\partial^{-1}(T)$ by

$$\theta(x,y) := \Theta(y)(x).$$

Recall from Lemma 6.5.5 and its proof that the pairing defined by Γ on $TH_1(\partial W;\Lambda_S)$ coincides with the Blanchfield pairing on $TH_1(X_L;\Lambda_S)$. Using these identifications, Lemma 6.5.5 implies the following proposition.

Proposition 6.5.6. *Let L be a μ-colored link and let W be the exterior of a pushed-in totally connected C-complex for L. The following diagram commutes:*

$$\begin{tikzcd} \partial^{-1}(TH_1(\partial W;\Lambda_S)) \times \partial^{-1}(TH_1(\partial W;\Lambda_S)) \arrow[r, "-\theta"] \arrow[d, "\partial\times\partial"] & Q_\mu \arrow[d] \\ TH_1(\partial W;\Lambda_S) \times TH_1(\partial W;\Lambda_S) \arrow[r, "\mathrm{Bl}(L)"] & Q_\mu/\Lambda_S. \end{tikzcd} \tag{6.19}$$

As (6.19) suggests, the computation of the Blanchfield pairing now boils down to the computation of θ. The remainder of the proof is devoted to this task.

From now on, we shall assume that W is the exterior of a pushed-in *totally connected* C-complex. Recall from Section 5.6 that the intersection form $\lambda_{\Lambda_S}(W)$ on W is defined as the adjoint of the composition

$$\Phi\colon H_2(W;\Lambda_S) \xrightarrow{i} H_2(W,\partial W;\Lambda_S) \xrightarrow{\mathrm{PD}} H^2(W;\Lambda_S) \xrightarrow{\mathrm{ev}} \overline{\mathrm{Hom}_{\Lambda_S}(H_2(W;\Lambda_S),\Lambda_S)}. \tag{6.20}$$

In other words, $\lambda_{\Lambda_S}(W)(x,y) := \Phi(y)(x)$. In particular, we notice that Φ vanishes on $\ker(i)$ and descends to a map on $H_2(W;\Lambda_S)/\ker(i)$ which we also denote by Φ.

Since we assumed that W is the exterior of a pushed-in totally connected C-complex, Corollary 6.3.4 implies that $H_1(W;\Lambda_S)=0$. Thus, there is an exact sequence

$$H_2(W;\Lambda_S) \xrightarrow{i} H_2(W,\partial W;\Lambda_S) \xrightarrow{\partial} H_1(\partial W;\Lambda_S) \to 0.$$

Consequently, from now on, we shall identify $H_1(\partial W;\Lambda_S)$ with the cokernel of the map i. In particular, elements of $H_1(\partial W;\Lambda_S)$ will be denoted by $[x]$, where x lies in $H_2(W,\partial W;\Lambda_S)$.

Furthermore, we shall identify the boundary map ∂ with the quotient map of $H_2(W,\partial W;\Lambda_S)$ onto $\operatorname{coker}(i)$. In other words, we allow ourselves to interchangeably write $\partial(x)$ and $[x]$.

Let Δ be the order of $TH_1(\partial W;\Lambda_S)$ and let x,y be in $\partial^{-1}(T)$. Since $[x]$ and $[y]$ are torsion, there exists x_0 and y_0 in $H_2(W;\Lambda_S)$ such that $\Delta x = i(x_0)$ and $\Delta y = i(y_0)$. Define a Q_μ-valued pairing ψ on $\partial^{-1}(T)$ by setting

$$\psi(x,y) := \frac{1}{\Delta^2}\lambda_{\Lambda_S}(W)(x_0,y_0).$$

Observe that ψ is well-defined: if x_0 and x_0' both satisfy $i(x_0) = \Delta x = i(x_0')$, then $x_0 - x_0'$ lies in $\ker(i)$ and thus $\lambda_{\Lambda_S}(W)(x_0 - x_0', y) = 0$, as we observed above. The same reasoning applies to the second variable. In particular, we could have very well taken x_0 and y_0 in $H_2(W;\Lambda_S)/\ker(i)$. Summarizing, we have two Q_μ-valued pairings defined on $\partial^{-1}(T)$ and we wish to show that they agree:

Proposition 6.5.7. *θ is equal to ψ.*

Before diving into the proof, let us set up some notation. First, we define a map $j\colon \partial^{-1}(T) \to \operatorname{im}(i)$ as follows. Given x in $\partial^{-1}(T)$, we set $j(x) := i(x_0)$, where x_0 is any element of $H_2(W;\Lambda_S)$ which satisfies $i(x_0) = \Delta x$. The map j is easily seen to be well-defined. Next, we set

$$K := \ker(H^2(W;\Lambda_S) \xrightarrow{i^{W,\partial W}_{\Lambda_S}} H^2(\partial W;\Lambda_S) \xrightarrow{i^{\partial W}_{\Lambda_S,Q_\mu}} H^2(\partial W;Q_\mu)).$$

Note that K already appeared in Lemma 6.5.2 as well as in the definition of θ. The discussion leading up to Lemma 6.5.2 also provided a homomorphism $(i^{(W,\partial W),W}_{Q_\mu})^{-1}$ whose domain was $i^W_{\Lambda_S,Q_\mu}(K)$. For the moment however, we shall rename it as

$$k^*\colon\quad i^W_{\Lambda_S,Q_\mu}(K) \to \frac{H^2(W,\partial W;Q_\mu)}{\ker(H^2(W,\partial W;Q_\mu)\to H^2(W;Q_\mu))}$$

and recall its definition. Given ϕ in K, the definition of K implies that $(i^{\partial W,W}_{Q_\mu} \circ i^W_{\Lambda_S,Q_\mu})(\phi)$ vanishes. Using the exactness of the long exact sequence of the pair $(W,\partial W)$ with Q_μ coefficients, it follows that $i^{(W,\partial W),W}_{Q_\mu}(\xi) = i^W_{\Lambda_S,Q_\mu}(\phi)$ for some $\xi \in H^2(W,\partial W;Q_\mu)$. The map k^* is defined by $k^*(i^W_{\Lambda_S,Q_\mu}(\phi)) = \xi$.

Remark 6.5.8. Note that if $\phi = i^{(W,\partial W),W}_{\Lambda_S}(\varphi)$ for some φ in $H^2(W,\partial W;\Lambda_S)$, then the description of k^* becomes more concrete. The reason is that we can pick ξ to be $i^{(W,\partial W)}_{\Lambda_S,Q_\mu}(\varphi)$. Indeed, we have

$$i^{(W,\partial W),W}_{Q_\mu}(\xi) = (i^{(W,\partial W)}_{Q_\mu} \circ i^{(W,\partial W)}_{\Lambda_S,Q_\mu})(\varphi) = (i^W_{\Lambda_S,Q_\mu} \circ i^{(W,\partial W),W}_{\Lambda_S})(\varphi) = i^W_{\Lambda_S,Q_\mu}(\phi),$$

where the second equality follows from the diagram below:

$$\begin{tikzcd}
H^2(W,\partial W;\Lambda_S) \arrow[r, "i^{(W,\partial W)}_{\Lambda_S,Q_\mu}"] \arrow[d, "i^{(W,\partial W),W}_{\Lambda_S}"] & H^2(W,\partial W;Q_\mu) \arrow[d, "i^{(W,\partial W),W}_{Q_\mu}"] \\
H^2(W;\Lambda_S) \arrow[r, "i^W_{\Lambda_S,Q_\mu}"] & H^2(W;Q_\mu).
\end{tikzcd} \tag{6.21}$$

Summarizing, we have $(k^* \circ i^W_{\Lambda_S,Q_\mu} \circ i^{(W,\partial W),W}_{\Lambda_S})(\varphi) = i^{(W,\partial W)}_{\Lambda_S,Q_\mu}(\varphi)$.

Let us temporarily write V instead of $H_2(W;\Lambda_S)$. Proposition 6.5.7 will follow if we manage to show that all the maps in (6.22) are well-defined and produce a commutative diagram. Indeed, in this diagram, there are two routes which lead from the upper right corner to the lower left corner. Taking the uppermost route produces the pairing ψ, while the lowermost route produces θ:

$$
\begin{tikzcd}
\frac{V}{\ker(i)} \arrow[r, "i", "\cong"'] \arrow[d, "\Phi"] & \operatorname{im}(i) \arrow[d, "PD"] & \partial^{-1}(T) \arrow[l, "j"'] \arrow[d, "\mathrm{PD}"] \\
\overline{\operatorname{Hom}_{\Lambda_S}(\frac{V}{\ker(i)},\Lambda_S)} \arrow[d] & i^{(W,\partial W),W}_{\Lambda_S}\circ \mathrm{PD}(V) \arrow[l, "\mathrm{ev}"'] \arrow[d, "i^W_{\Lambda_S,Q_\mu}"] & K \arrow[d, "i^W_{\Lambda_S,Q_\mu}"] \\
\overline{\operatorname{Hom}_{\Lambda_S}(\frac{V}{\ker(i)},Q_\mu)} \arrow[d, "\frac{1}{\Delta^2}j^*(i^{-1})^*"] & i^W_{\Lambda_S,Q_\mu}\circ i^{(W,\partial W),W}_{\Lambda_S}\circ PD(V) \arrow[l, "\mathrm{ev}"'] \arrow[r, "\cdot\frac{1}{\Delta}"] \arrow[d, "k^*"] & i^W_{\Lambda_S,Q_\mu}(K) \arrow[d, "k^*"] \\
\overline{\operatorname{Hom}_{\Lambda_S}(\partial^{-1}(T),Q_\mu)} & k^*\circ i^W_{\Lambda_S,Q_\mu}\circ i^{(W,\partial W),W}_{\Lambda_S}\circ PD(V) \arrow[l, "\frac{1}{\Delta}\mathrm{ev}"'] \arrow[r, "\cdot\frac{1}{\Delta}"] & \frac{H^2(W,\partial W;Q_\mu)}{\ker(H^2(W,\partial W;Q_\mu)\to H^2(W;Q_\mu))}. \arrow[ll, bend left, "\mathrm{ev}"]
\end{tikzcd}
\tag{6.22}
$$

We begin by arguing that all the maps in (6.22) are well-defined. We already checked that the rightmost vertical maps are well-defined, see Lemma 6.5.2 and Lemma 6.5.3. The middle Poincaré duality map is well-defined: this follows immediately from the equality $PD\circ i = i^{(\partial W,W),W}_{\Lambda_S}\circ PD$. Next, we deal with the two horizontal maps on the bottom right. First observe that $i^{(\partial W,W),W}_{\Lambda_S}\circ PD(V)$ is a subspace of K: indeed $K=\ker(H_2(W;\Lambda_S)\xrightarrow{i^{W,\partial W}_{\Lambda_S}} H_2(\partial W;\Lambda_S)\xrightarrow{i^{\partial W}_{\Lambda_S,Q_\mu}} H_2(\partial W;Q_\mu))$, and $i^{W,\partial W}_{\Lambda_S}\circ i^{(W,\partial W),W}_{\Lambda_S}=0$ by exactness. Consequently $i^W_{\Lambda_S,Q_\mu}\circ i^{(W,\partial W),W}_{\Lambda_S}\circ PD(V)$ is a subspace of $i^W_{\Lambda_S,Q_\mu}(K)$. It then follows that $k^*\circ i^W_{\Lambda_S,Q_\mu}\circ i^{(W,\partial W),W}_{\Lambda_S}\circ PD(V)$ is a subspace of the term in the lower right corner. Since these spaces are Q_μ-vector spaces, multiplication by $\frac{1}{\Delta}$ makes sense. It also follows from these observations and Lemma 6.5.4 that the lower two evaluation maps in (6.22) are well-defined. The upper two evaluation maps are well defined since induced maps commute with evaluations. The next lemma will conclude the proof of Proposition 6.5.7.

Lemma 6.5.9. *All the squares in (6.22) commute.*

Proof. The upper left square commutes by definition of Φ, see (6.20). The middle left square, the bottom right square and the bottom triangle all clearly commute. Let us now deal with the large rectangle on the upper right. Start with x in $\partial^{-1}(T)$. Using the definition of j, we have $j(x)=i(x_0)$, where x_0 lies in $H_2(W;\Lambda_S)$ and satisfies $i(x_0)=\Delta x$. The desired relation now follows readily:

$$\frac{1}{\Delta}(i^W_{\Lambda_S,Q_\mu}\circ \mathrm{PD}\circ j)(x)=\frac{1}{\Delta}(i^W_{\Lambda_S,Q_\mu}\circ \mathrm{PD}\circ i)(x_0)=(i^W_{\Lambda_S,Q_\mu}\circ PD)(x).$$

Finally, we deal with the lower left square. Let φ be in $H^2(W, \partial W; \Lambda_S)$ and let x be in $\partial^{-1}(T)$. Using once again the definition of j, we have $(i^{-1} \circ j)(x) = [x_0]$ where x_0 lies in $H_2(W; \Lambda_S)$ and satisfies $i(x_0) = \Delta x$. Consequently, we get the relation

$$\begin{aligned}\frac{1}{\Delta^2}\langle (i^W_{\Lambda_S,Q_\mu} \circ i^{(W,\partial W),W}_{\Lambda_S})(\varphi), (i^{-1} \circ j)(x)\rangle &= \frac{1}{\Delta^2}\langle (i^W_{\Lambda_S,Q_\mu} \circ i^{(W,\partial W),W}_{\Lambda_S})(\varphi), [x_0]\rangle = \frac{1}{\Delta^2}\langle \varphi, i([x_0])\rangle \\ &= \frac{1}{\Delta}\langle \varphi, x\rangle,\end{aligned}$$

where in the second equality, we simultaneously used that induced maps commute with evaluations and the fact that $i^W_{\Lambda_S,Q_\mu}$ changes the coefficients without affecting the expression involved. On the other hand, recalling the conclusion of Remark 6.5.8, we can compute the other term:

$$\frac{1}{\Delta}\langle (k^* \circ i^W_{\Lambda_S,Q_\mu} \circ i^{(W,\partial W),W}_{\Lambda_S})(\varphi), x\rangle = \frac{1}{\Delta}\langle i^{(W,\partial W)}_{\Lambda_S,Q_\mu}(\varphi), x\rangle = \frac{1}{\Delta}\langle \varphi, x\rangle.$$

Combining these observations, the lower left square of (6.22) commutes. This concludes the proof the lemma and thus the proof of Proposition 6.5.7. □

We now use Theorem 6.4.9 and the basis provided by Proposition 6.4.2 in order to show that the homomorphism $i\colon H_2(W; \Lambda_S) \to H_2(W, \partial W; \Lambda_S)$ can be represented by the transpose $H^T = \overline{H}$ of the C-complex matrix H.

Lemma 6.5.10. *If the C-complex S is totally connected, then there are bases with respect to which the homomorphism $i\colon H_2(W; \Lambda_S) \to H_2(W, \partial W; \Lambda_S)$ is represented by H^T.*

Proof. We first claim that the evaluation map $H^2(W; \Lambda_S) \to \overline{\mathrm{Hom}_{\Lambda_S}(H_2(W; \Lambda_S), \Lambda_S)}$ is an isomorphism. We use the universal coefficient spectral sequence of Theorem 5.4.4 where the nonsingular $\pi_1(W)$-invariant sesquilinear pairing is given by the multiplication $\Lambda_S \times \Lambda_S \to \Lambda_S, (x, y) \mapsto x\overline{y}$. This spectral sequence has $E^2_{p,q} = \mathrm{Ext}^q_{\Lambda_S}(H_p(W; \Lambda_S), \Lambda_S)$, has differentials of degree $(1-r, r)$ and converges to $H^*(W; \Lambda_S)$. Since we have $H_0(W; \Lambda_S) = H_1(W; \Lambda_S) = 0$ by Lemma 5.7.3 and Corollary 6.3.4, it follows that $E^{1,q}_\infty = 0 = E^{0,q}_\infty$ for each q, and $E^{2,0}_\infty = \overline{\mathrm{Hom}_{\Lambda_S}(H_2(W; \Lambda_S), \Lambda_S)}$. The claim now follows readily by studying the resulting filtration of $H^2(W; \Lambda_S)$.

Next, we pick our basis $\mathscr{B}$ for $H_1(S)$ (recall the beginning of Subsection 6.4.1). This basis yields generalized Seifert matrices A^ε for L and it induces a basis $\mathscr{C}$ of $H_2(W; \Lambda_S)$, recall Proposition 6.4.2. We endow $\overline{\mathrm{Hom}_{\Lambda_S}(H_2(W; \Lambda_S), \Lambda_S)}$ with the corresponding dual basis $\mathscr{C}^*$. Now we consider the following commutative diagram of Λ_S-homomorphisms

$$\begin{tikzcd} H_2(W;\Lambda_S) \arrow[r, "PD"] \arrow[d] & H^2(W,\partial W;\Lambda_S) \arrow[d] & \\ H_2(W,\partial W;\Lambda_S) \arrow[r, "PD"] & H^2(W;\Lambda_S) \arrow[r, "\mathrm{ev}"] & \overline{\mathrm{Hom}_{\Lambda_S}(H_2(W;\Lambda_S),\Lambda_S)}. \end{tikzcd} \tag{6.23}$$

Here the bottom-right map is an isomorphism thanks to the claim. We now assert that with respect to the bases $\mathscr{C}$ and $\mathscr{C}^*$, the homomorphism

$$\Theta : H_2(W; \Lambda_S) \to \overline{\mathrm{Hom}_{\Lambda_S}(H_2(W; \Lambda_S), \Lambda_S)}$$

is represented by the matrix H^T. By definition, the intersection pairing is given by $\lambda_{\Lambda_S}(W)(x,y) = \Theta(y)(x)$. In Theorem 6.4.9, we found out that the matrix representing $\lambda_{\Lambda_S}(W)$ with respect to the basis $\mathscr{C}$ is H. If $\mathscr{C} = \{x_1, \ldots, x_n\}$, we can hence write

$$\Theta(x_j) \;=\; \sum_{i=1}^{n} \overline{\Theta(x_j)(x_i)}\, x_i^* \;=\; \sum_{i=1}^{n} \overline{\lambda_{\Lambda_S}(W)(x_i, x_j)}\, x_i^* \;=\; \sum_{i=1}^{n} \overline{H_{ij}}\, x_i^* \;=\; \sum_{i=1}^{n} H_{ji}\, x_i^*,$$

which proves the claim. Notice that in the first equality, the bar appears because of the involuted structure of $\overline{\mathrm{Hom}_{\Lambda_S}(H_2(W;\Lambda_S),\Lambda_S)}$, while the last equality is true because H is a Hermitian matrix. Now we equip $H_2(W,\partial W;\Lambda_S)$ with the basis induced from $\mathscr{C}^*$ and the bottom two isomorphisms in the diagram of equation (6.23). Then, by commutativity of the diagram, the left vertical map is also represented by H^T. □

We are now in position to conclude the proof of Theorem 6.1.2.

proof of Theorem 6.1.2. Let L be a colored link and let W be the exterior of a pushed-in totally connected C-complex for L. Recall that i denotes the inclusion induced map from $H_2(W;\Lambda_S)$ to $H_2(W,\partial W;\Lambda_S)$ and that given torsion elements $[x]$ and $[y]$ in $H_1(X_L;\Lambda_S) \cong H_1(\partial W;\Lambda_S) \cong \mathrm{coker}(i)$, there exists x_0 and y_0 in $H_2(W;\Lambda_S)$ such that $i(x_0) = \Delta x$ and $i(y_0) = \Delta y$. Using Proposition 6.5.6, we already know that $\mathrm{Bl}(L)([x],[y]) = -\theta(x,y)$. Next, Proposition 6.5.7 implies that $\theta(x,y) = \psi(x,y) = \frac{1}{\Delta^2}\lambda_{\Lambda_S}(W)(x_0,y_0)$. Summarizing, we have

$$\mathrm{Bl}(L)([x],[y]) = -\theta(x,y) = -\psi(x,y) = -\frac{1}{\Delta^2}\lambda_{\Lambda_S}(W)(x_0,y_0). \tag{6.24}$$

Note that any choice of x_0, y_0 will do since $\lambda_{\Lambda_S}(W)$ vanishes on $\ker(i)$; this was already noticed in the definition of ψ. Furthermore, note that (6.24) holds independently of the chosen representatives x and y for the classes $[x]$ and $[y]$. Indeed if x and x' represent $[x]$, we claim that $\psi(x,y)$ and $\psi(x',y)$ coincide in Q_μ/Λ_S, i.e. that $\psi(x-x',y)$ lies in Λ_S; the same proof will hold for the second variable. Since x and x' both represent $[x]$, there is a v in $H_2(W;\Lambda_S)$ for which $x - x' = i(v)$. Consequently $i(\Delta v) = \Delta i(v)$. Picking y_0 such that $i(y_0) = \Delta y$ and using the definition of $\lambda_{\Lambda_S}(W)$, the following equalities prove our claim, since the rightmost term lies in Λ_S:

$$\psi(x-x',y) = \psi(i(v),y) = \frac{1}{\Delta^2}\lambda_{\Lambda_S}(W)(\Delta v, y_0) = \frac{1}{\Delta}\langle(\mathrm{PD}\circ i)(y_0), v\rangle = \langle \mathrm{PD}(y), v\rangle.$$

Using Theorem 6.4.9, we know that there are bases with respect to which the intersection pairing $\lambda_{\Lambda_S}(W)$ on $H_2(W;\Lambda_S)$ is represented by the C-complex matrix H described in the introduction. Furthermore, with respect to the same bases, it was observed in Lemma 6.5.10 that the map i is represented by $\overline{H} = H^T$. Consequently, Equation (6.24) can be reformulated as follows. Let n denote the rank of the Λ_S-module $H_2(W;\Lambda_S)$. Given $[x],[y] \in TH_1(X_L;\Lambda_S)$, we have $\mathrm{Bl}(L)([x],[y]) = -\frac{1}{\Delta^2}x_0^T H \overline{y_0}$ for any choice of $x_0, y_0 \in \Lambda_S^n$ such that $\overline{H}x_0 = \Delta x$ and $\overline{H}y_0 = \Delta y$. Using the notations of the introduction, this can be written as

$$\mathrm{Bl}(L)([x],[y]) = -\lambda_H([x],[y]).$$

Up to now, we always supposed that W arose by pushing in a totally connected C-complex. Thus, *a priori*, Theorem 6.1.2 only holds for C-complex matrices which arise from totally connected C-complexes. To conclude the proof of Theorem 6.1.2, it therefore only remains to check that the pairing λ_H is independent of the choice of a C-complex for L.

The proof follows closely the reasoning of Remark 3.3.12. As explained in [41, p. 1230], if S and S' are two C-complexes for isotopic links, then the corresponding C-complex matrices H and H' are related by a finite number of the following two moves:

$$H \mapsto H \oplus (0) \quad \text{and} \quad H \mapsto \begin{pmatrix} H & \xi & 0 \\ \xi^* & \lambda & \alpha \\ 0 & \overline{\alpha} & 0 \end{pmatrix},$$

with α a unit of Λ_S. In the first case, the Λ_S-module $\Lambda_S^n/\overline{H}\Lambda_S^n$ picks up a free rank 1 factor, so its torsion submodule is left unchanged. It can then be checked that λ_H and $\lambda_{H\oplus(0)}$ are canonically isometric. In the second case, since α is a unit in Λ_S, one can assume via the appropriate base change that H is transformed into $H \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. One can then once again check that the forms associated to these two Hermitian matrices are canonically isometric. $\square$

6.6 Applications of Theorem 6.1.2

In this section, we provide several applications of Theorem 6.1.2. First, in Subsection 6.6.1 we give a new proof that the Blanchfield pairing is Hermitian and nonsingular. Then, in Subsection 6.6.2 we give quick proofs of some elementary properties of the Blanchfield pairing. Finally, in Subsection 6.6.3 we apply Theorem 6.1.2 to boundary links.

6.6.1 The Blanchfield pairing is Hermitian

In this subsection, we prove Corollary 6.1.3 which states that the Blanchfield pairing is nonsingular, Hermitian and takes value in $(\Delta_L^{\mathrm{tor}})^{-1}\Lambda_S/\Lambda_S$, where Δ_L^{tor} denotes the first non-vanishing Alexander polynomial of L over Λ_S. Using Theorem 6.1.2, this reduces to showing the corresponding statement for λ_H, where H is any C-complex matrix for L. Since this is a purely algebraic statement, we shall prove it in a somewhat greater generality.

First we recall some terminology. Let R be an integral domain with involution and let $Q(R)$ be its field of fractions. Given an R-module V, a pairing $b\colon V \times V \to Q(R)/R$ is *sesquilinear* if it is linear in the first entry and antilinear in the second entry. A sesquilinear pairing b is *non-degenerate* (respectively *nonsingular*) if the adjoint map $V \to \overline{\mathrm{Hom}_R(V, Q(R)/R)}$, $p \mapsto (q \mapsto \lambda(p,q))$ is a monomorphism (respectively an isomorphism) and *Hermitian* if $\lambda(w,v) = \overline{\lambda(v,w)}$ for any $v, w \in V$.

From now on, we make the additional assumption that R is Noetherian and factorial. Let H be a Hermitian $n \times n$-matrix over R, and let Δ denote the order of the R-module $\mathrm{Tor}_R(R^n/\overline{H}R^n)$. Given classes $[v]$ and $[w]$ in $\mathrm{Tor}_R(R^n/\overline{H}R^n)$, there exists v_0, w_0 in R^n such that $\Delta v = \overline{H}v_0$ and $\Delta w = \overline{H}w_0$. Proposition 6.6.2 will show that setting

$$\lambda_H([v],[w]) := \frac{1}{\Delta^2} v_0^T H \overline{w_0}$$

gives rise to a well-defined, Hermitian and nonsingular $\Delta^{-1}R/R$-valued pairing on $\mathrm{Tor}_R(R^n/\overline{H}R^n)$. Before proving this result, we explain its connection to the Blanchfield pairing.

Remark 6.6.1. Let M be an R-module. For $k \geq 0$, let $\Delta^{(k)}(M)$ denote the greatest common divisor of all $(m-k)\times(m-k)$ minors of an $m\times n$ presentation matrix of M. Denoting by r the rank of M, it is known that the order of $\mathrm{Tor}_R(M)$ is equal to $\Delta^{(r)}(M)$, see [150, Lemma 4.9]. If M is presented by a Hermitian matrix H, the above discussion and the equality $\overline{H} = H^T$ guarantee that $\overline{\Delta} = \Delta$.

Taking R to be Λ_S and H to be a C-complex matrix for a link L, we now claim that Δ is equal to $\Delta_L^{\mathrm{tor}}(L)$, the first non-vanishing Alexander polynomial of L over Λ_S. First of all, note that while the Λ_S-module $\Lambda_S^n/\overline{H}\Lambda_S^n$ may not be equal to $H_1(X_L;\Lambda_S)$, their torsion parts agree, see Remark 3.3.12. The claim now follows from the fact that the order of $TH_1(X_L;\Lambda_S)$ is equal to the first non-vanishing Alexander polynomial of L, as mentioned above.

Combining Theorem 6.1.2 with Remark 6.6.1, the following proposition will immediately imply Corollary 6.1.3.

Proposition 6.6.2. *The assignment $(v,w) \mapsto \frac{1}{\Delta^2}v_0^T H\overline{w_0}$ induces a well-defined pairing*

$$\lambda_H\colon\, Tor_R(R^n/\overline{H}R^n) \times Tor_R(R^n/\overline{H}R^n) \to \Delta^{-1}R/R$$

which is Hermitian. Furthermore, if $\det(H)$ is non-zero, then this form is induced by $(v,w) \mapsto v^T H^{-1}\overline{w}$.

Proof. Let us first check that this definition is independent of the choice of v_0 in R^n such that $\Delta v = \overline{H}v_0$. Any other choice is of the form $v_0 + k$ with k in R^n such that $\overline{H}k = 0$. Since H is Hermitian, we have the equalities

$$\frac{1}{\Delta^2}k^T H\overline{w_0} = \frac{1}{\Delta^2}(\overline{H}k)^T\overline{w_0} = 0\,,$$

which give the result. A similar argument shows that the definition is independent of the choice of w_0 such that $\Delta w = \overline{H}w_0$. Next, let us check that it does not depend on the choice of v representing the class $[v]$. Any other choice is of the form $v + \overline{H}u$ where u lies in R^n; since $\Delta(v+\overline{H}u) = \overline{H}(v_0 + \Delta u)$ and $\overline{\Delta} = \Delta$, the element

$$\frac{1}{\Delta^2}(\Delta u)^T H\overline{w_0} = \frac{1}{\Delta}u^T H\overline{w_0} = u^T\overline{w}$$

belongs to R, so the class in $Q(R)/R$ is indeed well-defined. A similar argument shows that it does not depend on the choice of w representing the class $[w]$, thus concluding the proof that λ_H is well-defined. The fact that λ_H is sesquilinear is clear, and it is Hermitian since H is and $\overline{\Delta} = \Delta$.

To show the second claim, note that if $\det(H)$ is non-zero, then H is invertible over $Q(R)$ so the equation $\Delta v = \overline{H}v_0$ is equivalent to $v_0 = \Delta\overline{H}^{-1}v$ (and similarly for w_0). Replacing v_0 and w_0 by these values and using one last time the fact that H is Hermitian, we obtain the result. This concludes the proof of the proposition. □

6.6.2 Some properties of the Blanchfield pairing

Let R be a Noetherian factorial integral domain with involution. Before dealing with the properties of the Blanchfield pairing, we start by investigating the behavior of λ_H under direct sums and multiplication by norms.

Lemma 6.6.3. *Let $H_1, \dots, H_\mu$ and H be Hermitian matrices over R and let u be a unit of R.*

1. *Setting $B := H_1 \oplus \ldots \oplus H_\mu$, one has $\lambda_B = \bigoplus_{i=1}^{\mu} \lambda_{H_i}$.*
2. *The pairings $\lambda_{u\overline{u}H}$ and λ_H are isometric.*

Proof. We start by proving the first assertion. Assume that each H_i is of size k_i, set $k := k_1 + \ldots + k_\mu$ and observe that $R^k/\overline{B}R^k$ is equal to $R^{k_1}/\overline{H_1}R^{k_1} \oplus R^{k_2}/\overline{H_2}R^{k_2} \oplus \ldots \oplus R^{k_\mu}/\overline{H_\mu}R^{k_\mu}$. Since the torsion of the latter direct sum is equal to the direct sum of the torsion of the $R^{k_i}/\overline{H_i}R^{k_i}$, it follows that the order of $\mathrm{Tor}_R(R^k/\overline{B}R^k)$ is equal to the product of the orders of the $\mathrm{Tor}_R(R^{k_i}/\overline{H_i}R^{k_i})$. We shall write this as $\Delta = \Delta_1 \cdots \Delta_\mu$, where Δ_i denotes the order of $\mathrm{Tor}_R(R^{k_i}/\overline{H_i}R^{k_i})$.

Next, we compute the sum of the λ_{H_i}. Let $x = x^1 \oplus x^2 \oplus \ldots \oplus x^\mu$ and $y = y^1 \oplus y^2 \oplus \ldots \oplus y^\mu$ be torsion elements in $R^k/\overline{B}R^k$. Relying on the previous paragraph, the x^i and y^i are torsion in $R^{k_i}/\overline{H_i}R^{k_i}$, and so there exists x_0^i and y_0^i which satisfy $\overline{H_i}x_0^i = \Delta_i x^i$ and $\overline{H_i}y_0^i = \Delta_i y^i$. Thus, by definition we have

$$\bigoplus_{i=1}^{\mu} \lambda_{H_i}(x,y) = \sum_{i=1}^{\mu} \frac{1}{\Delta_i^2}(x_0^i)^T H_i \overline{y_0^i}. \tag{6.25}$$

In order to compute λ_B and conclude the proof we proceed as follows. We define an element x_0 in $R^k/\overline{B}R^k$ by requiring its i-th component to be equal to $\Delta\Delta_i^{-1}x_0^i$. This way, the i-th component of $\overline{B}x_0$ is $\overline{H_i}(\Delta\Delta_i^{-1}x_0^i) = \Delta\Delta_i^{-1}\overline{H_i}x_0^i = \Delta x^i$ and thus $\overline{B}x_0 = \Delta x$. We can therefore use x_0 and y_0 to compute $\lambda_B(x,y)$ and we get

$$\lambda_B(x,y) = \frac{1}{\Delta^2}x_0^T B\overline{y_0} = \frac{1}{\Delta^2}\sum_{i=1}^{n}(\Delta\Delta_i^{-1}x_0^i)^T H_i\overline{(\Delta\Delta_i^{-1}y_0^i)} = \sum_{i=1}^{n}\frac{1}{\Delta_i^2}(x_0^i)^T H_i\overline{y_0^i},$$

which agrees with (6.25). This concludes the proof of the first statement.

To deal with the second statement, first observe that since u is a unit, so are $\overline{u}$ and $u\overline{u}$. Consequently $R^n/\overline{H}R^n$ is equal to $R^n/(u\overline{u}\overline{H})R^n$ and thus the corresponding torsion submodule supports both the pairings λ_H and $\lambda_{u\overline{u}H}$. To prove the assertion, we wish to show that the automorphism φ defined by sending x to $u^{-1}x$ provides the desired isometry. To see this, start with torsion elements x and y in the cokernel of H and let x_0, y_0 be such that $\overline{H}x_0 = \Delta x$ and $\overline{H}y_0 = \Delta y$. Since u is a unit, $(u\overline{u})^{-1}$ lies in R, and thus $(u\overline{u}\overline{H})((u\overline{u})^{-1}x_0) = \Delta x$ and similarly for y. It follows that

$$\lambda_{u\overline{u}H}(x,y) = \frac{1}{\Delta^2}((u\overline{u})^{-1}x_0)^T(u\overline{u}H)\overline{((u\overline{u})^{-1}y_0)} = (u\overline{u})^{-1}\frac{1}{\Delta^2}x_0^T H\overline{y_0} = (u\overline{u})^{-1}\lambda_H(x,y).$$

On the other hand, the sesquilinearity of λ_H immediately implies that $\lambda_H(\varphi(x),\varphi(y)) = \lambda_H(u^{-1}x, u^{-1}y) = (u\overline{u})^{-1}\lambda_H(x,y)$. Consequently λ_H and $\lambda_{u\overline{u}H}$ are isometric, which concludes the proof. □

We can now apply Lemma 6.6.3 to obtain some results on the Blanchfield pairing. Before that however, we briefly recall some terminology from Section 3.2. Given a C-complex $F = F_1 \cup \ldots \cup F_\mu$ and a sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_\mu)$ of ± 1's, pushing curves off F_i in the ε_i-normal direction for $i = 1, \dots, \mu$ produces a map $i^\varepsilon \colon H_1(F) \to H_1(S^3 \backslash F)$. The assignment $\alpha^\varepsilon(x,y) := \ell k(i_\varepsilon(x), y)$ gives rise to a bilinear pairing on $H_1(F)$ and thus to a generalized Seifert matrix A^ε for L. In the next two propositions, we shall denote by $\mathrm{Bl}(L)(t_1, \dots, t_\mu)$ the Blanchfield pairing of a μ-colored link and similarly for the C-complex matrices.

Proposition 6.6.4. *Let $L' = L_1 \cup \ldots \cup L_{\nu-1} \cup L'_\nu$ and $L'' = L''_\nu \cup L_{\nu+1} \cup \ldots \cup L_\mu$ be two colored links. Consider a colored link $L = L_1 \cup \ldots \cup L_\mu$, where L_ν is a connected sum of L'_ν and L''_ν along any of their components. Then $\mathrm{Bl}(L)(t_1, \ldots, t_\mu)$ is isometric to $\mathrm{Bl}(L')(t_1, \ldots, t_\nu) \oplus \mathrm{Bl}(L'')(t_\nu, \ldots, t_\mu)$.*

Proof. Denote $\prod_{i>\nu}(1-t_i^{-1})(1-t_i)$ by u_1 and $\prod_{i<\nu}(1-t_i^{-1})(1-t_i)$ by u_2. Given a C-complex F' for L' and a C-complex F'' for L'', a C-complex for L is given by the band sum of F' and F'' along the corresponding components of F'_ν and F''_ν. Consequently, $A^\varepsilon_F = A^{\varepsilon'}_{F'} \oplus A^{\varepsilon''}_{F''}$, with $\varepsilon' = (\varepsilon_1, \ldots, \varepsilon_\nu)$ and $\varepsilon'' = (\varepsilon_\nu, \ldots, \varepsilon_\mu)$. It follows that a C-complex matrix H for L is given by

$$H = u_1 H'(t_1, \ldots, t_\nu) \oplus u_2 H''(t_\nu, \ldots, t_\mu).$$

Denoting these matrices by H' and H'', Theorem 6.1.2 and the first assertion of Lemma 6.6.3 imply that $\mathrm{Bl}(L)$ is isometric to $\lambda_{u_1 H'} \oplus \lambda_{u_2 H''}$. Since u_1 and u_2 are of the form $u\overline{u}$ with u a unit of Λ_S, the result follows from the second assertion of Lemma 6.6.3. □

Figure 6.7: Performing a trivial band clasping of the links L' and L''.

Recall that a *trivial band clasping* of two links is the operation depicted in Figure 6.7. A proof similar to the one of Proposition 6.6.4 yield the following result.

Proposition 6.6.5. *Let $L' = L_1 \cup \ldots \cup L_\nu$ and $L'' = L_{\nu+1} \cup \ldots \cup L_\mu$ be colored links with disjoint sets of colors.*

1. *Consider a colored link L obtained by trivially band clasping L_ν and $L_{\nu+1}$ along any of their components. Then $\mathrm{Bl}(L)(t)$ is isometric to $\mathrm{Bl}(L')(t') \oplus \mathrm{Bl}(L'')(t'')$.*
2. *Consider the colored link given by the disjoint sum of L' and L''. Then $\mathrm{Bl}(L)(t)$ is isometric to $\mathrm{Bl}(L')(t') \oplus \mathrm{Bl}(L'')(t'')$.*

We conclude this subsection by studying the effect of orientation reversal and taking the mirror image.

Proposition 6.6.6. *Let L be a colored link.*

1. *If $\overline{L}$ denotes the mirror image of L, then $\mathrm{Bl}(\overline{L})$ is isometric to $-\mathrm{Bl}(L)$.*
2. *If $-L$ denotes L with the opposite orientation, then $\mathrm{Bl}(-L)$ is isometric to $\mathrm{Bl}(L)$.*

Proof. If F is a C-complex for L, then the mirror image F' of F is a C-complex for $\overline{L}$. It follows that $H' = -H$. Since these two matrices present the same module, the corresponding torsion submodule supports both λ_H and λ_{-H}. We claim that the automorphism which sends x to $-x$ gives the required isometry. Indeed, if $\overline{H}x_0 = \Delta x$ and $\overline{H}y_0 = \Delta y$, then $(-\overline{H})x_0 = \Delta(-x)$ and $(-\overline{H})y_0 = \Delta(-y)$. Consequently $\lambda_{-H}(-x, -y)$ and $-\lambda_H(x, y)$ are both equal to $-x_0^T H \overline{y_0}$. The result now follows from Theorem 6.1.2. The second assertion follows similarly by noting that a C-complex matrix for $-L$ is given by $\overline{H}$ and by using the fact that λ_H is Hermitian. □

6.6.3 Boundary links

Before computing the Blanchfield pairing of a boundary link, we recall the terminology from Subsection 3.5.2. An n-component *boundary link* is a link $L = K_1 \cup \ldots \cup K_n$ whose n components bound n disjoint Seifert surfaces $F_1, \ldots, F_n$. Set $F = F_1 \sqcup \ldots \sqcup F_n$. Pushing curves off this *boundary Seifert surface* in the negative normal direction produces a homomorphism $i^- \colon H_1(F) \to H_1(S^3 \setminus F)$. The assignment $\theta(x, y) := \ell k(i^-(x), y)$ gives rise to a pairing on $H_1(F)$ and to a *boundary Seifert matrix for L*. Since $H_1(F)$ decomposes as the direct sum of the $H_1(F_i)$, the restriction of θ to $H_1(F_i) \times H_1(F_j)$ produces matrices A_{ij}. For $i \neq j$, these matrices satisfy $A_{ij} = A_{ji}^T$, while A_{ii} is nothing but a Seifert matrix for the knot K_i. Let g_i be the genus of F_i, let I_k be the $k \times k$ identity matrix, let τ be the block diagonal matrix whose diagonal blocks are $t_1 I_{2g_1}, t_2 I_{2g_2}, \ldots, t_n I_{2g_n}$ and set $g := g_1 + \ldots + g_n$. We use Theorem 6.1.2 in order to give a new proof of the following result, originally due to Hillman [88, pages 122-123], see also [46, Theorem 4.2].

Theorem 6.6.7. *Let $L = K_1 \cup \ldots \cup K_n$ be a boundary link. Assume that A is a boundary Seifert matrix for L of size $2g$. The Blanchfield pairing of L is isometric to*

$$\begin{aligned} \Lambda_S^{2g}/(A\tau - A^T)\Lambda_S^{2g} \times \Lambda_S^{2g}/(A\tau - A^T)\Lambda_S^{2g} &\to Q_\mu/\Lambda_S \\ (a, b) &\mapsto a^T(A - \tau A^T)^{-1}(\tau - I_{2g})\overline{b}. \end{aligned}$$

Proof. Let F be a boundary Seifert surface which gives rise to A. View F as a C-complex for L. We saw in the proof of Proposition 3.5.3 that a C-complex matrix for L is given by

$$H = u\overline{u}(I_{2g} - \tau)^{-1}(A - \tau A^T). \tag{6.26}$$

and that the module presented by $\overline{H} = H^T$ is canonically isomorphic to the module presented by $A\tau - A^T$. As the isomorphism is induced by the identity of Λ_S^{2g}, we shall slightly abuse notations and consider these modules as "equal", see the second left vertical arrow in (6.27). Furthermore, we also saw in Proposition 3.5.3 that $\Lambda_S^{2g}/\overline{H}\Lambda_S^{2g}$ is Λ_S-torsion. Now consider the following diagram:

$$\begin{tikzcd}
TH_1(X_L;\Lambda_S) \times TH_1(X_L;\Lambda_S) \arrow[r, "\mathrm{Bl}(L)"] \arrow[d, "\cong"] & Q_\mu/\Lambda_S \arrow[d, "="] \\
\frac{\Lambda_S^{2g}}{\overline{H}\Lambda_S^{2g}} \times \frac{\Lambda_S^{2g}}{\overline{H}\Lambda_S^{2g}} \arrow[r, "(a,b)\mapsto -a^T H^{-1}\overline{b}"] \arrow[d, "="] & Q_\mu/\Lambda_S \arrow[d, "="] \\
\frac{\Lambda_S^{2g}}{(A\tau - A^T)\Lambda_S^{2g}} \times \frac{\Lambda_S^{2g}}{(A\tau - A^T)\Lambda_S^{2g}} \arrow[r, "(a,b)\mapsto a^T(u\overline{u})^{-1}(A-\tau A^T)^{-1}(\tau - I_{2g})\overline{b}"] \arrow[d, "(a,b)\mapsto(u^{-1}a,u^{-1}b)"] & Q_\mu/\Lambda_S \arrow[d, "="] \\
\frac{\Lambda_S^{2g}}{(A\tau - A^T)\Lambda_S^{2g}} \times \frac{\Lambda_S^{2g}}{(A\tau - A^T)\Lambda_S^{2g}} \arrow[r, "(a,b)\mapsto a^T(A-\tau A^T)^{-1}(\tau - I_{2g})\overline{b}"] & Q_\mu/\Lambda_S.
\end{tikzcd} \tag{6.27}$$

The top square commutes by Theorem 6.1.2. Note that Corollary 6.1.3 ensures that $\lambda_H(a, b) = -a^T H^{-1}\overline{b}$: indeed we argued above that $\Lambda_S^{2g}/\overline{H}\Lambda_S^{2g}$ is torsion. The middle rectangle, commutes thanks to (6.26). Finally, the commutativity of the bottom square follows from a direct computation. This concludes the proof of the theorem. □

In the case of knots, we can actually get the result over Λ by a similar argument, see [52, Section 5.3] for details.

6.7 Further remarks

A key idea in the proof of Theorem 6.1.2 is that $H_1(\partial W_F; \Lambda_S) \cong H_1(X_L; \Lambda_S)$, where F is a pushed-in C-complex, i.e. a particular type of colored bounding surface, see Definition 14. Looking back at the proof of this fact, see Lemma 6.3.5, we observed that ∂W_F decomposes as the union of X_L and a 3-manifold M_F. This latter 3-manifold is a particular case of a so-called *plumbed 3-manifold*. This section reviews this concept, referring to [90, 131], as classical references as well as to [21, 82]. These concepts will then be used in Chapters 7 and 8.

6.7.1 Plumbed 3-manifolds

We begin by setting up notation. Let $G = (V, E)$ be an unoriented graph with no loops, where $s\colon E \to V$ and $t\colon E \to V$ are the source and the target maps and $i\colon E \to E$ is the involution of the edges, see e.g. [142, Section I.2]. We shall sometimes also denote $i(e)$ by $\bar{e}$, and we assume that the edges $e \in E$ are labeled by weights $\varepsilon(e) = \pm 1$.

Definition 20. For $v, w \in V$ denote by $E(v, w) = \{e \in E\colon s(e) = v, t(e) = w\}$ the set of all edges between v and w. A graph $G = (V, E)$ is *balanced* if for each pair (v, w) of distinct vertices, the sum $\sum_{\substack{s(e)=v,\\ t(e)=w}} \varepsilon(e)$ vanishes.

From now on, we shall assume that the set of vertices V consists of oriented, connected and compact surfaces F. Consider the disjoint union $\bigsqcup_{F \in V} F \times S^1$ and pick a collection of disjoint discs $D_e \subset s(e)$ for each edge $e \in E$. Denote the complement of these discs in $F \in V$ by

$$F^\circ = F \setminus \bigcup_{s(e)=F} D_e.$$

Furthermore, denote by $\sim$ the equivalence relation on $\bigsqcup_{F \in V} F^\circ \times S^1$ defined by

$$\begin{aligned} (-\partial D_e) \times S^1 &\to (-\partial D_{i(e)}) \times S^1 \\ (x, y) &\mapsto \begin{cases} (y^{-1}, x^{-1}) & \text{if } \varepsilon(e) = 1, \\ (y, x) & \text{if } \varepsilon(e) = -1. \end{cases} \end{aligned} \tag{6.28}$$

The main definition of this section is the following.

Definition 21. The *plumbed* 3-*manifold* associated to the graph $G = (V, E)$ consists of the 3-manifold

$$\mathrm{Pb}(G) := \left(\bigsqcup_{F \in V} F^\circ \times S^1 \right) / \sim$$

where for all e in E, the identifications are given by (6.28). Furthermore, we shall say that $\mathrm{Pb}(G)$ is *balanced* if the graph G is balanced.

Since the identifications of (6.28) make use of orientation reversing homeomorphisms, the 3-manifold $\mathrm{Pb}(G)$ carries an orientation which extends the orientation of each $F^\circ \times S^1$. Note that the orientation $-\partial D_e$ is the one obtained by considering the circle as a boundary component of F°. This is the opposite of the one induced by the boundary ∂D_e of the removed disk. In the general context of plumbing disk bundles, one trivializes over the removed disks, which causes the two formulas to flip, see e.g. [90, Chapter 8 p.67].

6.7.2 The boundary of W_F and plumbed manifolds

In the remainder of this chapter and in the next, plumbed 3-manifolds will mostly appear as boundaries of tubular neighborhoods of collections of surfaces in the 4-ball. To make this precise, we slightly generalize the notion of a colored bounding surface to the situation where the coloring of the link is irrelevant.

Definition 22. A *bounding surface* of a link L is a union $F = F_1 \cup \ldots \cup F_m$ of properly embedded, locally flat, compact, connected and oriented surfaces $F_i \subset D^4$ which only intersect each other transversally in double points, and $\partial F = L$.

Note that we require each F_i to be connected. Forgetting about the colors, a colored bounding surface turns into a bounding surface formed by the union of its connected pieces. Generalizing the observation we made for pushed-in C-complexes, we note that the exterior of a bounding surface contains a plumbed 3-manifold in its boundary.

Definition 23. The *intersection graph* (V, E) of a bounding surface $F = F_1 \cup \ldots \cup F_m$ has the vertices set $V = \{F_1, \ldots, F_m\}$. The set of edges E consists of triples $e = (x, F_i, F_j)$ where x is an intersection point between the components $F_i, F_j \in V$. The maps s, t, i are defined on e by

$$s(e) = F_i \quad t(e) = F_j \quad i(e) = (x, F_j, F_i).$$

Moreover, we assign a weight $\varepsilon(e) = \pm 1$ to each edge $e = (x, F_i, F_j)$ corresponding to the sign of the intersection at the point x.

Our interest in plumbed 3-manifolds essentially lies in the next example, which is only balanced if the link has pairwise vanishing linking numbers.

Example 6.7.1. Let $F \subset D^4$ be a bounding surface for a link L. Set $W_F := D^4 \setminus \nu F$ and $M_F := \overline{\nu F} \cap W_F$. This way, the boundary of W_F decomposes into $\partial W_F = X_L \cup_{L \times S^1} M_F$. Plumbing the trivialized disk bundles $F_i \times D^2$ by the intersection graph of F describes a neighborhood νF of F. In this model, the surfaces F_i are recovered as the zero sections $F_i \times \{0\}$ [90, Chapter 8]. As a consequence, we see that M_F is diffeomorphic to $\mathrm{Pb}(G)$, where G is the intersection graph of F.

Let L be a colored link. We conclude this section by a slightly offtopic remark, which nevertheless makes use of the decomposition $\partial W_F = X_L \cup_{L \times S^1} M_F$, for F a colored bounding surface. Namely, we observe that the untwisted signature of W_F vanishes. This result shall be used several times in Chapter 8.

Proposition 6.7.2. *If F is a colored bounding surface for a μ-colored link L, then the untwisted intersection form of W_F is trivial.*

Proof. Consider the portion $H_2(M_F) \to H_2(W_F) \oplus 0 \to 0$ of the Mayer-Vietoris sequence associated to the decomposition $D^4 = W_F \cup \overline{\nu F}$. It follows that the map $H_2(M_F) \to H_2(W_F)$ is surjective. Since M_F is contained in the boundary of W_F, the natural map $j \colon H_2(\partial W_F) \to H_2(W_F)$ must also be surjective. The statement follows immediately since elements of $\mathrm{im}(j)$ annihilate the intersection form, see Section 5.6. $\square$

Chapter 7

Technical results: Twisted signatures and plumbed 3-manifolds

7.1 Introduction and statement of the results

This chapter has two goals. First, Section 7.2 reviews a coefficient system which plays a crucial role in Chapter 8. Secondly, Sections 7.3, 7.4 and 7.5 deal with technical results which will be needed in some proofs of Chapter 8 and especially in Section 8.5. Consequently, we urge the reader to skim through the short Section 7.2 (or to read Example 5.6.3) but to skip the remainder of this introduction as well as Sections 7.3, 7.4 and 7.5, and only return to them if needed, e.g. after having read Chapter 8. Finally, we note that the results described here were obtained in joint work with Enrico Toffoli and Matthias Nagel [53].

Let $G = (V, E)$ be an unoriented graph with no loops and whose edges $e \in E$ are labeled by weights $\varepsilon(e) = \pm 1$. Assume that the set of vertices V consists of oriented, connected and compact surfaces F. Consider the disjoint union $\bigsqcup_{F \in V} F \times S^1$ and pick a collection of disjoint discs $D_e \subset s(e)$ for each edge $e \in E$. Denote the complement of these discs in $F \in V$ by $F^\circ = F \setminus \bigcup_{s(e)=F} D_e$. Recall from Subsection 6.7.1 that the *plumbed 3-manifold* associated to G consists of the 3-manifold

$$\mathrm{Pb}(G) := \left(\bigsqcup_{F \in V} F^\circ \times S^1 \right) / \sim$$

where for all $e \in E$, the identifications were given in (6.28). Recall furthermore that $\mathrm{Pb}(G)$ is *balanced* if G is balanced, see Definition 20. From now on, we assume that our plumbed 3-manifold $\mathrm{Pb}(G)$ comes with a homomorphism $\phi \colon H_1(\mathrm{Pb}(G)) \to \mathbb{Z}^\mu$. We call such a homomorphism *meridional* if, for each constituting piece $F^\circ \times S^1 \subseteq \mathrm{Pb}(G)$ with $F \in V$, the restriction of ϕ to $H_1(F^\circ \times S^1)$ sends the class of $\{pt\} \times S^1$ to one of the canonical generators $e_1, \ldots, e_\mu$ of $\mathbb{Z}^\mu$.

Now assume that $(\mathrm{Pb}(G), \phi)$ bounds a 4-manifold W over $\mathbb{Z}^\mu$. As in Examples 5.2.4 and 5.6.3, the choice of ω in $\mathbb{T}^\mu$ gives rise to twisted homology $\mathbb{C}$-vector spaces $H_k(W; \mathbb{C}^\omega)$ and to a twisted intersection form $\lambda_{\mathbb{C}^\omega}(W)$ on $H_2(W; \mathbb{C}^\omega)$. The signature of this intersection form is denoted by $\mathrm{sign}_\omega(W)$. The main goal of this chapter is to prove the following result.

Proposition 7.1.1. *Let $G = (V, E)$ be a balanced graph with vertices* closed *connected surfaces F. Suppose that $\phi \colon H_1(\mathrm{Pb}(G); \mathbb{Z}) \to \mathbb{Z}^\mu$ is a meridional homomorphism and that* $\mathrm{Pb}(G)$

bounds a 4-manifold W over $\mathbb{Z}^\mu$. Then, for all $\omega \in \mathbb{T}^\mu_*$,

$$\operatorname{sign}_\omega(W) - \operatorname{sign}(W) = 0.$$

The proof of Proposition 7.1.1 decomposes into two parts. First, in Section 7.3, we shall use the Atiyah-Patodi-Singer rho invariant to show that if V is a set of closed oriented connected surfaces and if W is a 4-manifold over $\mathbb{Z}^\mu$ with boundary $\partial W = \bigsqcup_{\Sigma \in V} \Sigma \times S^1$, then $\operatorname{sign}_\omega(W) - \operatorname{sign}(W) = 0$. Then, in Section 7.4, we shall build a cobordism Z from $\operatorname{Pb}(G)$ to $\bigsqcup_{\Sigma \in V} \Sigma \times S^1$ with vanishing signature defect. The proof of Proposition 7.1.1 will then follow by using the additivity of signatures.

Finally, in Section 7.5, we conclude this chapter with some further technical results which we shall need in Chapter 8; here is a brief outline. First, we apply the machinery developed in Section 7.4 in order to prove a lemma involving plumbed 3-manifolds. Then, the much more algebraic Subsection 7.5.2 will rely on the universal coefficient spectral sequences of Section 2.5 to prove some results on twisted homology with $\mathbb{C}^\omega$-coefficients.

7.2 $\mathbb{C}^\omega$-twisted intersection forms and signatures

Let W be a 4-dimensional manifold with (possibly empty) boundary. Let $\psi\colon \pi_1(W) \to \mathbb{Z}^\mu$ be a homomorphism and let $\omega = (\omega_1, \ldots, \omega_\mu)$ lie in $\mathbb{T}^\mu$. Composing the induced ring homomorphism $\mathbb{Z}[\pi_1(W)] \to \mathbb{Z}[\mathbb{Z}^\mu]$ with the map $\mathbb{Z}[\mathbb{Z}^\mu] \to \mathbb{C}$ which evaluates t_i at ω_i, produces a morphism $\phi\colon \mathbb{Z}[\pi_1(W)] \to \mathbb{C}$ of rings with involutions. In turn, ϕ endows $\mathbb{C}$ with a $(\mathbb{C}, \mathbb{Z}[\pi_1(X)])$-bimodule structure. As in Examples 5.2.4 and 5.6.3, we shall write $\mathbb{C}^\omega$ instead of $\mathbb{C}$ to emphasize the role of ω in the module structure. Since $\mathbb{C}^\omega$ is a $(\mathbb{C}, \mathbb{Z}[\pi_1(W)])$-bimodule, we may consider the complex vector spaces $H_k(W; \mathbb{C}^\omega)$ which we already encountered in Examples 5.2.4 and 5.6.3. As in this latter example, we then consider the twisted intersection form

$$\lambda_{\mathbb{C}^\omega}(W)\colon H_2(W; \mathbb{C}^\omega) \times H_2(W; \mathbb{C}^\omega) \to \mathbb{C}.$$

Still following the notation of Example 5.6.3, we write $\operatorname{sign}_\omega(W)$ for the signature of $\lambda_{\mathbb{C}^\omega}(W)$ and $\operatorname{sign}(W)$ for the signature of $\lambda_{\mathbb{Q}}(W)$. We will also study the *signature defect* $\operatorname{dsign}_\omega W := \operatorname{sign}_\omega(W) - \operatorname{sign}(W)$, see for instance Example 2.3.4 as well as Section 7.3.

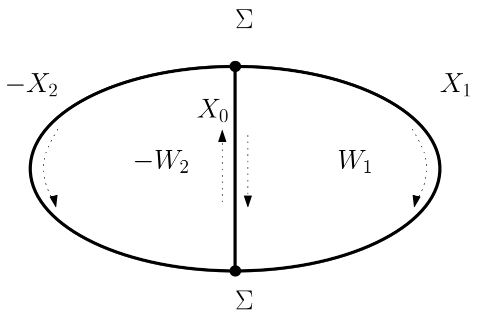

Figure 7.1: A 2-dimensional sketch of the Novikov-Wall set-up.

We now recall special cases of the Novikov-Wall additivity of signatures [156]. Assume X_0 is a properly embedded 3-manifold which splits W into two manifolds W_1 and W_2, i.e. $W = W_1 \cup_{X_0} (-W_2)$, where $\partial X_0 = X_0 \cap \partial W$. Set $\Sigma := \partial X_0$ and for $i = 1, 2$ orient $X_i = \partial W_i \setminus \operatorname{Int}(X_0)$

so as to obtain the decompositions $\partial W_1 = X_1 \cup_\Sigma (-X_0)$ and $\partial W_2 = X_2 \cup (-X_0)$, see Figure 7.1. Furthermore, consider the subspaces $\ker X_i := \ker(H_1(\Sigma;\mathbb{R}) \to H_1(X_i;\mathbb{R}))$ of $H_1(\Sigma;\mathbb{R})$ for $i = 0, 1, 2$.

The following result will often be referred to as the Novikov-Wall additivity of signatures.

Theorem 7.2.1. *Assume that W is decomposed as above as the union of W_1 and $-W_2$. If two of the three spaces $\ker X_0, \ker X_1, \ker X_2$ coincide, then the additivity of signatures holds:*

$$\operatorname{sign}(W) = \operatorname{sign}(W_1) - \operatorname{sign}(W_2).$$

In general, Wall proved that the (twisted) signature of W is equal to the sum of the (twisted) signatures of the W_1 and W_2 plus the so-called *Maslov index* of $\ker X_0, \ker X_1$ and $\ker X_2$ [156]. Since this section does not require the full generality of this statement, Theorem 7.2.1 imposes a condition for which the additivity of signatures does go through, i.e. for which the Maslov index vanishes. Section 16.2 will deal with a more general setting.

Passing to twisted coefficients, we once again avoid the utmost generality. Suppose we have a map $H_1(W) \to \mathbb{Z}^\mu$ so that we can construct the local coefficient systems $\mathbb{C}^\omega$ for every ω in $\mathbb{T}^\mu$. The following additivity result holds for the twisted signature.

Proposition 7.2.2. *Suppose that W is decomposed as above as the union of W_1 and $-W_2$. Then, for each $\omega \in \mathbb{T}^\mu$ such that $H_1(\Sigma;\mathbb{C}^\omega) = 0$, Novikov-Wall additivity holds for the twisted signature:*

$$\operatorname{sign}_\omega(W) = \operatorname{sign}_\omega(W_1) - \operatorname{sign}_\omega(W_2).$$

We now begin the proof of Proposition 7.1.1.

7.3 Proof of Proposition 7.1.1 part I: the Atiyah-Patodi-Singer rho invariant of a product $\Sigma \times S^1$

Given a smooth odd dimensional manifold M and a unitary representation $\alpha\colon \pi_1(M) \to U(n)$, Atiyah, Patodi and Singer use spectral analysis of elliptic differential operators to produce a real number $\rho(M,\alpha)$ which is called the *rho invariant* [7]. Just as in Remark 2.3.4, we do not attempt to define $\rho(M,\alpha)$ and focus on the special case where α is a homomorphism $H_1(M) \to U(1) = S^1$. The next proposition states some well known properties of $\rho(M,\alpha)$. In particular the first point can be thought of as a definition of $\rho(M,\alpha)$, see Remark 2.3.4.

Proposition 7.3.1.
1. *If Z is a smooth $2n$-manifold together with a homomorphism $\alpha\colon H_1(Z) \to U(1)$, then $\rho(\partial Z,\alpha) = -(\operatorname{sign}_\alpha(Z) - \operatorname{sign}(Z))$.*
2. *Given a closed surface Σ and homomorphisms $\alpha\colon H_1(\Sigma) \to U(1)$ and $\beta\colon H_1(S^1) \to U(1)$, we have*
$$\rho(\Sigma \times S^1, \alpha \otimes \beta) = \operatorname{sign}_\alpha(\Sigma) \cdot \rho(S^1,\beta) = \operatorname{sign}(\Sigma) \cdot \rho(S^1,\beta).$$

Proof. The first result is the specialisation to our setting of the Atiyah-Patodi-Singer index theorem [7, Theorem 2.4], while the second can be deduced from the definition and the classical Atiyah-Singer theorem. Both results can be found in [130, Theorem 1.2, (iii) and (v)], where it has to be observed that the invariant considered by the author is minus the rho invariant. The fact that $\operatorname{sign}_\alpha(\Sigma) = \operatorname{sign}(\Sigma)$ follows from (1), or alternatively from the Hirzebruch signature formula, see [15, Theorem 4.7]. □

Note that the following convention is implicit both in the previous proposition and in the next: if b is a skew-Hermitian form, then $\operatorname{sign}(b)$ is defined as the signature of the Hermitian form ib [130]. In particular, the (twisted) signature of an odd dimensional manifold is defined as the signature of its (twisted) skew-Hermitian intersection form.

Next, we restrict to manifolds M endowed with a homomorphism $H_1(M) \to \mathbb{Z}^\mu$. Since one-dimensional unitary representations of $H_1(M)$ factoring through $\mathbb{Z}^\mu$ are in bijection with values ω in $\mathbb{T}^\mu$, we will denote by $\rho_\omega(M)$ the rho invariant corresponding to the representation $\alpha\colon H_1(M) \to \mathbb{Z}^\mu \xrightarrow{\omega} S^1$, see also Remark 2.3.4. Using Proposition 7.3.1, we can prove the following lemma.

Lemma 7.3.2. *If Σ is a closed oriented connected surface and $\phi\colon H_1(\Sigma \times S^1) \to \mathbb{Z}^\mu$ is a homomorphism, then $\rho_\omega(\Sigma \times S^1) = 0$ for all $\omega \in \mathbb{T}^\mu_*$.*

Proof. Since $H_1(\Sigma \times S^1) = H_1(\Sigma) \oplus H_1(S^1)$, we may restrict $\phi\colon H_1(\Sigma \times S^1) \to \mathbb{Z}^\mu$ to each summand. This produces maps $\phi_\Sigma\colon H_1(\Sigma) \to \mathbb{Z}^\mu$ and $\phi_{S^1}\colon H_1(S^1) \to \mathbb{Z}^\mu$. Postcomposing each of these maps with the map $\mathbb{Z}^\mu \xrightarrow{\omega} S^1$ produces homomorphisms φ, φ_Σ and φ_{S^1}. Since these homomorphisms fit in the commutative diagram

$$\begin{tikzcd}
H_1(\Sigma \times S^1) \arrow[r, "\varphi"] \arrow[d, "pr_\Sigma \oplus pr_{S^1}"] & S^1 \\
H_1(\Sigma) \oplus H_1(S^1) \arrow[r, "\varphi_\Sigma \times \varphi_{S^1}"] & S^1 \times S^1, \arrow[u, "\cdot"']
\end{tikzcd}$$

it follows that $\varphi = \varphi_\Sigma \otimes \varphi_{S^1}$. Using point (2) of Proposition 7.3.1, one obtains

$$\rho_\omega(\Sigma \times S^1) = \rho(\Sigma \times S^1, \varphi_\Sigma \otimes \varphi_{S^1}) = -\operatorname{sign}(\Sigma)\rho(S^1, \varphi_{S^1}).$$

Since the ordinary signature of a closed oriented surface vanishes, we get $\rho_\omega(\Sigma \times S^1) = 0$ and the result is established. □

The following corollary is nearly immediate.

Corollary 7.3.3. *Let V be a set of closed oriented connected surfaces. If W is a 4-manifold over $\mathbb{Z}^\mu$ with boundary*

$$\partial W = \bigsqcup_{\Sigma \in V} \Sigma \times S^1,$$

then $\operatorname{sign}_\omega(W) - \operatorname{sign}(W) = 0$.

Proof. Thanks to point (1) of Proposition 7.3.1, $\operatorname{sign}_\omega(W) - \operatorname{sign}(W)$ coincides with minus the rho invariant of ∂W. By Lemma 7.3.2 and the additivity of the rho invariant under disjoint unions [130, Theorem 1.2.1], we get $\operatorname{sign}_\omega(W) - \operatorname{sign}(W) = 0$. □

Since Proposition 7.3.1 required the cobounding manifold to be smooth, one might worry about Corollary 7.3.3 only holding when W is a smooth 4-manifold. However [53, Remark 4.4] shows that this is not an issue.

7.4 Proof of Proposition 7.1.1 part II: building a cobordism

Looking back at Corollary 7.3.3, the idea is now to build a cobordism Z from $\mathrm{Pb}(G)$ to some $\bigsqcup_{\Sigma\in V}\Sigma\times S^1$, with vanishing signature defect, and then to use the additivity of signatures discussed in Section 7.2. However for the additivity statement of Theorem 7.2.1 to go through, we need to understand the kernel of the inclusion induced map $H_1(\partial\,\mathrm{Pb}(G);\mathbb{R})\to H_1(\mathrm{Pb}(G);\mathbb{R})$. For this reason, we slightly delay the construction of our cobordism Z.

To understand the aforementioned kernel, first note that the boundary of a plumbed 3-manifold $\mathrm{Pb}(G)$ is toroidal and the components correspond to the boundary components of the surfaces $F\in V$. Furthermore, by construction, the boundary components come with the product structure $\partial\,\mathrm{Pb}(G)=\bigsqcup_{F\in V}\partial F\times S^1$. In order to describe $\ker(H_1(\partial\,\mathrm{Pb}(G);\mathbb{R})\to H_1(\mathrm{Pb}(G);\mathbb{R}))$, we define the following homology class in $H_1(\partial\,\mathrm{Pb}(G))$:

$$[\partial F]:=[\partial F\times\{pt\}]$$

and introduce some more notation: for each surface $F\in V$ with boundary, label its boundary components $K_1,\dots,K_{n_F}$ and accordingly their meridians $\mu_1^F,\dots,\mu_{n_F}^F$ and longitudes $l_1^F,\dots,l_{n_F}^F$. We have the equality $[\partial F]=\sum_{k=1}^{n_F}[l_k^F]$.

Lemma 7.4.1. *The kernel of the inclusion induced map $H_1(\partial\,\mathrm{Pb}(G);\mathbb{R})\to H_1(\mathrm{Pb}(G);\mathbb{R})$ is freely generated by the elements*

$$[\partial F]-\sum_{s(e)=F}\varepsilon(e)\mu_1^{t(e)}\ \ \textit{and}\quad \mu_i^F-\mu_1^F,$$

for F varying over the elements in V with $\partial F\neq\emptyset$ and $2\le i\le n_F$.

Proof. From the construction of $\mathrm{Pb}(G)$, we see that for every edge $e\in E$ there is a torus $-\partial D_e\times S^1\subset s(e)\times S^1$ which is identified with $-\partial D_{\bar e}\times S^1\subset t(e)\times S^1$. We denote this torus by $T_e\subset\mathrm{Pb}(G)$. Hence, $T_e=-T_{\bar e}$.

Now pick an orientation $E'\subset E$ on the edges, i.e. for every $e\in E$, exactly one of the edges e and $\bar e$ is an element of E'. From the construction of $\mathrm{Pb}(G)$, we obtain a Mayer-Vietoris sequence

$$\cdots\to\bigoplus_{e\in E'}H_1(T_e;\mathbb{R})\xrightarrow{i_t-i_s}\bigoplus_{F\in V}H_1(F^\circ\times S^1;\mathbb{R})\to H_1(\mathrm{Pb}(G))\to\dots,\tag{7.1}$$

where i_t,i_s denote the maps induced by the inclusions of T_e into $t(e)\times S^1$ and $s(e)\times S^1$ respectively. For each F, the inclusion $\partial F\times S^1\to\mathrm{Pb}(G)$ factors through the space $\bigsqcup_{F\in V}F^\circ\times S^1$. Consequently, we have the commutative diagram of inclusion induced maps

$$\begin{tikzcd}\bigoplus_{e\in E'}H_1(T_e;\mathbb{R}) \arrow[r, "i_t-i_s"] & \bigoplus_{F\in V}H_1(F^\circ\times S^1;\mathbb{R}) \arrow[r, "h"] & H_1(\mathrm{Pb}(G);\mathbb{R}) \\ & & H_1(\partial\,\mathrm{Pb}(G);\mathbb{R}), \arrow[ul, "f"] \arrow[u, "j"']\end{tikzcd}$$

yielding $\ker(j)=\ker(h\circ f)=\{x\in H_1(\partial\,\mathrm{Pb}(G);\mathbb{R})\,|\,f(x)\in\mathrm{im}(i_t-i_s)\}$. We shall now restrict our attention to those surfaces F with $\partial F\neq\emptyset$, and prove that both $\mu_k^F-\mu_1^F$ and $[\partial F]-\sum_{s(e)=F}\varepsilon(e)\mu_1^{t(e)}$ belong to this set. As F° is connected, all elements μ_k^F for $1\le k\le n_F$

are equal in $H_1(\mathrm{Pb}(G);\mathbb{R})$, so the elements $\mu_k^F - \mu_1^F$ are in $\ker(f)$ and a fortiori in $\ker(j)$. Next, we check that an element of the form $[\partial F] - \sum_{s(e)=F} \varepsilon(e)\mu_1^{t(e)}$ is sent by f to the image of $i_s - i_t$. Note that $H_1(F^\circ \times S^1;\mathbb{R}) = H_1(F^\circ;\mathbb{R}) \oplus \mathbb{R}\langle \mu_1^F \rangle$, so that we have the relation $[\partial F] + \sum_{s(e)=F}[-\partial D_e] = 0$ in $H_1(F^\circ \times S^1;\mathbb{R})$. We thus obtain

$$f\Big([\partial F] - \sum_{s(e)=F} \varepsilon(e)\mu_1^{t(e)}\Big) = \sum_{s(e)=F} \Big([\partial D_e] - \varepsilon(e)\mu_1^{t(e)}\Big),$$

and the claim reduces to checking that this element is in the image of $i_t - i_s$. Consider the class $-[\partial D_e] \in H_1(T_e)$. We have $-i_s[-\partial D_e] = [\partial D_e]$ and, by the gluing map given in (6.28), $i_t[-\partial D_e] = -\varepsilon(e)\mu_{t(e)}$. As a result, the difference $[\partial D_e] - \varepsilon(e)\mu_1^{t(e)}$ is indeed in the image of $i_t - i_s$, and so $[\partial F] - \sum_{s(e)=F} \varepsilon(e)\mu_1^{t(e)}$ is in $\ker(j)$.

Note that the elements in the statement of the lemma span a subspace U, whose dimension is the number of boundary components of $\mathrm{Pb}(G)$, i.e. it is half the dimension of the space $H_1(\partial\,\mathrm{Pb}(G);\mathbb{R})$. By the half lives, half dies principle [112, Lemma 8.15], the kernel $\ker(j)$ has the same dimension as U and so coincides with U. □

We now return to the initial plan: namely, the next lemma shows that if G is balanced, then $\mathrm{Pb}(G)$ is cobordant to a disjoint union of trivial surface bundles, where the cobordism has vanishing signature defect. Note that we are dealing with plumbings of *closed* surfaces.

Lemma 7.4.2. *Let $G = (V, E)$ be a balanced graph with vertices closed connected surfaces. Suppose that $\phi\colon H_1(\mathrm{Pb}(G)) \to \mathbb{Z}^\mu$ is a meridional homomorphism. Then, there exists a smooth 4-manifold Z over $\mathbb{Z}^\mu$ such that:*

1. *the boundary of Z is a disjoint union*
$$\partial Z = -\mathrm{Pb}(G) \sqcup \bigsqcup_{F\in V} \Sigma_F \times S^1,$$
where Σ_F is a closed surface obtained from $F \in V$ by adding 1-handles;
2. *the restriction $H_1(\bigsqcup_{F\in V} \Sigma_F \times S^1) \to \mathbb{Z}^\mu$ is meridional;*
3. $\mathrm{dsign}_\omega Z = 0$ *for all $\omega \in \mathbb{T}_*^\mu$.*

Proof. Instead of proving the statement directly, we prove the following: if E is nonempty, then there exists a balanced graph $G' = (V', E')$ whose vertices are surfaces obtained from the vertices of V by adding 1-handles; the graph G' has fewer edges than G; furthermore there exists a manifold $Z_{G'}$ over $\mathbb{Z}^\mu$ with $\partial Z = -\mathrm{Pb}(G) \sqcup \mathrm{Pb}(G')$, which induces a meridional homomorphism on $\mathrm{Pb}(G')$. The signature defect of $Z_{G'}$ is zero.

The original statement can be recovered as follows: iterate the above to obtain a sequence of graphs $G = G_0, \dots, G_n$ such that the set of edges of G_n is empty. Consequently, $\mathrm{Pb}(G_n) = \bigsqcup_{F\in V} \Sigma_F \times S^1$ and Σ_F is obtained from surfaces of the original graph by adding 1-handles. We then glue the 4-manifolds together: $Z := Z_{G_1} \cup \ldots \cup Z_{G_n}$. We get $\partial Z = -\mathrm{Pb}(G) \sqcup \mathrm{Pb}(G_n)$ as required and by Novikov additivity $\mathrm{dsign}_\omega Z = \sum_{i=1}^n \mathrm{dsign}_\omega Z_{G_i} = 0$.

Now we proceed with the proof of the modified statement. Recall from Definition 21 and from the beginning of the proof of Lemma 7.4.1 that to each edge e corresponds the embedded torus $T_e = (-\partial D_e) \times S^1$. The complement of all of these tori is diffeomorphic to $\bigsqcup_{F \in V} F^\circ \times S^1 \subset \mathrm{Pb}(G)$. In order to produce the desired 4-manifold Z, our aim is to attach a $D^2 \times T^2$ to the trivial bordism $\mathrm{Pb}(G) \times I$.

Given two vertices $F_1, F_2 \in V$, we write $E(F_1, F_2) = \{e \in E \colon s(e) = F_1, t(e) = F_2\}$ as in Definition 20. Pick two vertices $F_1, F_2 \in V$ such that $E(F_1, F_2)$ is nonempty. As the graph is balanced, this implies we can also pick two edges $e, e' \in E(F_1, F_2)$ such that $\varepsilon(e) = 1$ and $\varepsilon(e') = -1$. Now set $X_{e,e'} := I \times I \times S^1 \times S^1$. Consider the corresponding tori $T_e = (-\partial D_e) \times S^1$ and $T_{e'} = (-\partial D_{e'}) \times S^1$, with oriented neighborhoods $I \times T_e$, $I \times T_{e'}$. We attach $X_{e,e'}$ to $\mathrm{Pb}(G) \times \{1\}$ along its vertical boundaries through a homeomorphism f given by the following formulas:

$$\begin{aligned} \{0\} \times I \times S^1 \times S^1 &\to I \times (-\partial D_e) \times S^1 \\ (0, t, x, y) &\mapsto (t, x, y), \end{aligned} \qquad \begin{aligned} \{1\} \times I \times S^1 \times S^1 &\to I \times (-\partial D_{e'}) \times S^1 \\ (1, t, x, y) &\mapsto (t, x^{-1}, y). \end{aligned}$$

The induced orientations on $\{0,1\} \times I \times S^1 \times S^1$ are such that the above map is orientation-reversing. As a consequence, the orientations of $\mathrm{Pb}(G) \times I$ and $X_{e,e'}$ extend to the resulting 4-manifold

$$Z := X_{e,e'} \cup_f \mathrm{Pb}(G) \times I.$$

Let $a_1, a_2 \in \mathbb{Z}^\mu$ be the images of the meridians of F_1 and F_2 under the map $H_1(\mathrm{Pb}(G)) \to \mathbb{Z}^\mu$. Recalling the construction of $\mathrm{Pb}(G)$ given in (6.28), we see that the induced maps to $\mathbb{Z}^\mu$ on T_e and $T_{e'}$ are given by

$$\begin{aligned} H_1(-\partial D_e \times S^1) &\to \mathbb{Z}^\mu \\ [\{p\} \times S^1] &\mapsto a_1 \\ [-\partial D_e \times \{p\}] &\mapsto a_2 \end{aligned} \qquad \begin{aligned} H_1(-\partial D_{e'} \times S^1) &\to \mathbb{Z}^\mu \\ [\{p\} \times S^1] &\mapsto a_1 \\ [-\partial D_{e'} \times \{p\}] &\mapsto -a_2. \end{aligned}$$

The difference in the sign of the image $[-\partial D_e \times \{p\}]$ is a consequence of the fact that the edges e, e' had opposite signs. This allows us to define a map $\phi_X \colon H_1(X_{e,e'}) \to \mathbb{Z}^\mu$ which glues with the map $\phi \colon H_1(\mathrm{Pb}(G)) \to \mathbb{Z}^\mu$, i.e. the following diagram commutes:

$$\begin{tikzcd} H_1(\{0,1\} \times I \times S^1 \times S^1) \arrow[rr, "f_*"] \arrow[dr, "\phi_X"'] & & H_1(I \times T_e) \oplus H_1(I \times T_e') \arrow[dl, "\phi"] \\ & \mathbb{Z}^\mu. & \end{tikzcd}$$

By making an additional choice of a splitting of the Mayer-Vietoris sequence

$$H_1(X_{e,e'}) \oplus H_1(\mathrm{Pb}(G) \times I) \to H_1(Z) \to H_0(\{0,1\} \times I \times T^2),$$

we obtain a map $H_1(Z) \to \mathbb{Z}^\mu$ which extends ϕ and ϕ_X on $H_1(\mathrm{Pb}(G) \times I)$ and $H_1(X_{e,e'})$.

The boundary of Z has two components. The bottom boundary is $-\mathrm{Pb}(G)$. The effect of adding $X_{e,e'}$ on the top boundary is that of cutting along T_e and $T_{e'}$ and gluing together the boundary component $-D_e \times S^1$ to $-D_{e'} \times S^1$ and on the other surface $-D_{i(e)} \times S^1$ to $-D_{i(e')} \times S^1$. This is the same as adding 1-handles to F_1 and F_2 and, consequently, the

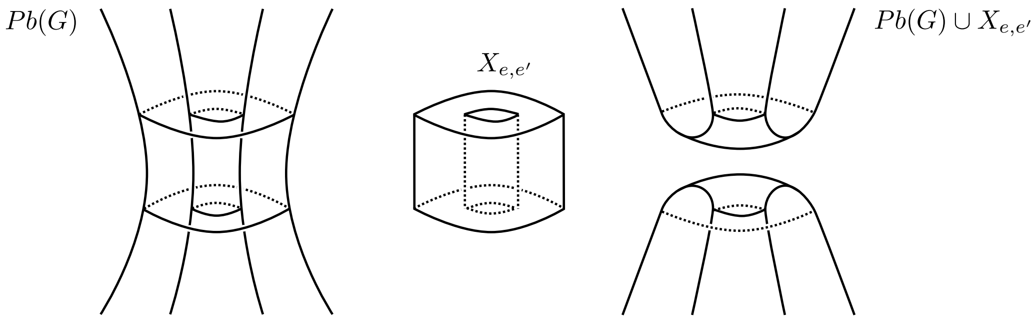

Figure 7.2: The effect of attaching $X_{e,e'}$ to $\mathrm{Pb}(G)$ depicted in reduced dimensions.

top boundary inherits a plumbed structure with modified F_1 and F_2 and the edges e and e' removed from the set of edges.

We have verified that Z fulfills the first statement. To conclude the proof of the proposition, it remains to prove that $\operatorname{dsign}_\omega Z = 0$. This is a consequence of the following claim.

Claim. *The twisted and untwisted signature of* $\mathrm{Pb}(G) \times I$ *and* $X_{e,e'}$ *vanish and Novikov-Wall additivity holds when gluing these two pieces together.*

To prove that the signatures vanish, note that both spaces are 4-manifolds W with the property that the inclusions of the boundary $H_2(\partial W) \to H_2(W)$ and $H_2(\partial W; \mathbb{C}^\omega) \to H_2(W; \mathbb{C}^\omega)$ surject. This implies that both the twisted and untwisted intersection forms vanish. In particular, the twisted and untwisted signatures of $\mathrm{Pb}(G) \times I$ and $X_{e,e'}$ are zero.

Next, we consider the additivity of the signatures. We are gluing $W_+ = X_{e,e'}$ to $W_- = \mathrm{Pb}(G) \times I$ along $Y = \nu T_e \sqcup \nu T_{e'} \subset \mathrm{Pb}(G) \times \{1\}$. The boundary of the gluing region is given by the four tori

$$-\partial D_e \times S^1 \sqcup -\partial D_{i(e)} \times S^1 \sqcup -\partial D_{e'} \times S^1 \sqcup -\partial D_{i(e')} \times S^1.$$

Let $hX = I \times \{0,1\} \times S^1 \times S^1$ denote the horizontal boundary of $X_{e,e'}$. In order to use Theorem 7.2.1, we now prove that the kernels $K = \ker(H_1(\Sigma; \mathbb{R}) \to H_1(hX; \mathbb{R}))$ and $K' = \ker H_1(\Sigma; \mathbb{R}) \to H_1(\mathrm{Pb}(G) \setminus Y; \mathbb{R})$ agree.

Observing the gluing maps above, we see that the kernel K admits the basis

$$[-\partial D_e] + [-\partial D_{e'}], [S^1_e] - [S^1_{e'}], \quad [-\partial D_{i(e)}] + [-\partial D_{i(e')}], [S^1_{i(e)}] - [S^1_{i(e')}]. \tag{7.2}$$

In order to compute the other kernel K' into $H_1(\mathrm{Pb}(G) \setminus Y; \mathbb{R})$, observe that $\mathrm{Pb}(G) \setminus Y$ also inherits a plumbed structure from $\mathrm{Pb}(G)$. It has the same surfaces as vertex set with F_1 and F_2 replaced by $F_1 \setminus (D_e \cup D_{e'})$ and $F_2 \setminus (D_{i(e)} \cup D_{i(e')})$. Its set of edges is obtained by removing e and e' from the set of edges of G. Note that $\Sigma = \partial \mathrm{Pb}(G) \setminus Y$ and we can use Lemma 7.4.1 to obtain a basis of its kernel. The difference of meridians give the basis elements $[S^1_e] - [S^1_{e'}], [S^1_{i(e)}] - [S^1_{i(e')}] \in K'$. The surface F_1 has boundary $-\partial D_e \sqcup -\partial D_{e'}$ and so

$$[-\partial D_e] + [-\partial D_{e'}] - \sum_{s(k)=v} \varepsilon(k)\mu_k = [-\partial D_e] + [-\partial D_{e'}] \in K',$$

are further elements of the basis, where the first equality uses that G is balanced. The analogous statements holds for the other surface F_2. Consequently, the kernel K' admits the same basis (7.2) and as a result $K' = K$. In particular, applying Theorem 7.2.1, the untwisted signature is additive.

For the twisted signature, thanks to Proposition 7.2.2, it is enough to prove that the twisted homology vanishes for Σ. This happens exactly if the induced $U(1)$-representation is nontrivial. This is the case, because ϕ is meridional and the coefficients of ω are taken to be different from 1. Consequently, the signature defect is additive and so

$$\operatorname{dsign}_\omega Z = \operatorname{dsign}_\omega \operatorname{Pb}(G) \times I + \operatorname{dsign}_\omega X_{e,e'} = 0,$$

concluding the proof of the lemma. □

Using Lemma 7.4.2, we can prove Proposition 7.1.1.

proof of Proposition 7.1.1. Since the graph G is balanced, Lemma 7.4.2 produces closed surfaces Σ_F and a 4-manifold Z over $\mathbb{Z}^\mu$ whose signature defect vanishes, with boundary

$$\partial Z = -\operatorname{Pb}(G) \sqcup \bigsqcup_{F \in V} \Sigma_F \times S^1.$$

One can now define $P := W \cup_{\operatorname{Pb}(G)} Z$. Since the boundary of P consists of a disjoint union of $\Sigma_F \times S^1$, Corollary 7.3.3 guaranties that $\operatorname{dsign}_\omega P = 0$. As we are gluing along a boundary component, Theorem 7.2.1 goes through trivially. It follows that $\operatorname{dsign}_\omega P = \operatorname{dsign}_\omega W + \operatorname{dsign}_\omega Z$. Since we know that $\operatorname{dsign}_\omega P$ and $\operatorname{dsign}_\omega Z$ both vanish, $\operatorname{dsign}_\omega W$ also vanishes. □

7.5 Further remarks

We conclude this chapter with some further technical results which we shall need in Chapter 8. Subsection 7.5.1 applies the machinery developed in Section 7.4 in order to prove a lemma involving plumbed 3-manifolds. Subsection 7.5.2 makes use of the universal coefficient spectral sequences of Section 2.5 to prove some results on twisted homology with $\mathbb{C}^\omega$-coefficients.

7.5.1 Another lemma involving plumbed 3-manifolds

Let $F = F_1 \cup \cdots \cup F_m$ be a bounding surface for a link L (recall Definition 22), and let L_i be the sublink given by ∂F_i, for $i = 1, \ldots, m$. Denote as usual the exterior of F by W_F and recall from Example 6.7.1 that $\partial W_F = X_L \cup_{L \times S^1} M_F$, where M_F is a plumbed 3-manifold. Enumerate the components of L_i and denote their meridians by $\mu_k^{L_i}$ for $1 \leq k \leq n_{L_i}$, where n_{L_i} is the number of components of L_i. The following computation will turn out to be useful when applying Novikov-Wall additivity. Note that we are using the convention according to which $\ell k(L_i, L_i) = 0$.

Lemma 7.5.1. *Let $F = F_1 \cup \cdots \cup F_m$ be a bounding surface for a link L. The vector space* $\ker M_F = \ker(H_1(L \times S^1; \mathbb{R}) \to H_1(M_F; \mathbb{R}))$ *is generated by the elements of the form*

$$[L_i] - \sum_{j=1}^{m} \ell k(L_i, L_j) \mu_1^{L_j} \quad \textit{and} \quad \mu_k^{L_i} - \mu_1^{L_i}.$$

Proof. Applying Lemma 7.4.1, the component F_i gives rise to the basis vectors

$$[L_i] - \sum_{s(e)=F_i} \varepsilon(e)\mu_1^{\partial t(e)} \quad \text{and} \quad \mu_k^{L_i} - \mu_1^{L_i}.$$

The result follows by observing that

$$\sum_{s(e)=F_i} \varepsilon(e)\mu_1^{\partial t(e)} = \sum_{j=1}^{m}(F_i \cdot F_j)\mu_1^{L_j} = \sum_{j=1}^{m} \ell k(L_i, L_j)\mu_1^{L_j}.$$

□

7.5.2 Two algebraic lemmas

We briefly recall the now familiar set-up of Example 5.2.4. Let $\psi\colon \pi_1(X) \to \mathbb{Z}^\mu$ be a homomorphism and let $\omega = (\omega_1, \ldots, \omega_\mu)$ lie in $\mathbb{T}^\mu$. Composing the induced ring homomorphism $\mathbb{Z}[\pi_1(X)] \to \mathbb{Z}[\mathbb{Z}^\mu]$ with the map $\mathbb{Z}[\mathbb{Z}^\mu] \to \mathbb{C}$ which evaluates t_i at ω_i, produces a ring homomorphism $\phi\colon \mathbb{Z}[\pi_1(W)] \to \mathbb{C}$. In turn, ϕ endows $\mathbb{C}$ with a $(\mathbb{C}, \mathbb{Z}[\pi_1(X)])$-bimodule structure. Since $\mathbb{C}^\omega$ is a $(\mathbb{C}, \mathbb{Z}[\pi_1(W)])$-bimodule, we may consider the complex vector spaces $H_k(X, Y; \mathbb{C}^\omega)$ and $H^k(X, Y; \mathbb{C}^\omega)$.

Observe that since none of the ω_i is equal to 1, the map $\phi\colon \mathbb{Z}[\pi_1(W)] \to \mathbb{C}$ factors through a map $\Lambda_S \to \mathbb{C}$. In particular, the homology $\mathbb{C}$-vector space $H_k(X, Y; \mathbb{C}^\omega)$ coincides with the k-th homology vector space of the chain complex $\mathbb{C}^\omega \otimes_{\Lambda_S} C_*(X, Y; \Lambda_S)$. Consequently Theorem 2.5.8 immediately yields the following result.

Proposition 7.5.2. *Given a CW-pair (X, Y) and $\omega \in \mathbb{T}^\mu_*$, there exists a spectral sequence*

1. *converging to $H_*(X, Y; \mathbb{C}^\omega)$,*
2. *with $E^2_{p,q} \cong Tor_q^{\Lambda_S}(H_p(X, Y; \Lambda_S), \mathbb{C}^\omega)$,*
3. *with differentials d^r of degree $(-r, r-1)$.*

More specifically, there is a filtration

$$0 \subset F_n^0 \subset F_n^1 \subset \cdots \subset F_n^n = H_n(X, Y; \mathbb{C}^\omega)$$

with $F_n^p / F_n^{p-1} \cong E^\infty_{p,n-p}$.

Proof. Given rings R, Z, a chain complex C_* of free left Z-modules and a (R, Z)-bimodule S, Theorem 2.5.8 produces a spectral sequence with $E^2_{p,q} \cong \mathrm{Tor}_p^Z(H_q(C_*), S)$ which converges to $H_*(S \otimes_Z C_*)$. To prove the proposition, we take $R = \mathbb{C}, Z = \Lambda_S, C_* = C_*(X, Y; \Lambda_S)$ and $S = \mathbb{C}^\omega$. □

The following lemma is a useful application of Proposition 7.5.2. Note that the key ingredient is once again the use of Λ_S-coefficients which has the effect of killing the zero-th homology module.

Lemma 7.5.3. *Let X be a connected CW-complex together with a homomorphism $H_1(X) \to \mathbb{Z}^\mu = \mathbb{Z}\langle e_1, \ldots, e_\mu\rangle$ whose image contains at least one generator e_i. If $\omega \in \mathbb{T}^\mu_*$, then $H_0(X; \mathbb{C}^\omega)$ vanishes. Furthermore, $H_1(X; \mathbb{C}^\omega)$ is isomorphic to $\mathbb{C}^\omega \otimes_{\Lambda_S} H_1(X; \Lambda_S)$.*

Proof. Using the assumption on the map $H_1(X) \to \mathbb{Z}^\mu$, the Λ_S-module $H_0(X;\Lambda_S)$ vanishes by Lemma 5.7.3. Thus Proposition 7.5.2 immediately implies that $H_0(X;\mathbb{C}^\omega) = 0$. Next, we prove the statement involving $H_1(X;\mathbb{C}^\omega)$. Using the notations of Proposition 7.5.2, the differential $0 = \operatorname{Tor}_2^{\Lambda_S}(H_0(X;\Lambda_S), \mathbb{C}^\omega) = E_{2,0} \to E_{0,1}$ is zero. Consequently, $E^\infty_{1,0} = E^2_{1,0} = 0$ and $E^\infty_{0,1} = E^2_{0,1} = \mathbb{C}^\omega \otimes_{\Lambda_S} H_1(X;\Lambda_S)$. It follows that $H_1(X;\mathbb{C}^\omega) \cong \mathbb{C}^\omega \otimes_{\Lambda_S} H_1(X;\Lambda_S)$, as desired. □

On the other hand, working with cohomology and applying Theorem 2.5.7 immediately yields the following result.

Proposition 7.5.4. *Given a CW-pair (X,Y) and $\omega \in \mathbb{T}^\mu$, for each k, evaluation provides the following isomorphism of left $\mathbb{C}$-vector spaces:*

$$H^k(X,Y;\mathbb{C}^\omega) \cong \overline{\operatorname{Hom}_{\mathbb{C}}(H_k(X,Y;\mathbb{C}^\omega),\mathbb{C})}.$$

Proof. Given a ring R with involution, $(R,\mathbb{Z}[\pi_1(X)])$-bimodules M,N, a (R,R)-bimodule S, and a nonsingular $\pi_1(X)$-invariant sesquilinear pairing $\langle -,-\rangle \colon M \times N \to S$, Theorem 5.4.4 produces a spectral sequence with $E_2^{p,q} \cong \operatorname{Ext}^q_R(H_p(X,Y;N),S)$ which converges to $H^*(X,Y;M)$. To prove the proposition, we take $R = \mathbb{C}$, $M = N = \mathbb{C}^\omega$, $S = \mathbb{C}$ and the pairing $(z,w) \mapsto z\overline{w}$. The result now follows: since $R = \mathbb{C}$ is a field, the Ext groups vanish. □

Here is an application of Proposition 7.5.4 which we shall frequently use in Chapter 8 and especially in Section 8.5.

Lemma 7.5.5. *Let $\omega \in \mathbb{T}^\mu$ and let W be a 4-dimensional manifold whose boundary decomposes as $\partial W = M \cup_\partial M'$, where M and M' are (possibly empty) connected 3-manifolds with $\partial M = \partial M'$. If W is equipped with a homomorphism $H_1(W) \to \mathbb{Z}^\mu$, then $\beta^\omega_{4-i}(W,M) = \beta^\omega_i(W,M')$ for $i = 0,1$.*

Proof. Using the Poincaré duality isomorphisms of Theorem 5.5.1, we have $H_{4-i}(W,M;\mathbb{C}^\omega) \cong H^i(W,M';\mathbb{C}^\omega)$ and Proposition 7.5.4 implies that $H^i(W,M';\mathbb{C}^\omega) \cong \overline{\operatorname{Hom}_{\mathbb{C}}(H_i(W,M';\mathbb{C}^\omega),\mathbb{C})}$ for $i = 0,1$. The result now follows immediately. □

Chapter 8

The multivariable signature via twisted coefficients

8.1 Introduction and statement of the results

As reviewed in Section 3.4, the multivariable signature admits both 3 and 4-dimensional interpretations. More precisely, given a colored link $L = L_1 \cup \ldots \cup L_\mu$, we saw in Section 3.4.1 that the multivariable signature and nullity

$$\sigma_L, \eta_L \colon \mathbb{T}^\mu \to \mathbb{Z},$$

were defined using C-complexes, while the 4-dimensional approach of Subsection 3.4.2 involved branched covers of the 4-ball. The latter set-up relies on smooth well-connected colored bounding surfaces, i.e. on unions $F = F_1 \cup \cdots \cup F_\mu \subset D^4$ of smoothly properly embedded connected oriented surfaces which only intersect each other transversally in double points. Considering the exterior W_F of a such a collection F of surfaces, the construction then involves building a finite branched cover $\overline{W}_F \to D^4$ and restricting the intersection form to the generalized eigenspace $H_2(\overline{W}_F;\mathbb{C})_\omega$. A result of Cimasoni-Florens [41], stated in Theorem 3.4.9, then shows that the signature of this restriction coincides with the 3-dimensional definition for all ω in $\mathbb{T}^\mu_* := (S^1 \setminus \{1\})^\mu$ of *finite order*.

This restriction on $\omega = (\omega_1, \ldots, \omega_\mu)$ has a clear cause: the use of finite branched covers. On the other hand, recent articles such as [48, 129, 134, 154] indicate that the following construction should produce a natural replacement for the generalized eigenspace $H_2(\overline{W}_F;\mathbb{C})_\omega$ and the subsequent signature. Since $H_1(W_F)$ is freely generated by the meridians of F (recall Lemma 3.4.8), sending the meridians of F_i to ω_i gives rise to a ring homomorphism $\mathbb{Z}[\pi_1(W_F)] \to \mathbb{Z}[H_1(W_F)] \to \mathbb{C}$ and thus to complex vector spaces $H_k(W_F;\mathbb{C}^\omega)$, see Example 5.2.4. As we saw in Section 5.6, $H_2(W_F;\mathbb{C}^\omega)$ is endowed with a twisted intersection form $\lambda_{\mathbb{C}^\omega}(W_F)$ and it is very natural to conjecture that its signature coincides with the multivariable signature:

$$\operatorname{sign}(\lambda_{\mathbb{C}^\omega}(W_F)) = \sigma_L(\omega).$$

In fact, Viro showed that $\operatorname{sign}(\lambda_{\mathbb{C}^\omega}(W_F))$ does not depend of the choice of a colored bounding surface, but did not prove that the resulting invariant coincides with the multivariable signature [154, Section 2.5]. With sufficient care, one should even be able to prove this equality in the topological category instead of the smooth category. Furthermore, since $\pi_1(W_F) \to \mathbb{Z}^\mu$

restricts to $\pi_1(X_L) \to \mathbb{Z}^\mu, \gamma \mapsto \ell k(\gamma, L_1) + \ldots + \ell k(\gamma, L_\mu)$ on the link exterior, one also ought to relate the multivariable nullity $\eta_L(\omega)$ to the dimension of $H_1(X_L; \mathbb{C}^\omega)$.

This is what we achieve in the first main result of this chapter (which is based on joint work with Enrico Toffoli and Matthias Nagel [53]), see Section 8.2.

Theorem 8.1.1. *Let L be a μ-colored link. For every colored bounding surface F and for all $\omega \in \mathbb{T}^\mu_*$, we have the equalities*

$$\sigma_L(\omega) = \operatorname{sign}_\omega(W_F) \quad \text{and} \quad \eta_L(\omega) = \dim_{\mathbb{C}} H_1(X_L; \mathbb{C}^\omega).$$

The restriction to $\omega \in \mathbb{T}^\mu_*$ of finite order implies that Cimasoni-Florens' main 4-dimensional results are proved for a certain subset $\mathbb{T}^\mu_P$ of roots of unity [41, Theorem 7.1 and 7.2]. Looking back at Theorem 8.1.1, it seems that these results should hold for a larger set of $\omega \in \mathbb{T}^\mu_*$, as well as in the topological category. In order to prove this kind of results, one more technical tool is needed. More precisely, generalizing work of Nagel-Powell [129], Section 8.3 introduces a subset

$$\mathbb{T}^\mu_P \subsetneq \mathbb{T}^\mu_! \subsetneq \mathbb{T}^\mu$$

given by those ω's which are not roots of any polynomial $p \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$ whose evaluation on $(1, \ldots, 1)$ is invertible. With this technical point taken care of, the remainder of the chapter deals with applications of Theorem 8.1.1.

A *colored cobordism* between two μ-colored links L and L' is a collection of properly embedded locally flat surfaces $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_\mu$ in $S^3 \times [0,1]$ which have the following properties: the surfaces only intersect each other transversally in double points, each surface Σ_i has boundary $L_i \sqcup -L'_i$, and each connected component of Σ_i has at least one boundary component in $S^3 \times \{0\}$ and one in $S^3 \times \{1\}$. The second main result of this chapter is a bound on the Euler characteristic and number of double points in such a cobordism, see Section 8.4.

Theorem 8.1.2. *If $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_\mu$ is a colored cobordism between two μ-colored links L and L' with c double points, then, for all $\omega \in \mathbb{T}^\mu_!$, we have*

$$|\sigma_L(\omega) - \sigma_{L'}(\omega)| + |\eta_L(\omega) - \eta_{L'}(\omega)| \leq c - \sum_{i=1}^{\mu} \chi(\Sigma_i).$$

Recall that two μ-colored links L and L' are *concordant* if there exists a μ-colored cobordism between L and L' which has no intersection points and consists exclusively of annuli. Note once again that we work in the topological category whereas Cimasoni-Florens work in the smooth setting. As a first application of Theorem 8.1.2, we obtain the following generalization of [41, Theorem 7.1], see Corollary 8.4.6 for a proof.

Corollary 8.1.3. *The multivariable signature and nullity are topological concordance invariants at all $\omega \in \mathbb{T}^\mu_!$*

As a second corollary, we obtain a generalization of [41, Theorem 7.2], see Corollary 8.4.7 for a proof and Remark 8.4.8 for a comparison with a similar but weaker result obtained by Viro [154, Section 4].

Corollary 8.1.4. *Let $F = F_1 \cup \cdots \cup F_\mu$ be a colored bounding surface for a μ-colored link L such that $F_1, \ldots, F_\mu$ have a total number of m connected components, intersecting in c double points. Then, for all $\omega \in \mathbb{T}^\mu_!$, we have*

$$|\sigma_L(\omega)| + |\eta_L(\omega) - m + 1| \leq \sum_{i=1}^{\mu} \beta_1(F_i) + c.$$

Returning to the study of link concordance, Section 8.5 deals with 1-solvable cobordisms. This notion was defined by Cha [30] giving a relative version of Cochran-Orr-Teichner's notion of n-solvability [47]. We refer to Section 8.5 for the precise definition of n-solvable cobordant links, however note that abelian link invariants are not expected to distinguish 1-solvable cobordant links. For instance, if two links are 1-solvable cobordant, then their first non-zero Alexander polynomials agree up to norms and their Blanchfield pairings are Witt equivalent [99, Theorems B and C]. Our final result is the corresponding statement for the multivariable signature and nullity.

Theorem 8.1.5. *If two μ-colored links L and L' are 1-solvable cobordant, then, for all $\omega \in \mathbb{T}^\mu_!$, we have*

$$\eta_L(\omega) = \eta_{L'}(\omega) \quad \textit{and} \quad \sigma_L(\omega) = \sigma_{L'}(\omega).$$

Since concordant links are n-solvable cobordant for all n, Theorem 8.1.5 can be viewed as a vast refinement of Corollary 8.1.3, see Remark 8.5.4. We also believe that Theorem 8.1.5 would be difficult to prove using branched coverings.

Remark 8.1.6. Note that the notion of n-solvable cobordism is related to Whitney tower/grope concordance, see [30] for the definition of these notions. In particular, using [30, Corollary 2.17], Theorem 8.1.5 implies that the multivariable signature and nullity are invariant under height 3 Whitney tower/grope concordance.

8.2 Proof of Theorem 8.1.1

We start by recalling our set-up. Let $L = L_1 \cup \ldots \cup L_\mu \subset S^3$ be a μ-colored link. Recall from Definition 14 that a colored bounding surface for L is a union $F = F_1 \cup \ldots \cup F_\mu$ of properly embedded, locally flat, compact oriented surfaces $F_i \subset D^4$ with $\partial F_i = L_i$ and which only intersect each other transversally in double points. As the surfaces F_i are required to be locally flat, they admit tubular neighborhoods. Given a colored bounding surface F of L, we denote by νF the union of some choice of tubular neighborhoods for its components. We then denote the exterior of F by $W_F := D^4 \setminus \nu F$.

Since we saw in Lemma 3.4.8 thats the abelian group $H_1(W_F)$ is freely generated by the meridians of the components F_i, there is a canonical homomorphism $\pi_1(W_F) \to H_1(W_F) \to \mathbb{Z}^\mu$ which restricts to $\pi_1(X_L) \to \mathbb{Z}^\mu, \gamma \mapsto \ell k(\gamma, L_1) + \ldots + \ell k(\gamma, L_\mu)$ on the link exterior. Indeed the inclusion $X_L \subset W_F$ sends the meridians of L to the meridians of F. As described in the introduction, mapping the meridians of F_i to ω_i gives rise to a homomorphism $\mathbb{Z}[\pi_1(W_F)] \to \mathbb{C}$ and thus to complex vector spaces $H_i(W_F; \mathbb{C}^\omega)$ and $H_i(X_L; \mathbb{C}^\omega)$, see Example 5.2.4. As we mentioned above, Viro [154, Theorem 2.A] shows that the twisted signature $\operatorname{sign}_\omega(W_F) := \operatorname{sign}(\lambda_{\mathbb{C}^\omega}(W_F))$ is a link invariant, i.e. it is independent of the choice of a colored bounding surface.

We can now prove Theorem 8.1.1 which shows that this invariant coincides with the multivariable signature and relates the multivariable nullity to the dimension of $H_1(X_L;\mathbb{C}^\omega)$.

proof of Theorem 8.1.1. Given a C-complex F, recall that the multivariable nullity is defined as $\operatorname{null} H(\omega)+\beta_0(F)-1$, where $H(\omega)$ is the evaluation at $t=\omega$ of some C-complex matrix H arising from F. Picking a connected C-complex ensures that $\beta_0(F)=1$ and thus $\eta_L(\omega)=\operatorname{null} H(\omega)$. Since S is connected, Remark 3.3.12 implies that H presents the Alexander module $H_1(X_L;\Lambda_S)$. Tensoring with $\mathbb{C}^\omega$, we deduce that $H(\omega)$ presents $\mathbb{C}^\omega\otimes_{\Lambda_S} H_1(X_L;\Lambda_S)$. Using Lemma 7.5.3, we obtain that $H_1(X_L;\mathbb{C}^\omega)\cong\mathbb{C}^\omega\otimes_{\Lambda_S} H_1(X_L;\Lambda_S)$, and consequently $H(\omega)$ also presents $H_1(X_L;\mathbb{C}^\omega)$. The claim involving the nullity now follows readily.

Since the colored signature is independent of the choice of a colored bounding surface, we take F to be a push-in of a C-complex in the 4-ball. By Theorem 6.1.4, the intersection pairing $\lambda_{\Lambda_S}(W_F)$ is represented by the C-complex matrix H. Since we wish to show that the intersection pairing $\lambda_{\mathbb{C}^\omega}(W_F)$ is represented by $H(\omega)$, the theorem will follow if we manage to produce the following commutative diagram

$$\begin{tikzcd}
\mathbb{C}^\omega\otimes_{\Lambda_S} H_2(W_F;\Lambda_S)\times\mathbb{C}^\omega\otimes_{\Lambda_S} H_2(W_F;\Lambda_S) \arrow[r] \arrow[d] & \mathbb{C}^\omega\otimes_{\Lambda_S}\Lambda_S \arrow[d] \\
H_2(W_F;\mathbb{C}^\omega)\times H_2(W_F;\mathbb{C}^\omega) \arrow[r] & \mathbb{C}.
\end{tikzcd} \tag{8.1}$$

Further assuming S to be totally connected implies that $H_i(W_F;\Lambda_S)$ vanishes for $i\neq 2$, and is a finitely generated free Λ_S-module for $i=2$, see Corollary 6.3.4 and Proposition 6.4.2. Consider the diagram below, where homology groups and tensor products without coefficients are over Λ_S. Applying the universal coefficient spectral sequences, as described in Propositions 7.5.4 and 7.5.2, the first three vertical maps in the following commutative diagram are isomorphisms:

$$\begin{tikzcd}
\mathbb{C}^\omega\otimes H_2(W_F) \arrow[r] \arrow[d, "\cong"] & \mathbb{C}^\omega\otimes H_2(W_F,\partial W_F) \arrow[r] \arrow[d, "\cong"] & \mathbb{C}^\omega\otimes H^2(W_F) \arrow[r] \arrow[d, "\cong"] & \mathbb{C}^\omega\otimes\overline{\operatorname{Hom}_{\Lambda_S}(H_2(W_F),\Lambda_S)} \arrow[d, "\cong"] \\
H_2(W_F;\mathbb{C}^\omega) \arrow[r] & H_2(W_F,\partial W_F;\mathbb{C}^\omega) \arrow[r] & H^2(W_F;\mathbb{C}^\omega) \arrow[r] & \overline{\operatorname{Hom}_{\mathbb{C}}(H_2(W_F;\mathbb{C}^\omega),\mathbb{C})}.
\end{tikzcd}$$

Here note that the horizontal maps are respectively inclusion induced, Poincaré duality and evaluation. Furthermore, the rightmost vertical map is an isomorphism since $H_2(W_F;\Lambda_S)$ is finitely generated and free. Considering the adjoints, we obtain the diagram of (8.1). □

8.3 Concordance roots and vanishing results

In order to make use of Theorem 8.1.1, we generalize the concept of *Knotennullstellen* introduced by Nagel-Powell [129]. After applying this concept to a variation of a well-known chain homotopy argument, we discuss some further properties of these elements.

Let $U\subset\mathbb{Z}[t_1^{\pm1},\dots,t_\mu^{\pm1}]$ be the multiplicative subset of Laurent polynomials $p(t_1,\dots,t_\mu)$ such that $p(1,\dots,1)=\pm1$.

Definition 24. An element $\omega\in\mathbb{T}^\mu_*=(S^1\backslash\{1\})^\mu$ is a *concordance root* if there is a polynomial $p\in U$ with $p(\omega)=0$. Define $\mathbb{T}^\mu_!$ to be the subset of all elements $\omega\in\mathbb{T}^\mu_*$ which are *not* concordance roots.

Definition 24 is a generalization of [129, Definition 1.1] to the multivariable case. The key property of non concordance roots is that they allow us to use a well-known chain homotopy argument, see [47, Proposition 2.10]. The result below is an adaptation of [129, Lemma 3.1]. Before we state it, we quickly recall the following property of the localization $U^{-1}\Lambda_\mu$ of $\Lambda_\mu := \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$ and of the augmentation map $\operatorname{aug}\colon \Lambda_\mu \to \mathbb{Z}$, which sends $p(t_1, \dots, t_\mu)$ to $p(1, \dots, 1)$: let $g\colon \Lambda_\mu^k \to \Lambda_\mu^k$ be a Λ_μ-module homomorphism with the property that $\mathbb{Z} \otimes_{\operatorname{aug}} g$ is an isomorphism. Then

$$U^{-1}\Lambda_\mu \otimes g\colon (U^{-1}\Lambda_\mu)^k \to (U^{-1}\Lambda_\mu)^k$$

is also an isomorphism. This can be obtained from considerations of determinants [129, Section 3].

Lemma 8.3.1. *Let k be a non-negative integer. Let (X, Y) be a pair of CW-complexes over $B\mathbb{Z}^\mu$ with $H_i(X, Y) = 0$ for $0 \le i \le k$. If ω is in $\mathbb{T}^\mu_!$, then $H_i(X, Y; \mathbb{C}^\omega) = 0$ for $0 \le i \le k$.*

Proof. We make the following abbreviations $C^{\mathbb{Z}} := C_*(X, Y; \mathbb{Z})$ and $C^{\Lambda_\mu} := C_*(X, Y; \Lambda_\mu)$ for the cellular chain complexes of the pairs (X, Y). For the remainder of the proof, i will be an arbitrary integer $0 \le i \le k$. The chain complex $C^{\mathbb{Z}}$ consists of finitely generated free $\mathbb{Z}$-modules, and as $H_i(C^{\mathbb{Z}}) = 0$, it admits a partial contraction, i.e. homomorphisms $s_i\colon C_i^{\mathbb{Z}} \to C_{i+1}^{\mathbb{Z}}$ with

$$id_i = s_{i-1} \circ d_i + d_{i+1} \circ s_i.$$

Consider the chain map $\varepsilon\colon C^{\Lambda_\mu} \to C^{\mathbb{Z}}$ of chain complexes over Λ_μ which is induced by tensoring with the augmentation map. Pick a lift $s_i^{\Lambda_\mu}$ of s_i under ε, which is a homomorphism $s_i^{\Lambda_\mu}\colon C_i^{\Lambda_\mu} \to C_{i+1}^{\Lambda_\mu}$ of Λ-modules such that the following diagram commutes:

$$\begin{array}{ccc} C_i^{\Lambda_\mu} & \xrightarrow{s_i^{\Lambda_\mu}} & C_{i+1}^{\Lambda_\mu} \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} \\ C_i^{\mathbb{Z}} & \xrightarrow{s_i} & C_{i+1}^{\mathbb{Z}}. \end{array}$$

Such a lift exists because $C_i^{\Lambda_\mu}$ consists of free modules and the map ε is surjective. Consider the partial chain map

$$f_i := s_{i-1}^{\Lambda_\mu} \circ d_i + d_{i+1} \circ s_i^{\Lambda_\mu}.$$

By construction, $\mathbb{Z} \otimes_{\Lambda_\mu} f_i = s_{i-1} \circ d_i + d_{i+1} \circ s_i = id_{C_i}$ and so $U^{-1}\Lambda_\mu \otimes_{\Lambda_\mu} f_i$ is also an isomorphism. We obtain that $U^{-1}\Lambda_\mu \otimes_{\Lambda_\mu} s_i^{\Lambda_\mu}$ is a partial chain contraction for $U^{-1}\Lambda_\mu \otimes_{\Lambda_\mu} C^{\Lambda_\mu}$ and

$$H_i(X, Y; U^{-1}\Lambda_\mu) = H_i(U^{-1}\Lambda_\mu \otimes_{\Lambda_\mu} C^{\Lambda_\mu}) = 0.$$

Now we tensor with the right Λ_μ-module $\mathbb{C}^\omega$. As ω is not a concordance root, the module $\mathbb{C}^\omega$ is in fact a right $U^{-1}\Lambda_\mu$-module. Note that $\mathbb{C}^\omega \otimes_{\Lambda_\mu} s_i^{\Lambda_\mu}$ is a partial chain contraction for $\mathbb{C}^\omega \otimes_{U^{-1}\Lambda_\mu} U^{-1}\Lambda_\mu \otimes_{\Lambda_\mu} C^{\Lambda_\mu}$ and so $H_i(X, Y; \mathbb{C}^\omega) = 0$. □

For the remainder of the section, we collect properties of the set $\mathbb{T}^\mu_!$ of non-concordance roots. For a prime p, define

$$\mathbb{T}^\mu_p := \{\omega \in \mathbb{T}^\mu : \omega_i \text{ is a } p^n\text{-root of unity for some } n\}$$

and $\mathbb{T}_P^\mu := \bigcup_p \mathbb{T}_p^\mu$. As we saw in Section 3.4, this is the set for which concordance invariance properties and genus bounds are proved in [41, Section 7]. The next result shows that the set $\mathbb{T}_!^\mu$ of non-concordance roots contains $\mathbb{T}_P^\mu$.

Proposition 8.3.2. *The set $\mathbb{T}_P^\mu$ is contained in $\mathbb{T}_!^\mu$.*

Proof. Let $\omega \in \mathbb{T}_P^\mu$ and $q(t_1, \ldots, t_\mu)$ be a polynomial such that $q(\omega) = 0$. We have to show that $q(1, \ldots, 1) \neq \pm 1$. We pick n large enough such that all ω_i are p^n-roots of unity. The subgroup consisting of the p^n-roots of unity is cyclic. Thus we write $\omega = (\zeta^{n_1}, \ldots, \zeta^{n_\mu})$ for a primitive p^n-root of unity ζ. Define the one variable polynomial $\overline{q}(t) := q(t^{n_1}, \ldots, t^{n_\mu})$. Hence, we have $\overline{q}(\zeta) = 0$, so $\overline{q}(t)$ is a multiple of the p^n-th cyclotomic polynomial, whose value at 1 equals p. It follows that p divides $q(1, \ldots, 1) = \overline{q}(1)$ and so cannot be equal to ± 1. $\square$

The following example shows that $\mathbb{T}_!^\mu$ also contains elements which are not in $\mathbb{T}_P^\mu$, but have algebraic coordinates.

Example 8.3.3. We claim that the algebraic element $\omega = (\frac{3+4i}{5}, -1)$ is in $\mathbb{T}_!^2$, but not contained in $\mathbb{T}_P^2$. The algebraic number $\omega_0 = \frac{3+4i}{5} \in S^1$, has minimal polynomial $p(t) = 5t^2 - 6t + 5$ and is not a root of unity [129, Lemma 2.1]. It follows that ω_0 is not an element of $\mathbb{T}_P^1$. To show that $\omega \in \mathbb{T}_!^2$, we prove that any polynomial $q(t_1, t_2)$ with $q(\omega) = 0$ has $q(1, 1) \neq \pm 1$. Consider $\overline{q}(t) := q(t, -1)$ and note that $\frac{3+4i}{5}$ is a root of $\overline{q}(t)$. As a consequence $4 = p(1)$ divides $\overline{q}(1)$ and $\overline{q}(1) = q(1, -1)$ is even. It follows that $q(1, 1)$ must also be even.

We conclude this section by showing how to produce elements which do not belong to $\mathbb{T}_!^\mu$, i.e. for which our main results do not apply.

Remark 8.3.4. Let $(\omega_1, \ldots, \omega_n) \in \mathbb{T}_!^n$ and suppose $\beta\colon \{1, \ldots, \mu\} \to \{1, \ldots, n\}$ is any map. We claim that $(\omega_{\beta(1)}, \ldots, \omega_{\beta(\mu)})$ is an element of $\mathbb{T}_!^\mu$. Now, given $\omega = (\omega_1, \ldots, \omega_\mu) \in \mathbb{T}_!^\mu$, this claim implies that all the coefficients ω_i belong to $\mathbb{T}_!^1$. Phrasing it differently, if any of the coefficients of ω is a concordance root, then ω itself is a concordance root.

We now prove our claim. Let $q(t_1, \ldots, t_\mu)$ be a polynomial such that $q(\omega_\beta) = 0$, where ω_β denotes $(\omega_{\beta(1)}, \ldots, \omega_{\beta(\mu)})$. Define a polynomial in n-variables by the equality $p(x_1, \ldots, x_n) = q(x_{\beta(1)}, \ldots, x_{\beta(\mu)})$. Note that $p(\omega_1, \ldots, \omega_n) = q(\omega_{\beta(1)}, \ldots, \omega_{\beta(\mu)}) = 0$ and as $(\omega_1, \ldots, \omega_n) \in \mathbb{T}_!^n$, we deduce that $q(1, \ldots, 1) = p(1, \ldots, 1) \neq \pm 1$. The claim follows.

8.4 The genus bound

For elements $\omega \in \mathbb{T}_P^\mu$, Cimasoni-Florens [41] prove that the multivariable signature and nullity give lower bounds on the genus of colored bounding surfaces, see Theorem 3.4.7. In this section, we prove a more general result for surfaces in $S^3 \times [0, 1]$. As corollaries, we extend two results of Cimasoni-Florens: the concordance invariance results mentioned in Theorem 3.4.6 and the aforementioned Theorem 3.4.7.

Definition 25. A *colored cobordism* between two μ-colored links L and L' is a collection of properly embedded locally flat surfaces $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_\mu$ in $S^3 \times [0, 1]$ which have the following properties: the surfaces only intersect each other in double points, each surface Σ_i has boundary $L_i \sqcup -L_i'$, and each connected component of Σ_i has a boundary both in $S^3 \times \{0\}$

and in $S^3 \times \{1\}$. We say that Σ *has m components* if the disjoint union of the surfaces $\Sigma_1, \ldots, \Sigma_\mu$ has m connected components.

The main result of this section was stated in Theorem 8.1.2, but we recall the statement here for the reader's convenience.

Theorem 8.1.2. *If $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_\mu$ is a colored cobordism between two μ-colored links L and L' with c double points, then for all $\omega \in \mathbb{T}^\mu_!$, we have*

$$|\sigma_L(\omega) - \sigma_{L'}(\omega)| + |\eta_L(\omega) - \eta_{L'}(\omega)| \leq c - \sum_{i=1}^{\mu} \chi(\Sigma_i).$$

Remark 8.4.1. The right-hand side of the inequality can equivalently be expressed in terms of the first Betti number or of the genus of the surfaces. Suppose that L is an n-component link, L' is an n'-component link, and that the cobordism Σ has m components (in the sense of Definition 25). Then, we have the following equalities:

$$c - \sum_{i=1}^{\mu} \chi(\Sigma_i) = c + \sum_{i=1}^{\mu} b_1(\Sigma_i) - m = c + \sum_{i=1}^{\mu} 2g_i(\Sigma_i) + n + n' - 2m.$$

For this reason, we will usually refer to the inequality of Theorem 8.1.2 as a *genus bound*, even if the genus does not appear explicitly in the formula.

8.4.1 Proof of Theorem 8.1.2

We proceed towards the proof of Theorem 8.1.2, starting with a series of preliminary results. First, we describe the Euler characteristic of the exterior W_Σ of a colored cobordism Σ in $S^3 \times [0,1]$ in terms of the Euler characteristic of the surfaces Σ_i.

Lemma 8.4.2. *Suppose Σ is a μ-colored cobordism between two colored links L and L'. Then the Euler characteristic of W_Σ is given by*

$$\chi(W_\Sigma) = c - \sum_{i=1}^{\mu} \chi(\Sigma_i).$$

Proof. First, we prove that $\chi(W_\Sigma) = -\chi(\nu\Sigma)$. Consider the decomposition $S^3 \times I = \overline{\nu\Sigma} \cup W_\Sigma$ and set $M_\Sigma := \overline{\nu\Sigma} \cap W_\Sigma$. Using the decomposition formula for the Euler characteristic yields $\chi(S^3 \times I) = \chi(W_\Sigma) + \chi(\overline{\nu\Sigma}) - \chi(M_\Sigma)$. As the Euler characteristic of a 3-manifold with toroidal boundary vanishes, $\chi(M_\Sigma) = 0$. Since $\chi(S^3 \times I)$ also vanishes, the claim follows. Now note that $\overline{\nu\Sigma}$ is homotopy equivalent to the union $A = \bigcup_i \Sigma_i \subset S^3$. Recall that the surfaces Σ_i intersect each other in c double points. We apply again the decomposition formula for A and obtain

$$\chi(A) = \sum_{i=1}^{\mu} \chi(\Sigma_i) - \chi\Big(\bigcup_{i \neq j} \Sigma_i \cap \Sigma_j\Big) = \sum_{i=1}^{\mu} \chi(\Sigma_i) - c.$$

□

Arguing as in Lemma 3.4.8, $H_1(W_\Sigma)$ is freely generated by the meridians of Σ. Consequently, there is a homomorphism $H_1(W_\Sigma) \to \mathbb{Z}^\mu$ which extends the maps on $H_1(X_L)$ and $H_1(X_{L'})$. Next, we observe that with $\mathbb{C}^\omega$ coefficients, the boundary of W_Σ behaves as the disjoint union of the link exteriors X_L and $X_{L'}$.

Lemma 8.4.3. *For all $\omega \in \mathbb{T}_*^\mu$, the inclusion of $X_L \sqcup X_{L'}$ into ∂W_Σ induces an isomorphism*

$$H_i(X_L;\mathbb{C}^\omega) \oplus H_i(X_{L'};\mathbb{C}^\omega) \cong H_i(\partial W_\Sigma;\mathbb{C}^\omega).$$

Proof. The boundary of W_Σ decomposes into the union of X_L, $X_{L'}$ and the plumbed 3-manifold M_Σ. The homology groups $H_*(M_\Sigma;\Lambda_S)$ are zero, see the proof of Lemma 6.3.5. The universal coefficient spectral sequence of Theorem 2.5.8 implies that $H_*(M_\Sigma;\mathbb{C}^\omega) = 0$. The result now follows from the Mayer-Vietoris exact sequence for ∂W_F. □

The next lemma provides some information on the twisted homology of W_Σ.

Lemma 8.4.4. *If $\Sigma \subset S^3 \times I$ is a μ-colored cobordism between L and L' and $\omega \in \mathbb{T}_!^\mu$, then*

1. *$\beta_1^\omega(W_\Sigma) \leq \eta_L(\omega)$ and $\beta_1^\omega(W_\Sigma) \leq \eta_{L'}(\omega)$,*
2. *$H_k(W_\Sigma;\mathbb{C}^\omega) = 0$ for $k = 0,3,4$.*

Proof. As W_Σ and X_L are both connected, there is an isomorphism $H_0(X_L) \cong H_0(W_\Sigma)$. Since the inclusion $X_L \subset W_\Sigma$ takes meridians to meridians, $H_1(X_L) \to H_1(W_\Sigma)$ is surjective. Combining these facts, $H_i(W_\Sigma, X_L) = 0$, so that Lemma 8.3.1 gives $H_i(W_\Sigma, X_L;\mathbb{C}^\omega) = 0$ for $i = 0,1$. It follows from the long exact sequence of the pair (W_Σ, X_L) that the inclusion induced map $H_1(X_L;\mathbb{C}^\omega) \to H_1(W_\Sigma;\mathbb{C}^\omega)$ is surjective, and thus $\beta_1^\omega(W_\Sigma) \leq \eta_L(\omega)$. Repeating the argument for $X_{L'}$, the first statement is proved.

Since the inclusion of X_L into W_Σ factors through ∂W_Σ, an analogous argument shows that $H_i(W_\Sigma, \partial W_\Sigma;\mathbb{C}^\omega) = 0$ for $i = 0,1$. Lemma 7.5.5 now implies that $H_i(W_\Sigma;\mathbb{C}^\omega) = 0$ for $i = 3,4$. The case $i = 0$ is immediate. □

We conclude this section with a dimension count which will prove itself useful to bound the twisted signature of W_Σ.

Lemma 8.4.5. *For $\omega \in \mathbb{T}_!^\mu$ and a μ-colored cobordism $\Sigma \subset S^3 \times [0,1]$, we have*

$$\dim_\mathbb{C}\left(\frac{H_2(W_\Sigma;\mathbb{C}^\omega)}{\operatorname{im}(i)}\right) = \beta_2^\omega(W_\Sigma) - \beta_2^\omega(\partial W_\Sigma) - \beta_1^\omega(W_\Sigma),$$

where $i\colon H_2(\partial W_\Sigma;\mathbb{C}^\omega) \to H_2(W_\Sigma;\mathbb{C}^\omega)$ is induced by the inclusion.

Proof. Recall that by Lemma 8.4.4, the vector space $H_3(W_\Sigma;\mathbb{C}^\omega)$ vanishes. Consider the following portion of the long exact sequence of the pair $(W_\Sigma, \partial W_\Sigma)$:

$$0 \to H_3(W_\Sigma, \partial W_\Sigma;\mathbb{C}^\omega) \xrightarrow{\delta} H_2(\partial W_\Sigma;\mathbb{C}^\omega) \xrightarrow{i} H_2(W_\Sigma;\mathbb{C}^\omega).$$

By exactness, the dimension of $\operatorname{im}(i)$ is equal to $\beta_2^\omega(\partial W_\Sigma) - \operatorname{im}(\delta)$. As δ is injective, one gets $\dim_\mathbb{C} \operatorname{im}(i) = \beta_2^\omega(\partial W_\Sigma) + \beta_3^\omega(W_\Sigma, \partial W_\Sigma)$. The result now follows since Lemma 7.5.5 implies that $\beta_3^\omega(W_\Sigma, \partial W_\Sigma) = \beta_1^\omega(W_\Sigma)$. □

We are now ready to conclude the proof of Theorem 8.1.2.

proof of Theorem 8.1.2. We start by proving the following inequality:

$$|\operatorname{sign}_\omega(W_\Sigma)| \leq \chi(W_\Sigma) - |\eta_L(\omega) - \eta_{L'}(\omega)|.$$

Since the twisted intersection form $\lambda_{\mathbb{C}^\omega}(W_\Sigma)$ descends to a pairing on $H_2(W_\Sigma; \mathbb{C}^\omega)/\operatorname{im}(i)$, an application of Lemma 8.4.5 yields

$$|\operatorname{sign}_\omega(W_\Sigma)| \leq \dim_{\mathbb{C}} \left(\frac{H_2(W_\Sigma; \mathbb{C}^\omega)}{\operatorname{im}(i)} \right) = \beta_2^\omega(W) - \beta_2^\omega(\partial W) - \beta_1^\omega(W). \tag{8.2}$$

Now, thanks to Lemma 8.4.4 and Remark 5.2.5, we have $\chi(W_\Sigma) = \beta_2^\omega(W_\Sigma) - \beta_1^\omega(W_\Sigma)$, and using Lemma 8.4.3 and Theorem 8.1.1, one gets $\beta_1^\omega(\partial W_\Sigma) = \eta_L(\omega) + \eta_{L'}(\omega)$. Using these last two identities, (8.2) can be rewritten as

$$|\operatorname{sign}_\omega(W)_F| \leq \chi(W_\Sigma) + 2\beta_1^\omega(W_\Sigma) - \eta_L(\omega) - \eta_{L'}(\omega).$$

The desired inequality is now obtained by using Lemma 8.4.4 to bound $\beta_1^\omega(W_\Sigma)$ above both by $\eta_L(\omega)$ and $\eta_{L'}(\omega)$. With the inequality above, Theorem 8.1.2 will follow from Lemma 8.4.2 once we have established that

$$\operatorname{sign}_\omega(W_\Sigma) = \sigma_{L'}(\omega) - \sigma_L(\omega).$$

Pick a colored bounding surface $F \subset D^4$ for L. Thanks to Proposition 6.7.2 and Theorem 8.1.1, we have $\sigma_L(\omega) = \operatorname{sign}_\omega(W_F)$. One can now form the surface with singularities $F \cup_L \Sigma \subset D^4 \cup_{S^3} S^3 \times I$. Using an orientation-preserving diffeomorphism between $D^4 \cup_{S^3} S^3 \times I$ and D^4, the surface $F \cup_L \Sigma$ is sent to a colored bounding surface for L'. Its exterior $W_{F'}$ is clearly homeomorphic to $W_F \cup_{X_L} W_\Sigma$. Once again thanks to Proposition 6.7.2, we have $\sigma_{L'}(\omega) = \operatorname{sign}_\omega(W_{F'})$. Since $H_1(L \times S^1; \mathbb{C}^\omega) = 0$, Proposition 7.2.2 implies that the additivity of twisted signature holds, yielding

$$\operatorname{sign}_\omega(W) = \operatorname{sign}_\omega(W_F) + \operatorname{sign}_\omega(W_\Sigma).$$

Summarizing, we have shown that $\sigma_{L'}(\omega) = \sigma_L(\omega) + \operatorname{sign}_\omega(W_\Sigma)$. Combining this with the inequality of Equation (8.2) concludes the proof of Theorem 8.1.2. □

8.4.2 Applications of Theorem 8.1.1

Recall from Definition 13 that two μ-colored links L and L' are *concordant* if there exists a μ-colored cobordism between L and L' which has no double points and consists exclusively of annuli. A first application of Theorem 8.1.2 is the following generalization of Theorem 3.4.6 which is due to Cimasoni-Florens [41].

Corollary 8.4.6. *If L_1 and L_2 are two concordant colored links, then, for all $\omega \in \mathbb{T}^\mu_!$, we have*

$$\sigma_{L_1}(\omega) = \sigma_{L_2}(\omega) \qquad \text{and} \qquad \eta_{L_1}(\omega) = \eta_{L_2}(\omega).$$

Proof. We apply Theorem 8.1.2 to the case where each Σ_i is a union of annuli and there are no double points. The result follows as all the terms on the right-hand side of the inequality are zero. □

We wish to emphasize once more that Cimasoni-Florens' result holds in the smooth category whereas Corollary 8.4.6 holds in the topological category. The same remark applies to the following application of Theorem 8.1.2, which is the corresponding generalization of Theorem 3.4.7, also due to Cimasoni-Florens.

Corollary 8.4.7. *Let $F = F_1 \cup \cdots \cup F_\mu$ be a colored bounding surface for L, and suppose that F has m components and c intersection points. Then, for all $\omega \in \mathbb{T}^\mu_!$, we have*

$$|\sigma_L(\omega)| + |\eta_L(\omega) - m + 1| \leq \sum_{i=1}^{\mu} \beta_1(F_i) + c.$$

Proof. Remove small 4-balls in the interior of D^4 on each component of F. With small enough balls, Σ will intersect the boundary spheres in unknots. Tubing the boundary spheres together, we have constructed a μ-colored cobordism Σ with m components between L and a μ-colored unlink L' of m components. Pick a disjoint union of m disks as a C-complex F for L'. The resulting generalized Seifert matrices are empty, yielding $\sigma_{L'}(\omega) = 0$ and $\eta_{L'}(\omega) = 0 + \beta_0(F) - 1 = m - 1$ for all $\omega \in \mathbb{T}^\mu$. Using Theorem 8.1.2 and Remark 8.4.1, we get

$$|\sigma_L(\omega)| + |\eta_L(\omega) - m + 1| \leq -\sum_{i=1}^{\mu} \chi(\Sigma_i) + c = \sum_{i=1}^{\mu} b_1(\Sigma_i) - m + c.$$

Now, if C is any of the m components of F, the corresponding component C' of Σ is obtained from C by removing a small disk, so that $\beta_1(C') = \beta_1(C) + 1$. Summing over all the components, we get $\sum_{i=1}^{\mu} \beta_1(\Sigma_i) = \sum_{i=1}^{\mu} \beta_1(F_i) + m$, whence the desired formula. □

The next remark compares Corollary 8.4.7 with similar results which appear in the literature.

Remark 8.4.8. As we mentioned above, Corollary 8.4.7 is a generalization of Theorem 3.4.7, which is due to Cimasoni-Florens [41]. The latter result is proved in the smooth setting and requires ω to be in the set $\mathbb{T}^\mu_P$, which is strictly smaller as $\mathbb{T}^\mu_!$, see Example 8.3.3. Furthermore, since these authors assume all the surfaces F_i to be connected, a μ appears instead of an m in their formula.

Finally, note that Viro proves inequalities similar to Corollary 8.4.7 in any odd dimension. In particular, for links in S^3, he obtains $|\sigma_L(\omega)| + \eta_L(\omega) \leq \beta_2(F, L) + \beta_1(F)$ and $|\sigma_L(\omega)| + \eta_L(\omega) \leq \beta_1(F, L) + \beta_0(F)$ [154, Theorem 4.C]. Reworking his equations leads to the inequality

$$|\sigma_L(\omega)| + \eta_L(\omega) - m \leq \sum_{i=1}^{\mu} \beta_1(F_i) + c,$$

which is slightly weaker than Corollary 8.4.7. The interested reader will note that while Viro essentially obtains his results for all $\omega \in \mathbb{T}^\mu_!$, his methods are quite different from the chain homotopy argument we rely on, see [154, Appendix C].

8.5 Invariance by 1-solvable cobordisms

The aim of this section is to prove that the multivariable signature and nullity are invariant under 1-solvable cobordism. Section 8.5.1 reviews the notion of an n-solvable cobordism. Section 8.5.2 tackles the invariance of the nullity. Section 8.5.3 proves some technical lemmas which are needed in Section 8.5.4 to complete the proof of the signature invariance.

8.5.1 Solvable cobordisms

We review the notion of n-solvable cobordism due to Cha [30]. For simplicity, we content ourselves with integral solvability and avoid defining the related concept of $n.5$-solvable cobordism.

A *cobordism* $(W; M, M', \phi)$ between two 3-manifolds M, M' with a preferred orientation-preserving diffeomorphism $\phi\colon \partial M \to \partial M'$ is a compact 4-manifold W with a decomposition $\partial W \cong -M \cup_\phi M'$. We will often supress ϕ from the notation. A cobordism $(W; M, M')$ is an H_1*-cobordism* if additionally the inclusions of M and M' into W induce isomorphisms $H_1(M) \xrightarrow{\cong} H_1(W) \xleftarrow{\cong} H_1(M')$.

We start by recalling some well-known facts about H_1-cobordisms.

Lemma 8.5.1. *If $(W; M, M')$ is an H_1-cobordism, then the following statements hold:*

1. *$H_i(W, M) = 0 = H_i(W, M')$ for all $i \neq 2$.*
2. *The groups $H_2(W, M)$ and $H_2(W, M')$ are isomorphic and free abelian.*
3. *Denote by $k\colon H_2(\partial W) \to H_2(W)$ the map induced by the inclusion. There exists a unique map $\varphi\colon H_2(W, M) \to H_2(W)/\operatorname{im}(k)$ such that*

$$\begin{tikzcd}
H_2(W)/\operatorname{im}(k) \arrow[r] & H_2(W, \partial W) \\
H_2(W) \arrow[u] \arrow[r] & H_2(W, M) \arrow[u] \arrow[ul, dashed, "\varphi"']
\end{tikzcd}$$

is commutative. The map φ is an isomorphism.

Proof. Since the first two assertions can be found in [30, Lemma 2.20], we only prove the third one. As a first step, we show that the map $i\colon H_2(W, M) \to H_2(W, \partial W)$ arising from the long exact sequence of the triple $(W, \partial W, M)$ is an injection. To prove this, consider the diagram

$$\begin{tikzcd}
\operatorname{Hom}(H_1(W), \mathbb{Z}) \arrow[r] & \operatorname{Hom}(H_1(M'), \mathbb{Z}) & & \\
H^1(W) \arrow[u, "\cong"', "\mathrm{ev}"] \arrow[r, "f"] \arrow[dd, "\cong"', "\mathrm{PD}"] & H^1(M') \arrow[u, "\cong"', "\mathrm{ev}"] \arrow[d, "\cong"', "\mathrm{PD}"] & & \\
 & H_2(M', \partial M') \arrow[d, "\cong"', "\mathrm{exc}"] & & \\
H_3(W, \partial W) \arrow[r] & H_2(\partial W, M) \arrow[r] & H_2(W, M) \arrow[r, "i"] & H_2(W, \partial W),
\end{tikzcd}$$

where exc denotes excision. The upper square clearly commutes, while the pentagon commutes by [23, Section VI.6, Problem 3]. Since $(W; M, M')$ is an H_1-cobordism, the uppermost horizontal map is an isomorphism. Consequently, the map f is an isomorphism and therefore so is the map $H_3(W, \partial W) \to H_2(\partial W, M)$. Exactness now implies that $i\colon H_2(W, M) \to H_2(W, \partial W)$ is injective.

As a second step, we show existence and uniqueness of $\varphi\colon H_2(W,M) \to \frac{H_2(W)}{\operatorname{im}(k)}$. The portion

$$H_2(\partial W) \xrightarrow{k} H_2(W) \xrightarrow{j} H_2(W,\partial W) \xrightarrow{\partial} H_1(\partial W) \xrightarrow{\ell} H_1(W)$$

of the long exact sequence of the pair $(W,\partial W)$ produces the short exact sequence in the top row of the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \frac{H_2(W)}{\operatorname{im}(k)} \arrow[r, "j"] & H_2(W,\partial W) \arrow[r, "\partial"] & \ker(\ell) \arrow[r] & 0 \\ & H_2(W) \arrow[r] \arrow[u, two heads] & H_2(W,M) \arrow[r] \arrow[u, "i"] \arrow[ul, dashed, "\varphi"] & \ker(H_1(M) \to H_1(W)). \arrow[u] & \end{tikzcd}$$

Since $(W;M,M')$ is an H_1-cobordism, the group $\ker(H_1(M) \to H_1(W))$ vanishes. Consequently, given $x \in H_2(W,M)$, the composition $i(\partial(x))$ is zero and so, by exactness, there exists $[y] \in \frac{H_2(W)}{\operatorname{im}(k)}$ such that $j([y]) = i(x)$. We therefore define $\varphi(x) := [y]$. As j is injective, φ is well-defined. By construction $\varphi j = i$.

Next, we show that φ is an isomorphism. Injectivity is immediate from the diagram above and the fact that i is injective. As $\ker(H_1(M) \to H_1(W)) = 0$, we obtain the following commutative diagram

$$\begin{tikzcd} H_2(W)/\operatorname{im}(k) \arrow[r] & H_2(W,\partial W) \\ H_2(W) \arrow[r, two heads] \arrow[u, two heads] & H_2(W,M) \arrow[u] \arrow[ul, "\varphi"] \end{tikzcd}$$

which shows the surjectivity of φ. $\square$

Let $(W;M,M')$ be an H_1-cobordism. Recall from Example 5.6.2 that $\pi^{(n)}$ denotes the n-th derived subgroup of $\pi := \pi_1(W)$ and that there is an intersection form

$$\lambda_n := \lambda_{\mathbb{Z}[\pi/\pi^{(n)}]}(W)\colon H_2(W;\mathbb{Z}[\pi/\pi^{(n)}]) \times H_2(W;\mathbb{Z}[\pi/\pi^{(n)}]) \to \mathbb{Z}[\pi/\pi^{(n)}]$$

which coincides with the ordinary intersection form when $n = 0$. Following Cha [30, Definition 2.8], we recall the following definition, which is inspired by Cochran-Orr-Teichner's work [47].

Definition 26. An H_1-cobordism $(W;M,M')$ is an *n-solvable cobordism* if there exists a submodule $L = \langle l_1, \ldots, l_r\rangle \subset H_2(W;\mathbb{Z}[\pi/\pi^{(n)}])$ together with homology classes $d_1, \ldots, d_r \in H_2(W;\mathbb{Z}[\pi/\pi^{(n)}])$ which satisfy the following properties:

1. the intersection form λ_n vanishes on L and the relation $\lambda_n(l_i, d_j) = \delta_{ij}$ holds for each i, j;
2. the image of L under the composition $H_2(W;\mathbb{Z}[\pi/\pi^{(n)}]) \to H_2(W) \to H_2(W,M)$ has half rank;
3. the restriction of the intersection form λ_0 to the image of L in $H_2(W)$ vanishes.

We shall sometimes refer to the submodule L as an *n-Lagrangian* and to the classes $d_1, \ldots, d_r$ as *n-duals*. It is worth noting that these notions are defined and studied in much greater generality in [30, Section 2]. For further reference, we make note of the following result, whose proof is outlined in [30, proof of Theorem 3.2].

Proposition 8.5.2. *If an H_1-cobordism W admits a 0-Lagrangian, then the ordinary signature* $\operatorname{sign}(W)$ *vanishes.*

Proof. Let $\varphi\colon H_2(W, M) \to H_2(W)/\operatorname{im}(H_2(\partial W))$ be the isomorphism of Lemma 8.5.1. Using the definition of a 1-solvable cobordism, the subspace $\varphi(L)$ gives rise to a Lagrangian of the nonsingular intersection pairing $\lambda_{\mathbb{Q}}(W)$ of W. Consequently, $\operatorname{sign}(W)$ vanishes. $\square$

The next definition is an adaptation to the colored framework of the one given by Cha [30]. Recall that the boundary $L \times S^1 = \partial X_L$ of a link exterior X_L inherits a product structure by longitudes and meridians. A bijection σ of the link components of two links L, L' induces an orientation-preserving diffeomorphism $\varphi_\sigma\colon L\times S^1 \to L'\times S^1$ preserving the product structures, which is unique up to isotopy.

Definition 27. Two colored links L, L' are *n-solvable cobordant* if there exists a bijection σ between the components of L and of L which preserves the colors and there is an n-solvable cobordism $(W; X_L, X_{L'}, \varphi_\sigma)$.

Remark 8.5.3. Suppose that two μ-colored links L and L' with m-components are n-solvable cobordant, we claim that they have the same pairwise linking numbers. To see this, first recall that the exterior X_L of a colored link L is equipped with a homomorphism $\beta_L\colon H_1(X_L) \to \mathbb{Z}^\mu$. An n-solvable cobordism between the links now fits into the commutative diagram

$$\begin{tikzcd}
H_1(X_L) \arrow[r, "i"] \arrow[dr, "\cong"'] \arrow[ddr, "\beta_L"'] & H_1(W) \arrow[d] & H_1(X_{L'}), \arrow[l, "i"'] \arrow[dl, "\cong"] \arrow[ddl, "\beta_{L'}"] \\
 & \mathbb{Z}^m \arrow[d] & \\
 & \mathbb{Z}^\mu &
\end{tikzcd} \tag{8.3}$$

where $\mathbb{Z}^m$ is indexed by the link components and $\mathbb{Z}^\mu$ by the colors. The maps $H_1(X_L) \to \mathbb{Z}^m$ and $H_1(X_{L'}) \to \mathbb{Z}^m$ send each meridian to the corresponding unit vector, and the map $\mathbb{Z}^m \to \mathbb{Z}^\mu$ is induced by sending a link component to its color. Now let $l \in H_1(X_L)$ be the longitude of some component K of L, and let $l' \in H_1(X_{L'})$ be the longitude of the corresponding component $K' = \sigma(K)$ of L'. Since the longitudes are glued together, we have $i(l) = i'(l')$ in Diagram (8.3). But the linking numbers of K and K' with the other components of L and L' respectively are nothing but the coordinates of the images of l and l' in $\mathbb{Z}^m$. Commutativity of the diagram above now implies the claim.

Remark 8.5.4. If L and L' are concordant, then they are n-solvable cobordant for each n. First, note that a concordance gives a bijection σ between the link components of L and the ones of L' by following the annuli. If W_C is the exterior of the concordance C, we have $\partial W_C \cong X_L \cup_{\varphi_\sigma} X_{L'}$, and the inclusion maps give rise to isomorphisms $H_*(X_L) \xrightarrow{\cong} H_*(W_C) \xleftarrow{\cong} H_*(X_{L'})$. Since the relative groups $H_2(W_C, X_L)$ and $H_2(W_C, X_L)$ vanish, the remaining conditions in the definition of an n-solvable cobordism are vacuous.

8.5.2 Nullities and 1-solvability

We now show that 1-solvable cobordant links have the same multivariable nullity. This proves half of Theorem 8.1.5.

Given a pair (X,Y) we denote by $\beta_i(X,Y)$ the rank of $H_i(X,Y)$ and by $\beta_i^\omega(X,Y)$ the dimension of $H_i(X,Y;\mathbb{C}^\omega)$. Recall that for an n-solvable cobordism $(W;M,M')$, we abbreviate $\pi_1(W)$ by π.

Lemma 8.5.5. *Let $(W;M,M')$ be a 1-solvable cobordism equipped with a homomorphism $\pi/\pi^{(1)} \to \mathbb{Z}^\mu$. Then both $H_i(W,M)$ and $H_i(W,M;\mathbb{C}^\omega)$ vanish for $i \neq 2$ and for all $\omega \in \mathbb{T}^\mu_!$. In particular, $\beta_2^\omega(W,M)$ is equal to $\beta_2(W,M)$.*

Proof. Since W is an H_1-cobordism, we get $H_i(W,M) = 0$ for $i = 0,1$. Since $\omega \in \mathbb{T}^\mu_!$, Lemma 8.3.1 implies that $H_i(W,M;\mathbb{C}^\omega) = 0$ for $i = 0,1$. In the untwisted case, for $i = 3,4$, applying duality and the universal coefficient theorem yields $H_i(W,M) = H^{4-i}(W,M') = \operatorname{Hom}(H_{4-i}(W,M'),\mathbb{Z}) = 0$. In the twisted case, the same conclusion follows by combining Lemma 7.5.5 with the fact that $H_i(W,M;\mathbb{C}^\omega) = 0$ for $i = 0,1$. The last claim now follows since the Euler characteristic of (W,M) may be computed with $\mathbb{Z}$-coefficients or $\mathbb{C}^\omega$-coefficients, see Remark 5.2.5. ☐

Next, we use each of the 1-Lagrangian and the 1-duals to produce a subspace of $H_2(W;\mathbb{C}^\omega)$ on which the intersection form $\lambda_{\mathbb{C}^\omega}(W)$ vanishes.

Lemma 8.5.6. *Given a 1-solvable cobordism $(W;M,M')$ equipped with a homomorphism $\pi/\pi^{(1)} \to \mathbb{Z}^\mu$, there exist subspaces $L_\mathbb{C}, D_\mathbb{C} \subset H_2(W;\mathbb{C}^\omega)$ such that*

1. *The intersection form $\lambda_{\mathbb{C}^\omega}(W)$ vanishes on $L_\mathbb{C}$.*
2. *The dimensions of $L_\mathbb{C}$ and $D_\mathbb{C}$ are equal to $r := \frac{1}{2}\operatorname{rank} H_2(W,M)$.*
3. *$L_\mathbb{C}$ and $D_\mathbb{C}$ are respectively generated by classes $l'_1,\ldots,l'_r$ and $d'_1,\ldots,d'_r$ which satisfy $\lambda_{\mathbb{C}^\omega}(W)(l'_i,d'_j) = \delta_{ij}$.*

Proof. We first claim that the $l_1,\ldots,l_r$ freely generate the 1-Lagrangian of W. Since W is a 1-Lagrangian there exist 1-duals $d_1,\ldots,d_r$ satisfying $\lambda_1(l_i,d_j) = \delta_{ij}$. If $\sum_{i=1}^r a_i l_i = 0$, then applying $\lambda_1(-,d_j)$ proves that a_j vanishes for each j, showing the linear independence of the l_i. Next, the composition $\alpha\colon \pi/\pi^{(1)} \to \mathbb{Z}^\mu \xrightarrow{\omega} \mathbb{C}$ gives rise to a map $\alpha_*\colon H_*(W;\mathbb{Z}[\pi/\pi^{(1)}]) \to H_*(W;\mathbb{C}^\omega)$ on homology, and we set $L_\mathbb{C} := \alpha_*(L)$. We abbreviate $\mathbb{Z}[\pi/\pi^{(1)}]$ by Λ. Since the following diagram (in which the horizontal maps are inclusion induced, Poincaré duality and evaluation) commutes

$$\begin{array}{ccccccc}
H_2(W;\Lambda) & \longrightarrow & H_2(W,\partial W;\Lambda) & \longrightarrow & H^2(W;\Lambda) & \longrightarrow & \overline{\operatorname{Hom}_\Lambda(H_2(W;\Lambda),\Lambda)} \\
\downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} \\
H_2(W;\mathbb{C}^\omega) & \longrightarrow & H_2(W,\partial W;\mathbb{C}^\omega) & \longrightarrow & H^2(W;\mathbb{C}^\omega) & \longrightarrow & \overline{\operatorname{Hom}_\mathbb{C}(H_2(W;\mathbb{C}^\omega),\mathbb{C})},
\end{array}$$

we deduce that $\lambda_{\mathbb{C}^\omega}(W)(\alpha_*(x),\alpha_*(y)) = \lambda_1(x,y)$. From this, it follows that the elements $l'_i := \alpha_*(l_i)$ and $d'_i := \alpha_*(d_i)$ are dual for $1 \leq i \leq r$, showing the last property. Using the previous argument, we see that they freely span $L_\mathbb{C}$ and $D_\mathbb{C} := \alpha_*(D)$ and so both have dimension r. Since λ_1 vanishes on L, it also follows that $\lambda_{\mathbb{C}^\omega}(W)$ vanishes on $L_\mathbb{C}$. ☐

We are now ready to prove the invariance of the nullity under 1-solvable cobordism.

Proposition 8.5.7. *If two μ-colored links L and L' are* 1*-solvable cobordant, then, for all $\omega \in \mathbb{T}^\mu_!$ we have*

$$\eta_L(\omega) = \eta_{L'}(\omega).$$

Proof. Recall from Theorem 8.1.1 that the nullity $\eta_L(\omega)$ is equal to the dimension of $H_1(X_L; \mathbb{C}^\omega)$, and the same goes for L'. To prove the proposition, we shall show that $H_1(X_L; \mathbb{C}^\omega) \cong H_1(W; \mathbb{C}^\omega) \cong H_1(X_{L'}; \mathbb{C}^\omega)$. Define M to be either X_L or $X_{L'}$, and let r be the rank of $H_2(W, M)$. Since we proved in Lemma 8.5.5 that $H_1(W, M; \mathbb{C}^\omega) = 0$, the long exact sequence of the pair (W, M) gives

$$H_2(M; \mathbb{C}^\omega) \to H_2(W; \mathbb{C}^\omega) \to H_2(W, M; \mathbb{C}^\omega) \to H_1(M; \mathbb{C}^\omega) \to H_1(W; \mathbb{C}^\omega) \to 0.$$

Consequently, to prove the statement, it is enough to show that the map $H_2(W; \mathbb{C}^\omega) \to H_2(W, M; \mathbb{C}^\omega)$ is surjective. Let $L_\mathbb{C}$ and $D_\mathbb{C}$ be the $r/2$-dimensional vector subspaces of $H_2(W; \mathbb{C}^\omega)$ provided by Lemma 8.5.6. By the intersection properties of the vectors in $L_\mathbb{C} \oplus D_\mathbb{C}$, one can check that the space $L_\mathbb{C} \oplus D_\mathbb{C}$ intersects trivially the image of $H_2(M; \mathbb{C}^\omega) \to H_2(W; \mathbb{C}^\omega)$. By Lemma 8.5.5 the space $H_2(W, M; \mathbb{C}^\omega)$ has dimension r, which is also the dimension of $L_\mathbb{C} \oplus D_\mathbb{C} \subset H_2(W; \mathbb{C}^\omega)$. Consequently, the map $H_2(W; \mathbb{C}^\omega) \to H_2(W, M; \mathbb{C}^\omega)$ must be surjective. This concludes the proof of the proposition. $\square$

8.5.3 Technical lemmas

We collect some technical lemmas needed to prove the invariance of the signature under 1-solvable cobordism. More precisely, we use various arguments from homological algebra in order to estimate the dimension of the space $L_\mathbb{C}$ given by Lemma 8.5.6.

We start by computing some twisted Betti numbers.

Lemma 8.5.8. *If two μ-colored links L and L' are* 1*-solvable cobordant via W, then for $i = 1, 2$ and for all $\omega \in \mathbb{T}^\mu_!$*

1. $\beta_1^\omega(X_L) = \beta_2^\omega(X_{L'})$,
2. $\beta_1^\omega(\partial W) = \beta_2^\omega(\partial W)$,
3. $\beta_2^\omega(W, \partial W) = \beta_2^\omega(W, X_L) + \beta_3^\omega(W, \partial W) = \beta_2^\omega(W, X_{L'}) + \beta_3^\omega(W, \partial W)$.

Proof. Set $M := X_L$ and $M' := X_{L'}$ and note that since M is a link exterior, its Euler characteristic vanishes. Since $\beta_0^\omega(M)$ and $\beta_3^\omega(M)$ vanish and since the Euler characteristic can be computed with any coefficients (recall Remark 5.2.5), the first statement follows immediately from the equality $\chi^\omega(M) = \chi(M) = 0$. Arguing as in Lemma 8.4.3, one deduces that $\beta_i^\omega(\partial W) = \beta_i^\omega(M) + \beta_i^\omega(M')$. As the nullity is invariant under 1-solvable cobordism thanks to Proposition 8.5.7, the second statement now follows from the first. We now turn to the third statement. As we showed in Lemma 8.5.5 that $H_3(W, M; \mathbb{C}^\omega) = 0$, the long exact sequence of the triple $(W, \partial W, M)$ gives

$$\begin{aligned} 0 \to H_3(W, \partial W; \mathbb{C}^\omega) \to H_2(\partial W, M; \mathbb{C}^\omega) &\to H_2(W, M; \mathbb{C}^\omega) \to H_2(W, \partial W; \mathbb{C}^\omega) \\ &\to H_1(\partial W, M; \mathbb{C}^\omega) \to H_1(W, M; \mathbb{C}^\omega) \to H_1(W, \partial W; \mathbb{C}^\omega) \to 0. \end{aligned}$$

Lemma 8.5.5 implies that $H_1(W, M; \mathbb{C}^\omega)$ vanishes and consequently so does $H_1(W, \partial W; \mathbb{C}^\omega)$. Since the alternating sum of dimensions of an exact sequence vanishes, one gets

$$\beta_2^\omega(W, \partial W) = \beta_2^\omega(W, M) + \beta_3^\omega(W, \partial W) + \beta_1^\omega(\partial W, M) - \beta_2^\omega(\partial W, M),$$

and so the third statement reduces to proving the equality $\beta_1^\omega(\partial W, M) = \beta_2^\omega(\partial W, M)$. To achieve this, consider the long exact sequence of the pair $(\partial W, M)$:

$$\begin{aligned} 0 \to H_2(M; \mathbb{C}^\omega) \to H_2(\partial W; \mathbb{C}^\omega) \to H_2(\partial W, M; \mathbb{C}^\omega) \\ \to H_1(M; \mathbb{C}^\omega) \to H_1(\partial W; \mathbb{C}^\omega) \to H_1(\partial W, M; \mathbb{C}^\omega) \to 0. \end{aligned}$$

Note that $H_3(\partial W, M; \mathbb{C}^\omega) = 0$ because of the long exact sequence of $(W, \partial W, M)$ together with the fact that $H_3(W, M; \mathbb{C}^\omega) = 0$, see Lemma 8.5.5. Using the vanishing of the alternate sum of dimensions, the desired equality now follows by combining the first two statements. $\square$

Next, we prove an inequality on the twisted Betti numbers of a 1-solvable cobordism.

Lemma 8.5.9. *If two μ-colored links L and L' are 1-solvable concordant via W, then*

$$\beta_3^\omega(W, \partial W) - \beta_1^\omega(\partial W) + \beta_1^\omega(W) \leq 0$$

for all $\omega \in \mathbb{T}_!^\mu$.

Proof. By Lemma 7.5.5, one gets $\beta_3^\omega(W, \partial W) = \beta_1^\omega(W)$. Setting $M := X_L$ and $M' := X_{L'}$, arguing as in the proof of Lemma 8.5.8 and using Proposition 8.5.7, we see that $\beta_1^\omega(\partial W) = 2\beta_1^\omega(M)$. Since Lemma 8.5.5 implies that $H_1(W, M; \mathbb{C}^\omega) = 0$, the map $H_1(M; \mathbb{C}^\omega) \to H_1(W; \mathbb{C}^\omega)$ is surjective, and so $\beta_1^\omega(W) - \beta_1^\omega(M) \leq 0$. Combining these facts, $\beta_3^\omega(W, \partial W) - \beta_1^\omega(\partial W) + \beta_1^\omega(W)$ is equal to $2(\beta_1^\omega(W) - \beta_1^\omega(M)) \leq 0$, as desired. $\square$

Finally, we estimate the dimension of the space which supports the twisted intersection form.

Proposition 8.5.10. *Assume two μ-colored links L and L' are 1-solvable cobordant via W. Then, for all $\omega \in \mathbb{T}_!^\mu$, the subspace $L_\mathbb{C} \subset H_2(W; \mathbb{C}^\omega)$ of Proposition 8.5.6 satisfies*

$$\frac{1}{2} \dim_\mathbb{C} \left(\frac{H_2(W; \mathbb{C}^\omega)}{\operatorname{im}(H_2(\partial W; \mathbb{C}^\omega) \to H_2(W; \mathbb{C}))} \right) \leq \dim_\mathbb{C}(L_\mathbb{C}).$$

Proof. Invoking Lemma 8.5.6, the dimension of $L_\mathbb{C}$ is equal to half the rank of $H_2(W, M)$. Using Lemma 8.5.5, $\beta_2^\omega(W, M) = \beta_2(W, M)$, and so the proposition reduces to showing the inequality

$$d := \dim_\mathbb{C} \left(\frac{H_2(W; \mathbb{C}^\omega)}{\operatorname{im}(H_2(\partial W; \mathbb{C}^\omega) \to H_2(W; \mathbb{C}^\omega))} \right) \leq \beta_2^\omega(W, M).$$

Set $V := \operatorname{im}(H_2(\partial W; \mathbb{C}^\omega) \to H_2(W; \mathbb{C}^\omega))$. Since we proved in Lemma 8.5.5 that $H_1(W, \partial W; \mathbb{C}^\omega)$ vanishes, the long exact sequence of the pair $(W, \partial W)$ now takes the form

$$0 \to V \to H_2(W; \mathbb{C}^\omega) \to H_2(W, \partial W; \mathbb{C}^\omega) \to H_1(\partial W; \mathbb{C}^\omega) \to H_1(W; \mathbb{C}^\omega) \to 0.$$

Finally, using the fact that the alternating dimensions of an exact sequence sum up to zero, one gets

$$\begin{aligned} d &= \beta_2^\omega(W, \partial W) - \beta_1^\omega(\partial W) + \beta_1^\omega(W) \\ &= \beta_2^\omega(W, M) + \beta_3^\omega(W, \partial W) - \beta_1^\omega(\partial W) + \beta_1^\omega(W) \\ &\leq \beta_2^\omega(W, M), \end{aligned}$$

where the last two steps use respectively Lemma 8.5.8 and Lemma 8.5.9. $\square$

8.5.4 Proof of Theorem 8.1.5

We prove that 1-solvable cobordant links have the same multivariable signature, concluding the proof of Theorem 8.1.5.

Let L and L' be 1-solvable cobordant μ-colored links. Let $F, F' \subset D^4$ be colored bounding surfaces for L and L' respectively, with the additional requirement that they have only a single component per color. We denote by W_F and $W_{F'}$ their respective exteriors. Setting as usual $M_F := \overline{\nu F} \cap W_F$, we see that the boundary ∂W_F decomposes into $X_L \cup_{L \times S^1} M_F$. An analogous decomposition holds for $\partial W_{F'}$. Given a 1-solvable cobordism W, we consider the 4-manifold

$$V := W_F \cup_{X_L} W \cup_{X_{L'}} W_{F'},$$

which has boundary $M_F \cup_\Sigma M_{F'}$, where Σ is a disjoint union of tori. By the diagram (8.3), the coefficient systems on the link exteriors X_L and $X_{L'}$ extend over W and thus over V. We shall now compute $\operatorname{dsign}_\omega(V) = \operatorname{sign}_\omega(V) - \operatorname{sign}(V)$ using the Novikov-Wall additivity of signatures.

Claim. $\operatorname{dsign}_\omega(V) = \operatorname{dsign}_\omega(W_F) - \operatorname{dsign}_\omega(W_{F'}) + \operatorname{dsign}_\omega(W)$.

To prove the claim, the first step is to establish the equality $\operatorname{dsign}_\omega(V) = \operatorname{dsign}_\omega(W_F) + \operatorname{dsign}_\omega(W \cup_{X_{L'}} W_{F'})$. For this, consider V as the union of W_F with $W \cup_{X_{L'}} W_{F'}$ along X_L. Since $H_1(L \times S^1; \mathbb{C}^\omega)$ vanishes, the twisted additivity follows immediately from Proposition 7.2.2. To deal with the untwisted signature, we need to consider the kernels

$$\begin{aligned} \ker M_F &= \ker(H_1(L \times S^1) \to H_1(M_F)), \\ \ker M_{F'} &= \ker(H_1(L \times S^1) \to H_1(M_{F'})), \\ \ker X_L &= \ker(H_1(L \times S^1) \to H_1(X_L)). \end{aligned}$$

As L and L' are 1-solvable concordant, they have the same pairwise linking numbers between their components, see Remark 8.5.3. Since we assumed that F and F' have exactly one component for each color they can be seen as bounding surfaces in the uncolored sense, and applying Lemma 7.5.1 to the two links shows that $\ker M_F = \ker M_{F'}$. As a consequence, Theorem 7.2.1 yields $\operatorname{dsign}_\omega(V) = \operatorname{dsign}_\omega(W_F) + \operatorname{dsign}_\omega(W \cup_{X_{L'}} W_{F'})$.

To prove the claim, it remains to show that $\operatorname{dsign}_\omega(W \cup_{X_{L'}} W_{F'}) = \operatorname{dsign}_\omega(W) - \operatorname{dsign}_\omega(W_{F'})$. Consider the two kernels $\ker(H_1(L' \times S^1) \to H_1(X_L))$ and $\ker(H_1(L' \times S^1) \to H_1(X_{L'}))$ involved in Novikov-Wall additivity. We have a commutative diagram

$$\begin{tikzcd} & H_1(L' \times S^1) \arrow[dl] \arrow[dr] & \\ H_1(X_L) \arrow[rr, "\cong"] & & H_1(X_{L'}), \end{tikzcd}$$

where the isomorphism is given by the H_1-cobordism W. It follows that the two kernels agree and thus the Novikov-Wall additivity of Theorem 7.2.1 goes through. The twisted additivity follows immediately from Proposition 7.2.2. This completes the proof of the claim.

We now show that $\operatorname{dsign}_\omega(V) = \sigma_L(\omega) - \sigma_{L'}(\omega)$. The combination of Proposition 6.7.2 and Theorem 8.1.1 implies that $\operatorname{dsign}_\omega(W_F) = \sigma_L(\omega)$ and $\operatorname{dsign}_\omega(W_{F'}) = \sigma_{L'}(\omega)$, it follows from the claim that

$$\operatorname{dsign}_\omega(V) = \sigma_L(\omega) - \sigma_{L'}(\omega) + \operatorname{dsign}_\omega(W)$$

Next, we argue that the signature defect $\operatorname{dsign}_\omega(W)$ vanishes. By Proposition 8.5.2, the ordinary signature of W vanishes. Combining Propositions 8.5.6 and 8.5.10, there exists a Lagrangian $L_{\mathbb{C}} \subset H_2(W;\mathbb{C}^\omega)$ for the nonsingular intersection form on $H_2(W;\mathbb{C}^\omega)/\operatorname{im}(H_2(\partial W;\mathbb{C}^\omega) \to H_2(W;\mathbb{C}^\omega))$ and thus the twisted signature of W vanishes.

To conclude the proof, it only remains to show that $\operatorname{dsign}_\omega(V)$ vanishes. Recall that $\partial V = M_F \cup_\Sigma M_{F'}$, where Σ is a disjoint union of tori. Gluing the components of F, F' and Σ along their respective boundaries expresses ∂V as a plumbed 3-manifold. Since L and L' are 1-solvable cobordant, their linking numbers match up, see Remark 8.5.3. Thus ∂V is balanced. Proposition 7.1.1 now implies that $\operatorname{dsign}_\omega(V) = 0$, as desired.

Part II

The Burau representation and its generalizations

Chapter 9

The Burau representation of the braid group

9.1 Introduction

This introductory chapter deals with the braid group, the Burau representation and the colored Gassner representation. A *braid* roughly consists of n monotonic disjoint strands in the cylinder $D^2 \times [0,1]$. Given two braids β_1, β_2, one can form their *composition* $\beta_1\beta_2$ by stacking β_1 on top of β_2. Using this operation, isotopy classes of braids form a group B_n which is referred to as the *braid group* and which was first introduced by Emil Artin [4]. The braid group also admits various other definitions some of which will be briefly discussed in Section 9.2.

From the perspective of a knot theorist, the interest in braids comes from the following fact, due to Alexander [2]: every link can be obtained as the closure of a braid. A natural question then arises: what link invariants can be recovered from the braid group? In the case of the Alexander polynomial, the answer is well known and relies on the so-called *reduced Burau representation* of the braid group. This representation, which we shall review in Section 9.3 takes the form of a homomorphism

$$\overline{\mathscr{B}}_t \colon B_n \to GL_{n-1}(\mathbb{Z}[t^{\pm 1}]),$$

and its relation to the Alexander polynomial, which is due to Burau [25], can be described as follows. If one uses Alexander's theorem to view a link L as the closure $\widehat{\alpha}$ of an n-stranded braid α, then the Alexander polynomial of L satisfies

$$\Delta_L(t)(t^n - 1) \doteq (t-1)\det(\overline{\mathscr{B}}_t(\alpha) - I_{n-1}). \tag{9.1}$$

What about the multivariable Alexander polynomial? To explain the answer to this question, we introduce some more terminology. A braid is *pure* if the permutation it induces is trivial, and the resulting *pure braid group* P_n admits a representation $\overline{\mathscr{G}}_{t_1,\dots t_n} \colon P_n \to GL_{n-1}(\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}])$ known as the *reduced Gassner representation.* In Birman's classical book [17], it is then shown that for any pure braid α with n strands, one has

$$\Delta_{\widehat{\alpha}}(t_1, \dots, t_n)(t_1 t_2 \cdots t_n - 1) \doteq \det(\overline{\mathscr{G}}_{t_1,\dots,t_n}(\alpha) - I_{n-1}).$$

In order to interpolate between the Gassner representation and the Burau representation, Section 9.4 deals with the colored braid groups B_c and the (algebraic) *reduced colored Gassner representation*

$$\overline{\mathscr{B}}_{\psi_c} \colon B_c \to GL_{n-1}(\mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]).$$

Although this representation is sometimes hinted at (see [42, 103, 126]), it seems that a thorough study has never been carried out. Both for this reason and to prepare the reader for Chapter 11, a fair amount of time is devoted to this task. Finally, Section 9.5 will provide explicit computations of the (un)reduced colored Gassner representation.

Although most of the material presented here is well known, several results and proofs appear to be somewhat folkore. Moreover, some remarks concerning the colored Gassner representation do appear to be novel. Furthermore, note that we have tried to make this chapter accessible for a reader who has only skimmed through the first part of this thesis. In particular, we chose to start this chapter with the language of covering spaces, delaying the use of twisted homology. Exceptions nevertheless occur in some remarks which might require taking a look at Section 2.5 and Chapter 5. Indeed, our opinion is that the Burau representation should be understood first and foremost as an induced map on twisted homology.

9.2 The braid group

This section reviews the braid group. The classical reference on the subject is Birman's book [17], but we also refer the reader to Turaev and Kassel's monograph [92].

Let D^2 be the closed unit disk in $\mathbb{R}^2$. Fix a set of $n \geq 1$ punctures $p_1, p_2, \ldots, p_n$ in the interior of D^2. We shall assume that the p_i lie in $(-1, 1) = \mathrm{Int}(D^2) \cap \mathbb{R}$ and $p_1 < p_2 < \ldots < p_n$. A *braid with n strands* is an oriented n-component one-dimensional submanifold β of the cylinder $D^2 \times [0,1]$ whose boundary is $\bigsqcup_{i=1}^n (p_i \times \{0\}) \sqcup (-\bigsqcup_{i=1}^n (p_i \times \{1\}))$, and where the projection to $[0,1]$ maps each component of β homeomorphically onto $[0,1]$, see Figure 9.2. Two braids β_1 and β_2 are *isotopic* if there is a self-homeomorphism of $D^2 \times [0,1]$ which keeps $\partial(D^2 \times [0,1])$ fixed, such that $h(\beta_1) = \beta_2$ and $h|_{\beta_1} : \beta_1 \simeq \beta_2$ is orientation preserving.

Definition 28. The *braid group B_n* consists of the set of isotopy classes of braids. The identity element is given by the *trivial braid $id_n := \{p_1, p_2, \ldots, p_n\} \times [0,1]$* while the composition of $\beta_1 \beta_2$ consists in gluing β_1 on top of β_2 and shrinking the result by a factor 2 as in Figure 9.1.

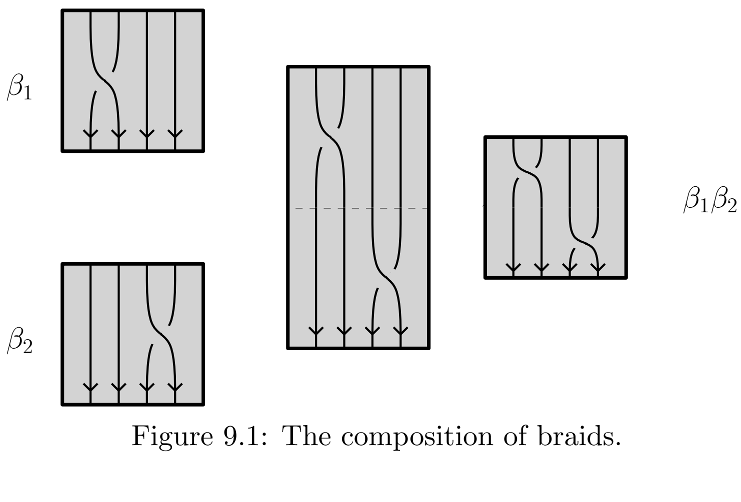

Figure 9.1: The composition of braids.

The group B_n admits a presentation with $n-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ subject to the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for each i, and $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 2$ [17]. Topologically, the generator σ_i is the braid whose i-th component passes over the $(i+1)$-th component as in Figure 9.2. The braid group B_n can also be seen as the fundamental group of the

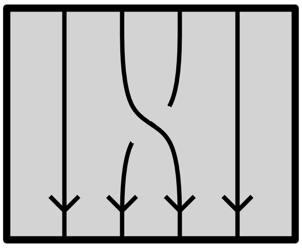

Figure 9.2: The braid σ_2: an element of the braid group on 4 strands B_4.

configuration space of n points in the plane [17]. However it is the interpretation of B_n as a mapping class group of $D_n := D^2 \setminus \{p_1, \ldots, p_n\}$ [17] which interests us the most. To understand this fact, first note that a braid β induces a deformation retract of its exterior $X_\beta := (D^2 \times [0,1]) \setminus \nu\beta$ onto $D_n \times \{0\}$. Denoting this retract by $H_\beta \colon X_\beta \times [0,1] \to X_\beta$, it turns out that the isotopy class (rel ∂D^2) of the orientation-preserving homeomorphism $h_\beta \colon D_n \times \{1\} \to D_n \times \{0\}, x \mapsto H_\beta(x,1)$ depends only on the isotopy class of the braid (see [17] and [42, Section 5.1] for details):

Theorem 9.2.1. *The braid group coincides with the set of isotopy classes of orientation-preserving homeomorphisms of D_n fixing ∂D^2 pointwise.*

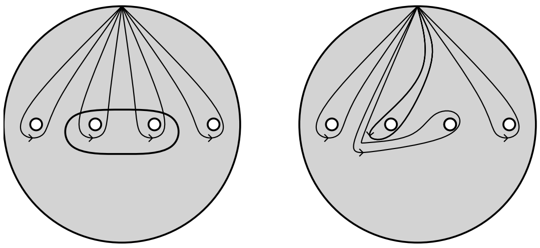

Figure 9.3: The effect of the homeomorphism h_{σ_2} on the punctured disk D_4.

Fix a base point z of D_n and denote by x_i the simple loop based at z turning once around p_i counterclockwise for $i = 1, 2, \ldots, n$ as in Figure 9.3. The group $\pi_1(D_n)$ can then be identified with the free group F_n on the x_i. Using Theorem 9.2.1, it follows that if h_β is a homeomorphism of D_n representing a braid β, then the induced automorphism of the free group F_n only depends on β. Slightly abusing notation, this induced automorphism shall also be denoted by h_β. It follows from the way we compose braids that $h_{\gamma\beta} = h_\beta h_\gamma$, and the resulting *right* action of B_n on F_n can be explicitly described (see also Figure 9.3) by

$$x_j h_{\sigma_i} = \begin{cases} x_i x_{i+1} x_i^{-1} & \text{if } j = i, \\ x_i & \text{if } j = i+1, \\ x_j & \text{otherwise.} \end{cases}$$

For this reason, $\theta_1\theta_2$ denotes the *left to right* composition of $\theta_1, \theta_2 \in \mathrm{Aut}(F_n)$ i.e. if $x \in F_n$, then $(x)\theta_1\theta_2 = ((x)\theta_1)\theta_2$. Moreover, if $f : F_n \to G$ is a group homomorphism, then $\beta_* f$ will denote the composition of f with the automorphism induced by β. With these conventions, if β and γ are two braids, then $(\beta\gamma)_* f = \beta_* \gamma_* f$. Finally, the *closure* of a braid β is the oriented link $\widehat{\beta}$ obtained from β by adding parallel strands in $S^3 \setminus (D^2 \times [0,1])$ as in Figure 9.4. As we mentioned in the introduction, every link can be obtained as the closure of a braid [2].

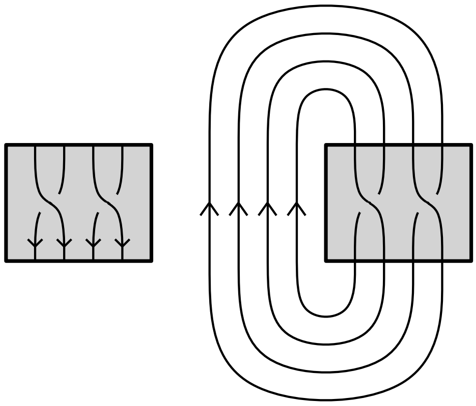

Figure 9.4: A braid α and its closure $\widehat{\alpha}$.

9.3 The Burau representation

The unreduced Burau representation is a homomorphism $\mathscr{B}_t \colon B_n \to GL_n(\mathbb{Z}[t^{\pm 1}])$. If one uses the previously mentioned generators $\sigma_1, \ldots, \sigma_{n-1}$ of the braid group B_n, then $\mathscr{B}_t$ is represented by the *unreduced Burau matrices*

$$\mathscr{B}_t(\sigma_i) = I_{(i-1)} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{(n-i-1)}. \tag{9.2}$$

This representation is not irreducible and a particularly interesting subrepresentation is given by the reduced Burau representation $\overline{\mathscr{B}}_t \colon B_n \to GL_{n-1}(\mathbb{Z}[t^{\pm 1}])$. Using the same generators of the braid group as above, $\overline{\mathscr{B}}_t$ is represented by the *reduced Burau matrices*

$$\overline{\mathscr{B}}_t(\sigma_i) = I_{(i-2)k} \oplus \begin{pmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{(n-i-2)} \tag{9.3}$$

for $1 < i < n-1$, and for σ_1 and σ_{n-1} it is represented by

$$\overline{\mathscr{B}}_t(\sigma_1) = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{(n-3)}, \qquad \overline{\mathscr{B}}_t(\sigma_{n-1}) = I_{(n-3)} \oplus \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}.$$

The aim of the following subsections is to review a more intrinsic construction of the Burau representation which involves covering spaces. More precisely, this section is organized as follows. Subsection 9.3.1 recalls the definition of the unreduced Burau representation. Subsection 9.3.2 reviews the definition of the reduced Burau representation. Subsection 9.3.3 recalls why the Burau representation is unitary. References include [17, 92, 151].

9.3.1 The unreduced Burau representation

Fix a basepoint z of the punctured disk D_n. Consider the map $\psi\colon \pi_1(D_n) \to \mathbb{Z} = \langle t \rangle$ which sends each x_i to t. Let $\pi\colon D_n^\infty \to D_n$ be the regular cover corresponding to $\ker(\psi)$, and let $P = \pi^{-1}(z)$ be the fiber over z. The homology groups of D_n^∞ are naturally modules over $\Lambda = \mathbb{Z}[t^{\pm 1}]$.

Lemma 9.3.1. *The homology Λ-module $H_1(D_n^\infty, P)$ is free of rank n.*

Proof. The punctured disk D_n is homotopy equivalent to the wedge of the n loops representing the generators of $\pi_1(D_n)$ described in Section 9.2. Choose a cellular decomposition of this latter space consisting of the 0-cell z (the basepoint of the wedge) and one 1-cell x_i for each loop. For $i = 1, 2, \dots, n$, let $\widetilde{x}_i$ be the lift of x_i starting at an (arbitrary) fixed lift $\widetilde{z}$ of z. With this cell structure, the Λ-module $C_1(D_n^\infty, P)$ is freely generated by the $\widetilde{x}_i$, see Figure 9.5. As $C_0(D_n^\infty, P)$ vanishes, $H_1(D_n^\infty, P) = C_1(D_n^\infty, P)$ and the claim follows. $\square$

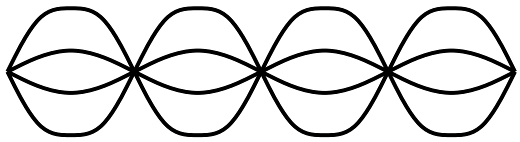

Figure 9.5: The covering space D_4^∞ has the homotopy type of a 1-dimensional complex.

Using Theorem 9.2.1, let $h_\alpha\colon D_n \to D_n$ be a homeomorphism representing a braid α. Covering space theory implies that the homeomorphism h_α lifts to a unique homeomorphism $\widetilde{h}_\alpha\colon D_n^\infty \to D_n^\infty$ fixing the fiber over z pointwise. The map induced by h_α on $H_1(D_n^\infty, P) = C_1(D_n^\infty, P)$ only depends on the braid α. Indeed, two isotopic choices of representatives for a given braid produce homotopic maps in the cover. As the braid group acts by right automorphisms on the free group $\pi_1(D_n)$, the left Λ-module $C_1(D_n^\infty, P)$ inherits a *right* action of the braid group. As in Section 9.2, the composition of automorphisms on $C_1(D_n^\infty, P)$ will be read from *left to right.*

Definition 29. The *unreduced Burau representation* $\mathscr{B}_t\colon B_n \to \mathrm{Aut}_\Lambda(H_1(D_n^\infty, P))$ is the representation obtained by sending α to the Λ-linear automorphism $H_1(D_n^\infty, P) \to H_1(D_n^\infty, P)$ induced by $\widetilde{h}_\alpha$.

Note that the unreduced Burau representation is n-dimensional, see Lemma 9.3.1. We now argue that Definition 29 does indeed produce a representation. If $\beta, \gamma \in B_n$ are two braids, then the composition $H_1(D_n^\infty, P) \xrightarrow{\mathscr{B}_t(\beta)} H_1(D_n^\infty, P) \xrightarrow{\mathscr{B}_t(\gamma)} H_1(D_n^\infty, P)$ of the maps induced by β and γ coincides with the map induced by $\beta\gamma$. As the braid group acts on homology from the right, and composition is read from left to right, the claim follows.

Despite the description we gave in Definition 29, we believe that the Burau representation ought to be understood using twisted homology. The next remark, which elaborates on this opinion, assumes some familiarity with the material of Chapter 5. It will also play a crucial role in Chapter 11.

Remark 9.3.2. Using Lemma 5.2.1, the homology Λ-module $H_1(D_n^\infty, P)$ coincides with the twisted homology Λ-module $H_1(D_n, z; \Lambda)$, where Λ inherits its right $\mathbb{Z}[\pi_1(D_n)]$-module from the aforementioned map $\psi\colon \pi_1(D_n) \to \mathbb{Z}$. Using this language, the Burau representation of a braid β is nothing but the map induced by h_β on $H_1(D_n, z; \Lambda)$, see Lemma 5.7.1. Furthermore, the equality $\mathscr{B}_t(\alpha\beta) = \mathscr{B}_t(\alpha)\mathscr{B}_t(\beta)$ follows immediately from Corollary 5.7.2, see Section 11.3 for further details on this point of view.

Next, we briefly review some Fox calculus. Recall from Subsection 2.2.2 that if F_n denotes the free group on the n generators $x_1, \ldots, x_n$, there exists a unique Fox derivative $\frac{\partial}{\partial x_j} \colon \mathbb{Z}[F_n] \to \mathbb{Z}[F_n]$ satisfying $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$. Since $\frac{\partial}{\partial x_j}$ is a derivative, it satisfies $\frac{\partial(uv)}{\partial x_j} = \frac{\partial u}{\partial x_j} + u\frac{\partial v}{\partial x_j}$ and $\frac{\partial x_i^{-1}}{\partial x_j} = -\delta_{ij}x_i^{-1}$. The next remark explains the well known topological relevance of the Fox derivative. It will also allow us to relate the unreduced Burau representation to the unreduced Burau matrices given in (9.2).

Remark 9.3.3. Assume a CW complex X admits a cell structure with one 0-cell v, n oriented 1-cells labeled $x_1, x_2, \ldots, x_n$ having all their endpoints identified with v to form n loops, and m oriented 2-cells $c_1, c_2, \ldots, c_m$ with each ∂c_i glued to the 1-cells according to a word r_i. The fundamental group of X then admits a presentation with generators $x_1, x_2, \ldots, x_n$ and relators $r_1, r_2, \ldots, r_m$. Let $\widetilde{v}, \widetilde{x}_i$ and $\widetilde{c}_i$ be corresponding lifts to the universal cover $p : \widetilde{X} \to X$.

Let $pr : \mathbb{Z}[F_n] \to \mathbb{Z}[\pi_1(X)]$ denote the ring homomorphism induced by the quotient map. The chain $\mathbb{Z}[\pi_1(X)]$-module $C_1(\widetilde{X}, p^{-1}(v))$ is freely generated by the $\widetilde{x}_i$, and if w is a word in the x_i, then its lift $\widetilde{w}$ (viewed as a 1-chain in the universal cover) can be written as

$$\widetilde{w} = \sum_{j=1}^{n} pr\left(\frac{\partial w}{\partial x_j}\right)\widetilde{x}_j. \tag{9.4}$$

When the context is clear, we shall often omit the ring homomorphism pr from the notation. Although (9.4) is all we need for the remainder of this section, we conclude this remark with a well know fact. Namely if follows from (9.4) that the boundary map ∂_2 of the chain complex $C_*(\widetilde{X}, p^{-1}(v))$ sends $\widetilde{c}_i$ to the lift of r_i beginning at $\widetilde{v}$. Therefore (9.4) specializes to

$$\partial_2(\widetilde{c}_i) = \sum_{j=1}^{n} \frac{\partial r_i}{\partial x_j}\widetilde{x}_j.$$

Consequently, ∂_2 is represented by the $(m \times n)$ Fox matrix whose (i,j)-coefficient is $\frac{\partial r_i}{\partial x_j}$. Note that here we are following a common convention, according to which the elements in the chain complex $C_*(\widetilde{X}, p^{-1}(v))$ of free left $\mathbb{Z}[\pi_1(X)]$-modules are *row* vectors and that the matrices of the differentials act by *right* multiplication, see [102, 150].

Proposition 9.3.4. *With respect to the basis $\widetilde{x}_1, \ldots, \widetilde{x}_n$ described in Lemma 9.3.1, the unreduced Burau representation is given by the unreduced Burau matrices of (9.2).*

Proof. Fix a lift of z to D_n^∞. Given a homeomorphism h_β representing a braid β, let $\widetilde{h}_\beta$ be the Λ-linear homomorphism induced by the lift of h_β on the chain Λ-module $C_1(D_n^\infty, P)$. As $H_1(D_n^\infty, P) = C_1(D_n^\infty, P)$, the Burau representation is given by the homomorphism $\mathscr{B}_t(\beta) = \widetilde{h}_\beta$. Clearly $\widetilde{x}_i\widetilde{h}_\beta$ is the lift of a loop representing x_ih_β to D_n^∞. Remark 9.3.3 and in particular (9.4) then show that on the chain group level

$$\widetilde{x}_i\widetilde{h}_\beta = \sum_{j=1}^{n} \psi\left(\frac{\partial(x_ih_\beta)}{\partial x_j}\right)\widetilde{x}_j.$$

As we view elements of the left Λ-module $C_1(D_n^\infty, P)$ as row vectors, $\widetilde{h}_\beta$ is represented by the $(n \times n)$ matrix whose (i,j) component is $\psi\left(\frac{\partial(x_i h_\beta)}{\partial x_j}\right)$. Since we wish to recover the Burau matrices, we specialize this computation to the generators of B_n. First, performing Fox calculus yields

$$\frac{\partial(x_i h_{\sigma_i})}{\partial x_i} = \frac{\partial(x_i x_{i+1} x_i^{-1})}{\partial x_i} = 1 - x_i x_{i+1} x_i^{-1} \quad \text{and} \quad \frac{\partial(x_i h_{\sigma_i})}{\partial x_{i+1}} = \frac{\partial(x_i x_{i+1} x_i^{-1})}{\partial x_{i+1}} = x_i.$$

The result now follows by applying ψ. $\square$

Let us make some remarks on the various conventions involved in the definition of the unreduced Burau representation and in Proposition 9.3.4.

Remark 9.3.5. Since $h_{\alpha\beta} = h_\beta h_\alpha$ for any two braids $\alpha, \beta \in B_n$, we made the choice to read the composition from left to right. This led us both to the property $\mathscr{B}_t(\alpha\beta) = \mathscr{B}_t(\alpha)\mathscr{B}_t(\beta)$ and to view elements of $C_1(D_n^\infty, P)$ as row vectors. Fox calculus thus produced the Burau matrices, and not their transposes. We will keep to these choices in the current chapter and in Chapter 11, see also [17].

On the other hand if we had chosen to read $h_{\alpha\beta} = h_\beta h_\alpha$ from right to left, we would have gotten an *anti-representation* (i.e. a map satisfying $f(\alpha\beta) = f(\beta)f(\alpha)$). Similarly, we would have had to view elements of $C_1(D_n^\infty, P)$ as column vectors and thus Fox calculus would have produced the transpose of the Burau matrices. We will make this choice in Chapter 12, see also [42].

9.3.2 The reduced Burau representation

We now follow a similar procedure in order to define the reduced Burau representation. Namely, given a braid β, we shall consider the map induced by h_β on the absolute homology group $H_1(D_n^\infty)$ instead of the relative group $H_1(D_n^\infty, P)$. To obtains matrices, the first step is to check that $H_1(D_n^\infty)$ is a free Λ-module.

Lemma 9.3.6. *The homology Λ-module $H_1(D_n^\infty)$ is free of rank $n-1$.*

Proof. Fix a lift $\widetilde{z}$ of z to D_n^∞. Arguing as in Lemma 9.3.1, D_n is homotopy equivalent to the wedge of n circles. It follows that the boundary map $\partial\colon C_1(D_n^\infty) \to C_0(D_n^\infty)$ is given by $\widetilde{x}_i \mapsto (t-1)\widetilde{z}$. As D_n^∞ has the homotopy type of a one-dimensional complex, $H_1(D_n^\infty)$ is isomorphic to $\ker(\partial)$ and has a Λ-basis given by elements $v_i := \widetilde{x}_{i+1} - \widetilde{x}_i$ for $i = 1, \ldots, n-1$. $\square$

Let $h_\alpha\colon D_n \to D_n$ be a homeomorphism representing a braid α and let $\widetilde{h}_\alpha\colon D_n^\infty \to D_n^\infty$ be the lift fixing the fiber P over z pointwise.

Definition 30. The *reduced Burau representation* $\overline{\mathscr{B}}_t\colon B_n \to \mathrm{Aut}_\Lambda(H_1(D_n^\infty))$ is the representation obtained by sending α to the Λ-linear automorphism $H_1(D_n^\infty) \to H_1(D_n^\infty)$ induced by $\widetilde{h}_\alpha$.

Note that the reduced Burau representation is $(n-1)$-dimensional, see Lemma 9.3.1. The fact that $\overline{\mathscr{B}}_t$ is a representation follows as in the unreduced case. Furthermore, just as in Remark 9.3.2, the reduced Burau representation can be viewed as the homomorphism induced by h_β on the twisted homology Λ-module $H_1(D_n; \Lambda)$. The next proposition relates $\overline{\mathscr{B}}_t$ to the reduced Burau matrices.

Proposition 9.3.7. *With respect to the basis $v_1, \ldots, v_{n-1}$ described in Lemma 9.3.6, the reduced Burau representation is given by the reduced Burau matrices of (9.3).*

Proof. The naturality of the long exact sequence of the pair (D_n^∞, P), implies that the (injective) inclusion induced map $i : H_1(D_n^\infty) \to H_1(D_n^\infty, P)$ satisfies $i_* \mathscr{B}_t(\alpha) = \overline{\mathscr{B}}_t(\alpha) i_*$. Using the basis $\widetilde{x}_1, \ldots, \widetilde{x}_n$ for $H_1(D_n^\infty, P)$ (recall Lemma 9.3.1) and the basis $v_1, \ldots, v_{n-1}$ for $H_1(D_n^\infty)$ (recall Lemma 9.3.6), a simple computation involving Proposition 9.3.4 implies the result, see also Section 9.5. □

Following Birman [17], we shall now define another version of the reduced Burau representation. Instead of working with the free generators $x_1, x_2 \ldots, x_n$ of $\pi_1(D_n)$, consider the elements $g_1, g_2, \ldots, g_n$, where $g_i = x_1 x_2 \cdots x_i$. The action of the braid group B_n on this new set of free generators for $\pi_1(D_n)$ is given by

$$g_j \sigma_i = \begin{cases} g_j & \text{if } j \neq i, \\ g_{i+1} g_i^{-1} g_{i-1} & \text{if } j = i \neq 1, \\ g_2 g_1^{-1} & \text{if } j = i = 1. \end{cases} \tag{9.5}$$

Let $\widetilde{g}_i$ be the lift of g_i starting at a fixed lift of z. Using the same argument as in Lemma 9.3.1, one obtains the splitting $H_1(D_n^\infty, P) = \bigoplus_{i=1}^{n-1} \Lambda \widetilde{g}_i \oplus \Lambda \widetilde{g}_n$. As g_n is always fixed by the action of the braid group, its lift $\widetilde{g}_n$ is fixed by the lift $\widetilde{h}_\beta$ of a homeomorphism h_β representing a braid β.

Definition 31. The *algebraic reduced Burau representation* $\overline{\mathscr{B}}_t \colon B_n \to GL_{n-1}(\Lambda)$ sends a braid β to the restriction $\overline{\mathscr{B}}_t(\beta)$ of the Burau representation to the free Λ-module of rank $(n-1)$ generated by $\widetilde{g}_1, \ldots, \widetilde{g}_{n-1}$.

If $\widetilde{\mathscr{B}}_t(\beta)$ denotes the Burau matrix of a braid β with respect to the basis described above, then

$$\widetilde{\mathscr{B}}_t(\beta) = \begin{pmatrix} \overline{\mathscr{B}}_t(\beta) & V \\ 0 & 1 \end{pmatrix}$$

for some $((n-1) \times 1)$-matrix V. In particular, one can check that the algebraic reduced Burau representation is also represented by the reduced Burau matrices.

Remark 9.3.8. We therefore have two homomorphisms $B_n \to GL_{n-1}(\Lambda)$ represented by the reduced Burau matrices of (9.3). These two representations are therefore isomorphic. Note however that they are not defined on the same module: lifts of the g_i do not belong to $H_1(D_n^\infty)$.

We conclude this subsection by mentioning Burau's result which relates the reduced Burau representation to the Alexander polynomial [25].

Theorem 9.3.9. *For any braid β with n strands, we have*

$$\Delta_{\widehat{\beta}}(t)(t^n - 1) \doteq (t-1) \det(\overline{\mathscr{B}}_t(\beta) - I_{n-1}).$$

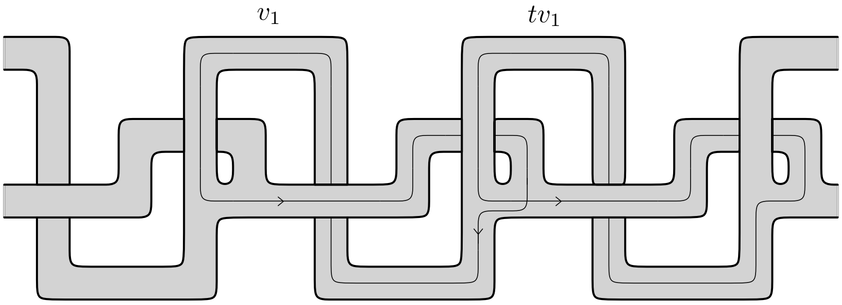

Figure 9.6: The intersection of v_1 and tv_1 in D_2^∞.

9.3.3 The reduced Burau representation is unitary

Let H be a Λ-module and let $\lambda\colon H \times H \to \Lambda$ be a skew-Hermitian form. A Λ-linear endomorphism $f\colon H \to H$ is *unitary* if $\lambda(f(x), f(y)) = \lambda(x, y)$ for all x, y in H. Using a purely algebraic computation, Squier was the first to observe that the reduced Burau representation is unitary [144]. This fact was then reinterpreted in more geometrical terms using intersection pairings [103, 116, 125, 151]. It is this latter viewpoint which we shall now present, first following the exposition of Cimasoni-Turaev [42] before making use of the machinery developed in Chapter 5.

Let $\langle\ ,\ \rangle\colon H_1(D_n^\infty) \times H_1(D_n^\infty) \to \mathbb{Z}$ be the skew-symmetric algebraic intersection pairing obtained by lifting the orientation of D_n to D_n^∞. Consider the skew-Hermitian form $\lambda\colon H_1(D_n^\infty) \times H_1(D_n^\infty) \to \Lambda$ defined by

$$\lambda(x, y) = \sum_{k \in \mathbb{Z}} \langle t^k x, y \rangle t^{-k}.$$

The reader familiar with Section 5.6 might have observed that λ is the twisted intersection pairing on $H_1(D_n; \Lambda)$. Remark 9.3.12 will expand on this point of view, but first, let us start by building a concrete understanding of the pairing λ.

Example 9.3.10. Consider the case $n = 2$. The Λ-module $H_1(D_2^\infty)$ is freely generated by $v_1 = \widetilde{x}_1 - \widetilde{x}_2$ and so the pairing reduces to $\langle tv_1, v_1 \rangle t^{-1} + \langle t^{-1} v_1, v_1 \rangle t = -(t - t^{-1})\langle tv_1, v_1 \rangle$. Looking at Figure 9.6, one sees that $\langle tv_1, v_1 \rangle = -1$ and so the form is represented by the 1×1-matrix $(t - t^{-1})$.

In the general case, we will use a more schematic interpretation of the cover D_n^∞ in which the $\mathbb{Z}$ sheets of the cover are more apparent, see Figures 9.7 and 9.8. If $\widetilde{x}_1, \ldots, \widetilde{x}_n$ are the lifts of the loops $x_1, \ldots, x_n$ starting at some fixed lift $\widetilde{z}$ of the basepoint z, we saw in Lemma 9.3.6 that a basis of the free Λ-module $H_1(D_n^\infty)$ is given by $v_i = \widetilde{x}_i - \widetilde{x}_{i+1}$ for $i = 1, \ldots, n-1$.

Example 9.3.11. With respect to the basis $v_1, \ldots, v_{n-1}$ of $H_1(D_n^\infty)$, a matrix for the skew-Hermitian intersection form λ is

$$\begin{pmatrix} t - t^{-1} & 1 - t & 0 & \ldots & 0 \\ t^{-1} - 1 & t - t^{-1} & & \ddots & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & \ddots & & & 1 - t \\ 0 & \ldots & 0 & t^{-1} - 1 & t - t^{-1} \end{pmatrix}. \tag{9.6}$$

For instance, let us compute in full detail $\lambda(v_i, v_{i+1}) = \langle v_i, v_{i+1}\rangle + \langle t^{-1}v_i, v_{i+1}\rangle t$. As one can see from Figures 9.7 and 9.8, $\langle v_i, v_{i+1}\rangle = 1$ while $\langle t^{-1}v_i, v_{i+1}\rangle = \langle v_i, tv_{i+1}\rangle = -1$, and consequently $\lambda(v_i, v_{i+1}) = 1-t$, as desired. The other intersection numbers can be computed similarly.

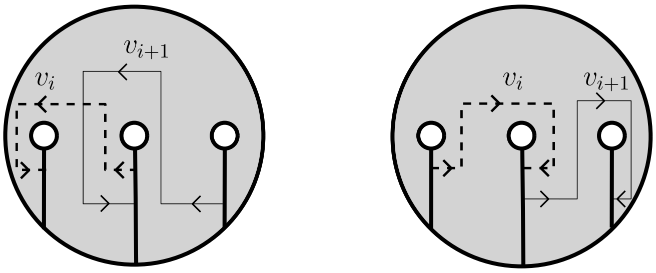

Figure 9.7: Computing $\langle v_i, v_{i+1}\rangle$.

Since the determinant of the matrix in (9.6) is non-zero, the form λ is non-degenerate. There are also more coordinate-free ways of understanding this fact: for instance, Cimasoni and Turaev [42, Lemma 3.2] rely on a theorem of Blanchfield. The next remark, which assumes familiarity with Subsection 2.5 and Chapter 5, gives another proof that λ is non-degenerate. The idea is to rely on the more algebraic definition of the pairing λ which we reviewed in Section 5.6.

Remark 9.3.12. As we observed in Remark 9.3.2, the homology Λ-module $H_1(D_n^\infty, P)$ is nothing but the twisted homology Λ-module $H_1(D_n, z; \Lambda)$. Using this viewpoint, as we saw in Section 5.6, the adjoint of the intersection pairing λ is given by the composition

$$\Phi \colon H_1(D_n;\Lambda) \xrightarrow{i} H_1(D_n, \partial D_n; \Lambda) \xrightarrow{PD} H^1(D_n;\Lambda) \xrightarrow{\text{ev}} \overline{\operatorname{Hom}_\Lambda(H_1(D_n;\Lambda),\Lambda)},$$

where i is the inclusion induced homomorphism. In order to show that λ is non-degenerate, we must prove that Φ is injective. To achieve this, we first argue that i is injective. To start with, observe that since the boundary of D_n^∞ consists of a disjoint union of spaces homeomorphic to the real line, $H_1(\partial D_n;\Lambda)$ vanishes. The injectivity of i now follows from the long exact sequence of the pair $(D_n, \partial D_n)$.

To deal with the evaluation map, we first claim that $H^1(D_n;\Lambda)$ is isomorphic to the Λ-module $\overline{\operatorname{Hom}_\Lambda(H_1(D_n;\Lambda),\Lambda)} \oplus \mathbb{Z}$. Since D_n has the homotopy type of a 1-dimensional complex, $H_p(D_n;\Lambda) = 0$ for $p \geq 2$. Next, observe that for $q > 0$, one has

$$\operatorname{Ext}^q_\Lambda(H_0(D_n;\Lambda),\Lambda) \cong \operatorname{Ext}^q_{\mathbb{Z}[\mathbb{Z}]}(\mathbb{Z};\Lambda) \cong H^q(\mathbb{Z};\Lambda) \cong H^q(B\mathbb{Z};\Lambda) \cong H^q(S^1;\Lambda) \cong H_{1-q}(S^1;\Lambda)$$

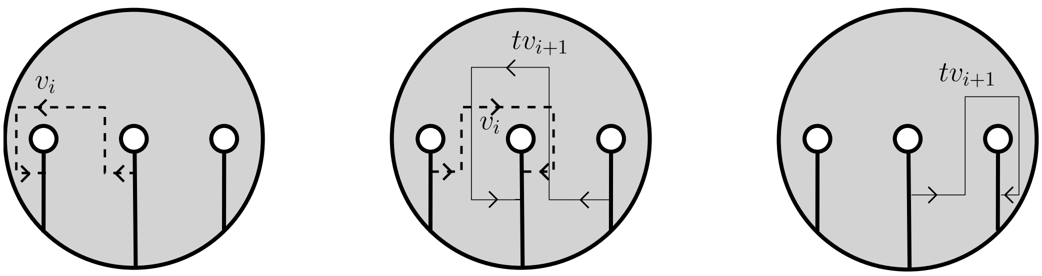

Figure 9.8: Computing $\langle v_i, tv_{i+1}\rangle$.

which equals $\mathbb{Z}$ for $q = 1$ and vanishes for $q \geq 2$. Consequently the spectral sequence of Theorem 2.5.7 collapses at the second page and one obtains the short exact sequence

$$0 \to \operatorname{Ext}^1_\Lambda(H_0(D_n;\Lambda),\Lambda) \to H^1(D_n;\Lambda) \to \overline{\operatorname{Hom}_\Lambda(H_1(D_n;\Lambda),\Lambda)} \to 0.$$

By Lemma 9.3.6, $H_1(D_n;\Lambda)$ is free. Thus $\overline{\operatorname{Hom}_\Lambda(H_1(D_n;\Lambda),\Lambda)}$ is also free and so this exact sequence splits. The claim follows.

Since $H_1(D_n;\Lambda)$ is free and the composition $H_1(D_n;\Lambda) \xrightarrow{\Psi} \overline{\operatorname{Hom}_\Lambda(H_1(D_n;\Lambda),\Lambda)} \oplus \mathbb{Z}$ is injective, the fact that λ is injective will follow from the following algebraic claim. If R is a domain, F is a free R-module, M is an arbitrary R-module, T is a torsion R-module and $\Psi\colon F \to M \oplus T$ is injective, then the composition $\Phi\colon F \xrightarrow{\Psi} M \oplus T \to M$ is also injective. To prove this, assume $x \in F$ satisfies $\Phi(x) = 0$; the goal is to show that $x = 0$. By definition of Ψ, we have $\Psi(x) = (\Phi(x), t) = (0, t)$ for some $t \in T$. Since T is torsion, there exists $a \neq 0$ such that $at = 0$ and so $0 = a(0,t) = a\Psi(x) = \Psi(ax)$. Since Ψ is injective, $ax = 0$ which implies that $x = 0$, since R is a domain and F is free.

Last but not least, we note that the Burau representation does indeed preserve λ, i.e. that it is unitary, as first observed by Squier [144].

Proposition 9.3.13. *The reduced Burau representation preserves λ.*

Proof. Fix an orientation preserving homeomorphism h_α representing a braid α. Since the lift $\widetilde{h}_\alpha$ is also an orientation preserving homeomorphism, it preserves the algebraic intersection pairing $\langle\ ,\ \rangle$ and thus it also preserves λ. □

9.4 A study of the colored Gassner representation

In this section, we review the definition of colored braids and study the colored Gassner representation. Although some results may be folkore, to the best of our knowledge, the systematic study of this representation using twisted homology is new. Contrarily to Section 9.3, we increasingly assume familiarity with Chapter 5 and gradually introduce the use of twisted homology. This provides a transition between Section 9.3 and the more technically demanding Chapter 11. Finally, note that to make the exposition more fluid, several explicit computations are delayed to Section 9.5.

An n-stranded braid $\beta \subset D^2 \times [0,1]$ is *μ-colored* if each of its components is assigned an element in $\{1, 2, \dots, \mu\}$ via a surjective map. A μ-colored braid induces a coloring on the punctures of $D^2 \times \{0,1\}$. For emphasis, we shall denote the resulting punctured disks by D_c and $D_{c'}$, and call a μ-colored braid a (c,c')-braid, where $c = (c_1, \dots, c_n)$ and $c' = (c'_1, \dots, c'_n)$ are the sequences of $1, 2, \dots, \mu$ induced by the coloring of the braid. Two colored braids are *isotopic* if the underlying isotopy is color preserving, and we shall denote by id_c the isotopy class of the trivial (c,c)-braid. The composition of a (c,c')-colored braid β_1 with a (c',c'')-colored braid β_2 is the (c,c'')-braid $\beta_1\beta_2$, see Figure 9.9. In particular note that isotopy classes of (c,c)-braids form a subgroup of the braid group:

Definition 32. Given a sequence c of integers, the *c-colored braid group* is the group B_c of isotopy classes of (c,c)-braids.

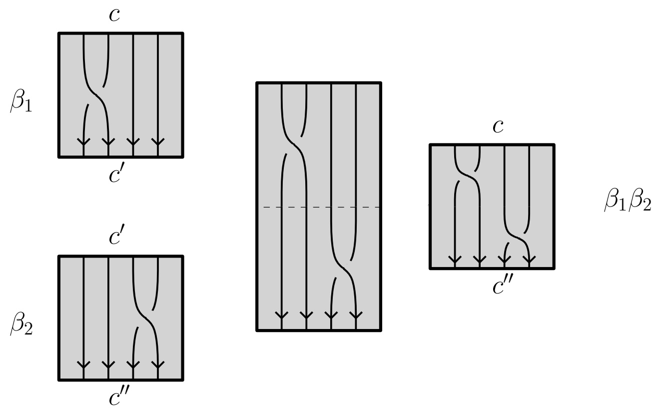

Figure 9.9: The composition of colored braids.

In other words, the colored braid groups are subgroups of the braid group B_n which interpolate between $B_n = B_{(1,1,\dots,1)}$ and the pure braid group $P_n := B_{(1,2,\dots,n)}$. The *closure* of a μ-colored braid $\beta \in B_c$ is the μ-colored link $\widehat{\beta}$ obtained from β by adding colored parallel strands in $S^3 \setminus (D^2 \times [0,1])$.

We now move towards the definition of the unreduced colored Gassner representation. Fix a basepoint z of the punctured disk D_c. Consider the map $\psi_c \colon \pi_1(D_c) \to \mathbb{Z}^\mu = \langle t_1, \dots, t_\mu \rangle$ which sends each x_i to t_{c_i}. Let $\widehat{D}_c \to D_c$ be the regular covering space corresponding to $\ker(\psi_c)$ and let P be the fiber over z. The homology groups of $\widehat{D}_c$ are naturally modules over $\Lambda_\mu = \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$. As in Section 9.3, the first step in defining the unreduced colored Gassner representation is to ensure that $H_1(\widehat{D}_c, P)$ is a free Λ_μ-module.

Lemma 9.4.1. *The homology Λ_μ-module $H_1(\widehat{D}_c, P)$ is free of rank n.*

Proof. Arguing word for word as in Lemma 9.3.1, lifts $\widetilde{x}_1, \dots, \widetilde{x}_n$ of the loops $x_1, \dots, x_n$ provide a basis for $H_1(\widehat{D}_c, P)$. □

Let $h_\alpha \colon D_c \to D_c$ be a homeomorphism representing a colored braid α. One can check that h_α lifts to a homeomorphism $\widetilde{h}_\alpha \colon \widehat{D}_c \to \widehat{D}_c$ fixing P pointwise. As the braid group acts by right automorphisms on the free group $\pi_1(D_c)$, the left Λ_μ-module $C_1(\widehat{D}_c, P)$ inherits a *right* action of the colored braid group B_c. As in Section 9.2, the composition of automorphisms on $C_1(\widehat{D}_c, P)$ will be read from *left to right.*

Definition 33. The *unreduced colored Gassner representation* is the representation $\mathscr{B}_{\psi_c} \colon B_c \to \mathrm{Aut}_{\Lambda_\mu}(H_1(\widehat{D}_c, P))$ obtained by sending α to the Λ_μ-automorphism $H_1(\widehat{D}_c, P) \to H_1(\widehat{D}_c, P)$ induced by $\widetilde{h}_\alpha$.

Note that the unreduced colored Gassner representation is n-dimensional thanks to Lemma 9.4.1. Observe that if $\mu = 1$, Definition 33 recovers the unreduced Burau representation of the braid group B_n, see Definition 29. On the other hand, it can be shown that if $\mu = n$, we get the unreduced Gassner representation of the pure braid group described in [17]. Finally, the emphasis on the homomorphism ψ_c in Definition 33 should become clear in Remark 9.4.2 below.

In order to deal with (c,c')-braids, it is more convenient to work with twisted homology. Consequently, from now on, we assume familiarity with Chapter 5. Denote by $\tilde{p}\colon \widetilde{D}_c \to D_c$ the universal cover of D_c, so that $C_*(\widetilde{D}_c, \tilde{p}^{-1}(z))$ forms a chain complex of free left $\mathbb{Z}[\pi_1(D_c)]$-modules. Since the homomorphism ψ_c endows Λ_μ with a right $\mathbb{Z}[\pi_1(D_c)]$-module structure, one may consider the chain complex $\Lambda_\mu \otimes_{\mathbb{Z}[\pi_1(D_c)]} C_*(\widetilde{D}_c, \tilde{p}^{-1}(z))$ and its homology left Λ_μ-modules $H_*(D_c, z; \Lambda_\mu)$. Recall from Lemma 5.2.1 that $H_1(D_c, z; \Lambda_\mu)$ is isomorphic to $H_1(\widehat{D}_c, P)$. Using this language and generalizing Remark 9.3.2, the unreduced colored Gassner representation of $\alpha \in B_c$ is the automorphism induced by h_α on twisted homology, see also Proposition 9.4.3 below.

Remark 9.4.2. The choice of the notation $\mathscr{B}_{\psi_c}$ for the unreduced colored Gassner representation reflects that Λ_μ inherits its right $\mathbb{Z}[\pi_1(D_c)]$-module structure from ψ_c. More generally, an arbitrary homomorphism $\varphi\colon \pi_1(D_c) \to \mathbb{Z}^\mu$ induces such a $\mathbb{Z}[\pi_1(D_c)]$-module structure on Λ_μ and, given $\alpha \in B_c$, we shall denote by $\mathscr{B}_\varphi(\alpha)$ the resulting Λ_μ-linear homomorphism. In this setting, we also write $H_1^\varphi(D_c, z; \Lambda_\mu)$ for emphasis. For instance, juggling with these notations, a (c,c')-colored braid α induces a Λ_μ-linear homomorphism

$$H_1^{\psi_c}(D_c, z; \Lambda_\mu) = H_1^{\alpha_* \psi_{c'}}(D_c, z; \Lambda_\mu) \to H_1^{\psi_{c'}}(D_{c'}, z; \Lambda_\mu)$$

We also refer to Chapter 11, and in particular to Section 11.3 and (11.3), for similar remarks in a more general setting.

The next proposition immediately follows from Remark 9.4.2 and Corollary 5.7.2 but we give some details nonetheless.

Proposition 9.4.3. *Each (c,c')-braid α induces a well-defined Λ_μ-homomorphism $\mathscr{B}_{\psi_{c'}}(\alpha)$ which coincides with the unreduced colored Gassner representation on (c,c)-braids. Furthermore, given a (c,c')-braid β and a (c',c'')-braid γ, we have*

$$\mathscr{B}_{\psi_{c''}}(\beta\gamma) = \mathscr{B}_{\psi_{c'}}(\beta)\mathscr{B}_{\psi_{c''}}(\gamma).$$

In particular, restricting to (c,c)-braids, $\mathscr{B}_{\psi_c}$ is a representation.

Proof. Let α be a (c,c')-colored braid. Using Remark 9.4.2, $h_\alpha\colon D_c \to D_{c'}$ induces a well defined map $H_1^{\alpha_*\psi_{c'}}(D_c, z; \Lambda_\mu) \to H_1^{\psi_{c'}}(D_{c'}, z; \Lambda_\mu)$; this is our definition of $\mathscr{B}_{\psi_{c'}}$. The second claim now follows by adapting the notations of Corollary 5.7.2 to those introduced in Remark 9.4.2. The result is given by

$$\mathscr{B}_{\psi_{c''}}(\beta\gamma) = \mathscr{B}_{\gamma_*\psi_{c''}}(\beta)\mathscr{B}_{\psi_{c''}}(\gamma) = \mathscr{B}_{\psi_{c'}}(\beta)\mathscr{B}_{\psi_{c''}}(\gamma)$$

for any (c,c')-colored braid β and (c',c'')-colored braid γ. The lemma follows. $\square$

Adapting the proof of Proposition 9.3.4, the unreduced colored Gassner representation can be computed using Fox calculus. Namely, with respect to the basis $\widetilde{x}_1, \ldots, \widetilde{x}_n$ described in Lemma 9.4.1, the unreduced colored Gassner representation of σ_i (viewed as a (c,c')-colored braid) is given by

$$\mathscr{B}_{\psi_{c'}}(\sigma_i) = I_{(i-1)} \oplus \begin{pmatrix} 1 - t_{c'_{i+1}} & t_{c'_i} \\ 1 & 0 \end{pmatrix} \oplus I_{(n-i-1)}. \tag{9.7}$$

Observe that if $\mu = 1$, then one recovers Proposition 9.3.4. The following example illustrates both Proposition 9.4.3 and how colored braids can be useful to make computations involving pure braids.

Example 9.4.4. Write $c = (1, 2)$, $c' = (2, 1)$ and decompose the pure braid $\sigma_1^2 \in P_2 = B_c$ as $\sigma_1 \sigma_1'$, where σ_1 is viewed as a (c, c')-braid and σ_1' is the braid σ_1 viewed as a (c', c)-braid. Applying Proposition 9.4.3 and (9.7) yields

$$\mathscr{B}_{\psi_c}(\sigma_1^2) = \mathscr{B}_{\psi_{c'}}(\sigma_1)\mathscr{B}_{\psi_c}(\sigma_1') = \begin{pmatrix} 1-t_1 & t_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1-t_2 & t_1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1-t_1+t_1t_2 & t_1(1-t_1) \\ 1-t_2 & t_1 \end{pmatrix}.$$

On the other hand, the action of σ_1^2 on the free group F_2 is given by

$$\begin{aligned} x_1 h_{\sigma_1^2} &= (x_1 x_2 x_1^{-1}) h_{\sigma_1} = (x_1 x_2 x_1^{-1}) x_1 (x_1 x_2^{-1} x_1^{-1}) = x_1 x_2 x_1 x_2^{-1} x_1^{-1}, \\ x_2 h_{\sigma_1^2} &= x_1 h_{\sigma_1} = x_1 x_2 x_1^{-1}, \end{aligned}$$

and Fox calculus yields

$$\begin{aligned} \frac{\partial (x_1 h_{\sigma_1^2})}{\partial x_1} &= \frac{\partial (x_1 x_2 x_1 x_2^{-1} x_1^{-1})}{\partial x_1} = 1 + x_1 x_2 - x_1 x_2 x_1 x_2^{-1} x_1^{-1}, \\ \frac{\partial (x_1 h_{\sigma_1^2})}{\partial x_2} &= \frac{\partial (x_1 x_2 x_1 x_2^{-1} x_1^{-1})}{\partial x_2} = x_1 (1 - x_2 x_1 x_2^{-1}). \end{aligned}$$

Consequently, applying ψ_c, one obtains the same matrix as above, namely the unreduced Gassner matrix of the pure braid σ_1^2.

Note that (9.7) also shows that the unreduced Burau matrices of (9.2) can be recovered from the unreduced colored Gassner matrices by setting $t_i = t$ for each i. This can be understood on the homological level as follows.

Remark 9.4.5. The map $\mathbb{Z}^\mu \to \mathbb{Z}$ sending each t_i to t gives rise to a map $\mathrm{ev}\colon \Lambda_\mu \to \Lambda$ which endows Λ with a Λ_μ-module structure. As $\mathrm{ev} \circ \psi_c$ coincides with the map ψ whose kernel produces the cover of D_n^∞ of Section 9.3, the Burau representation is induced by the map $H_1^{\mathrm{ev} \circ \psi_c \circ h_\alpha}(D_c; \Lambda \otimes_{\Lambda_\mu} \Lambda_\mu) \to H_1^{\mathrm{ev} \circ \psi_c}(D_c; \Lambda \otimes_{\Lambda_\mu} \Lambda_\mu)$. Since the latter map also has the effect of evaluating the colored Gassner representation at t, the claim follows.

From now on, we return to our usual notation: we write $H_1(D_c; \Lambda_\mu)$ instead of $H_1^{\psi_c}(D_c; \Lambda_\mu)$. Let us move on to the reduced colored Gassner representation. Although most of the construction resembles the one variable case, there is one notable difference: for $\mu \geq 3$, the Λ_μ-module $H_1(D_c; \Lambda_\mu)$ is not free, see Lemma 9.4.6 below. For this reason, we shall use the now familiar ring $\Lambda_S := \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}, (1-t_1)^{-1}, \dots, (1-t_\mu)^{-1}]$, see for instance Subsection 5.7.3. In particular, recall that since Λ_S is flat over Λ_μ, the left Λ_S-modules $H_1(D_c; \Lambda_S)$ and $\Lambda_S \otimes_{\Lambda_\mu} H_1(\widehat{D}_c)$ are isomorphic. The next lemma illustrates once more the usefulness of Λ_S-coefficients.

Lemma 9.4.6. *Given a sequence $c = (c_1, \dots, c_n)$,*

1. *the Λ_μ-module $H_1(D_c; \Lambda_\mu)$ is of rank $n-1$ and is free for $\mu = 1, 2$;*
2. *the Λ_S-module $H_1(D_c; \Lambda_S)$ is free of rank $n-1$ for all μ.*

Proof. Arguing as in Lemma 9.3.6, the punctured disk D_c is homotopy equivalent to the wedge of n circles and the boundary map $C_1(\widehat{D}_c) \to C_0(\widehat{D}_c)$ is given by $\widetilde{x}_i \mapsto (t_{c_i} - 1)\widetilde{z}$. It follows that $H_1(D_c; \Lambda_\mu)$, which coincides with the kernel of this map, has a spanning set given by the $n-1$ elements $(1 - t_{c_i})\widetilde{x}_{i+1} - (1 - t_{c_{i+1}})\widetilde{x}_i$. Some linear algebra shows that if $\mu = 1, 2$, then this is indeed a basis over Λ_μ, while in general, it is only a basis over Λ_S. □

Let $h_\alpha\colon D_n \to D_n$ be a homeomorphism representing a colored braid $\alpha \in B_c$ and let $\widetilde{h}_\alpha\colon \widehat{D}_c \to \widehat{D}_c$ be the lift fixing the fiber P over z pointwise.

Definition 34. The *reduced colored Gassner representation* is the representation $B_c \to \mathrm{Aut}_{\Lambda_\mu}(H_1(\widehat{D}_c))$ obtained by sending α to the automorphism $H_1(\widehat{D}_c) \to H_1(\widehat{D}_c)$ induced by $\widetilde{h}_\alpha$. The *localized reduced colored Gassner representation* is obtained by using the induced map on $\Lambda_S \otimes_{\Lambda_\mu} H_1(\widehat{D}_c)$.

Note that the localized reduced colored Gassner representation is $(n-1)$-dimensional and the same remark holds in the unlocalized case for $\mu = 1, 2$, see Lemma 9.4.6. The fact that both of these reduced colored Gassner representations are indeed homomorphisms follows just as in the unreduced and one variable cases. Furthermore, these representations should once again be understood as induced maps on twisted homology, see Remarks 9.3.2 and 9.4.2. The upshot of Definition 34 and of Lemma 9.4.6 is that localized coefficients are necessary for matrix computations: they ensure that the twisted homology of $\widehat{D}_c$ is free. Naturally, using the first point of Lemma 9.4.6, if $\mu = 1, 2$, localizing is altogether unnecessary.

Remark 9.4.7. With respect to the basis $v_1, \ldots, v_{n-1}$ described in Lemma 9.4.6, matrices for the localized reduced colored Gassner representation $\overline{\mathscr{B}}^{\mathrm{loc}}_{\psi_c}$ can be computed explicitly. Indeed, following step by step the proof of Proposition 9.3.7, one observes that the inclusion induced map $i_*\colon H_1(D_c;\Lambda_S) \to H_1(D_c, z; \Lambda_S)$ is injective and satisfies $i_*\mathscr{B}_{\psi_c}(\alpha) = \overline{\mathscr{B}}^{\mathrm{loc}}_{\psi_c} i_*$. We refer to Section 9.5 for explicit computations.

Before considering the colored analogue of the "algebraic reduced Burau representation" described in Definition 31, we first observe that the reduced colored Gassner representation is unitary, generalizing the corresponding statement for the reduced Burau representation, see Subsection 9.3.3. Let $\langle\ ,\ \rangle\colon H_1(\widehat{D}_c) \times H_1(\widehat{D}_c) \to \mathbb{Z}$ be the skew-symmetric algebraic intersection pairing obtained by lifting the orientation of D_c to $\widehat{D}_c$. Consider the pairing $\lambda_c\colon H_1(\widehat{D}_c) \times H_1(\widehat{D}_c) \to \Lambda_\mu$ given by $\lambda_c(x,y) = \sum_{g \in \mathbb{Z}^\mu} \langle gx, y\rangle g^{-1}$. The pairing is skew-Hermitian and adapting the proof of Proposition 9.3.13 shows that the reduced colored Gassner representation preserves λ_c. Tensoring with Λ_S, this is also the case for the localized reduced colored Gassner representation. As in Remark 9.3.12, these forms are perhaps best understood using the machinery of twisted intersection forms, see Section 5.6. Be that as it may, we start by forging some intuition with an example.

Example 9.4.8. Consider the case $n = \mu = 2$. $H_1(\widehat{D}_{(12)})$ is freely generated by $v_1 = (1-t_2)\widetilde{x}_1 - (1-t_1)\widetilde{x}_2$, see Lemma 9.4.6. With respect to this basis, a matrix for the skew-Hermitian intersection form λ_c is $\left((t_1 - t_1^{-1}) + (t_2 - t_2^{-1}) - (t_1t_2 - t_1^{-1}t_2^{-1})\right)$. An illustration of part of this computation is shown in Figure 9.4.8.

In the case of the Burau representation, the pairing λ was non-degenerate. The statement generalizes to the colored case.

Remark 9.4.9. In order to see that the pairing λ_c is non-degenerate on $H_1(D_c;\Lambda_S)$, one can either argue as in Cimasoni-Turaev [42, Lemma 3.2 and Section 6] or adapt Remark 9.3.12 to the colored case. Note that with Λ_S coefficients, the latter argument is actually less technical than its one variable counterpart since $H_0(D_c;\Lambda_S) = 0$ by Lemma 5.7.3.

We now deal with the colored generalization of the algebraic reduced Burau representation. Recall from Subsection 9.3.2 that instead of working with the free generators $x_1, x_2 \ldots, x_n$

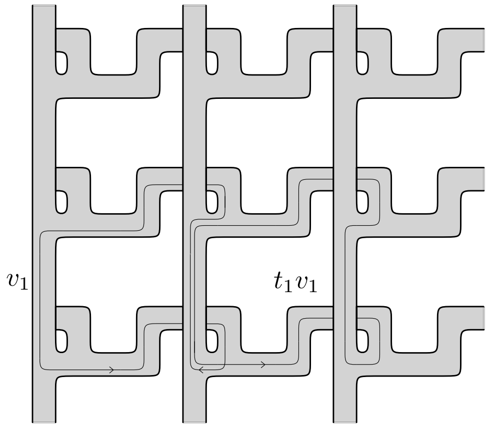

Figure 9.10: Computing the algebraic intersection of v_1 and t_1v_1 in $\widehat{D}_{(12)}$.

of $\pi_1(D_n)$, one can consider the elements $g_1, g_2, \ldots, g_n$, defined by $g_i = x_1x_2\cdots x_i$. Let $\widetilde{g}_i$ be the lift of g_i to $\widehat{D}_c$ starting at a fixed lift of z. Using the same argument as in Lemma 9.4.1, one obtains the splitting $H_1(\widehat{D}_c, P) = \bigoplus_{i=1}^{n-1} \Lambda_\mu \widetilde{g}_i \oplus \Lambda_\mu \widetilde{g}_n$. As g_n is always fixed by the action of the braid group, its lift $\widetilde{g}_n$ is fixed by the lift $\widetilde{h}_\beta$ of a homeomorphism h_β representing a colored braid β.

Definition 35. The *algebraic reduced colored Gassner homomorphism* $\overline{\mathscr{B}}^{\mathrm{alg}}_{\psi_c} \colon B_n \to GL_{n-1}(\Lambda_\mu)$ sends a braid β to the restriction $\overline{\mathscr{B}}^{\mathrm{alg}}_{\psi_c}(\beta)$ of the unreduced colored Gassner map to the free Λ_μ-module of rank $(n-1)$ generated by $\widetilde{g}_1, \ldots, \widetilde{g}_{n-1}$.

If $\widetilde{\mathscr{B}}_{\psi_c}(\beta)$ denotes the matrix of the unreduced colored Gassner representation of a braid β with respect to the basis $\widetilde{g}_1, \ldots, \widetilde{g}_n$, then

$$\widetilde{\mathscr{B}}_{\psi_c}(\beta) = \begin{pmatrix} \overline{\mathscr{B}}^{\mathrm{alg}}_{\psi_c}(\beta) & V \\ 0 & 1 \end{pmatrix} \tag{9.8}$$

for some $((n-1)\times 1)$-matrix V. In particular, the algebraic reduced colored Gassner representation can be computed via Fox calculus, see (9.7) and Example 9.4.4. Explicit computations will be performed in Section 9.5.

Remark 9.4.10. As in the one variable case, we have two "reduced colored Gassner representations" which are defined on two different modules (the lifts of the g_i do not belong to $H_1(\widehat{D}_c)$). This time however, they do not to agree over Λ.

As in Section 9.3, the reduced colored Gassner representation is also related to the multivariable Alexander polynomial. In the case of the (localized) reduced colored Gassner representation, we refer to [103]. In the case of the algebraic reduced colored Gassner representation, the following theorem is a slight generalization of a result of Birman [17], see also Morton [126].

Theorem 9.4.11. *Given a μ-colored braid $\alpha \in B_c$ with n strands, we have*

$$\Delta_{\widehat{\alpha}}(t_1, \ldots, t_\mu)(t_{c_1}\ldots t_{c_n} - 1) \doteq \begin{cases} (t_1 - 1)\det(\overline{\mathscr{B}}^{\mathrm{alg}}_{t_1}(\alpha) - I_{n-1}) & \text{if } \mu = 1, \\ \det(\overline{\mathscr{B}}^{\mathrm{alg}}_{\psi_c}(\alpha) - I_{n-1}) & \text{if } \mu > 1. \end{cases}$$

.

Since a much more general result will be proved in Chapter 11, the proof of Theorem 9.4.11 is omitted. Finally, note that Theorem 9.4.11 recovers Burau's original result in the case $\mu = 1$, recall Theorem 9.3.9.

9.5 Further remarks

This section provides several computations of the objects introduced in Section 9.4. In Subsection 9.5.1, we give the matrices for the algebraic reduced colored Gassner representation of (c, c')-braids and show how they can be used to efficiently recover the usual Gassner representation of the pure braid group. In Subsection 9.5.2, we carry out explicit computations of the localized reduced colored Gassner representation.

9.5.1 Computations of the algebraic reduced colored Gassner representation

Recall from Example 9.7 that a matrix for the unreduced colored Gassner representation of σ_i, viewed as a (c, c')-colored braid, is given by

$$\mathscr{B}_{\psi_{c'}}(\sigma_i) = I_{(i-1)} \oplus \begin{pmatrix} 1 - t_{c'_{i+1}} & t_{c'_i} \\ 1 & 0 \end{pmatrix} \oplus I_{(n-i-1)}. \tag{9.9}$$

Let $\widetilde{\mathscr{B}}_{\psi_{c'}}(\beta)$ denote the matrix of the unreduced colored Gassner representation of a (c, c')-braid β with respect to the basis $\widetilde{g}_1, \ldots, \widetilde{g}_n$. We saw in (9.8), that $\widetilde{\mathscr{B}}_{\psi_{c'}}(\beta) = \begin{bmatrix} \overline{\mathscr{B}}^{\mathrm{alg}}_{\psi_{c'}}(\beta) & V \\ 0 & 1 \end{bmatrix}$, where $\overline{\mathscr{B}}^{\mathrm{alg}}_{\psi_{c'}}(\beta)$ denotes the algebraic reduced colored Gassner representation and V is some $((n-1) \times 1)$-matrix. A short computation involving (9.9), then shows that a matrix for the algebraic reduced colored Gassner representation of σ_i, viewed as a (c, c')-colored braid, is given by

$$\overline{\mathscr{B}}^{\mathrm{alg}}_{\psi_{c'}}(\sigma_i) = I_{(i-2)k} \oplus \begin{pmatrix} 1 & 0 & 0 \\ t_{c'_{i+1}} & -t_{c'_{i+1}} & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{(n-i-2)} \tag{9.10}$$

for $1 < i < n-1$, and for σ_1 and σ_{n-1} it is represented by

$$\overline{\mathscr{B}}^{\mathrm{alg}}_{\psi_{c'}}(\sigma_1) = \begin{pmatrix} -t_{c'_2} & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{(n-3)}, \qquad \overline{\mathscr{B}}^{\mathrm{alg}}_{\psi_{c'}}(\sigma_{n-1}) = I_{(n-3)} \oplus \begin{pmatrix} 1 & 0 \\ t_{c'_n} & -t_{c'_n} \end{pmatrix}.$$

Observe that, as expected, setting $t_i = t$ recovers the reduced Burau matrices of (9.3). In general, working in the colored setting adds some extra flexibility. As an illustration, we shall now use the *reduced colored Gassner matrices* of (9.10) in order to recover the well known Gassner matrices of the pure braid group. First of all, it is known (see for instance [17, Lemma 1.8.2]) that generators of the pure braid group P_n are given by

$$A_{ij} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}. \tag{9.11}$$

Explicit matrices for the unreduced Gassner representation of A_{ij} can then be computed using Fox calculus [17, Section 3.2]. On the other hand, a combination of Proposition 9.4.3 and (9.10) provides a practical alternate method of computation. However, since inverses of the σ_i also appear in (9.11), we first collect a consequence of Proposition 9.4.3.

Proposition 9.5.1. *Given a (c,c')-colored braid β with n strands, the unreduced colored Gassner representation satisfies the following relation:*

$$\mathscr{B}_{\psi_c}(\beta^{-1}) = \mathscr{B}_{\psi_{c'}}(\beta)^{-1}.$$

The same equation holds for the algebraic reduced colored Gassner representation.

Proof. Let id_c denote the trivial (c,c)-braid with n strands. Since $I_n = \mathscr{B}_{\psi_c}(id_c) = \mathscr{B}_{\psi_c}(\beta\beta^{-1})$, Proposition 9.4.3 implies that $I_n = \mathscr{B}_{\psi_{c'}}(\beta)\mathscr{B}_{\psi_c}(\beta^{-1})$ and the result follows. □

For convenience, let us introduce some notation. Given a (c,c')-braid β, we shall temporarily write $[\beta]^c_{c'}$ for emphasize. Applying Proposition 9.5.1 to the generators $[\sigma_i]^c_{c'}$, we obtain:

$$\overline{\mathscr{B}}^{\mathrm{alg}}_{\psi_{c'}}([\sigma_i]^{c_1,\dots,c_i,c_{i+1},\dots,c_n}_{c_1,\dots,c_{i+1}c_i,\dots c_n})^{-1} = \overline{\mathscr{B}}^{\mathrm{alg}}_{\psi_c}([\sigma_i^{-1}]^{c_1,\dots,c_{i+1},c_i,\dots,c_n}_{c_1,\dots,c_ic_{i+1},\dots c_n}). \tag{9.12}$$

We now combine these observations in order to show how the algebraic reduced Gassner representation $\overline{\mathscr{G}}^{\mathrm{alg}}_{t_1,\dots,t_n}$ of the pure braid group can be computed using colored braids.

Example 9.5.2. Consider the pure braid A_{13} in P_3. Using the notation introduced above, A_{13} can be decomposed as $[\sigma_2]^{123}_{132}[\sigma_1]^{132}_{312}[\sigma_1]^{312}_{132}[\sigma_2^{-1}]^{132}_{123}$. Using (9.12) and the reduced colored Gassner matrices of (9.10), we get

$$\begin{aligned}
\overline{\mathscr{G}}^{\mathrm{alg}}_{t_1,t_2,t_3}(A_{13}) &= \overline{\mathscr{B}}^{\mathrm{alg}}_{\psi_{(132)}}([\sigma_2]^{123}_{132})\overline{\mathscr{B}}^{\mathrm{alg}}_{\psi_{(312)}}([\sigma_1]^{132}_{312})\overline{\mathscr{B}}^{\mathrm{alg}}_{\psi_{(132)}}([\sigma_1]^{312}_{132})\overline{\mathscr{B}}^{\mathrm{alg}}_{\psi_{(132)}}([\sigma_2]^{123}_{132})^{-1} \\
&= \begin{pmatrix} 1 & 0 \\ t_2 & -t_2 \end{pmatrix}\begin{pmatrix} -t_1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -t_3 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t_2 & -t_2 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1-t_1+t_1t_3 & t_2^{-1}(t_1-1) \\ t_1t_2(t_3-1) & t_1 \end{pmatrix}.
\end{aligned}$$

9.5.2 Computations of the localized reduced colored Gassner representation

We give examples of the computation of the (localized) reduced colored Gassner representation, expanding on Remark 9.4.7. Instead of burdening the reader with the general case, we shall carry out the computations in the cases $n = 2$ and $n = 3$.

Example 9.5.3. Consider once again the pure braid A_{13} in P_3 and let $\mathscr{G}_{t_1,t_2,t_3}$ denote the unreduced colored Gassner representation of the pure braid group P_3. Using the same reasoning as in Example 9.5.2, we obtain

$$\begin{aligned}
\mathscr{G}_{t_1,t_2,t_3}(A_{13}) &= \mathscr{B}_{\psi_{(132)}}([\sigma_2]^{123}_{132})\mathscr{B}_{\psi_{(312)}}([\sigma_1]^{132}_{312})\mathscr{B}_{\psi_{(132)}}([\sigma_1]^{312}_{132})\mathscr{B}_{\psi_{(132)}}([\sigma_2]^{123}_{132})^{-1} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t_2 & t_3 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1-t_1 & t_3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1-t_3 & t_1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t_2 & t_3 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1-t_1+t_1t_3 & 0 & (1-t_1)t_1 \\ (1-t_2)(1-t_3) & 1 & (1-t_2)(t_1-1) \\ 1-t_3 & 0 & t_1 \end{pmatrix}.
\end{aligned}$$

We now use this result in order to compute the localized reduced colored Gassner representation. Using Lemmas 9.4.1 and 9.4.6, we endow $H_1(D_3, z; \Lambda_S)$ with the basis $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3$ and $H_1(D_3; \Lambda_S)$ with the basis $v_1 = \widetilde{x}_1(1-t_2) + \widetilde{x}_2(t_1-1), v_2 = \widetilde{x}_2(1-t_3) + \widetilde{x}_3(t_2-1)$. With respect to these bases, the matrix for the inclusion induced homomorphism $H_1(D_3; \Lambda_S) \to H_1(D_3, z; \Lambda_S)$ is given by

$$\mathrm{incl}_3 = \begin{pmatrix} 1-t_2 & t_1-1 & 0 \\ 0 & 1-t_3 & t_2-1 \end{pmatrix}. \tag{9.13}$$

Here, we recall that elements of our modules are row vectors, and multiplication is read from left to right. Using (9.13) and Example 9.5.2, we can now solve the equation $\overline{\mathscr{G}}^{\mathrm{loc}}_{t_1,t_2,t_3}(A_{13})\mathrm{incl}_3 = \mathrm{incl}_3\mathscr{G}_{t_1,t_2,t_3}(A_{13})$ for $\overline{\mathscr{G}}^{\mathrm{loc}}_{t_1,t_2,t_3}(A_{13})$. The result of this explicit computation is

$$\overline{\mathscr{G}}^{\mathrm{loc}}_{t_1,t_2,t_3}(A_{13}) = \begin{pmatrix} t_3 & t_1-1 \\ (t_3-1)t_3 & 1-t_3+t_1t_3 \end{pmatrix}.$$

We conclude this chapter with an easier example which we shall use in the third part of this thesis and more specifically in Chapter 16.

Example 9.5.4. Consider the pure braid A_{12} in P_2. Using the notation above, A_{12} can be decomposed as $[\sigma_1]^{12}_{21}[\sigma_1]^{21}_{12}$. As we saw in Example 9.4.4, the unreduced colored Gassner representation of A_{12} is given by

$$\mathscr{G}_{t_1,t_2}(A_{12}) = \begin{pmatrix} 1-t_1+t_1t_2 & t_1(1-t_1) \\ 1-t_2 & t_1 \end{pmatrix}. \tag{9.14}$$

We now use (9.14) to compute the reduced colored Gassner representation of A_{12}. Note that since $\mu < 3$, we do not need to localize, recall Lemma 9.4.6. Using Lemmas 9.4.1 and 9.4.6, we endow $H_1(D_2, z; \Lambda_2)$ with the basis $\widetilde{x}_1, \widetilde{x}_2$ and $H_1(D_2; \Lambda_2)$ with the basis $v_1 = \widetilde{x}_1(1-t_2)+\widetilde{x}_2(t_1-1)$. As in Example 9.5.3, a matrix for the inclusion induced homomorphism $H_1(D_2; \Lambda_2) \to H_1(D_2, z; \Lambda_2)$ is given by

$$\mathrm{incl}_2 = \begin{pmatrix} 1-t_2 & t_1-1 \end{pmatrix}. \tag{9.15}$$

Using (9.14) and (9.15), we can now solve the equation $\overline{\mathscr{G}}_{t_1,t_2}(A_{12})\mathrm{incl}_2 = \mathrm{incl}_2\mathscr{G}_{t_1,t_2}(A_{12})$ for $\overline{\mathscr{G}}_{t_1,t_2}(A_{12})$. The result of this explicit computation is $\mathscr{G}_{t_1,t_2}(A_{12}) = \begin{pmatrix} t_1t_2 \end{pmatrix}$.

Chapter 10

A short review of Reidemeister torsion

10.1 Introduction

In Sections 2.2 and 3.3, we reviewed several constructions of the Alexander polynomial using orders of modules, Fox calculus and (generalized) Seifert surfaces. We purposefully delayed the connection to the so-called Reidemeister torsion to this short chapter. The reason for this is the following: although this chapter is fairly introductory and contains no new results, it assumes some familiarity with twisted homology.

The chapter is organized as follows. Section 10.2 reviews the Reidemeister torsion of a chain complex, while Section 10.3 presents its relation with the Alexander polynomial. Note that this organization also reflects the historical development of the subject: while Reidemeister [139] originally introduced torsion invariants in 1935 in order to classify lens spaces, the connection with knot theory was only made in 1962 by Milnor [122, 123] before being further developed by Turaev [149].

10.2 Torsion of chain complexes

This section briefly reviews the torsion of a chain complex. References include [149, 150].

Given two bases $\mathbf{c}, \mathbf{c}'$ of a finite dimensional vector space over a field F, let $[\mathbf{c}/\mathbf{c}'] \in F \setminus \{0\}$ be the determinant of the matrix expressing the vectors of the basis $\mathbf{c}$ as a linear combination of vectors in $\mathbf{c}'$. Furthermore, if $0 \to E \to D \xrightarrow{\beta} G \to 0$ is a short exact sequence of vector spaces, then bases $\mathbf{e} = (e_1, \dots, e_k)$ of E and $\mathbf{g} = (g_1, \dots, g_l)$ of G give rise to a basis of D. Indeed, since β is surjective, we may lift each g_i to some $\tilde{g}_i \in D$ and $\mathbf{eg} := (e_1, \dots, e_k, \tilde{g}_1, \dots, \tilde{g}_l)$ is a basis of D.

Let $C = (0 \to C_m \to C_{m-1} \to \cdots \to C_0 \to 0)$ be a chain complex of vector spaces over F such that for $i = 1, 2, \dots, m$ each C_i has a distinguished basis $\mathbf{c}_i$. If C is not acyclic, then we set $\tau(C) = 0$. Otherwise, let $\mathbf{b}_i$ be a sequence of vectors in C_i such that $\partial_{i-1}(\mathbf{b}_i)$ forms a basis of $\mathrm{Im}(\partial_{i-1})$. Clearly the sequence $\partial_i(\mathbf{b}_{i+1})\mathbf{b}_i$ is a basis of C_i. The *torsion* of the based chain complex C is defined as

$$\tau(C) = \prod_{i=0}^{m} [\partial_i(\mathbf{b}_{i+1})\mathbf{b}_i/\mathbf{c}_i]^{(-1)^{i+1}} \in F.$$

It turns out that $\tau(C)$ depends on the choice of basis for C_i but does not depend on the choice of $\mathbf{b}_i$. For the proof of the next proposition, see [149, Theorem 0.1.1].

Proposition 10.2.1. *Let $0 \to C' \to C \to C'' \to 0$ be a short exact sequence of finite-dimensional chain complexes over F. Assume that C' or C'' is acyclic, and that C_i, C'_i, C''_i have distinguished bases $\mathbf{c}_i, \mathbf{c}'_i, \mathbf{c}''_i$ such that $[\mathbf{c}_i/\mathbf{c}'_i\mathbf{c}''_i] = 1$. Then $\tau(C) = \pm\tau(C')\tau(C'')$.*

In order to compute the Reidemeister torsion, one of the most efficient methods relies on so-called matrix τ-chains [150, Section 2.1]. Let C be a chain complex and let A_i be the matrix of the boundary map ∂_i of C. A *matrix chain* for C is a collection of sets $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m)$, where $\alpha_i \subset \{1, 2, \ldots, \dim C_i\}$, so that $\alpha_0 = \emptyset$. Let $S_i = S_i(\alpha)$ be the submatrix of A_i formed by the entries a_{jk} with $j \in \alpha_{i+1}$ and $k \notin \alpha_i$. The matrix chain α is called a *τ-chain* if $S_0, S_1, \ldots, S_{m-1}$ are square matrices. The τ-chain α is said to be *non-degenerate* if $\det(S_i) \neq 0$ for all i. The proof of the following proposition can be found in [150, Theorem 2.2].

Proposition 10.2.2. *Any matrix τ-chain $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m)$ for C with $\det S_i(\alpha) \neq 0$ for all even i is non-degenerate and*

$$\tau(C) = \pm \prod_{i=0}^{m-1} (\det S_i(\alpha))^{(-1)^{i+1}}.$$

Here is an example of Proposition 10.2.2 which occurs frequently in practice.

Example 10.2.3. Consider the chain complex complex $C = (0 \to C_2 \to C_1 \to C_0 \to 0)$ where C_0 is one-dimensional, C_1 is n-dimensional and C_2 is $(n-1)$-dimensional. Consider the τ-chain $\alpha = (\emptyset, \alpha_1, \alpha_2)$, where $\alpha_1 = \{j\}$ and α_2 consists of the set $\{1, \ldots, n-1\}$ with j removed. It immediately follows that $\tau(C) = \pm \det(B_1)/\det(B_0)$, where B_1 denotes the square matrix A_1 with its j column removed and B_0 consists of the j-th entry of A_0.

The following result, whose proof relies on Proposition 10.2.2, is also fundamental: indeed it relates the torsion of a chain complex to its homological invariants. We refer to [150, Theorem 4.7] for a proof.

Proposition 10.2.4. *Let R be a Noetherian unique factorization domain and let C be a based free chain complex of finite rank over R such that each homology group $H_i(C)$ is torsion as a R-module. Let $Q(R)$ be the field of fractions of R. Then the based complex $Q(R) \otimes_R C$ is acyclic and*

$$\tau(Q(R) \otimes_R C) \doteq \prod_{i=0}^{m} (\mathrm{Ord}\ \ H_i(C))^{(-1)^{i+1}}, \tag{10.1}$$

where $\mathrm{Ord}\, H_i(C)$ *denotes the order of the R-module $H_i(C)$.*

Observe that if $m = 2$ and both $H_0(C), H_2(C)$ are torsion, then both terms in (10.1) vanish if and only if $H_1(C)$ is not torsion.

10.3 The Alexander torsion of a colored link

In this section, we review how the Alexander polynomial of a link can be interpreted using Reidemeister torsion. We purposefully avoid working in a too great generality to make the connection with the multivariable Alexander polynomial more apparent. More general considerations can be found in Section 11.2. Once again an excellent reference is [150].

Let $L = L_1 \cup \cdots \cup L_\mu$ be a colored link, let X_L denote its exterior, and let $\psi\colon \pi_1(X_L) \to \mathbb{Z}^\mu$ be the epimorphism given by $\gamma \mapsto (\ell k(\gamma, L_1), \ldots, \ell k(\gamma, L_\mu))$. Furthermore, we denote by Q_μ the field of fractions of Λ_μ. Since ψ endows Λ_μ with the structure of a right $\mathbb{Z}[\pi_1(X_L)]$-module, Q_μ naturally becomes a $(Q_\mu, \mathbb{Z}[\pi_1(X_L)])$-bimodule. In order to base the chain complexes, as required by Section 10.2, choose a lift $\widetilde{x}_i^q$ of each q-cell x_i^q of X_L to the universal cover $\widetilde{X}_L$. This yields a basis for each Q_μ-vector space $Q_\mu \otimes_{\mathbb{Z}[\pi_1(X_L)]} C_*(\widetilde{X}_L)$.

Definition 36. The *Alexander torsion* $\tau(L)$ of the colored link L is the torsion of the chain complex $Q_\mu \otimes_{\mathbb{Z}[\pi_1(X_L)]} C_*(\widetilde{X}_L)$.

It is known that the Alexander torsion is well-defined up to multiplication by units of Λ_μ, see [122, 150]. Arguing as in [150, Section 11.4] and using Proposition 10.2.4, one obtains the following well known relation between the multivariable Alexander polynomial and the Reidemeister torsion:

$$\tau(L) \doteq \begin{cases} \Delta_L(t_1)/(t_1 - 1) & \text{if } \mu = 1, \\ \Delta_L(t_1, \ldots, t_\mu) & \text{if } \mu > 1. \end{cases} \tag{10.2}$$

Although the indeterminacy of the torsion is the same as the one of the Alexander polynomial, the use of $\tau(L)$ often leads to shorter proofs which are more amenable to generalizations, see [149, 150].

Since we shall be working with braids, note that (10.2) can be combined with Theorem 9.4.11 to produce the following statement:

Proposition 10.3.1. *Given a (c, c)-braid β with n strands, we have*

$$\tau(\widehat{\beta})(t_{c_1} \cdots t_{c_n} - 1) \doteq \det(\overline{\mathscr{B}}_{\psi_c}(\beta) - I_{n-1}).$$

We conclude this section by reviewing the well known relation between the Alexander torsion and Fox calculus. Let $P = \langle x_1, \ldots, x_n | r_1, \ldots, r_{n-1} \rangle$ be a Wirtinger presentation for L, let F_n be the free group on $x_1, \ldots, x_n$ and let $pr\colon \mathbb{Z}[F_n] \to \mathbb{Z}[\pi_1(X_L)]$ be the map induced by the canonical projection. Let A be the $(n-1) \times n$ matrix with coefficients in Λ_μ obtained by applying $\psi \circ pr$ componentwise to the matrix of Fox derivatives arising from P, see Subsection 2.2.2 and Remark 9.3.3. Finally, let A_j denote the matrix obtained by removing the j-th column of A. The following result is well known [95, Lemma 7.3.2], but here is a quick proof.

Proposition 10.3.2. *Let $P = \langle x_1, \ldots, x_n | r_1, \ldots, r_{n-1} \rangle$ be a Wirtinger presentation for a μ-colored link L. Then, for each $j = 1, \ldots, n$, we have*

$$\tau(L) \doteq \det(A_j)/(t_{\psi(x_j)} - 1).$$

Proof. Consider the 2-dimensional complex W_P arising from the presentation P of $\pi_1(X_L)$, see Remark 9.3.3. Since $\pi_1(W_P) = \pi_1(X_L)$, Remark 5.2.3 implies that Δ_L (and thus τ_L thanks to (10.2)) can be computed from W_P. The chain complex $C_*(W_P; \Lambda_\mu)$ takes the form $0 \to \Lambda_\mu^{n-1} \xrightarrow{\partial_2} \Lambda_\mu^n \xrightarrow{\partial_1} \Lambda_\mu \to 0$, where Remark 9.3.3 implies that ∂_2 is represented by the Fox matrix A, while ∂_1 is represented by a column vector of $(t_{\psi(x_i)} - 1)$'s. The result now follows from Example 10.2.3. $\square$

Chapter 11

Twisted Burau maps and twisted Alexander polynomials

11.1 Introduction and statement of the results

The multivariable Alexander polynomial Δ_L of an n-component ordered link L is extracted from an abelian cover of the exterior X_L. Reformulating, Δ_L is closely related to the abelianization homomorphism $\pi_1(X_L) \to H_1(X_L)$. The idea of twisted Alexander polynomials is to keep track of the fundamental group $\pi_1(X_L)$ via a representation $\rho\colon \pi_1(X_L) \to GL_k(R)$, where R is an integral domain. As we saw in Sections 3.3 and 10.3, the Alexander polynomial can be interpreted via orders, Fox calculus, and Reidemeister torsion. Keeping this in mind, the history of twisted Alexander polynomials followed a somewhat unexpected path. Lin first defined these objects by combining Fox calculus with the notion of a "regular surface" [113]. Wada simplified this process and introduced like-minded invariants via Fox calculus [155]. Since Wada's approach is the most practical for computations, let us outline his construction. Let $\psi\colon \pi_1(X_L) \to \mathbb{Z}^n = \langle t_1, \ldots, t_n \rangle$ denote the abelianization homomorphism and consider the representation

$$\begin{aligned} \rho \otimes \psi\colon \pi_1(X_L) &\to GL_k(R[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]) \\ \gamma &\mapsto \psi(\gamma)\rho(\gamma). \end{aligned}$$

Given a deficiency one presentation P for $\pi_1(X_L)$, consider the $(n-1) \times n$ matrix $A^{\rho\otimes\psi}$ obtained by applying $\rho \otimes \psi$ to each coefficient of the Fox matrix. For each j, regard the matrix $A_j^{\rho\otimes\psi}$ obtained by removing the j-th column of $A^{\rho\otimes\psi}$, as a $((n-1)k \times (n-1)k)$ matrix with coefficients in $R[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. Wada [155] defined the twisted Alexander polynomial as

$$\frac{\det(A_j^{\rho\otimes\psi})}{\det\left((\rho \otimes \psi)(x_j - 1)\right)} \tag{11.1}$$

and proved that $\det\left((\rho \otimes \psi)(x_j - 1)\right)$ is non-zero for all $j = 1, 2, \ldots, n$. Additionally, Wada also showed that (11.1) only depends on the link L and on the representation ρ. Looking back to Chapter 10, one might expect this definition to involve Reidemeister torsion. As we shall see in Section 11.2, this step was taken by Kitano [104] who recast Wada's invariant by considering

$$\tau^{\rho\otimes\psi}(L) := \tau\left(Q^k \otimes_{\mathbb{Z}[\pi_1(X_L)]} C_*(\widetilde{X}_L)\right),$$

where Q is the field of fractions of $R[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. Independently, a more complete picture was given by Kirk-Livingston who interpreted Wada's invariant using both Reidemeister torsion and orders of twisted Alexander modules [102].

Since then, the activity around twisted Alexander polynomials gradually increased. Therefore, instead of trying to provide an exhaustive list of the literature, we only mention some notable results and refer to the survey [77] for details: twisted Alexander polynomials are closely related to the genus [73], produce sliceness obstructions [102, Section 5] and have a particular behavior for periodic knots [89]. In other words, twisted Alexander polynomials satisfy most of the properties of the classical Alexander polynomial. It therefore makes sense to ask whether they can also be computed using braids.

In this chapter (which is based on our paper [50]), we answer this question affirmatively by introducing *twisted Burau maps*. More precisely, given an integral domain R and a representation $\rho : F_n \to GL_k(R)$ of the free group, Section 11.3 introduces unreduced twisted Burau maps, reduced twisted Burau maps

$$\overline{\mathscr{B}}_\rho \colon B_c \to GL_{(n-1)k}(R[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}])$$

and studies their properties. Although these maps are not representations, they remain computable via Fox calculus, see Propositions 11.3.3 and 11.3.4. The main theorem of Chapter 11 then relates the reduced Burau map $\overline{\mathscr{B}}_\rho$ of a braid β to the twisted torsion $\tau^\rho(\widehat{\beta})$ of its closure, see Section 11.4 for the proof.

Theorem 11.1.1. *Let F_n be the free group on $x_1, \ldots, x_n$ and let $\beta \in B_c$ be a μ-colored braid with n strands. If $\rho \colon F_n \to GL_k(R)$ is a representation which extends to $\pi_1(S^3 \setminus \widehat{\beta})$, then*

$$\tau^\rho(\widehat{\beta})(t_1, t_2, \ldots, t_\mu) \det\left(\rho(x_1 x_2 \cdots x_n) t_{c_1} t_{c_2} \cdots t_{c_n} - I_k\right) \doteq \det(\overline{\mathscr{B}}_\rho(\beta) - I_{(n-1)k}),$$

where $\doteq$ refers to the indeterminacy of the twisted torsion, as explained in Remark 11.2.1.

Note that when ρ is the trivial representation, Theorem 11.1.1 recovers the results of Burau and Birman which were stated in Theorems 9.3.9, 9.4.11 and Proposition 10.3.1. Finally in Section 11.5, we give an example of Theorem 11.1.1 and discuss some technical issues related to the reduced twisted Burau map.

11.2 Twisted torsion of links

Following the exposition of Cha-Friedl [31] and Kitano [104], we review twisted torsion invariants of links. This section requires some familiarity with Chapters 5 and 10.

Let X be a CW complex and let $Y \subset X$ be a possibly empty subcomplex. Denote by $p \colon \widetilde{X} \to X$ the universal cover of X and set $\widetilde{Y} := p^{-1}(X)$. The left action of $\pi_1(X)$ on $\widetilde{X}$ endows the chain complex $C_*(\widetilde{X}, \widetilde{Y})$ with the structure of a left $\mathbb{Z}[\pi_1(X)]$-module. Any representation $\rho : \pi_1(X) \to GL_k(R)$ induces a $(R, \mathbb{Z}[\pi_1(X)])$-bimodule structure on R^k, where the right action is given by right multiplication on row vectors. Let $\psi : \pi_1(X) \to H$ be an epimorphism onto a free abelian group H and define

$$\begin{aligned} \rho \otimes \psi \colon \pi_1(X_L) &\to GL_k(R[H]) \\ \gamma &\mapsto \psi(\gamma)\rho(\gamma). \end{aligned}$$

Since $\rho\otimes\psi$ endows $R[H]^k$ with the structure of a $(R[H], \mathbb{Z}[\pi_1(X)])$-bimodule, one may consider the chain complex of left $R[H]$-modules $R[H]^k \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X}, \widetilde{Y})$ which was denoted by $C_*(X, Y; R[H]^k)$ in Section 5.2. However, in this section, in order to keep further track of the representations, we shall instead denote the twisted chain complex by

$$C_*^{\rho\otimes\psi}(X, Y; R[H]^k) := R[H]^k \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X}, \widetilde{Y}) \tag{11.2}$$

and the corresponding twisted homology left $R[H]^k$-modules by $H_*^{\rho\otimes\psi}(X, Y; R[H]^k)$.

From now on, assume that R is an integral domain. Let $Q(H)$ denote the field of fractions of the integral domain $R[H]$. As the representation $\rho\otimes\psi$ extends to a k-dimensional representation of $Q(H)$, one may also consider the $Q(H)$-vector spaces $H_*^{\rho\otimes\psi}(X; Q(H)^k)$. Choose a lift $\widetilde{x}_i^q$ of each q-cell x_i^q of X to the universal cover $\widetilde{X}$ and denote by $e_1, e_2, \ldots, e_k$ the canonical basis of $Q(H)^k$. For each q, this yields a basis $\{e_j \otimes \widetilde{x}_i^q\}$ over $Q(H)$ for $C_q^{\rho\otimes\psi}(X; Q(H)^k)$.

Definition 37. The *twisted torsion* of X is defined as

$$\tau^{\rho\otimes\psi}(X) := \tau(C_*^{\rho\otimes\psi}(X; Q(H)^k)) \in Q(H).$$

It is known that the twisted torsion $\tau^{\rho\otimes\psi}(X)$ is well-defined up to multiplication by an element in $\pm\det(\rho\otimes\psi(\pi_1(X)))$ and is invariant under simple homotopy [31, 74, 102, 104, 122, 150]. Since $\det(\rho\otimes\psi(\pi_1(X)))$ is contained in $\det(\rho(\pi_1(X)))\cdot H$, one often considers $\tau^{\rho\otimes\psi}(X)$ up to multiplication by $\pm dh$ for $d \in \det(\rho(\pi_1(X)))$ and $h \in H$.

Remark 11.2.1. From now on, it will be understood that we write $\tau^{\rho\otimes\psi}(X) \doteq w$ if there exists a representative of $\tau^{\rho\otimes\psi}(X)$ which equals w. For latter use, we also recall that in Chapter 10, we adopted the following convention: if a chain complex is not acyclic, then its torsion is defined to be zero.

By Chapman's theorem [35], $\tau^{\rho\otimes\psi}(X)$ only depends on the homeomorphism type of X. In particular, when M is a manifold, one can define $\tau^{\rho\otimes\psi}(M)$ by picking any CW-structure for M. When the context is clear, we shall drop the ψ both in the notation of twisted homology and twisted torsion.

In practice, twisted torsion invariants are often computed by using Wada's set-up, see (11.1). Let us briefly recall this approach.

Remark 11.2.2. Assume (for the sake of exposition) that $\pi_1(X)$ admits a deficiency one presentation with n generators. In other words, there are n generators $x_1, \ldots, x_n$ and $n-1$ relators $r_1, \ldots, r_{n-1}$. Let $A^{\rho\otimes\psi}$ be the $((n-1)\times n)$ matrix whose (i,j) coefficient is the $k\times k$ invertible matrix $\rho\otimes\psi\left(\frac{\partial r_i}{\partial x_j}\right)$ with coefficients in $R[H]$. For $j = 1, 2, \ldots, n$, regard the matrix $A_j^{\rho\otimes\psi}$, obtained by removing the j-th column of A, as a $((n-1)k \times (n-1)k)$ matrix with coefficients in $R[H]$. Kitano [104] showed that

$$\tau^\rho(X) = \frac{\det(A_j^{\rho\otimes\psi})}{\det\left((\rho\otimes\psi)(x_j - 1)\right)}.$$

The proof can also be understood using Example 10.2.3, see [77, Theorem 2].

We now restrict our attention to links. Let L be a μ-colored link and let ψ be the epimorphism $\pi_1(X_L) \to H$ mapping γ to $t_1^{\ell k(\gamma, L_1)} \cdots t_\mu^{\ell k(\gamma, L_\mu)}$. The *twisted torsion* $\tau^{\rho\otimes\psi}(L)$ *of the colored link* L is defined as the twisted torsion of the exterior of L.

Remark 11.2.3. Note that if ρ is the trivial one-dimensional representation, then the twisted torsion recovers the Alexander torsion of Definition 36. In particular it encompasses both the classical one-variable Alexander polynomial and the multivariable Alexander polynomial.

We conclude this section with a well known lemma.

Lemma 11.2.4. *If the $R[H]$-module $H_1^\rho(X_L; R[H]^k)$ is torsion, then $C_*^\rho(X_L; Q(H)^k)$ is acyclic.*

Proof. As the link exterior X_L is homotopy equivalent to a 2-complex, $H_3^\rho(X_L; R[H]^k)$ vanishes. If $\widehat{X}_L$ denotes the covering of X_L corresponding to the kernel of ψ, then Lemma 5.2.1 implies that $H_0^\rho(X_L; R[H]^k)$ is isomorphic to $H_0^\rho(\widehat{X}_L; R^k)$. Since ψ is surjective, $\widehat{X}_L$ is connected and consequently $H_0^\rho(X_L; R[H]^k)$ is $R[H]$-torsion. Using Remark 5.2.5, an Euler characteristic argument then shows that $H_2^\rho(X_L; R[H]^k)$ is torsion over $R[H]$. As all the twisted homology modules of X_L are torsion over $R[H]$, the chain complex $C_*^\rho(X_L; Q(H)^k)$ is acyclic and the claim follows. □

11.3 Twisted Burau maps

In this section we introduce the (reduced) twisted Burau maps. Namely, Subsection 11.3.1 defines unreduced twisted Burau maps, while Subsection 11.3.2 defines reduced twisted Burau maps.

11.3.1 The unreduced twisted Burau map

Fix a sequence $c = (c_1, c_2, \ldots, c_n)$ of elements in $\{1, 2, \ldots, \mu\}$ and a representation $\rho\colon \pi_1(D_c) \to GL_k(R)$. If H denotes the free abelian group on $t_1, t_2, \ldots, t_\mu$, then we let $\psi_c : \pi_1(D_c) \to H$ be the epimorphism defined by $x_i \mapsto t_{c_i}$. Given a homeomorphism h_β representing a (c, c)-braid β, we saw in Lemma 5.7.1 that there is an induced map

$$H_1(D_c, z; h_\beta^*(R[H]^k)) \to H_1(D_c, z; R[H]^k), \tag{11.3}$$

on twisted homology. Here, $h_\beta^*(R[H]^k)$ denotes the right $\mathbb{Z}[\pi_1(D_c)]$-module whose underlying abelian group is $R[H]^k$, and whose right $\mathbb{Z}[\pi_1(D_c)]$-module structure is obtained by restriction of scalars, i.e. $x \cdot \gamma = x(\rho \circ h_\beta)(\gamma)$, where γ lies in $\pi_1(D_c)$ and x lies in $R[H]^k$.

Remark 11.3.1. At this stage, we observe a key difference with the untwisted case. If we wish to mimic Section 9.3, a twisted Burau map of a braid β ought to be an *auto*morphism induced by h_β on twisted homology. Unfortunately, (11.3) indicates that this has no chance of occurring in general: the domain and target of induced maps in twisted homology differ.

Nevertheless, we could simply define the twisted Burau map as the homomorphism displayed in (11.3). However, for the sake of concreteness, we choose to pick canonical bases and express the twisted Burau maps as matrices.

In order to carry out the idea outlined in Remark 11.3.1, we first set-up some conventions: we wish our notations to match those of [50]. First, as in Section 9.2, $x(\rho \circ h_\beta)(\gamma)$ can be rewritten as $x\beta_*(\rho)(\gamma)$. Next, given a continuous map $f\colon X \to Y$ and a representation

$\rho\colon \pi_1(Y) \to GL_k(R)$ we write $H_*^{\rho f_*}(X; R^k)$ instead of $H_*(X; f^*R^k)$. In particular, with these conventions, the $R[H]$-linear map of (11.3) can be rewritten as

$$H_1^{\beta_*\rho}(D_c, z; R[H]^k) \to H_1^{\rho}(D_c, z; R[H]^k). \tag{11.4}$$

This notation coincides with [50, Example 2.3]. In order to obtain a matrix out of this homomorphism, we need to make sure that $H_1^{\beta_*\rho}(D_c, z; R[H]^k)$ and $H_1^{\rho}(D_c, z; R[H]^k)$ are free $R[H]$-modules, and pick appropriate bases.

The proof of the next lemma is a generalization of Lemmas 9.3.1 and 9.4.1.

Lemma 11.3.2. *Let $z \in D_n$, let S be a commutative ring, and let M be a $(S, \mathbb{Z}[\pi_1(D_n)])$-bimodule. If M is free of rank k as a left S-module, then the left S-module $H_1(D_n, z; M)$ is free of rank nk.*

Proof. The punctured disk D_n is homotopy equivalent to the wedge of the n loops representing the generators of $\pi_1(D_n)$ described in Section 9.2. Choose a cellular decomposition of this latter space X consisting of the 0-cell z (the basepoint of the wedge) and one 1-cell x_i for each loop. For $i = 1, 2, \ldots, n$, let $\widetilde{x}_i$ be the lift of x_i starting at an (arbitrary) fixed lift of z. With this cell structure, the twisted chain complex of (X, z) is

$$C_1(X, z; M) = M \otimes_{\mathbb{Z}[\pi_1(X)]} C_1(\widetilde{X}, \widetilde{z}) = M \otimes_{\mathbb{Z}[\pi_1(X)]} \bigoplus_{i=1}^{n} \mathbb{Z}[\pi_1(X)]\widetilde{x}_i \cong \bigoplus_{i=1}^{n} M\widetilde{x}_i.$$

As the chain group $C_0(\widetilde{X}, \widetilde{z})$ vanishes, $H_1(D_n, z; M) = C_1(X, z; M)$. The claim now follows from the assumption that M is free of rank k. $\square$

Returning to the case $M = R[H]^k$, Lemma 11.3.2 provides a canonical isomorphism $H_1^{\rho}(D_n, z; R[H]^k) \cong \bigoplus_{i=1}^{n} R[H]^k \widetilde{x}_i$ and the resulting basis shall be called the *good basis* of $H_1^{\rho}(D_n, z; R[H]^k)$. With respect to the good bases of $H_1^{\beta_*\rho}(D_n, z; R[H]^k)$ and $H_1^{\rho}(D_n, z; R[H]^k)$, the homomorphism of (11.4) gives rise to a $kn \times kn$ matrix with coefficients in $R[H]$. This matrix will be denoted by $\mathscr{B}_\rho(\beta)$.

Definition 38. The *twisted Burau map* $\mathscr{B}_\rho\colon B_c \to GL_{nk}(R[H])$ sends a colored braid β to the matrix $\mathscr{B}_\rho(\beta) \in GL_{nk}(R[H])$ defined above.

If $R = \mathbb{Z}$ and ρ is the trivial one-dimensional representation, then one gets a homomorphism $B_c \to GL_n(\mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}])$ which coincides with the unreduced colored Gassner representation described in Section 9.4. In particular, Definition 38 generalizes both the unreduced Burau representation and the unreduced Gassner representation, see also Example 11.3.5 below.

The next proposition shows that while the twisted Burau maps are generally not representations, the situation remains quite manageable. Since the result was essentially proved in Corollary 5.7.2 (see also Proposition 9.4.3), we shall be quite brief.

Proposition 11.3.3. *If $\beta, \gamma \in B_c$ are two μ-colored braids, then $\mathscr{B}_\rho(\beta\gamma) = \mathscr{B}_{\gamma_*\rho}(\beta)\mathscr{B}_\rho(\gamma)$ for each representation ρ of the free group $\pi_1(D_n)$.*

Proof. We saw in Remark 11.3.1 that the twisted Burau maps arises as induced homomorphisms on twisted homology. The proposition now follows from Corollary 5.7.2. $\square$

While a result similar to Proposition 11.3.3 was already observed in [19, Equation (12)] by means of Fox calculus, in our setting it is simply a particular case of the behavior of induced maps on the twisted homology.

The next proposition provides a formula to compute the twisted Burau maps. The idea is to adapt the proof of Proposition 9.3.4 and (9.7) to the twisted setting.

Proposition 11.3.4. *Let $\beta \in B_c$ be a colored braid. Consider the $(n \times n)$-matrix A whose (i,j) component is*

$$(\rho \otimes \psi_c)\left(\frac{\partial(x_i h_\beta)}{\partial x_j}\right) \in M_k(R[H]).$$

If one views A as a $(nk \times nk)$-matrix with coefficients in $R[H]$, then $\mathscr{B}_\rho(\beta)$ is equal to A.

Proof. Fix a lift of z to the universal cover. Given a homeomorphism h_β representing a braid β, let $\widetilde{h}_\beta$ be the map induced by the lift of h_β on the chain group $C_1(\widetilde{D}_n, \widetilde{z})$. As $H_1^\rho(D_n, z; R[H]^k) = R[H]^k \otimes_{\mathbb{Z}[\pi_1(D_n)]} C_1(\widetilde{D}_n, \widetilde{z})$, the twisted Burau map is given by the homomorphism $\mathscr{B}_\rho(\beta) = id \otimes \widetilde{h}_\beta$. Clearly $\widetilde{x}_i \widetilde{h}_\beta$ is the lift of a loop representing $x_i h_\beta$ to the universal cover. Remark 9.3.3 then shows that on the chain group level

$$\widetilde{x}_i \widetilde{h}_\beta = \sum_{j=1}^{n} \frac{\partial(x_i h_\beta)}{\partial x_j} \widetilde{x}_j.$$

As we view elements of the left $\mathbb{Z}[\pi_1(D_n)]$-module $C_1(\widetilde{D}_n, \widetilde{z})$ as row vectors, $\widetilde{h}_\beta$ is represented by the $(n \times n)$ matrix whose (i,j) component is $\frac{\partial(x_i h_\beta)}{\partial x_j}$. The claim now follows from the right $\mathbb{Z}[\pi_1(D_n)]$-module structure of $R[H]^k$. □

As in Remark 9.4.2 and Proposition 9.4.3, the definitions and propositions of this subsection can easily be adapted to include (c, c')-braids. Namely, to each (c, c')-braid, one may associate a $(kn \times kn)$-matrix which coincides with the $(n \times n)$-matrix whose (i,j) coefficient is $(\rho \otimes \psi_{c'})\left(\frac{\partial(x_i h_\beta)}{\partial x_j}\right) \in M_k(R[H])$. Proposition 11.3.4 also generalizes to this setting and the following example provides a concrete example.

Example 11.3.5. Performing the same Fox calculus computation as in the proof of Proposition 9.3.4, we see that with respect to the good bases, the twisted Burau map of σ_i (viewed as a (c, c')-colored braid) is given by

$$\mathscr{B}_\rho(\sigma_i) = I_{(i-1)k} \oplus \begin{pmatrix} I_k - \rho(x_i x_{i+1} x_i^{-1}) t_{c'_{i+1}} & \rho(x_i) t_{c'_i} \\ I_k & 0 \end{pmatrix} \oplus I_{(n-i-1)k}.$$

If ρ is the trivial one-dimensional representation, $\mu = 1$ and $R = \mathbb{Z}$, then one recovers the unreduced colored Gassner matrices of (9.7) and in particular the unreduced Burau matrices of (9.2).

Combining Proposition 11.3.3 with Example 11.3.5, we see that the twisted Burau maps of (c, c)-colored braids can be computed using (c, c')-colored braids. This observation is a generalization both of Example 9.4.4 and of the computations carried out in Section 9.5.

11.3.2 The reduced twisted Burau map

In the classical case, we saw in Subsection 9.3.2 that the reduced Burau matrices could be recovered using two distinct constructions: the first involved induced maps on absolute homology, while the second made use of a different set of generators for the free group. Surprisingly enough, we saw in Section 9.4 that in the colored case these two constructions do not coincide. Along the way, we observed that the module $H_1(D_c; \Lambda_\mu)$ is not free for $\mu > 2$. In the twisted case, these problems only get worse. Therefore we start by generalizing the "algebraic reduced colored Gassner representation".

Proposition 11.3.6. *Fix a basepoint $z \in \partial D_n$. For each n-stranded (c, c)-braid β, the twisted Burau map $\mathscr{B}_\rho(\beta)$ fixes a free submodule of $H_1^{\beta_*\rho}(D_c, z; R[H]^k)$ of rank k.*

Proof. As in Subsection 9.3.2 and Section 9.4, instead of working with the free generators $x_1, x_2 \dots, x_n$ of $\pi_1(D_n)$, we consider the elements $g_1, g_2, \dots, g_n$, where $g_i = x_1 x_2 \cdots x_i$. Let $\widetilde{g}_i$ be the lift of g_i starting at a fixed lift of z. Using the same argument as in Lemma 11.3.2, one obtains the splitting $H_1^{\beta_*\rho}(D_n, z; R[H]^k) = \bigoplus_{i=1}^{n-1} R[H]^k \widetilde{g}_i \oplus R[H]^k \widetilde{g}_n$. As g_n is always fixed by the action of the braid group, its lift $\widetilde{g}_n$ is fixed by the lift $\widetilde{h}_\beta$ of a homeomorphism h_β representing a braid β. This concludes the proof of the proposition. □

The following definition generalizes the definiton of the algebraic reduced colored Gassner representation and in particular of the reduced Burau representation, see Definition 35.

Definition 39. The *reduced twisted Burau map* $\overline{\mathscr{B}}_\rho \colon B_c \to GL_{(n-1)k}(R[H])$ sends a braid β to the restriction $\overline{\mathscr{B}}_\rho(\beta) \in GL_{(n-1)k}(R[H])$ of the twisted Burau map to the free $R[H]$-module of rank $k(n-1)$ given by the proof of Proposition 11.3.6.

As we mentioned above, we encounter some problems when trying to generalize Definition 30 to the twisted case. Indeed, the $R[H]$-module $H_1^\rho(D_n; R[H])$ need not be free: recall Lemma 9.4.6 and see Subsection 11.5.2 below. Thus, while the usual "induced map on absolute homology" approach can be applied, the result seems hard to compute in general.

On the other hand, the reduced twisted Burau maps of Definition 39 remain computable. Indeed, if $\widetilde{\mathscr{B}}_\rho(\beta)$ denotes the twisted Burau matrix of a braid β with respect to the basis described in the proof of Proposition 11.3.6, then Proposition 11.3.6 implies that

$$\widetilde{\mathscr{B}}_\rho(\beta) = \begin{pmatrix} \overline{\mathscr{B}}_\rho(\beta) & V \\ 0 & I_k \end{pmatrix} \tag{11.5}$$

for some $(k(n-1) \times k)$-matrix V. As in the unreduced case, the definition of the reduced twisted Burau maps may easily be adapted to include the case of (c, c')-braids. We illustrate this with a concrete example.

Example 11.3.7. Combining Proposition 11.3.4 and (11.5), the reduced twisted Burau map of σ_i, viewed as a (c, c')-colored braid, is given by

$$\overline{\mathscr{B}}_\rho(\sigma_i) = I_{(i-2)k} \oplus \begin{pmatrix} I_k & 0 & 0 \\ \rho(g_{i+1}g_i^{-1})t_{c'_{i+1}} & -\rho(g_{i+1}g_i^{-1})t_{c'_{i+1}} & I_k \\ 0 & 0 & I_k \end{pmatrix} \oplus I_{(n-i-2)k}$$

for $1 < i < n-1$, and for σ_1 and σ_{n-1} it is represented by

$$\overline{\mathscr{B}}_\rho(\sigma_1) = \begin{pmatrix} -\rho(g_2g_1^{-1})t_{c_2'} & I_k \\ 0 & I_k \end{pmatrix} \oplus I_{(n-3)k},$$

$$\overline{\mathscr{B}}_\rho(\sigma_{n-1}) = I_{(n-3)k} \oplus \begin{pmatrix} I_k & 0 \\ \rho(g_ng_{n-1}^{-1})t_{c_n'} & -\rho(g_ng_{n-1}^{-1})t_{c_n'} \end{pmatrix}.$$

If ρ is the trivial one-dimensional representation and $R = \mathbb{Z}$, then one recovers the reduced colored Gassner matrices of (9.10) and in particular the reduced Burau matrices of (9.3).

Finally, note that the reduced twisted Burau maps clearly satisfy the same property with respect to braid composition as in the unreduced case, see Proposition 11.3.3. We conclude this subsection with a concrete example.

Example 11.3.8. Setting $\mu = 1$, we will compute the reduced Burau map of $\sigma_1^3 \in B_2$ for any representation ρ. Using Example 11.3.7, $\overline{\mathscr{B}}_\rho(\sigma_1) = -\rho(g_2g_1^{-1})t$. As $(g_2g_1^{-1})h_{\sigma_1} = g_2g_1g_2^{-1}$, it follows from Proposition 11.3.3 that

$$\overline{\mathscr{B}}_\rho(\sigma_1^3) = \overline{\mathscr{B}}_{\sigma_{1*}\rho}\overline{\mathscr{B}}_{\sigma_{1*}\rho}(\sigma_1)\overline{\mathscr{B}}_\rho(\sigma_1) = -\rho(g_2g_1g_2^{-1})\rho(g_2g_1g_2^{-1})\rho(g_2g_1^{-1})t^3 = -\rho(g_2g_1)t^3,$$

and consequently one gets $\overline{\mathscr{B}}_\rho(\sigma_1^3) = -\rho(x_1x_2x_1)t^3$.

11.4 Proof of Theorem 11.1.1

In this section, we relate the reduced twisted Burau maps to twisted torsion invariants of colored links. This will generalize both Burau and Birman's results to the twisted setting.

Let R be a Noetherian factorial domain. If β is a braid with n strands, then $\pi_1(S^3 \setminus \widehat{\beta})$ admits a presentation where the n generators $x_1, x_2, \ldots, x_n$ are subject to the relations $x_i = x_ih_\beta$ for $i = 1, 2, \ldots, n$. Although we will come back to this fact in the proof of the theorem, we wish to point out that in general, a representation $\rho\colon F_n \to GL_k(R)$ need do not extend to a representation of $\pi_1(S^3 \setminus \widehat{\beta})$.

Using this set-up, we can now prove Theorem 11.1.1 whose statement we recall for the reader's convenience.

Theorem 11.1.1. *Let F_n be the free group on $x_1, x_2, \ldots, x_n$ and let $\beta \in B_c$ be a μ-colored braid with n strands. If $\rho\colon F_n \to GL_k(R)$ is a representation which extends to $\pi_1(S^3 \setminus \widehat{\beta})$, then we have*

$$\tau^\rho(\widehat{\beta})(t_1, t_2, \ldots, t_\mu) \det\left(\rho(x_1x_2\cdots x_n)t_{c_1}t_{c_2}\cdots t_{c_n} - I_k\right) \doteq \det(\overline{\mathscr{B}}_\rho(\beta) - I_{(n-1)k}). \qquad (11.6)$$

Proof. Let X_β be the exterior of the braid β in the cylinder $D^2 \times [0,1]$. The manifold obtained by gluing X_β and X_{id_c} along $D_c \sqcup D_c$ is nothing but the exterior of the link $\widehat{\beta} \cup \partial D_c$ in S^3. Consequently the exterior $X_{\widehat{\beta}}$ of $\widehat{\beta}$ can be obtained by gluing the solid torus $D^2 \times \partial D_c$ to $X_{\widehat{\beta}\cup\partial D_c}$ along $\partial D^2 \times \partial D_c$. Identify the free group F_n with $\pi_1(D_c)$ so that the free generators x_i correspond to the loops described in Section 9.2. As in Subsection 11.3.2, the elements $g_1, g_2, \ldots, g_n$ then also form a free generating set of $\pi_1(D_c)$. If x is a meridian of ∂D_c, then van Kampen's theorem implies that $\pi_1(X_{\widehat{\beta}\cup\partial D_c})$ admits a presentation P' where the $n+1$

generators $g_1, g_2, \ldots, g_n, x$ are subject to the n relations $x^{-1}g_ix = g_i\beta$. The representation ρ extends to $\pi_1(X_{\widehat{\beta}\cup\partial D_c})$ by setting $\rho(x) = I_k$. A second application of van Kampen's theorem ensures that this extension of ρ coincides with the representation induced by ρ on $\pi_1(S^3 \setminus \widehat{\beta})$. The same remark goes for ψ_c by setting $\psi_c(x) = 1$.

Since g_n freely generates $\pi_1(D^2 \times \partial D_c)$, the chain complex $C_*^\rho(D^2 \times \partial D_c; Q(H)^k)$ is acyclic and the twisted torsion of $D^2 \times \partial D_c$ is equal to $1/\det(\rho(g_n)\psi_c(g_n) - I_k)$. Using excision, one observes that $H_2^\rho(X_{\widehat{\beta}}, X_{\widehat{\beta}\cup\partial D_c}; R[H]^k)$ is torsion and $H_1^\rho(X_{\widehat{\beta}}, X_{\widehat{\beta}\cup\partial D_c}; R[H]^k)$ vanishes. Consequently, the long exact sequence of the pair $(X_{\widehat{\beta}}, X_{\widehat{\beta}\cup\partial D_c})$ with coefficients in $Q(H)^k$ reduces to

$$0 \to H_1^\rho(X_{\widehat{\beta}\cup\partial D_c}; Q(H)^k) \to H_1^\rho(X_{\widehat{\beta}}; Q(H)^k) \to 0. \tag{11.7}$$

Therefore, if $H_1^\rho(X_{\widehat{\beta}}; R[H]^k)$ is torsion, then so is $H_1^\rho(X_{\widehat{\beta}\cup\partial D_c}; R[H]^k)$. In this case, Lemma 11.2.4 implies that the twisted chain complexes of $X_{\widehat{\beta}}$ and $X_{\widehat{\beta}\cup\partial D_c}$ are acyclic over $Q(H)^k$. Applying Proposition 10.2.1 to the short exact sequence of chain complexes resulting from the decomposition $X_{\widehat{\beta}} = X_{\widehat{\beta}\cup\partial D_c} \cup (D^2 \times \partial D_c)$ yields

$$\tau^{\rho\otimes\psi_c}(\widehat{\beta})(t_1, \ldots, t_\mu) \det\left(\rho(x_1x_2\cdots x_n)t_{c_1}t_{c_2}\cdots t_{c_n} - I_k\right) \doteq \tau^{\rho\otimes\psi_c}(\widehat{\beta} \cup \partial D_c)(t_1, \ldots, t_\mu).$$

Let us now color the trivial knot ∂D_c so that $\widehat{\beta} \cup \partial D_c$ becomes a $\mu + 1$-colored link via a sequence c'. If H' is the free abelian group on $t_1, \ldots, t_\mu, t_{\mu+1}$ and $\psi_{c'}$ is the homomorphism which coincides with ψ_c on the g_i and sends x to $t_{\mu+1}$, then one obtains

$$\tau^{\rho\otimes\psi_c}(\widehat{\beta} \cup \partial D_c)(t_1, \ldots, t_\mu) \doteq \tau^{\rho\otimes\psi_{c'}}(\widehat{\beta} \cup \partial D_c)(t_1, \ldots, t_\mu, 1).$$

Let $A^{\rho\otimes\psi_{c'}}$ be the $(n \times (n+1))$ matrix obtained by performing Fox calculus on the previously described deficiency one presentation of $\pi_1(X_{\widehat{\beta}\cup\partial D_c})$. A short computation shows that $\frac{\partial(g_ih_\beta - x^{-1}g_ix)}{\partial g_j} = \frac{\partial(g_ih_\beta)}{\partial g_j} - x^{-1}\delta_{ij}$. Consequently, using (11.5), the $(nk \times nk)$ matrix resulting from the deletion of the $(n+1)$-th column of $A^{\rho\otimes\psi_{c'}}$ is

$$A_{n+1}^{\rho\otimes\psi_{c'}} = \begin{pmatrix} \overline{\mathscr{B}}_\rho(\beta) - t_{\mu+1}^{-1}I_{(n-1)k} & V \\ 0 & I_k(1 - t_{\mu+1}^{-1}) \end{pmatrix}$$

for some $(k(n-1) \times k)$-matrix V. As $A_{n+1}^{\rho\otimes\psi_{c'}}$ is an upper triangular block matrix, its determinant is the product of the diagonal blocks. Using Wada's characterization of the twisted torsion and simplifying the $I_k(1 - t_{\mu+1}^{-1})$ terms, the twisted torsion of $\widehat{\beta} \cup \partial D_c$ is equal (up to the indeterminacy of the twisted torsion) to $\det\left(\overline{\mathscr{B}}_\rho(\beta) - t_{\mu+1}^{-1}I_{(n-1)k}\right)$. This concludes the proof when $H_1^\rho(X_{\widehat{\beta}}; R[H]^k)$ is torsion.

Finally, if $H_1^\rho(X_{\widehat{\beta}}; R[H]^k)$ is not torsion, then neither is $H_1^\rho(X_{\widehat{\beta}\cup\partial D_c}; R[H]^k)$ by (11.7). Thus the left hand side of (11.6) vanishes, see Remark 11.2.1. We claim that the right hand side of (11.6) also vanishes. Note that $\det(\overline{\mathscr{B}}_\rho(\beta) - I_{(n-1)k})$ is equal to $\tau^\rho(W_{P'})$, where P' is the presentation we chose for $\pi_1(X_{\widehat{\beta}\cup\partial D_c})$. Thus to prove the claim, it is enough to show that $\tau^\rho(W_{P'})$ is equal to $\tau^\rho(\widehat{\beta} \cup \partial D_c)$: indeed since $H_1^\rho(X_{\widehat{\beta}\cup\partial D_c}; R[H]^k)$ is not torsion, $\tau^\rho(\widehat{\beta} \cup \partial D_c)$ vanishes. Since the twisted torsion can be computed from orders of modules [102, Section 4], $\tau^\rho(W_{P'})$ only depends on $\pi_1(X_{\widehat{\beta}\cup\partial D_c}) = \pi_1(W_{P'})$. Consequently $\tau^\rho(W_{P'})$ is equal to $\tau^\rho(\widehat{\beta} \cup \partial D_c)$, as desired. □

While this is the proof which appeared in [50], the first part of the argument can actually be simplified quite a bit. Namely, as explained in Section 12.5 below, the presentation for $\pi_1(S^3 \setminus \widehat{\beta})$ actually reduces to a deficiency one presentation P, see (12.2). The result then follows promptly by using the 2-complex W_P and Example 10.2.3. This idea is exploited at the begining of the proof of Theorem 12.1.1.

11.5 Further remarks

This section is organized as follows. In Subsection 11.5.1, we give an example of Theorem 11.1.1. In Subsection 11.5.2, we explain why the "induced map" approach to reduced twisted Burau maps leads to some difficulties.

11.5.1 An example of Theorem 11.1.1

Set $\mu = 1$ and consider the braid $\sigma_1^3 \in B_2$ whose closure is the trefoil knot T. The group $\pi_1(S^3 \setminus T)$ admits a presentation with two generators x_1, x_2 and a unique relation $x_1x_2x_1 = x_2x_1x_2$. If r denotes the element $x_1x_2x_1 - x_2x_1x_2$ of $\mathbb{Z}[\pi_1(S^3 \setminus T)]$, then Fox calculus shows that

$$\frac{\partial r}{\partial x_1} = 1 - x_2 + x_1x_2 \quad \text{and} \quad \frac{\partial r}{\partial x_2} = -1 - x_1 + x_2x_1.$$

Let $\rho : F_2 \to GL_2(\mathbb{Z}[s^{\pm 1}])$ be the representation given by

$$\rho(x_1) = \begin{pmatrix} -s & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(x_2) = \begin{pmatrix} 1 & 0 \\ s & -s \end{pmatrix}.$$

If $\psi : \pi_1(S^3 \setminus T) \to \mathbb{Z} = \langle t \rangle$ is the abelianization homomorphism sending x_i to t for $i = 1, 2$, then a short computation shows that

$$\det\left(\rho \otimes \psi\left(\frac{\partial r}{\partial x_1}\right)\right) = \det\begin{pmatrix} 1-t & -st^2 \\ -st+st^2 & 1+st-st^2 \end{pmatrix} = (1-t)(1+st)(1-st^2)$$

and

$$\det((\rho \otimes \psi)(1 - x_1)) = \det\begin{pmatrix} 1+st & -t \\ 0 & 1-t \end{pmatrix} = (1-t)(1+st).$$

Therefore, up to the indeterminacy, which in this case is $\pm s^m t^n$, where $m, n \in \mathbb{Z}$, the twisted torsion of T is

$$\tau^\rho(T)(t) \doteq 1 - st^2.$$

Now let us compute the remaining expressions involved in (11.6). Using Example 11.3.8, the reduced twisted Burau map of the braid σ_1^3 with respect to ρ is given by

$$\overline{\mathscr{B}}_\rho(\sigma_1^3) = -\rho(x_1)\rho(x_2)\rho(x_1)t^3 = \begin{pmatrix} 0 & st^3 \\ s^2t^3 & 0 \end{pmatrix}.$$

Consequently, one obtains

$$\det(\overline{\mathscr{B}}_\rho(\sigma_1^3) - I_2) = \det\left(\begin{pmatrix} 0 & st^3 \\ s^2t^3 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 1 - s^3t^6,$$

and

$$\det(\rho(x_1)\rho(x_2)t^2 - I_2) = \det\left(\begin{pmatrix} -s & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s & -s \end{pmatrix} t^2 - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 1 + st^2 + s^2t^4.$$

Combining these observations, its follows that

$$\frac{\det(\overline{\mathscr{B}}_\rho(\sigma_1^3))}{\det(\rho(x_1)\rho(x_2)t^2 - I_2)} = \frac{1 - s^3t^6}{1 + st^2 + s^2t^4} = 1 - st^2.$$

This coincides with our computation of $\tau^\rho(\widehat{\sigma_1^3})(t)$, as predicted by Theorem 11.1.1.

11.5.2 Freeness issues

Let h_β be a homeomorphism representing a (c, c)-braid β. Inspired by the untwisted case, it is tempting to define the reduced twisted Burau of β as the map $H_1^{\beta_*\rho}(D_c; R[H]^k) \to H_1^\rho(D_c; R[H]^k)$ induced by h_β on twisted homology. Unfortunately, in general, $H_1^\rho(D_c; R[H]^k)$ has no reason of being free as a left $R[H]$-module. The next proposition summarizes what we know about this situation.

Proposition 11.5.1. *Let $\rho\colon F_n \to GL_k(R)$ be a representation and let $\psi_c\colon F_n \to H$ be the epimorphism mapping the generator x_i of F_n to t_{c_i}.*

(a) The dimension of the $Q(H)$-vector space $H_1^\rho(D_n; Q(H)^k)$ is $k(n-1)$.

(b) If $\mu = 1$ and R is a principal ideal domain, then $H_1^\rho(D_n; R[H]^k)$ is a free $R[H]$-module of rank $k(n-1)$.

Proof. Fix a basepoint z of D_n and consider the portion

$$0 \to H_1^\rho(D_n; R[H]^k) \to H_1^\rho(D_n, z; R[H]^k) \to H_0^\rho(z; R[H]^k) \to H_0^\rho(D_n; R[H]^k)$$

of the long exact sequence of the pair (D_n, z). As $H_1^\rho(D_n, z; R[H]^k)$ and $H_0^\rho(z; R[H]^k)$ are free $R[H]$-modules, and $H_0^\rho(D_n; R[H]^k)$ is torsion, the first assertion follows by taking the tensor product with $Q(H)$. The second assertion is an immediate consequence of the following algebraic claim: if R is a principal ideal domain and one has a sequence of $R[t^{\pm 1}]$-modules

$$0 \to K \to P \to F,$$

where P and F are free and finitely-generated, then K is also free. To prove the claim, first note that since R is principal, the ring $R[t^{\pm 1}]$ has global dimension 2 [158, Theorem 4.3.7]. Following word for word the proof of [42, Lemma 3.7] (with R instead of $\mathbb{Z}$), it then follows that K is projective. Since P is finitely-generated over the Noetherian ring $R[t^{\pm 1}]$, K is also finitely generated. The conclusion now follows from the fact that if R is a principal ideal domain, then every finitely-generated projective $R[t^{\pm 1}]$-module is free [145]. □

Chapter 12

L^2-Burau maps and L^2-Alexander torsions

12.1 Introduction and statement of the results

As we saw in Chapter 11, twisted invariants of a space X aim to refine classical invariants via the additional data of a *finite dimensional* representation of the fundamental group $\pi_1(X)$. In the infinite dimensional case, one could therefore expect the action of $\pi_1(X)$ on the Hilbert space $\ell^2(\pi_1(X))$ to lead to new link invariants. Historically however, this is not the way L^2-invariants were born. Indeed, it was Atiyah, in his work on the L^2-index theorem, who first defined L^2-Betti numbers via analytical methods [6]. A wealth of activity followed, leading to a vast literature on the subject. Consequently, we refer to [117] for an overview of L^2-theory (some basics are reviewed in Section 12.2) and promptly return to knots.

The notion of L^2-Alexander polynomials took some time to crystallize. First, Li and Zhang defined an L^2-Alexander polynomial for a knot via Fox calculus [111]. Further work on this knot invariant was carried out by Ben Aribi [12] and Dubois-Wegner [64]. Recently, Dubois-Friedl-Luck [63] introduced the so-called L^2-Alexander torsion. As we shall review in Section 12.3, this invariant generalizes the L^2-Alexander polynomial in the same way as the Reidemeister torsion generalizes the classical Alexander polynomial. More precisely, if one fixes a homomorphism $\phi\colon \pi_1(X) \to \mathbb{Z}$ and a second homomorphism $\gamma\colon \pi_1(X) \to G$ through which ϕ factors, then the L^2-Alexander torsion is a real valued function (defined up to some indeterminacy):

$$T^{(2)}(X, \phi, \gamma)\colon \mathbb{R}_{>0} \to \mathbb{R}.$$

The similarities with the classical theory are now striking. For instance the torsion enjoys symmetry properties [62], is related to the genus [63] and can be computed via Fox calculus. Taking these similarities into account, it is natural to wonder whether some L^2-Alexander torsions can be computed from braids. The aim of this chapter (which is based on joint work with Fathi Ben Aribi [13]) is to develop a theory of L^2-Burau representations and to provide such relations.

Let us outline briefly the data required to define the *L^2-Burau maps*, referring to Section 12.4 for details. First, recall that F_n denotes the free group on $x_1, \ldots, x_n$. Next, fix $t > 0$ and a homomorphism $\gamma\colon F_n \to G$ through which the epimorphism $\phi\colon F_n \to \mathbb{Z}, x_i \mapsto 1$ factors. Denoting by $B(\ell^2(G))$, the algebra of bounded operators on $\ell^2(G)$, we define an *unreduced*

L^2*-Burau map* and a *reduced* L^2*-Burau map*

$$\overline{\mathscr{B}}^{(2)}_{t,\gamma}\colon B_n \to M_{n,n}(B(\ell^2(G)))$$

and study their properties, see for instance Propositions 12.4.3, 12.4.4 and 12.4.7. The main theorem of this chapter relates the reduced Burau map $\overline{\mathscr{B}}^{(2)}_{t,\gamma}$ of a braid β to a certain $L^{(2)}$-torsion of its closure. To state this result, whose proof can be found in Section 12.5, we introduce some further notation.

Let β be a braid with closure L. Identifying the free group F_n with $\pi_1(D_n)$, there is a canonical epimorphism $F_n \to \pi_1(X_L)$ which we denote by γ_L. In particular, one may consider the corresponding reduced L^2-Burau map $\overline{\mathscr{B}}^{(2)}_{t,\gamma_L}$. Finally, let $\phi\colon \pi_1(X_L) \to \mathbb{Z}$ be the homomorphism which sends each meridian to one. The main result of this chapter reads as follows.

Theorem 12.1.1. *Given an oriented link L obtained as the closure of a braid $\beta \in B_n$, one has*

$$T^{(2)}(X_L, \phi, id)(t) \cdot \max(1,t)^n \doteq \det{}^r_{\mathscr{N}(G_L)}\left(\overline{\mathscr{B}}^{(2)}_{t,\gamma_L}(\beta) - Id^{\oplus(n-1)}\right)$$

for all $t > 0$.

In Corollary 12.5.1, we shall apply Theorem 12.1.1 in order to show that our L^2-Burau maps distinguish more braids that the classical Burau representation.

We conclude this introduction with some additional remarks. The definition of the L^2-Burau maps follows closely the definition of the twisted Burau maps of Section 11.3. In order to avoid repetitions, we shall therefore give slightly less details than in Chapter 11. On the other hand, despite a similar statement, the proof of Theorem 12.1.1 has a somewhat different flavor than the proof of Theorem 11.1.1: some additional technical difficulties must be overcome (e.g. the L^2-torsion is only known to be invariant under *simple* homotopy). Finally, note that Section 12.2 is not meant to be an introductory text on L^2 invariants. The same remark goes for Section 12.3 and we respectively refer to [117] and [11, 63] for detailed accounts of these subjects.

12.2 Hilbert $\mathscr{N}(G)$-modules and the L^2-torsion.

In this section, we briefly review some theory of L^2-invariants: Subsection 12.2.1 begins with the von Neumann dimension of a finitely generated Hilbert $\mathscr{N}(G)$-module, Subsection 12.2.2 deals with the Fuglede-Kadison determinant and Subsection 12.2.3 discusses the L^2-torsion. We mostly follow [117] and [63].

12.2.1 The von Neumann dimension

Given a countable discrete group G, the completion of the algebra $\mathbb{C}[G]$ endowed with the scalar product $\left\langle \sum_{g\in G}\lambda_g g, \sum_{g\in G}\mu_g g\right\rangle := \sum_{g\in G}\lambda_g\overline{\mu_g}$ is the Hilbert space

$$\ell^2(G) := \left\{ \sum_{g\in G}\lambda_g g \mid \lambda_g \in \mathbb{C}, \sum_{g\in G}|\lambda_g|^2 < \infty \right\}$$

of square-summable complex functions on G. We denoted by $B(\ell^2(G))$ the algebra of operators on $\ell^2(G)$ that are bounded with respect to the operator norm.

Given $h \in G$, we define the corresponding *left-* and *right-multiplication operators* L_h and R_h in $B(\ell^2(G))$ as extensions of the automorphisms $(g \mapsto hg)$ and $(g \mapsto gh)$ of G. One can extend the operators R_h $\mathbb{C}$-linearly to operators $R_w \colon \ell^2(G) \to \ell^2(G)$ for any $w \in \mathbb{C}[G]$. Moreover, if $\ell^2(G)^n$ is endowed with its usual Hilbert space structure and $A = (a_{i,j}) \in M_{p,q}(\mathbb{C}[G])$ is a $\mathbb{C}[G]$-valued $p \times q$ matrix, then the right multiplication

$$R_A := \left(R_{a_{i,j}}\right)_{1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q}$$

provides a bounded operator $\ell^2(G)^q \to \ell^2(G)^p$. Note that we shall consider elements of $\ell^2(G)^n$ as *column vectors* and suppose that matrices with coefficients in $B(\ell^2(G))$ act on the *left* (even though the coefficients may be *right*-multiplication operators).

The *von Neumann algebra* $\mathscr{N}(G)$ of the group G is the sub-algebra of $B(\ell^2(G))$ made up of G-equivariant operators (i.e. operators that commute with all left multiplications L_h). A *finitely generated Hilbert* $\mathscr{N}(G)$*-module* consists of a Hilbert space V together with a left G-action by isometries such that there exists a positive integer m and an embedding φ of V into $\ell^2(G)^m$. A *morphism of finitely generated Hilbert* $\mathscr{N}(G)$*-modules* $f \colon U \to V$ is a linear bounded map which is G-equivariant.

Denoting by e the neutral element of G, the von Neuman algebra of G is endowed with the *trace* $\mathrm{tr}_{\mathscr{N}(G)} \colon \mathscr{N}(G) \to \mathbb{C}, \phi \mapsto \langle \phi(e), e \rangle$ which extends to $\mathrm{tr}_{\mathscr{N}(G)} \colon M_{n,n}(\mathscr{N}(G)) \to \mathbb{C}$ by summing up the traces of the diagonal elements.

Definition 40. The *von Neumann dimension* of a finitely generated Hilbert $\mathscr{N}(G)$-module V is defined as

$$\dim_{\mathscr{N}(G)}(V) := \mathrm{tr}_{\mathscr{N}(G)}(\mathrm{pr}_{\varphi(V)}) \in \mathbb{R}_{\geqslant 0},$$

where $\mathrm{pr}_{\varphi(V)} \colon \ell^2(G)^m \to \ell^2(G)^m$ is the orthogonal projection onto $\varphi(V)$.

The von Neumann dimension does not depend on the embedding of V into the finite direct sum of copies of $\ell^2(G)$.

12.2.2 The Fuglede-Kadison determinant

The *spectral density* $F(f) \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ of a morphism $f \colon U \to V$ of finitely generated Hilbert $\mathscr{N}(G)$-modules maps $\lambda \in \mathbb{R}_{\geqslant 0}$ to

$$F(f)(\lambda) := \sup\{\dim_{\mathscr{N}(G)}(L) | L \in \mathscr{L}(f, \lambda)\},$$

where $\mathscr{L}(f, \lambda)$ is the set of finitely generated Hilbert $\mathscr{N}(G)$-submodules of U on which the restriction of f has a norm smaller or equal to λ. Since $F(f)(\lambda)$ is monotonous and right-continuous, it defines a measure $dF(f)$ on the Borel set of $\mathbb{R}_{\geqslant 0}$ solely determined by the equation $dF(f)(]a, b]) = F(f)(b) - F(f)(a)$ for all $a < b$.

Definition 41. The *Fuglede-Kadison determinant of* f is defined by

$$\det_{\mathscr{N}(G)}(f) = \begin{cases} \exp\left(\int_{0^+}^{\infty} \ln(\lambda)\, dF(f)(\lambda)\right) & \text{if } \int_{0^+}^{\infty} \ln(\lambda)\, dF(f)(\lambda) > -\infty, \\ 0 & \text{otherwise .} \end{cases}$$

Moreover, when $\int_{0^+}^{\infty} \ln(\lambda)\, dF(f)(\lambda) > -\infty$, one says that f *is of determinant class.*

If U and V have the same von Neumann dimension, we define the *regular Fuglede-Kadison determinant of* f, denoted by $\det^r_{\mathscr{N}(G)}(f)$, as $\det_{\mathscr{N}(G)}(f)$ when f is injective, and zero otherwise. For later use, let us mention the following property of the determinant, see [117] for the proof.

Proposition 12.2.1. *Let G be a countable discrete group. If $g \in G$ is of infinite order, then for all $t \in \mathbb{C}$ the operator $Id - tR_g$ is injective and $\det_{\mathscr{N}(G)}(Id - tR_g) = \max(1, |t|)$.*

The next subsection reviews the L^2 analogue of the torsion of a chain complex.

12.2.3 L^2-torsion of a finite Hilbert $\mathscr{N}(G)$-chain complex

A *finite Hilbert $\mathscr{N}(G)$-chain complex* C_* is a sequence of morphisms of finitely generated Hilbert $\mathscr{N}(G)$-modules $0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$ such that $\partial_p \circ \partial_{p+1} = 0$ for all p. The *p-th L^2-homology* of C_* is the finitely generated Hilbert $\mathscr{N}(G)$-module

$$H_p^{(2)}(C_*) := \operatorname{Ker}(\partial_p)/\overline{\operatorname{Im}(\partial_{p+1})}.$$

The *p-th L^2-Betti number of* C_* is defined as $b_p^{(2)}(C_*) := \dim_{\mathscr{N}(G)}(H_p^{(2)}(C_*))$. A finite Hilbert $\mathscr{N}(G)$-chain complex C_* is *weakly acyclic* if its L^2-homology is trivial (i.e. if all its L^2-Betti numbers vanish) and of *determinant class* if all the operators ∂_p are of determinant class. The following result is a reformulation of [117, Theorem 1.21 and Theorem 3.35 (1)]:

Proposition 12.2.2. *Let $0 \to C_* \xrightarrow{\iota_*} D_* \xrightarrow{\rho_*} E_* \to 0$ be an exact sequence of finite Hilbert $\mathscr{N}(G)$-chain complexes. If two of the finite Hilbert $\mathscr{N}(G)$-chain complexes C_*, D_*, E_* are weakly acyclic (respectively weakly acyclic and of determinant class), then the third is as well.*

The main definition of this section is the following.

Definition 42. The L^2-*torsion* of a finite Hilbert $\mathscr{N}(G)$-chain complex C_* is defined as

$$T^{(2)}(C_*) := \prod_{i=1}^{n} \det{}_{\mathscr{N}(G)}(\partial_i)^{(-1)^i} \in \mathbb{R}_{>0}$$

when C_* is weakly acyclic and of determinant class, and as $T^{(2)}(C_*) := 0$ otherwise.

Let $C_* = (0 \to \ell^2(G)^k \xrightarrow{\partial_2} \ell^2(G)^{k+l} \xrightarrow{\partial_1} \ell^2(G)^l \to 0)$ be a finite Hilbert $\mathscr{N}(G)$-chain complex and let $J \subset \{1, \dots, k+l\}$ be a subset of size l. Viewing ∂_1, ∂_2 as matrices with coefficients in $B(\ell^2(G))$, denote by $\partial_1(J)$ the operator composed of the columns of ∂_1 indexed by J, and by $\partial_2(J)$ the operator obtained from ∂_2 by deleting the rows indexed by J. We refer to [63, Lemma 3.1] for the proof of the following proposition.

Proposition 12.2.3. *Assume that $\partial_1(J)$ is injective and of determinant class. Then*

$$T^{(2)}(C_*) = \frac{\det^r_{\mathscr{N}(G)}(\partial_2(J))}{\det^r_{\mathscr{N}(G)}(\partial_1(J))}.$$

In particular, $\partial_2(J)$ is injective and of determinant class if and only if C_* is weakly acyclic and of determinant class, and in this case one can write

$$T^{(2)}(C_*) = \frac{\det_{\mathscr{N}(G)}(\partial_2)}{\det_{\mathscr{N}(G)}(\partial_1)} = \frac{\det_{\mathscr{N}(G)}(\partial_2(J))}{\det_{\mathscr{N}(G)}(\partial_1(J))}.$$

The reader might want to compare Proposition 12.2.3 to Example 10.2.3.

12.3 The L^2-Alexander torsion

In this section, we briefly review the L^2-Alexander torsion introduced by Dubois, Friedl and Luck [63]. Along the way, we recall some notions related to the L^2-homology of CW-complexes.

Let X be a compact connected CW-complex endowed with a basepoint z. Setting $\pi = \pi_1(X, z)$, an *admissible triple* (π, ϕ, γ) consists of homomorphisms $\phi\colon \pi \to \mathbb{Z}$ and $\gamma\colon \pi \to G$ such that ϕ factors through γ. Given such a triple and $t > 0$, if we denote by

$$\kappa(\pi, \phi, \gamma, t)\colon \mathbb{Z}[\pi] \to \mathbb{R}[G]$$

the ring homomorphism determined by $g \mapsto t^{\phi(g)}\gamma(g)$ for $g \in \pi$, then there is a right action of π on $\ell^2(G)$ given by $a \cdot g = R_{\kappa(\pi,\phi,\gamma,t)(g)}(a)$, where $a \in \ell^2(G)$ and $g \in \pi$. This turns $\ell^2(G)$ into a right $\mathbb{Z}[\pi]$-module.

Let Y be a connected CW-subcomplex of X, denote by $p\colon \widetilde{X} \to X$ the universal cover of X and write $\widetilde{Y} = p^{-1}(Y)$. Since the cellular chain complex $C_*(\widetilde{X}, \widetilde{Y})$ consists of left $\mathbb{Z}[\pi]$-modules, we can consider the *$\mathscr{N}(G)$-cellular chain complex* of the pair (X, Y) associated to (ϕ, γ, t) which is the finite Hilbert $\mathscr{N}(G)$-chain complex

$$C_*^{(2)}(X, Y; \phi, \gamma, t) := \ell^2(G) \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}, \widetilde{Y}).$$

The *L^2-homology* of (X, Y) associated to (ϕ, γ, t), denoted $H_*^{(2)}(X, Y; \phi, \gamma, t)$, is obtained by taking the L^2-homology of $C_*^{(2)}(X, Y; \phi, \gamma, t)$. The next remark deals with induced maps on L^2-homology.

Remark 12.3.1. Given an admissible triple $(\pi_1(X', z'), \phi, \gamma\colon \pi_1(X', z') \to G)$, note that a basepoint-preserving homeomorphism of pairs $F\colon (X, Y) \to (X', Y')$ induces an isomorphism $f\colon \pi_1(X) \to \pi_1(X')$ and isomorphisms of finitely generated Hilbert $\mathscr{N}(G)$-modules

$$H_i^{(2)}(F)\colon H_i^{(2)}(X, Y; \phi \circ f, \gamma \circ f, t) \to H_i^{(2)}(X', Y'; \phi, \gamma, t).$$

The proof follows from the same considerations as in the twisted case and so we refer to Subsection 5.7.1 for details.

We now move towards the definition of the L^2-Alexander torsion of Dubois-Friedl-Luck [63]. Recall that an *admissible triple* (π, ϕ, γ) consists of homomorphisms $\phi\colon \pi \to \mathbb{Z}$ and $\gamma\colon \pi \to G$ such that ϕ factors through γ. The main definition of this section is the following.

Definition 43. The *L^2-Alexander torsion* of (X, ϕ, γ) at $t > 0$ is defined as

$$T^{(2)}(X, \phi, \gamma)(t) := T^{(2)}\left(C_*^{(2)}(X; \phi, \gamma, t)\right).$$

Note that L^2-Alexander torsions are only defined up to multiplication by $(t \mapsto t^k)$ with $k \in \mathbb{Z}$. For this reason, we shall write $f(t) \doteq g(t)$ if f is equal to g up to multiplication by $(t \mapsto t^k)$ for $k \in \mathbb{Z}$. Moreover the L^2-Alexander torsions are invariant by simple homotopy equivalence [11, 35, 63, 117].

Remark 12.3.2. Looking back at Definition 42, we note that $T^{(2)}(X, \phi, \gamma)(t) \neq 0$ if and only if $C_*^{(2)}(X; \phi, \gamma, t)$ is weakly acyclic and of determinant class.

Next, we briefly review how Fox calculus can be used to compute the L^2-Alexander torsion. Recall that the 2-complex W_P associated to a presentation $P = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ of a group π is constructed as follows: there is one 0-cell v, n oriented 1-cells labeled $x_1, x_2, \ldots, x_n$ and m oriented 2-cells $c_1, c_2, \ldots, c_m$ with each ∂c_j glued to the 1-cells according to the word r_i. Note that $\pi_1(W_P) = \pi$ and that the $\mathbb{Z}[\pi]$-module $C_1(\widetilde{W_P})$ is freely generated by the $\widetilde{x}_i$.

Remark 12.3.3. We shall assume that the elements in the chain complex $C_*(\widetilde{W_P})$ of free left $\mathbb{Z}[\pi]$-modules are *column* vectors and that the matrices of the differentials ∂_i act by *left* multiplication. These conventions differ from the ones used up to now, compare in particular with Chapters 9 and 11.

Denote by $\mathrm{pr} : \mathbb{Z}[F_n] \to \mathbb{Z}[\pi]$ the ring homomorphism induced by the quotient map. Recall from Remark 9.3.3 that ∂_2 is represented by the $(n \times m)$ matrix whose (i, j)-coefficient is $\mathrm{pr}\left(\frac{\partial r_j}{\partial x_i}\right)$. Combining these remarks with Propositions 12.2.1 and 12.2.3, together with the fact that for any integer k and any $t > 0$, $\max\left(1, t^k\right) = t^{\frac{k-|k|}{2}} \max(1, t)^{|k|} \doteq \max(1, t)^{|k|}$, the following result is immediate.

Proposition 12.3.4. *Let $P = \langle x_1, \ldots, x_n | r_1, \ldots, r_{n-1} \rangle$ be a deficiency one presentation of a group π, fix $t > 0$ and let $(\pi, \phi\colon \pi \to \mathbb{Z}, \gamma\colon \pi \to G)$ be an admissible triple. If one denotes by A the matrix in $M_{n-1,n-1}(\mathbb{C}[G])$ whose (i, j) component is*

$$\kappa(\pi, \phi, \gamma, t)\left(\mathrm{pr}\left(\frac{\partial r_j}{\partial x_i}\right)\right)$$

and one assumes that $\gamma(x_n)$ has infinite order in G, then

$$T^{(2)}(W_P, \phi, \gamma)(t) \doteq \frac{\det^r_{\mathcal{N}(G)}(R_A)}{\det^r_{\mathcal{N}(G)}\left(t^{\phi(x_n)} R_{\gamma(x_n)} - Id\right)} \doteq \frac{\det^r_{\mathcal{N}(G)}(R_A)}{\max(1, t)^{|\phi(x_n)|}}.$$

Moreover, if M is an irreducible 3-manifold with non-empty toroidal boundary and infinite $\pi = \pi_1(M)$, then

$$T^{(2)}(M, \phi, \gamma)(t) \doteq \frac{\det^r_{\mathcal{N}(G)}(R_A)}{\det^r_{\mathcal{N}(G)}(t^{\phi(x_n)} R_{\gamma(x_n)} - Id)} \doteq \frac{\det^r_{\mathcal{N}(G)}(R_A)}{\max(1, t)^{|\phi(x_n)|}}.$$

The second part of the above proposition uses the fact that the L^2-Alexander torsions are invariant under simple homotopy equivalence and the following lemma, see [5, Section 3.2] for a proof (or [13, Lemma 3.4] for an argument involving L^2-Betti numbers).

Lemma 12.3.5. *Let M be an irreducible 3-manifold with non-empty toroidal boundary and infinite fundamental group. If P is a deficiency one presentation of $\pi_1(M)$, then M is simple homotopy equivalent to W_P.*

Next, we apply this machinery to links. Given an oriented link $L = K_1 \cup \ldots \cup K_\mu$ in S^3, denote by X_L its exterior and by $G_L = \pi_1(X_L)$ its group. Since any homomorphism $\phi\colon G_L \to \mathbb{Z}$ factors through the abelianization $\alpha_L\colon G_L \to H_1(X_L) \cong \mathbb{Z}^\mu$, it is determined by integers $n_1, \ldots, n_\mu$. Following the notation of [11], we denote by $(n_1, \ldots, n_\mu)\colon H_1(X_L) \to \mathbb{Z}$ the map sending the i-th meridian of L to n_i (thus $\phi = (n_1, \ldots, n_\mu) \circ \alpha_L$) and we call

$$T^{(2)}_{L,(n_1,\ldots,n_\mu)}(\gamma)(t) := T^{(2)}(X_L, \phi, \gamma)(t)$$

the *L^2-Alexander torsion associated to the link L* and the morphism $\gamma\colon G_L \to G$ at the value $t > 0$. The next lemma underlines a crucial difference between the L^2-Alexander torsion and the classical Alexander polynomial, and should be compared with Proposition 3.5.4.

Lemma 12.3.6. *Let L be a link, $t > 0$ and $n_1, \ldots, n_\mu \in \mathbb{Z}$. The following assertions are equivalent:*

1. *L is split.*
2. *$C^{(2)}(X_L, (n_1, \ldots, n_\mu) \circ \alpha_L, id, t)$ is not weakly acyclic.*
3. *The L^2-Alexander torsion $T^{(2)}_{L,(n_1,\ldots,n_\mu)}(id)(t)$ vanishes.*

Proof. If the μ-component link L is not split, then its exterior X_L is irreducible, and it follows from [118] that for all integers $n_1, \ldots, n_\mu$ and all $t > 0$, $C^{(2)}(X_L, (n_1, \ldots, n_\mu) \circ \alpha_L, id, t)$ is weakly acyclic and of determinant class. Thus, in this case $T^{(2)}_{L,(n_1,\ldots,n_\mu)}(id)(t)$ is non-zero, proving that (3) implies (1). The fact that (2) implies (3) is immediate.

We conclude by proving that (1) implies (2). If L is split, then X_L is not irreducible, and one can write $X_L = X_1 \# \ldots \# X_r$, where the X_i are irreducible link exteriors in S^3. Let us order the X_i so that

$$X_L = (X_1 \setminus B^3) \cup \left(\bigcup_{i=2}^{r-1} (X_i \setminus (B^3 \sqcup B^3)) \right) \cup (X_r \setminus B^3),$$

where the intersection is a disjoint union of $r-1$ spheres S^2. Fix $t > 0$, integers $n_1, \ldots, n_\mu \in \mathbb{Z}$ (we denote $\phi_L = (n_1, \ldots, n_\mu) \circ \alpha_L$) and let j_i be the group monomorphism induced by the inclusion of X_i minus one or two balls into X_L. An immediate generalization of the proof of [11, Theorem 3.1] (see also [117]) implies that

$$0 \to \bigoplus_{i=1}^{r-1} C^{(2)}_*(S^2, 1, 1, t) \to \begin{array}{c} C^{(2)}_*(X_1 \setminus B^3, \phi_L \circ j_1, j_1, t) \oplus \\ \bigoplus_{i=2}^{r-1} C^{(2)}_*(X_i \setminus (B^3 \sqcup B^3), \phi_L \circ j_i, j_i, t) \\ \oplus C^{(2)}_*(X_r \setminus B^3, \phi_L \circ j_r, j_r, t) \end{array} \to C^{(2)}_*(X_L, \phi_L, id, t) \to 0$$

is an exact sequence of finite Hilbert $\mathscr{N}(G_L)$-chain complexes.

Now, for all $i = 1, \ldots, r-1$, we add a term $\ell^2(G_L)\widetilde{B^3} \oplus \ell^2(G_L)\widetilde{B^3}$ to the i-th summand of the left part of the sequence and to the i-th and $(i+1)$-th summands of the middle part (one ball for each), where the boundary operators send one $\widetilde{B^3}$ to the corresponding $\widetilde{S^2}$ and the other to a corresponding $-\widetilde{S^2}$. Since this process does not change exactness of the sequence, it follows that

$$0 \to \bigoplus_{i=1}^{r-1} C^{(2)}_*(S^3, 1, 1, t) \to \bigoplus_{i=1}^{r} C^{(2)}_*(X_i, \phi_L \circ j_i, j_i, t) \to C^{(2)}_*(X_L, \phi_L, id, t) \to 0$$

remains an exact sequence of finite Hilbert $\mathcal{N}(G_L)$-chain complexes. Each j_i is an injective group homomorphism and thus induces an induction functor, as explained in [117, Section 1.1.5]. As weak acyclicity is unaffected by these induction functors, the first part of the proof applied to the irreducible pieces X_i shows that $\bigoplus_{i=1}^{r} C_*^{(2)}(X_i, \phi_L \circ j_i, j_i, t)$ is weakly acyclic. Since the left part of the above short exact sequence is not weakly acyclic (see [117, Theorem 1.35 (8)]), neither is $C^{(2)}(X_L, (n_1, \ldots, n_\mu) \circ \alpha_L, id, t)$ by Proposition 12.2.2. □

12.4 The L^2-Burau maps

In this section, we define L^2-Burau maps and reduced L^2-Burau maps.

Fix a basepoint $z \in \partial D_n$ and let $h_\beta : D_n \to D_n$ be a homeomorphism representing a braid $\beta \in B_n$. As h_β fixes the boundary of the disk D^2, it lifts uniquely to a homeomorphism $\widetilde{h}_\beta \colon \widetilde{D}_n \to \widetilde{D}_n$ which preserves a fixed lift of z. Denote by $\phi \colon \pi_1(D_n) \to \mathbb{Z}$ the epimorphism defined by $x_i \mapsto 1$. Fixing $t > 0$ and a homomorphism $\gamma \colon \pi_1(D_n) \to G$ through which ϕ factors, Remark 12.3.1 implies that h_β induces a well-defined isomorphism of finitely generated Hilbert $\mathcal{N}(G)$-modules:

$$H_1^{(2)}(h_\beta) \colon H_1^{(2)}(D_n, z; \phi \circ h_\beta, \gamma \circ h_\beta, t) \to H_1^{(2)}(D_n, z; \phi, \gamma, t). \tag{12.1}$$

Since $\phi \circ h_\beta = \phi$ holds for all braids $\beta \in B_n$, from now on we shall write ϕ instead of $\phi \circ h_\beta$. Before moving on the L^2-Burau map, let us fix some conventions.

Remark 12.4.1. In contrast with Chapters 9 and 11, compositions of maps shall be read from right to left. Consequently, the right action of B_n on F_n produces an anti-representation $B_n \to \mathrm{Aut}(F_n)$.

If we wish to define an L^2-Burau map as an induced map on homology, then (12.1) shows that the result cannot be an automorphism: the domain and target of induced maps in L^2 homology differ. For this reason, we shall proceed as in Chapter 11: instead of defining the L^2-Burau maps as the homomorphisms of (12.1), we pick canonical bases to make the situation more concrete. The proof of the following lemma proceeds as in the twisted case, see Lemma 11.3.2.

Lemma 12.4.2. *Given $z \in D_n$, for all admissible (π, ϕ, γ) and all $t > 0$, the finitely generated Hilbert $\mathcal{N}(G)$-module $H_1^{(2)}(D_n, z; \phi, \gamma, t)$ has von Neumann dimension n.*

Proof. The punctured disk D_n is simple homotopy equivalent to X, the wedge of the n loops representing the generators of $\pi_1(D_n)$ described in Section 9.2. As a consequence, it follows from [117, Theorem 1.21] and the proof of [11, Theorem 2.12] that $H_1^{(2)}(X, z; \phi, \gamma, t)$ and $H_1^{(2)}(D_n, z; \phi, \gamma, t)$ have same von Neumann dimension. Thus it suffices to prove the claim for X.

Choose a cellular decomposition of this latter space X consisting of the 0-cell z (the basepoint of the wedge) and one 1-cell x_i for each loop. For $i = 1, 2, \ldots, n$, let $\widetilde{x}_i$ be the lift of x_i starting at an (arbitrary) fixed lift of z. With this cell structure, the $\mathcal{N}(G)$-cellular chain complex of the pair (X, z) associated to (ϕ, γ, t) is $0 \to C_1^{(2)}(X, z; \phi, \gamma, t) \to C_0^{(2)}(X, z; \phi, \gamma, t) \to 0$, where $C_1^{(2)}(X, z; \phi, \gamma, t) = \ell^2(G) \otimes_{\mathbb{Z}[\pi]} C_1(\widetilde{X}, \widetilde{z}) \cong \bigoplus_{i=1}^{n} \ell^2(G)\widetilde{x}_i$. Since $C_0^{(2)}(\widetilde{X}, \widetilde{z})$ vanishes, $H_1^{(2)}(X, z; \phi, \gamma, t) = C_1^{(2)}(X, z; \phi, \gamma, t)$ and the claim follows. □

Using the same notations as in the proof of Lemma 12.4.2, we shall call the basis resulting from the isomorphism $H_1^{(2)}(D_n,z;\phi,\gamma,t)\cong\bigoplus_{i=1}^n \ell^2(G)\widetilde{x}_i$ the *good basis* of $H_1^{(2)}(D_n,z;\phi,\gamma,t)$. With respect to the good bases of $H_1^{(2)}(D_n,z;\phi,\gamma\circ h_\beta,t)$ and $H_1^{(2)}(D_n,z;\phi,\gamma,t)$, the isomorphism of finitely generated $\mathscr{N}(G)$-modules $H_1^{(2)}(h_\beta)$ gives rise to a $n\times n$ matrix $\mathscr{B}^{(2)}_{t,\gamma}(\beta)$ with coefficients in $B(\ell^2(G))$.

Definition 44. The L^2*-Burau map* $\mathscr{B}^{(2)}_{t,\gamma}$ associated to the value $t>0$ and the homomorphism γ sends a braid $\beta\in B_n$ to the matrix $\mathscr{B}^{(2)}_{t,\gamma}(\beta)\in M_{n,n}\big(B(\ell^2(G))\big)$ representing the isomorphism of finitely generated Hilbert $\mathscr{N}(G)$-modules defined above.

The next lemma shows that while the L^2-Burau map is generally not an anti-representation, it is nevertheless determined by the generators of B_n. The proof is omitted since it proceeds just as in the twisted case, see Proposition 11.3.3 and Corollary 5.7.2.

Proposition 12.4.3. *Given two braids $\alpha,\beta\in B_n$, the equation*

$$\mathscr{B}^{(2)}_{t,\gamma}(\alpha\beta)=\mathscr{B}^{(2)}_{t,\gamma}(\beta)\circ\mathscr{B}^{(2)}_{t,\gamma\circ h_\beta}(\alpha)$$

holds for all $t>0$ and for all $\gamma\colon \pi_1(D_n)\to G$ through which ϕ factors.

In particular, Proposition 11.3.3 shows that if one picks a homomorphism γ satisfying $\gamma\circ h_\beta=\gamma$ for each $\beta\in B_n$, then the L^2-Burau maps $\mathscr{B}^{(2)}_{t,\gamma}$ yield anti-representations of the braid group. More generally, fixing $\gamma\colon\pi_1(D_n)\to G$, the L^2-Burau maps $\mathscr{B}^{(2)}_{t,\gamma}$ provide anti-representations of $B_n^\gamma:=\{\beta\in B_n\mid \gamma\circ h_\beta=\gamma\}$.

The next proposition shows that the L^2-Burau map can be computed via Fox calculus.

Proposition 12.4.4. *Let $\beta\in B_n$ be a braid. If one denotes by A the $(n\times n)$-matrix whose (i,j) component is*

$$\kappa(\pi_1(D_n),\phi,\gamma,t)\left(\frac{\partial(h_\beta(x_j))}{\partial x_i}\right)\in\mathbb{C}[G],$$

then $\mathscr{B}^{(2)}_{t,\gamma}(\beta)$ is equal to R_A.

Proof. The proof follows the same lines as Proposition 11.3.4. Fix a lift of z to the universal cover $p\colon\widetilde{D}_n\to D_n$. Given a homeomorphism h_β representing a braid β, let $\widetilde{h}_\beta$ be the map induced by the lift of h_β on the chain group $C_1(\widetilde{D}_n,\widetilde{z})$ (where $\widetilde{z}=p^{-1}(z)$). As $H_1^{(2)}(D_n,z;\phi,\gamma,t)\cong\ell^2(G)\otimes_{\mathbb{Z}[\pi_1(D_n)]}C_1(\widetilde{D}_n,\widetilde{z})$, it remains to compute the operator $id\otimes\widetilde{h}_\beta$. Clearly $\widetilde{h}_\beta(\widetilde{x}_j)$ is the lift of a loop representing $h_\beta(x_j)$ to the universal cover. Remark 9.3.3 then shows that on the chain level

$$\widetilde{h}_\beta(\widetilde{x}_j)=\sum_{i=1}^n\frac{\partial(h_\beta(x_j))}{\partial x_i}\widetilde{x}_i.$$

As we view elements of the left $\mathbb{Z}[\pi_1(D_n)]$-module $C_1(\widetilde{D}_n,\widetilde{z})$ as column vectors (recall Remark 12.3.3), $\widetilde{h}_\beta$ is represented by the $(n\times n)$ matrix whose (i,j) component is $\frac{\partial(h_\beta(x_j))}{\partial x_i}$. The claim now follows from the right $\mathbb{Z}[\pi_1(D_n)]$-module structures of $\ell^2(G)$. □

We wish to emphasize once again that Propositions 12.4.3 and 12.4.4 use different conventions than Chapters 9 and 11, see in particular Remarks 9.3.5 and 12.4.1.

Remark 12.4.5. In this section, we compose maps on the L^2-chain complexes from right to left (and not from left to right) and the elements of these chain complexes are viewed as column vectors (and not row vectors). This explains the differences between Propositions 12.4.3 and 11.3.3 and Propositions 12.4.4 and 11.3.4.

We now illustrate Proposition 12.4.4 and Proposition 12.4.3 with some concrete examples.

Example 12.4.6. A short computation involving Fox calculus shows that with respect to the good bases, the L^2-Burau maps of σ_i are given by

$$\mathscr{B}^{(2)}_{t,\gamma}(\sigma_i) = Id^{\oplus(i-1)} \oplus \begin{pmatrix} Id - tR_{\gamma(x_i x_{i+1} x_i^{-1})} & Id \\ tR_{\gamma(x_i)} & 0 \end{pmatrix} \oplus Id^{\oplus(n-i-1)}.$$

Let us now use these computations together with Proposition 12.4.4 in order to illustrate Proposition 12.4.3 with an example. For $\sigma_1, \sigma_2 \in B_3$, we get the matrices

$$\mathscr{B}^{(2)}_{t,\gamma}(\sigma_2) = \begin{pmatrix} Id & 0 & 0 \\ 0 & Id - tR_{\gamma(x_2 x_3 x_2^{-1})} & Id \\ 0 & tR_{\gamma(x_2)} & 0 \end{pmatrix}, \mathscr{B}^{(2)}_{t,\gamma\circ h_{\sigma_2}}(\sigma_1) = \begin{pmatrix} Id - tR_{\gamma(x_1 x_2 x_3 x_2^{-1} x_1^{-1})} & Id & 0 \\ tR_{\gamma(x_1)} & 0 & 0 \\ 0 & 0 & Id \end{pmatrix},$$

whose composition is equal to

$$\mathscr{B}^{(2)}_{t,\gamma}(\sigma_2) \circ \mathscr{B}^{(2)}_{t,\gamma\circ h_{\sigma_2}}(\sigma_1) = \begin{pmatrix} Id - tR_{\gamma(x_1 x_2 x_3 x_2^{-1} x_1^{-1})} & Id & 0 \\ tR_{\gamma(x_1)} - t^2 R_{\gamma(x_1 x_2 x_3 x_2^{-1})} & 0 & Id \\ t^2 R_{\gamma(x_1 x_2)} & 0 & 0 \end{pmatrix},$$

which coincides with $\mathscr{B}^{(2)}_{t,\gamma}(\sigma_1\sigma_2)$.

Let us now relate the L^2-Burau maps to the classical Burau representation $\mathscr{B}_T$. Note that contrarily to Chapter 9, we use a capital "T": in this chapter, "t" already denotes a positive real number. Given $\beta \in B_n$, recall that a matrix for $\mathscr{B}_T(\beta) \in M_{n,n}(\mathbb{Z}[T, T^{-1}])$ can be obtained by computing $T^\phi\left(\dfrac{\partial(h_\beta(x_i))}{\partial x_j}\right)$, where the ring homomorphism $T^\phi \colon \mathbb{Z}[F_n] \to \mathbb{Z}[T, T^{-1}]$ sends x_i to the indeterminate T. For any given γ and t, the L^2-Burau map $\mathscr{B}^{(2)}_{t,\gamma}$ holds at least as much information as the classical Burau representation, in the following sense:

Proposition 12.4.7. *Given $\beta \in B_n$, for any $t > 0$ and $\gamma \colon F_n \to G$, there exists a map $\Theta \colon B(\ell^2(G)^n) \to M_{n,n}(\mathbb{Z}[T, T^{-1}])$ such that $\Theta\left(\mathscr{B}^{(2)}_{t,\gamma}(\beta)\right) = \mathscr{B}_T(\beta)$. In particular, if $\alpha, \beta \in B_n$ and $\mathscr{B}^{(2)}_{t,\gamma}(\alpha) = \mathscr{B}^{(2)}_{t,\gamma}(\beta)$, then $\mathscr{B}_T(\alpha) = \mathscr{B}_T(\beta)$.*

Proof. Using Proposition 12.4.4, $\mathscr{B}^{(2)}_{t,\gamma}(\beta)$ is the right-multiplication operator R_A where the matrix A has $\kappa(F_n, \phi, \gamma, t)\left(\frac{\partial(h_\beta(x_j))}{\partial x_i}\right) \in \mathbb{C}[G]$ as its (i,j)-coefficient. By considering the map $\theta \colon B(\ell^2(G)^n) \to M_{n,n}(\ell^2(G))$ which evaluates an operator S on the n canonical (column) vectors of $\ell^2(G)^n$, one can extract $A = \theta(R_A)$ from R_A. Thus it only remains to recover $\mathscr{B}_T(\beta)$ from A.

Since $(\pi_1(D_n), \phi, \gamma)$ is an admissible triple, there exists a homomorphism $\psi\colon G \to \mathbb{Z}$ such that $\phi = \psi \circ \gamma$. Defining the homomorphism $T^\psi\colon G \to \{T^m; m \in \mathbb{Z}\} \subset \mathbb{Z}[T, T^{-1}]$ by $g \mapsto T^{\psi(g)}$, the (i,j)-coefficient of $\kappa(G, \psi, T^\psi, t^{-1})(A)$ is

$$\left(\kappa(G, \psi, T^\psi, t^{-1}) \circ \kappa(F_n, \phi, \gamma, t)\right)\left(\frac{\partial(h_\beta(x_j))}{\partial x_i}\right) = T^\phi\left(\frac{\partial(h_\beta(x_j))}{\partial x_i}\right),$$

which is precisely the (j,i)-coefficient of $\mathscr{B}_T(\beta)$. The map $\Theta = \mathsf{tra} \circ \kappa(G, \psi, T^\psi, t^{-1}) \circ \theta$ thus satisfies the assumptions of the proposition; here tra is the transpose operator. $\square$

Remark 12.4.8. Although all L^2-Burau maps recover the Burau representation, different choices of $\gamma\colon F_n \to G$ produce various effects on the injectivity of the resulting maps and their defect to being anti-representations. On one end of the spectrum, if γ is the identity, the L^2-Burau maps $\mathscr{B}^{(2)}_{t,id}\colon B_n \to B(\ell^2(F_n)^n)$ are injective for all $t > 0$ (since $B_n \to \mathrm{Aut}(F_n), \beta \mapsto h_\beta$ is injective and automorphisms of the free group are determined by their Fox jacobian [26, Proposition 9.8]). As G becomes smaller, the L^2-Burau maps $\mathscr{B}^{(2)}_{t,\gamma}$ lose in injectivity but edge closer to being actual anti-representations. As the proof of Proposition 12.4.7 demonstrates, a critical step appears when γ reaches T^ϕ: in this case, $\mathscr{B}^{(2)}_{t,T^\phi}(\beta)$ is an anti-representation which is equal to $R_{\mathsf{tra}(\mathscr{B}_T(\beta))}$ up to a change of variable; in particular it is known not to be faithful for $n \geq 5$ [16, 116].

Summarizing, the various L^2-Burau maps distinguish at least as many braids as the Burau representation (as shown in Proposition 12.4.7) but sometimes do better as Corollary 12.5.1 will show.

We now define reduced L^2-Burau maps. As in Subsection 11.3.2, instead of working with the free generators $x_1, x_2 \ldots, x_n$ of $\pi_1(D_n)$, consider the elements $g_1, g_2, \ldots, g_n$, where $g_i = x_1 x_2 \cdots x_i$. Let $\widetilde{g}_i$ be the lift of g_i starting at a fixed lift of z. Using the same argument as in Lemma 12.4.2, one obtains the splitting $H^{(2)}_1(D_n, z; \phi, \gamma, t) = \bigoplus_{i=1}^{n-1} \ell^2(G)\widetilde{g}_i \oplus \ell^2(G)\widetilde{g}_n$ for any $\gamma\colon F_n \to G$ through which ϕ factors. As g_n is always fixed by the action of the braid group, its lift $\widetilde{g}_n$ is fixed by the lift $\widetilde{h}_\beta$ of a homeomorphism h_β representing a braid β.

Definition 45. The *reduced L^2-Burau map* sends a braid β to the restriction $\overline{\mathscr{B}}^{(2)}_{t,\gamma}(\beta)$ of the L^2-Burau map to the subspace of $H_1(D_n, z; \phi, \gamma \circ h_\beta, t)$ generated by $\widetilde{g}_1, \ldots, \widetilde{g}_{n-1}$.

If $\widetilde{\mathscr{B}}^{(2)}_{t,\gamma}(\beta)$ denotes the L^2-Burau matrix of a braid $\beta \in B_n$ with respect to the basis of the $\widetilde{g}_i$, then

$$\widetilde{\mathscr{B}}^{(2)}_{t,\gamma}(\beta) = \begin{pmatrix} \overline{\mathscr{B}}^{(2)}_{t,\gamma}(\beta) & 0 \\ V & Id \end{pmatrix},$$

where $V \in M_{1,n-1}(B(\ell^2(G)))$. Furthermore, note that the reduced L^2-Burau map also satisfies the property of Proposition 11.3.3. Combining these observations, the reduced L^2-Burau maps can be computed by using the same methods as outlined in Example 12.4.6, see [13, Example 4.8].

12.5 Proof of Theorem 12.1.1

In this section, we show how a particular L^2-Alexander torsion associated to a link can be computed from some reduced L^2-Burau maps. As an application, we exhibit two braids which are distinguished by the L^2-Burau maps but cannot be told apart by the classical Burau representation.

We start by building the admissible triple involved in the statement of the theorem. Let X_β be the exterior of a braid $\beta \in B_n$ in the cylinder $D^2 \times [0,1]$. The manifold obtained by gluing X_β and X_{id_n} along $D_n \sqcup D_n$ is nothing but the exterior of the link $L' := \widehat{\beta} \cup \partial D_n$ in S^3. Identify the free group F_n with $\pi_1(D_n)$ so that the free generators x_i correspond to the loops described in Section 9.2. An application of van Kampen's theorem shows that $G_{L'}$ admits the presentation

$$P' = \langle g_1, \dots, g_n, x | h_\beta(g_1) = x g_1 x^{-1}, \dots, h_\beta(g_n) = x g_n x^{-1} \rangle.$$

The exterior X_L of $L = \widehat{\beta}$ can now be recovered by canonically pasting a solid torus on the boundary component of $X_{L'}$ corresponding to ∂D_n. Since $h_\beta(g_n) = g_n$ in the free group F_n, G_L thus admits the following deficiency one presentation:

$$P = \langle g_1, \dots, g_n | h_\beta(g_1) = g_1, \dots, h_\beta(g_{n-1}) = g_{n-1} \rangle. \tag{12.2}$$

Finally, denote by $\gamma_L \colon F_n \to G_L$ the resulting quotient map. This way, if one sets $\phi_L := (1, \dots, 1) \circ \alpha_L$, then the map $\phi \colon \pi_1(D_n) \to \mathbb{Z}$ described in Subsection 12.4 factors as $\phi_L \circ \gamma_L$.

Using this set-up, we can now prove Theorem 12.1.1 whose statement we recall for the reader's convenience.

Theorem 12.1.1. *Given an oriented link L obtained as the closure of a braid $\beta \in B_n$, one has*

$$T^{(2)}_{L,(1,\dots,1)}(id)(t) \cdot \max(1,t)^n \doteq {\det}^r_{\mathscr{N}(G_L)} \left(\overline{\mathscr{B}}^{(2)}_{t,\gamma_L}(\beta) - Id^{\oplus(n-1)} \right)$$

for all $t > 0$.

Proof. Fix $t > 0$ and assume that L is non-split. Performing Fox calculus on the presentation P yields $\frac{\partial(h_\beta(g_j)g_j^{-1})}{\partial g_i} = \frac{\partial(h_\beta(g_j))}{\partial g_i} - \delta_{ij}$. Since X_L is irreducible and the previously described presentation P of G_L has deficiency one, combining Proposition 12.3.4 with the definition of the reduced Burau representation then gives

$$T^{(2)}(X_L, \phi_L, id)(t) \doteq \frac{{\det}^r_{\mathscr{N}(G_L)} \left(\overline{\mathscr{B}}^{(2)}_{t,\gamma_L}(\beta) - Id^{\oplus(n-1)} \right)}{{\det}^r_{\mathscr{N}(G_L)}(t^n R_{g_n} - Id)} = \frac{{\det}^r_{\mathscr{N}(G_L)} \left(\overline{\mathscr{B}}^{(2)}_{t,\gamma_L}(\beta) - Id^{\oplus(n-1)} \right)}{\max(1,t)^n},$$

which proves the theorem in the non-split case.

Next, assume that L is split. Since Lemma 12.3.6 implies that $T^{(2)}_{L,(1,\dots,1)}(id)(t) = 0$, it only remains to prove that ${\det}^r_{\mathscr{N}(G_L)} \left(\overline{\mathscr{B}}^{(2)}_{t,\gamma_L}(\beta) - Id^{\oplus(n-1)} \right)$ also vanishes. By Proposition 12.3.4, the latter claim reduces to proving that $C^{(2)}_*(W_P, \phi_L, id, t)$ is not weakly acyclic. As $C^{(2)}_*(X_L, \phi_L, id, t)$ is not weakly acyclic by Lemma 12.3.6, the L^2-version of the Torres formula [11, Theorem 3.8] implies that $C^{(2)}_*(X_{L'}, \phi_L \circ Q, Q, t)$ is not weakly acyclic either, where $Q \colon G_{L'} \to G_L$ is the epimorphism induced by the inclusion $X_{L'} \subset X_L$. Since L' is non split, $X_{L'}$ is simply homotopy equivalent to $W_{P'}$ by Lemma 12.3.5 and it follows that $C^{(2)}_*(W_{P'}, \phi_L \circ Q, Q, t)$ is not weakly acyclic, by [11, Theorem 2.12]. Let v be the 0-cell of W_P, $g_1, \dots, g_n$ be its 1-cells and $r_1, \dots, r_{n-1}$ be its 2-cells. Similarly let v' be the 0-cell of $W_{P'}$, $g'_1, \dots, g'_n, x$ be its 1-cells and $r'_1, \dots, r'_n$ be its 2-cells. Denote the lifts to the universal covers and set $D_1 = \ell^2(G)\widetilde{x}$, $D_2 = \ell^2(G)\widetilde{r}'_n$. A straightforward matrix computation involving Fox calculus now shows that

$$0 \to C_*^{(2)}(W_P, \phi_L, id, t) \xrightarrow{\iota} C_*^{(2)}(W_{P'}, \phi_L \circ Q, Q, t) \xrightarrow{\rho} D_* \to 0$$

is an exact sequence of finite Hilbert $\mathcal{N}(G_L)$-chain complexes, where $\iota_1(\widetilde{g}_i) = \widetilde{g}'_i, \iota_2(\widetilde{r}_i) = \widetilde{r}'_i$ for $i = 1, \dots, n-1$ and ρ_1, ρ_2 are the obvious projections. As the boundary operator $D_2 \to D_1$ is given by the injective operator $Id - t^n R_{g_n}$, the chain complex D_* is weakly acyclic. Since $C^{(2)}(W_{P'}, \phi_L \circ Q, Q, t)$ is not weakly acyclic, neither is $C^{(2)}(W_P, \phi_L, id, t)$ by Proposition 12.2.2. This concludes the proof. □

Note that if L is a knot K, then Theorem 12.1.1 show that $\Delta_K^{(2)}(t) \cdot \max(1,t)^{n-1}$ is equal to $\det{}^r_{\mathcal{N}(G_K)}\left(\overline{\mathscr{B}}^{(2)}_{t,\gamma_K}(\beta) - Id^{\oplus(n-1)}\right)$, where $\Delta_K^{(2)}(t)$ is the L^2-Alexander invariant of K defined by Li-Zhang [111]. We conclude with an application of Theorem 12.1.1.

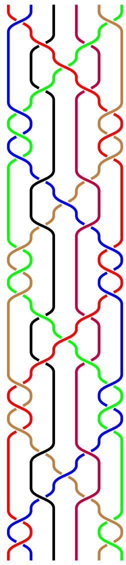

Figure 12.1: The braid $\beta \in B_6$.

Corollary 12.5.1. *There exist two braids which have the same image under the classical Burau representation but have different images under an L^2-Burau map with a non-injective γ.*

Proof. Long and Paton [116] proved that the braid $\beta \in B_6$ depicted in Figure 12.1 has the same image under the classical Burau representation as the trivial braid $id_6 \in B_6$. Taking any $t > 0$, we will prove that $\mathscr{B}^{(2)}_{t,\gamma_{\widehat{\beta}}}(\beta) \neq \mathscr{B}^{(2)}_{t,\gamma_{\widehat{\beta}}}(id_6)$. To achieve this, we will show that $\overline{\mathscr{B}}^{(2)}_{t,\gamma_{\widehat{\beta}}}(\beta) \neq \overline{\mathscr{B}}^{(2)}_{t,\gamma_{\widehat{\beta}}}(id_6)$: this is enough since the reduced L^2-Burau map is the upper left matricial part of the L^2-Burau map expressed in the basis of the $\widetilde{g}_i$. We claim that the closure L of β is a 6-component non-split link. To see this, define $\Gamma(L)$ to be the graph whose vertices are the components L_i of L and such that there is an edge between L_i and L_j when there exists a third component L_k such that $L_i \cup L_j \cup L_k$ is a non-split link. Since L being split implies $\Gamma(L)$ being disconnected, it suffices to show that $\Gamma(L)$ is connected. One can observe that all sublinks of L with three components are either trivial or the non-split link L_{10a140}, and there are enough of the second type so that $\Gamma(L)$ is connected.

Consequently, as L is non-split, $T^{(2)}_{L,(1,\dots,1)}(t)$ is non-zero for all $t > 0$ by Lemma 12.3.6, and thus Theorem 12.1.1 implies that the operator $\overline{\mathscr{B}}^{(2)}_{t,\gamma_L}(\beta) - Id^{\oplus(n-1)}$ has non-zero regular Fuglede-Kadison determinant and is thus injective. The result follows immediately. $\square$

Chapter 13

Colored tangles and the Lagrangian functor

13.1 Introduction

The aim of this chapter is to review a construction, due to Cimasoni and Turaev [42], which extends the Burau representation from braids to tangles. Recall from Section 9.2 that an n-stranded braid consists of sn monotonous intervals in the cylinder $D^2 \times [0,1]$. Moreover, the isotopy classes of these braids give rise to the braid group B_n. Dropping the monotonicity condition yields so-called n-stranded *string links*. Isotopy classes of string links no longer form a group but a monoid. Even more generally, a *tangle* consists of a (particular type of) properly embedded 1-submanifold of the cylinder; examples are depicted in Figure 13.1. This time, isotopy classes of tangles form the morphisms of a category **Tangles** as we shall see in Section 13.2.

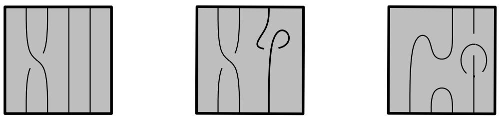

Figure 13.1: From left to right: a braid, a string link which is not a braid, and a tangle which is neither a braid nor a string link.

The reduced Burau representations, which we encountered in Chapter 9, take the form of homomorphisms $B_n \to GL_{n-1}(\Lambda)$ with $\Lambda = \mathbb{Z}[t^{\pm 1}]$, which preserve some non-degenerate skew-Hermitian form on Λ^{n-1}. The construction of these homomorphisms easily extends to *oriented braids*, i.e. braids where different strands can be oriented in different directions. In a slightly pedantic style, one can therefore say that the reduced Burau representations constitute a *reduced Burau functor* $\overline{b}$ from the groupoid **Braids**, with objects finite sequences of signs ± 1 and morphisms oriented braids, to the groupoid $\mathbf{U}_\Lambda$, with objects Λ-modules equipped with a non-degenerate skew-Hermitian form and morphisms unitary Λ-isomorphisms.

In this context, it is natural to ask whether this reduced Burau functor $\overline{b}$ extends to the category **Tangles** of oriented tangles. Such an extension was constructed by Cimasoni-Turaev in [42] and takes the form of the *Lagrangian functor* $\mathbf{Tangles} \to \mathbf{Lagr}_\Lambda$. As we shall see in

Section 13.3, $\mathbf{Lagr}_\Lambda$ is the category of *Lagrangian relations* in which $\mathbf{U}_\Lambda$ embeds via the graph functor: if f is a unitary isomorphism, then its graph Γ_f is a Lagrangian relation. In Section 13.4, we shall review the construction of the Lagrangian functor and recall one of its key features: for any oriented braid β, the Lagrangian relation $\mathscr{F}(\beta)$ coincides with the graph $\Gamma_{\overline{\mathscr{B}}_t(\beta)}$ of the reduced Burau representation $\overline{\mathscr{B}}_t(\beta)$. Finally, as outlined in [42, Section 6], we shall see that the Lagrangian functor also admits a multivariable generalization which extends the (localized) reduced colored Gassner representation.

13.2 The category of colored tangles

In this section, we review the category $\mathbf{Tangles}_\mu$ of μ-colored tangles. References include [42, 152].

Let D^2 be the closed unit disk in $\mathbb{R}^2$. Given a positive integer n, let p_j be the point $((2j-n-1)/n, 0)$ in D^2, for $j = 1, \ldots, n$. Let ε and ε' be sequences of ± 1's of respective length n and n'. An $(\varepsilon, \varepsilon')$-*tangle* is the pair consisting of the cylinder $D^2 \times [0,1]$ and an oriented piecewise linear 1-submanifold τ whose oriented boundary is $\sum_{j=1}^{n'} \varepsilon'_j (p'_j, 1) - \sum_{j=1}^{n} \varepsilon_i (p_j, 0)$.

Example 13.2.1. Observe that tangles encompass both braids and links. Indeed τ is an oriented braid if every component of τ is strictly increasing or strictly decreasing with respect to the projection onto $[0,1]$. On the other hand, a $(\emptyset, \emptyset)$-tangle is nothing but an oriented link.

An oriented tangle τ is called μ-*colored* if each of its components is assigned an element in $\{1, \ldots, \mu\}$. We shall call a μ-colored $(\varepsilon, \varepsilon')$-tangle a (c, c')-*tangle*, where c and c' are the sequences of $\pm 1, \pm 2, \ldots, \pm \mu$ induced by the orientation and coloring of the tangle. Given such a sequence c, we denote by $\ell(c)$ the element in $\mathbb{Z}^\mu$ whose i^{th} coordinate is equal to $\ell(c)_i = \sum_{j; c_j = \pm i} \varepsilon_j$. Note that for a (c, c')-tangle to exist, we must have $\ell(c) = \ell(c')$.

Example 13.2.2. A (c, c')-tangle $\tau \subset D^2 \times [0,1]$ is a colored braid if every component is strictly increasing or strictly decreasing with respect to the projection to $[0,1]$. As in Example 13.2.1, a $(\emptyset, \emptyset)$-colored tangle is nothing but a colored link.

Two (c, c')-tangles τ_1 and τ_2 are *isotopic* if there exists an auto-homeomorphism h of $D^2 \times [0,1]$, keeping $\partial \left(D^2 \times [0,1]\right)$ fixed, such that $h(\tau_1) = \tau_2$ and $h|_{\tau_1} \colon \tau_1 \simeq \tau_2$ is orientation and color-preserving. We shall denote by $T_\mu(c, c')$ the set of isotopy classes of (c, c')-tangles, and by id_c the isotopy class of the trivial (c, c)-tangle $\{p_1, \ldots, p_n\} \times [0,1]$. Given a (c, c')-tangle τ_1 and a (c', c'')-tangle τ_2, their *composition* is the (c, c'')-tangle $\tau_2 \tau_1$ obtained by gluing the two cylinders along the disk corresponding to c' and shrinking the length of the resulting cylinder by a factor 2, see Figure 13.2. Clearly, the composition of tangles induces a composition $T_\mu(c, c') \times T_\mu(c', c'') \to T_\mu(c, c'')$ on the isotopy classes of μ-colored tangles.

Definition 46. The category $\mathbf{Tangles}_\mu$ of μ-colored tangles is defined as follows: the objects are the finite sequences c of elements in $\{\pm 1, \pm 2, \ldots, \pm \mu\}$, and the morphisms are given by $\mathrm{Hom}(c, c') = T_\mu(c, c')$.

The composition is clearly associative, and the trivial tangle id_c plays the role of the identity endomorphism of c. Finite sequences of elements in $\{\pm 1, \pm 2, \ldots, \pm \mu\}$, as objects, and

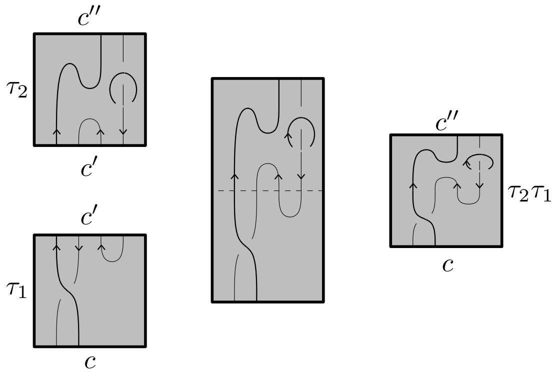

Figure 13.2: A (c, c')-tangle τ_1 with $c = (-1, +2)$ and $c' = (+2, -1, +1, -1)$, a (c', c'')-tangle τ_2 with $c'' = (+2, -1)$, and their composition, the (c, c'')-tangle $\tau_2\tau_1$.

isotopy classes of colored braids, as morphisms, form a subcategory $\mathbf{Braids}_\mu \subset \mathbf{Tangles}_\mu$. The same remark goes for isotopy classes of colored string links, leading to subcategories $\mathbf{Braids}_\mu \subset \mathbf{String}_\mu \subset \mathbf{Tangles}_\mu$. In the case $\mu = 1$, these are simply the categories of oriented braids, oriented string links and oriented tangles.

Remark 13.2.3. Note that the conventions adopted in the following chapters differ from those of Chapters 9, 11 and 12: what we used to call a (c', c)-braid is now a (c, c')-braid.

Finally, given an endomorphism $\tau \in T_\mu(c, c)$, one can define its *closure* as the μ-colored link $\widehat{\tau}$ obtained from τ by adding oriented colored parallel strands in $S^3 \setminus (D^2 \times [0, 1])$, just as in the case of braids.

13.3 The category of Lagrangian relations

In this section, we review the category of Lagrangian relations over an integral domain Λ. We shall closely follow [42, 43], but we also refer to [152] for the easier case where Λ is a field.

Fix an integral domain Λ endowed with a ring involution $a \mapsto \overline{a}$. A *skew-Hermitian form* on a Λ-module H is a map $\lambda\colon H \times H \to \Lambda$ such that for all $x, y, z \in H$ and all $a, a' \in \Lambda$,

1. $\lambda(ax + a'y, z) = a\lambda(x, z) + a'\lambda(y, z)$,
2. $\lambda(x, y) = -\overline{\lambda(y, x)}$,
3. if $\lambda(x, y) = 0$ for all $y \in H$, then $x = 0$.

A *Hermitian Λ-module* H is a finitely generated Λ-module endowed with a skew-Hermitian form λ. The same module H with the opposite form $-\lambda$ will be denoted by $-H$. The *annihilator* of a submodule $A \subset H$ is the submodule $\mathrm{Ann}(A) = \{x \in H \mid \lambda(v, x) = 0 \text{ for all } v \in A\}$. The submodule A is called *Lagrangian* if it is equal to its annihilator. Given a submodule A of a Hermitian Λ-module H, set

$$\overline{A} = \{x \in H \mid ax \in A \text{ for a non-zero } a \in \Lambda\}.$$

Observe that if A is Lagrangian, then $\overline{A} = A$. Moreover, if Λ is a field, then $\overline{A} = A$ for all A. If H and H' are Hermitian Λ-modules, a *Lagrangian relation* from H to H' is a Lagrangian submodule of $(-H) \oplus H'$. For instance, given a Hermitian Λ-module H, the *diagonal relation*

$$\Delta_H = \{h \oplus h \in H \oplus H\}$$

is a Lagrangian relation from H to H. Given two Lagrangian relations N_1 from H to H' and N_2 from H' to H'', their *composition* is defined as $N_2 \circ N_1 := \overline{N_2 N_1} \subset (-H) \oplus H''$, where

$$N_2 N_1 = \{x \oplus z \mid x \oplus y \in N_1 \text{ and } y \oplus z \in N_2 \text{ for some } y \in H'\}.$$

The proof of the next theorem can be found in [42, Theorem 2.7].

Theorem 13.3.1. *Hermitian Λ-modules, as objects, and Lagrangian relations, as morphisms, form a category.*

Following [42], we shall denote this category by $\mathbf{Lagr}_\Lambda$ and call it the *category of Lagrangian relations over* Λ. Let us now briefly recall why Lagrangian relations can be understood as a generalization of unitary Λ-isomorphisms and unitary Q-isomorphisms, where Q is the field of fractions of Λ.

Given two Hermitian Λ-modules (H_1, λ_1) and (H_2, λ_2), a Λ-linear map $\varphi\colon H_1 \to H_2$ is said to be *unitary* if it satisfies $\lambda_2(\varphi(x), \varphi(y)) = \lambda_1(x, y)$ for all $x, y \in H_1$. Let $\mathbf{U}_\Lambda$ be the category of Hermitian Λ-modules and unitary Λ-isomorphisms. Also, let $\mathbf{U}^0_\Lambda$ be the category of Hermitian Λ-modules, where the morphisms between H and H' are the unitary Q-isomorphisms between $H \otimes Q$ and $H' \otimes Q$. The *graph* of a Λ-linear map $f\colon H \to H'$ is the submodule $\Gamma_f = \{x \oplus f(x)\}$ of $H \oplus H'$. Similarly the *restricted graph* of a Q-linear map $\varphi\colon H \otimes Q \to H' \otimes Q$ is $\Gamma^0_\varphi = \Gamma_\varphi \cap (H \oplus H')$. The proof of the following theorem can be found in [42, Theorem 2.9].

Theorem 13.3.2. *The maps $f \mapsto f \otimes id_Q$, $f \mapsto \Gamma_f$ and $\varphi \mapsto \Gamma^0_\varphi$ define faithful functors which are the identity on objects, and which fit in the commutative diagram*

$$\begin{tikzcd} \mathbf{U}_\Lambda \arrow[r, "-\otimes Q"] \arrow[rr, bend right, "\Gamma"'] & \mathbf{U}^0_\Lambda \arrow[r, "\Gamma^0"] & \mathbf{Lagr}_\Lambda . \end{tikzcd}$$

We shall call such functors *embeddings of categories*.

13.4 The Lagrangian functor

In this section, we review the Lagrangian functor of Cimasoni-Turaev [42]. Namely, Subsection 13.4.1 deals with the one-variable case, while Subsection 13.4.2 is concerned with the multivariable case.

13.4.1 The one variable case

Denote by $\mathscr{N}(\{p_1, \ldots, p_n\})$ an open tubular neighborhood of $\{p_1, \ldots, p_n\}$ in $D^2 \subset \mathbb{R}^2$, and let S^2 be the 2-sphere obtained by the one-point compactification of $\mathbb{R}^2$. Given a sequence $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ of ± 1's, set $\ell_\varepsilon = \sum_{i=1}^n \varepsilon_i$ and endow the compact surface

$$D_\varepsilon = \begin{cases} D^2 \setminus \mathscr{N}(\{p_1, \ldots, p_n\}) & \text{if } \ell_\varepsilon \neq 0 \\ S^2 \setminus \mathscr{N}(\{p_1, \ldots, p_n\}) & \text{if } \ell_\varepsilon = 0 \end{cases}$$

with an orientation (pictured counterclockwise), a base point z, and the generating family $\{x_1, \dots, x_n\}$ of $\pi_1(D_\varepsilon, z)$, where x_i is a simple loop turning once around p_i counterclockwise if $\varepsilon_i = +1$, clockwise if $\varepsilon_i = -1$. The same space with the opposite orientation will be denoted by $-D_\varepsilon$.

The natural epimomorphism $H_1(D_\varepsilon) \to \mathbb{Z}$ given by $x_j \mapsto 1$ induces an infinite cyclic covering $\widehat{D}_\varepsilon \to D_\varepsilon$ whose homology is endowed with a structure of module over $\Lambda = \mathbb{Z}[t^{\pm 1}]$. If $\ell_\varepsilon \neq 0$, then D_ε retracts by deformation on the wedge of n circles representing the generators $x_1, \dots, x_n$ of $\pi_1(D_\varepsilon, z)$, and $H_1(\widehat{D}_\varepsilon)$ is a free Λ-module of rank $n-1$, see Lemma 9.3.6. (It is free of rank $n-2$ if ℓ_ε vanishes.) Furthermore, given $x, y \in H_1(\widehat{D}_\varepsilon)$, we saw in Section 9.3.3 that the formula

$$\lambda_\varepsilon(x, y) = \sum_{k \in \mathbb{Z}} \langle t^k x, y \rangle t^{-k}$$

endows the Λ-module $H_1(\widehat{D}_\varepsilon)$ with a Λ-valued skew-Hermitian form. Since we observed in Section 9.3.3 that this form is non-degenerate, following the terminology of Section 13.3, $(H_1(\widehat{D}_\varepsilon), \lambda_\varepsilon)$ is a free Hermitian Λ-module for any object ε of the category of oriented tangles. As the next example illustrates, the skew-Hermitian form λ_ε is in fact quite explicit.

Example 13.4.1. Endow the Λ-module $H_1(\widehat{D}_\varepsilon)$ with the basis $v_1, \dots, v_{n-1}$ described in Lemma 9.3.6. Namely, pick lifts $\widetilde{x}_i$ of the x_i and set $v_i = \widetilde{x}_{i+1} - \widetilde{x}_i$ for $i = 1, \dots, n-1$. When all the ε_i are equal to one, we already computed the matrix for λ_ε in Example 9.3.11. Generalizing the computation to arbitrary ε produces the following matrix:

$$\begin{pmatrix} \frac{1}{2}(\varepsilon_1+\varepsilon_2)(t-t^{-1}) & 1-t^{\varepsilon_2} & 0 & \dots & 0 \\ t^{-\varepsilon_2}-1 & \frac{1}{2}(\varepsilon_2+\varepsilon_3)(t-t^{-1}) & & \ddots & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & \ddots & & & 1-t^{\varepsilon_n} \\ 0 & \dots & 0 & t^{-\varepsilon_n}-1 & \frac{1}{2}(\varepsilon_{n-1}+\varepsilon_n)(t-t^{-1}) \end{pmatrix}. \tag{13.1}$$

Let us now turn to morphisms. First recall that the existence of an $(\varepsilon, \varepsilon')$-tangle $\tau \subset D^2 \times [0,1]$ implies that $\ell_\varepsilon = \ell_{\varepsilon'}$. Denote by $\mathscr{N}(\tau)$ an open tubular neighborhood of τ in $D^2 \times [0,1]$. We shall orient the exterior

$$X_\tau = \begin{cases} (D^2 \times [0,1]) \setminus \mathscr{N}(\tau) & \text{if } \ell_\varepsilon \neq 0 \\ (S^2 \times [0,1]) \setminus \mathscr{N}(\tau) & \text{if } \ell_\varepsilon = 0 \end{cases}$$

of τ so that the induced orientation on ∂X_τ extends the orientation on $(-D_\varepsilon) \sqcup D_{\varepsilon'}$. Clearly, the abelian group $H_1(X_\tau)$ is freely generated by the oriented meridians of the connected components of τ. The homomorphism $H_1(X_\tau) \to \mathbb{Z}$ mapping these meridians to 1 extends the previously defined homomorphisms $H_1(D_\varepsilon) \to \mathbb{Z}$ and $H_1(D_{\varepsilon'}) \to \mathbb{Z}$. It determines an infinite cyclic covering $\widehat{X}_\tau \to X_\tau$ whose homology is endowed with a structure of module over Λ.

Let $i_\tau \colon H_1(\widehat{D}_\varepsilon) \to H_1(\widehat{X}_\tau)$ and $i'_\tau \colon H_1(\widehat{D}_{\varepsilon'}) \to H_1(\widehat{X}_\tau)$ be the homomorphisms induced by the inclusions of $\widehat{D}_c$ and $\widehat{D}_{\varepsilon'}$ into $\widehat{X}_\tau$. Denote by $j_\tau \colon H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}) \to H_1(\widehat{X}_\tau)$ the homomorphism given by $j_\tau(x, x') = i'_\tau(x') - i_\tau(x)$. Finally consider

$$\ker(j_\tau) \subset H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'}) \,.$$

It is proved in [42] that for any tangle τ, the module $\overline{\ker(j_\tau)}$ is Lagrangian. It can also be checked that $\ker(j_{\tau_1\tau_2}) = \ker(j_{\tau_1})\ker(j_{\tau_2})$ for any tangles τ_1, τ_2. This leads to the main result of [42].

Theorem 13.4.2. *Let $\mathscr{F}$ assign to each map ε the pair $(H_1(\widehat{D}_\varepsilon), \lambda_\varepsilon)$ and to each $(\varepsilon, \varepsilon')$-tangle τ the submodule $\overline{\ker(j_\tau)}$ of $H_1(\widehat{D}_\varepsilon) \oplus H_1(\widehat{D}_{\varepsilon'})$. Then $\mathscr{F}$ is a functor which fits in the commutative diagram*

$$\begin{tikzcd}
\mathbf{Braids} \arrow[r] \arrow[d, "\overline{b}"] & \mathbf{String} \arrow[r] \arrow[d] & \mathbf{Tangles} \arrow[d, "\mathscr{F}"] \\
\mathbf{U}_\Lambda \arrow[r, "-\otimes Q"] \arrow[rr, bend right, "\Gamma"'] & \mathbf{U}^0_\Lambda \arrow[r, "\Gamma^0"] & \mathbf{Lagr}_\Lambda,
\end{tikzcd}$$

where the left-most vertical arrow is the reduced Burau functor and the horizontal arrows are the embeddings of categories described in Sections 13.2 and 13.3.

In fact, as in Chapter 12, Cimasoni and Turaev think of the right action of B_n on F_n as an *anti*-representation, recall Remark 9.3.5. Consequently, there is slight subtlety involved in the statement of Theorem 13.4.2.

Remark 13.4.3. Picking the bases which we have been using repeatedly (see Lemmas 9.3.1 and 9.3.6), it can be checked that both $\sigma_i \mapsto \overline{\mathscr{B}}_t(\sigma_i)$ and $\overline{\mathscr{B}}_t(\sigma_i)^T$ give rise to representations of the braid group which we denote by $\overline{\mathscr{B}}_t$ and $\overline{\mathscr{B}}'_t$. Taking the transposes of these matrices we therefore obtain two anti-representations $\overline{\mathscr{B}}_t^T$ and $(\overline{\mathscr{B}}'_t)^T$.

It is worth noting that while $\overline{\mathscr{B}}_t$ and $(\overline{\mathscr{B}}'_t)^T$ are represented by the same matrices, the former is a representation while the later is an anti-representation. Using all these conventions, Cimasoni and Turaev in fact show that the restriction of the Lagrangian functor to B_n gives the anti-representation $(\overline{\mathscr{B}}'_t)^T$ [42, Proposition 5.2].

From now on, it shall implicitly be understood that the Lagrangian functor generalizes the Burau representation in the sense of Remark 13.4.3. We conclude this subsection with a last remark regarding conventions:

Remark 13.4.4. Cimasoni and Turaev read the composition of tangles from top to bottom and the composition of braids from bottom to top [42, proof of Proposition 5.2]. On the other hand, for these authors, the generators σ_i of B_n corresponds to our σ_i^{-1} (see [43, Figure 2]), i.e. following Birman, our σ_i has a negative crossing, while their σ_i has a positive crossing. Finally, we remind the reader that from now on, the conventions regarding colored braids differ from those of Chapters 9, 11 and 12, see Remark 13.2.3.

13.4.2 The multivariable case

In all this subsection, we shall assume for simplicity that the maps $c\colon \{1, \ldots, n\} \to \{\pm 1, \ldots, \pm\mu\}$ are such that $\ell(c)$ is nowhere vanishing. Given such a map c, the homomorphism $H_1(D_c) \to \mathbb{Z}^\mu$, $e_j \mapsto t_{|c_j|}$ induces a free abelian covering $\widehat{D}_c \to D_c$ whose homology is endowed with a structure of module over $\Lambda_\mu = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$. As we saw in Lemma 9.4.6 $H_1(\widehat{D}_c)$ is a module of rank $n-1$ which is free over Λ_μ if $\mu < 3$ and free of rank $n-1$ after tensoring with Λ_S.

Furthermore, as we recalled in Section 9.4, the pairing

$$\lambda_c(x,y) = \sum_{g\in\mathbb{Z}^\mu} \langle gx, y\rangle g^{-1}$$

defines a skew-Hermitian Λ_μ-valued pairing on $H_1(\widehat{D}_c)$ which is non-degenerate. Therefore, following the terminology of Subsection 13.3, $(H_1(\widehat{D}_c),\lambda_c)$ is a Hermitian Λ_μ-module. The same statement holds over Λ_S.

The homomorphism $H_1(X_\tau)\to\mathbb{Z}^\mu$, $m_j\mapsto t_{|c_j|}$ extends the previously defined homomorphisms $H_1(D_c)\to\mathbb{Z}^\mu$ and $H_1(D_{c'})\to\mathbb{Z}^\mu$. It determines a free abelian covering $\widehat{X}_\tau\to X_\tau$ whose homology is also endowed with a structure of module over Λ_μ. Let $i_\tau\colon H_1(\widehat{D}_c)\to H_1(\widehat{X}_\tau)$ and $i'_\tau\colon H_1(\widehat{D}_{c'})\to H_1(\widehat{X}_\tau)$ be the homomorphisms induced by the inclusions of $\widehat{D}_c$ and $\widehat{D}_{c'}$ into $\widehat{X}_\tau$. Denote by $j_\tau\colon H_1(\widehat{D}_c)\oplus H_1(\widehat{D}_{c'})\to H_1(\widehat{X}_\tau)$ the homomorphism given by $j_\tau(x,x')=i'_\tau(x')-i_\tau(x)$. Finally consider $\ker(j_\tau)\subset H_1(\widehat{D}_c)\oplus H_1(\widehat{D}_{c'})$.

The following theorem is the multivariable analogue of Theorem 13.4.2; note that it also holds over Λ_S.

Theorem 13.4.5. *Let $\mathscr{F}$ assign to each coloring map $c\colon\{1,\dots,n\}\to\{\pm1,\dots,\pm\mu\}$ the pair $(H_1(\widehat{D}_c),\lambda_c)$ and to each (c,c')-tangle τ the submodule $\overline{\ker(j_\tau)}$ of $H_1(\widehat{D}_c)\oplus H_1(\widehat{D}_{c'})$. Then $\mathscr{F}$ is a functor which fits in the commutative diagram*

$$\begin{tikzcd}
\mathbf{Braids}_\mu \arrow[r] \arrow[d] & \mathbf{String}_\mu \arrow[r] \arrow[d] & \mathbf{Tangles}_\mu \arrow[d, "\mathscr{F}"] \\
\mathbf{U}_{\Lambda_\mu} \arrow[r, "-\otimes Q_\mu"] \arrow[rr, bend right, "\Gamma"'] & \mathbf{U}^0_{\Lambda_\mu} \arrow[r, "\Gamma^0"] & \mathbf{Lagr}_{\Lambda_\mu},
\end{tikzcd}$$

where the horizontal arrows are the embeddings of categories described in Sections 13.2 and 13.3.

In particular, note that if $\alpha\in B_c$ is a colored braid, then $\mathscr{F}(\alpha)$ is precisely the graph of the unitary automorphism of $H_1(\widetilde{D}_c)$ given by the colored Gassner automorphism $\mathscr{B}_t(\alpha)$.

Chapter 14

Extending the Burau representation to a 2-functor

14.1 Introduction and statement of the results

Chapter 13 reviewed the construction of Cimasoni and Turaev's *Lagrangian functor* $\mathscr{F}\colon \mathbf{Tangles} \to \mathbf{Lagr}_\Lambda$ which extends the Burau representation from braids to tangles [42]. However, the category **Tangles** comes with additional structure. Indeed, roughly speaking, oriented surfaces between oriented tangles turn **Tangles** into a bicategory, i.e. a category in which there are "morphisms between morphisms". Thus we are led to the following question: can the Lagrangian functor $\mathscr{F}$ be promoted to a (weak) 2-functor? The aim of this chapter (which is based on joint work with David Cimasoni [38]) is to construct such a *Burau-Alexander 2-functor*; let us briefly outline its construction.

The first roadblock lies in the target category: *a priori*, the Lagrangian category $\mathbf{Lagr}_\Lambda$ does not admit the structure of a bicategory. In order to circumvent this issue, the idea is to consider *cospans* [14, 84] of Λ-modules, i.e. diagrams of the form $H \to T \leftarrow H'$. As we shall see in Section 14.2, there is a category $\mathbf{L}_\Lambda$ of *Lagrangian cospans* which should be understood as a generalization of the category of Lagrangian relations, in the sense that there is a full (non-faithful) functor $F\colon \mathbf{L}_\Lambda \to \mathbf{Lagr}_\Lambda$ which is the identity on objects. Naturally, the point of considering $\mathbf{L}_\Lambda$ instead of $\mathbf{Lagr}_\Lambda$ lies in the fact that (a slight modification of) the former is endowed with the structure of a bicategory. Apart from dealing with this fact, Section 14.3 will also explain why tangles naturally lead to a bicategory.

There is, however, a key point which we have been glossing over. Informally, a bicategory $\mathscr{C}$ consists of a set of objects and, for each pair of objects (X, Y), a category $\mathscr{C}(X, Y)$ whose objects are called 1-morphisms and whose morphisms are called 2-morphisms. Furthermore, there is a composition law for 1-morphisms and two composition laws for 2-morphisms. The subtlety is that on the level of 1-morphisms the composition is only associative up to isomorphism. In particular the objects and 1-morphisms of a bicategory do not provide the data for a category. The same remark goes for functors: restricting a weak 2-functor to objects and morphisms does not produce a functor.

For this reason, we will construct our Burau-Alexander 2-functor in two steps. First, we will define a functor $\overline{\mathscr{B}}\colon \mathbf{Tangles} \to \mathbf{L}_\Lambda$ which generalizes the Lagrangian functor. Although a much more precise a detailed statement can be found in Section 14.4, an approximate statement goes as follows:

Theorem 14.1.1 (A non-technical statement of Theorem 14.4.1)**.** *There is a functor* $\overline{\mathscr{B}}\colon \mathbf{Tangles} \to \mathbf{L}_\Lambda$ *which satisfies* $F \circ \overline{\mathscr{B}} = \mathscr{F}$*, where* $\mathscr{F}$ *is the Lagrangian functor described in Section 13.4, and* $F\colon \mathbf{L}_\Lambda \to \mathbf{Lagr}_\Lambda$ *is a functor described in Section 14.2 below. Furthermore, if* τ *is an oriented link, then* $\overline{\mathscr{B}}(\tau)$ *is nothing but its Alexander module.*

Our Burau-Alexander 2-functor is then built from $\overline{\mathscr{B}}$ by taking into account the subtlety described above. Namely, slight modifications of the categories **Tangles** and $\mathbf{L}_\Lambda$ produce bicategories of tangles and Lagrangian 2-cospans, and similar changes give rise to our weak 2-functor. Once again, we delay precise formulations to Section 14.4 but a short perusal of the following statement should give the reader the gist of our main result.

Theorem 14.1.2 (A non-technical statement of Theorem 14.4.2)**.** *The functor* $\overline{\mathscr{B}}$ *of Theorem 14.1.2 gives rise to a weak* 2*-functor from the bicategory of oriented tangles to the bicategory of Lagrangian cospans, whose restriction to oriented surfaces is given by the Alexander module.*

Along the way, we shall discuss the *core* of the various categories we have introduced, i.e. their maximal subgroupoid. More precisely, as an additional feature of our theory, we shall observe that the restriction of $\overline{\mathscr{B}}$ to **Braids** (the core of **Tangle**) actually takes value in the core of $\mathbf{L}_\Lambda$. Moreover $core(\mathbf{L}_\Lambda)$ turns out to be equivalent to the category $\mathbf{U}_\Lambda$ of unitary morphisms, see Proposition 14.2.5.

14.2 Lagrangian cospans

The aim of this section is to introduce the various algebraic categories that appear in our construction. In Subsection 14.2.1, we recall the theory of cospans in a category with pushouts. In Subsection 14.2.2, we define the category $\mathbf{L}_\Lambda$ of Lagrangian cospans, and relate it to the category of Lagrangian relations via a full functor $F\colon \mathbf{L}_\Lambda \to \mathbf{Lagr}_\Lambda$. In Subsection 14.2.3, we show that this functor restricts to an equivalence of categories between the core groupoid of $\mathbf{L}_\Lambda$ and $\mathbf{U}_\Lambda$.

14.2.1 Cospans in a category with pushouts

Among the arguments that will be used in this chapter, some are well-known and of purely categorical nature. This subsection contains a quick review of these results, see [14, 138] for further detail.

Let us fix a category $\mathbf{C}$. Throughout this subsection, all objects, morphisms, diagrams, and the like will be in this fixed category $\mathbf{C}$. Recall that a *span* is a diagram of the form $T_1 \xleftarrow{i_1} H \xrightarrow{i_2} T_2$. A *pushout* of such a span is an object P together with morphisms $T_1 \xrightarrow{j_1} P \xleftarrow{j_2} T_2$ such that $j_1 i_1 = j_2 i_2$, which satisfies the following universal property: for any $T_1 \xrightarrow{k_1} Q \xleftarrow{k_2} T_2$ such that $k_1 i_1 = k_2 i_2$, there exists a unique morphism $u\colon P \to Q$ with $u j_1 = k_1$ and $u j_2 = k_2$.

This is illustrated in the following commutative diagram:

$$\begin{tikzcd}
 & Q & \\
 & P \arrow[u, dotted, "u"] & \\
T_1 \arrow[ur, "j_1"] \arrow[uur, bend left, "k_1"] & & T_2 . \arrow[ul, "j_2"'] \arrow[uul, bend right, "k_2"'] \\
 & H \arrow[ul, "i_1"] \arrow[ur, "i_2"'] &
\end{tikzcd}$$

If a span admits a pushout, then the latter is unique up to canonical isomorphism. However, not all spans admit pushouts in general. From now on, we shall assume that $\mathbf{C}$ is a *category with pushouts*, i.e. that any span admits a pushout. Moreover, we fix for each span a pushout.

Let H, H' be two objects. A *cospan* from H to H' is a diagram $H \xrightarrow{i} T \xleftarrow{i'} H'$. Two cospans $H \xrightarrow{i_1} T_1 \xleftarrow{i'_1} H'$ and $H \xrightarrow{i_2} T_2 \xleftarrow{i'_2} H'$ are *isomorphic* if there is an isomorphism $f\colon T_1 \to T_2$ such that $fi_1 = i_2$ and $fi'_1 = i'_2$. The *composition* of two cospans $H \xrightarrow{i_1} T_1 \xleftarrow{i'_1} H'$ and $H' \xrightarrow{i'_2} T_2 \xleftarrow{i''_2} H''$ is the cospan from H to H'' given by the (fixed) pushout diagram

$$\begin{tikzcd}
 & & T_2 \circ T_1 & & \\
 & T_1 \arrow[ur, "j_1"] & & T_2 \arrow[ul, "j_2"'] & \\
H \arrow[ur, "i_1"] & & H' \arrow[ul, "i'_1"'] \arrow[ur, "i'_2"] & & H'' . \arrow[ul, "i''_2"']
\end{tikzcd}$$

Finally, the *identity cospan* of an object H is defined as the cospan $I_H := (H \xrightarrow{id} H \xleftarrow{id} H)$.

Remark 14.2.1. Given any morphism $H' \xrightarrow{i'} H''$, one easily checks that

$$\begin{tikzcd}
 & H'' & \\
H' \arrow[ur, "i'"] & & H'' \arrow[ul, "id"'] \\
 & H' \arrow[ul, "id"] \arrow[ur, "i'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & H'' & \\
H'' \arrow[ur, "id"] & & H' \arrow[ul, "i'"'] \\
 & H' \arrow[ul, "i'"] \arrow[ur, "id"'] &
\end{tikzcd}$$

are pushout diagrams. Therefore, if one makes this choice of pushout for spans of the form $H' \xleftarrow{id} H' \xrightarrow{i'} H''$ and $H'' \xleftarrow{i'} H' \xrightarrow{id} H'$, then the composition of $H \xrightarrow{i} H' \xleftarrow{id} H'$ and $H' \xrightarrow{i'} T \xleftarrow{i''} H''$ is given by $H \xrightarrow{i'i} T \xleftarrow{i''} H''$, and the composition of $H \xrightarrow{i} T \xleftarrow{i'} H'$ and $H' \xrightarrow{id} H' \xleftarrow{i''} H''$ is given by $H \xrightarrow{i} T \xleftarrow{i'i''} H''$. For this reason, cospans should be understood as generalizing morphisms in the category $\mathbf{C}$.

Note that the composition of cospans depends on the choice of a pushout for each span; therefore, it cannot be associative for all such choices. For the same reason, the composition does not admit I_H as a two-sided unit in general. However, for any fixed choice of pushouts, the corresponding composition does satisfy these properties up to canonical isomorphisms of cospans. We refer the reader to [138] for a proof of this standard fact in the dual context of spans.

There are two possible strategies at this point. The first one, which we will use in the remaining part of Section 13.3, is to consider the category given by the objects of $\mathbf{C}$, as objects, and the isomorphism classes of cospans in $\mathbf{C}$, as morphisms. The second one, which we will use in the next sections, is to follow the "main principle of category theory " as stated in [91, p.179], that is: not to identify isomorphic cospans, but to view these canonical isomorphisms as part of the (higher) structure. This naturally leads to the concept of a *bicategory*, that will be reviewed in Subsection 14.3.1 and used in Subsections 14.3.2 and 14.4.2.

14.2.2 The category $\mathbf{L}_\Lambda$ of Lagrangian cospans

We now take $\mathbf{C}$ to be the category of Λ-modules, with Λ any integral domain. After observing that this is a category with pushouts, we impose further conditions on our cospans and work with isomorphism classes thereof.

We begin with the following standard result, whose easy proof is left to the reader.

Lemma 14.2.2. *The square*

$$\begin{tikzcd} & P & \\ T_1 \arrow[ur, "j_1"] & & T_2 \arrow[ul, "j_2"'] \\ & H \arrow[ul, "i_1"] \arrow[ur, "i_2"'] & \end{tikzcd}$$

is a pushout diagram in the category of Λ-modules if and only if the sequence

$$H \xrightarrow{(-i_1, i_2)} T_1 \oplus T_2 \xrightarrow{\binom{j_1}{j_2}} P \longrightarrow 0$$

is exact.

In particular, a pushout is given by the cokernel of the map $(-i_1, i_2)\colon H \to T_1 \oplus T_2$ sending x to $(-i_1(x)) \oplus i_2(x)$, so this is a category with pushouts.

By abuse of notation, we shall sometimes simply denote by T (the isomorphism class of) a cospan of the form $H \to T \leftarrow H'$. For a cospan $H \xrightarrow{i} T \xleftarrow{i'} H'$, consider the submodule $N_T := \ker \binom{-i}{i'}$ of $H \oplus H'$, where $\binom{-i}{i'}\colon H \oplus H' \to T$ maps (x, y) to $i'(y) - i(x)$. Note that if T_1 and T_2 are isomorphic cospans, then N_{T_1} and N_{T_2} are equal.

Lemma 14.2.3. *For any two composable cospans T_1 and T_2, we have $N_{T_2 \circ T_1} = N_{T_2} N_{T_1}$.*

Proof. Consider two cospans $H \xrightarrow{i_1} T_1 \xleftarrow{i_1'} H'$ and $H' \xrightarrow{i_2'} T_2 \xleftarrow{i_2''} H''$. By definition, $N_{T_2 \circ T_1}$ is the kernel of the map $H \oplus H'' \to T_2 \circ T_1$ given by $(x, z) \mapsto j_2(i_2''(z)) - j_1(i_1(x))$. Since $T_2 \circ T_1$ is represented by the cokernel of the map $(-i_1', i_2')\colon H' \to T_1 \oplus T_2$, $N_{T_2 \circ T_1}$ consists of the elements $x \oplus z \in H \oplus H''$ for which $(-i_1(x)) \oplus i_2''(z)$ lies in the image of $(-i_1', i_2')$. Therefore, $N_{T_2 \circ T_1}$ is equal to

$$\{x \oplus z \in H \oplus H'' |\ i_1(x) = i_1'(y) \text{ and } i_2'(y) = i_2''(z) \text{ for some } y \in H'\}\,.$$

In other words, $N_{T_2 \circ T_1}$ is equal to $\ker \binom{-i_2'}{i_2''} \ker \binom{-i_1}{i_1'} = N_{T_2} N_{T_1}$. □

Recall from Section 13.3 that if A is a submodule of a Hermitian Λ-module H, then $\overline{A}$ consists of all $x \in H$ such that $\lambda x \in A$ for a non-zero $\lambda \in \Lambda$. We shall say that a cospan $H \xrightarrow{i} T \xleftarrow{i'} H'$ is *Lagrangian* if $\overline{N_T}$ is a Lagrangian submodule of $(-H) \oplus H'$. For instance, the identity cospan I_H is a Lagrangian cospan, since $\overline{N_H} = N_H$ is equal to the diagonal relation Δ_H.

Proposition 14.2.4. *Hermitian Λ-modules, as objects, and isomorphism classes of Lagrangian cospans, as morphisms, form a category $\mathbf{L}_\Lambda$. Moreover, the map $T \mapsto \overline{N_T}$ gives rise to a full functor $F \colon \mathbf{L}_\Lambda \to \mathbf{Lagr}_\Lambda$.*

Proof. As explained in Subsection 14.2.1, it is a standard fact that the composition of isomorphism classes of cospans by pushouts is well-defined (i.e. does not depend on the choice of the pushouts), is associative, with the identity cospan acting trivially. Therefore, we only need to check that the composition of two Lagrangian cospans $H \to T_1 \leftarrow H'$ and $H' \to T_2 \leftarrow H''$ is also Lagrangian, and that F is a full functor. By Lemma 14.2.3, we have

$$\overline{N_{T_2 \circ T_1}} = \overline{N_{T_2} N_{T_1}} = N_{T_2} \circ N_{T_1} \,.$$

Since N_{T_2} are N_{T_1} are Lagrangian, $N_{T_2} = \overline{N_{T_2}}, N_{T_1} = \overline{N_{T_1}}$ and $N_{T_2} \circ N_{T_1}$ is also Lagrangian by Theorem 13.3.1. Therefore, the cospan $H \to T_2 \circ T_1 \leftarrow H''$ is Lagrangian and F is a functor. Finally, given a Lagrangian relation N from H to H', consider the cospan $H \xrightarrow{i} T \xleftarrow{i'} H'$ where $T = (H \oplus H')/N$ and i (resp. i') is the inclusion of H (resp. H') into $H \oplus H'$ composed with the canonical projection. By construction, it is a Lagrangian cospan with $\overline{N_T} = \overline{N} = N$, so the functor F is full. $\square$

Let us conclude this subsection by noting that the functor $F \colon \mathbf{L}_\Lambda \to \mathbf{Lagr}_\Lambda$ is *not* faithful. Indeed, given any cospan $H \xrightarrow{i} T \xleftarrow{i'} H'$ and any Λ-module $\widetilde{T}$, consider the cospan given by $H \xrightarrow{(i,0)} T \oplus \widetilde{T} \xleftarrow{(i',0)} H'$. One immediately checks the equality $N_{T \oplus \widetilde{T}} = N_T$. Therefore, if the first cospan is Lagrangian and $\widetilde{T}$ is non-trivial, then these two cospans represent different morphisms in $\mathbf{L}_\Lambda$ mapped by F to the same morphism in $\mathbf{Lagr}_\Lambda$.

14.2.3 The core of the category $\mathbf{L}_\Lambda$

Recall that the *core* of a category $\mathscr{C}$ is the maximal sub-groupoid of $\mathscr{C}$. In other words, $\mathit{core}(\mathscr{C})$ is the subcategory of $\mathscr{C}$ consisting of all objects of $\mathscr{C}$ and with morphisms all the isomorphisms of $\mathscr{C}$.

We shall say that a cospan $H \xrightarrow{i} T \xleftarrow{i'} H'$ is *invertible* if i and i' are Λ-isomorphisms.

Proposition 14.2.5. *The core of $\mathbf{L}_\Lambda$ consists of Hermitian Λ-modules, as objects, and isomorphism classes of invertible Lagrangian cospans, as morphisms. Furthermore, the map assigning to such a cospan $H \xrightarrow{i} T \xleftarrow{i'} H'$ the Λ-isomorphism $i'^{-1} i \colon H \to H'$ gives rise to an equivalence of categories $\mathit{core}(\mathbf{L}_\Lambda) \xrightarrow{\simeq} \mathbf{U}_\Lambda$ which fits in the commutative diagram*

$$\begin{tikzcd} \mathit{core}(\mathbf{L}_\Lambda) \arrow[r] \arrow[d, "\simeq"'] & \mathbf{L}_\Lambda \arrow[d, "F"] \\ \mathbf{U}_\Lambda \arrow[r, "\Gamma"] & \mathbf{Lagr}_\Lambda \,. \end{tikzcd}$$

Proof. Given a cospan $H \xrightarrow{i} T \xleftarrow{i'} H'$ with i an isomorphism, one easily checks that the diagram

$$\begin{tikzcd}
& & H & & \\
& T \arrow[ur, "i^{-1}"] & & T \arrow[ul, "i^{-1}"'] & \\
H \arrow[ur, "i"] & & H' \arrow[ul, "i'"'] \arrow[ur, "i'"] & & H \arrow[ul, "i"']
\end{tikzcd}$$

satisfies the universal property for the pushout defining the composition of $H \xrightarrow{i} T \xleftarrow{i'} H'$ with $H' \xrightarrow{i'} T \xleftarrow{i} H$. Hence, if both i and i' are isomorphisms, then these cospans are inverse of one another, and therefore isomorphisms in $\mathbf{L}_\Lambda$, i.e. morphisms in $\mathit{core}(\mathbf{L}_\Lambda)$. Conversely, let $H \xrightarrow{i_1} T_1 \xleftarrow{i_1'} H'$ be a morphism in $\mathit{core}(\mathbf{L}_\Lambda)$, and let $H' \xrightarrow{i_2'} T_2 \xleftarrow{i_2} H$ be its inverse. Working with the cokernel representatives of $T_2 \circ T_1$ and $T_1 \circ T_2$, this means that there exist Λ-isomorphisms $C := \operatorname{coker}(-i_1', i_2') \xrightarrow{\varphi} H$ and $C' := \operatorname{coker}(-i_2, i_1) \xrightarrow{\varphi'} H'$ such that the following diagrams commute:

$$\begin{tikzcd}
& & & H & & & \\
& & & C \arrow[u, "\varphi", "\simeq"'] & & & \\
& T_1 \arrow[urr, "j_1"] & & & & T_2 \arrow[ull, "j_2"'] & \\
H \arrow[ur, "i_1"] \arrow[uuurrr, bend left, "id_H"] & & & H' \arrow[ull, "i_1'"'] \arrow[urr, "i_2'"] & & & H \arrow[ul, "i_2"'] \arrow[uuulll, bend right, "id_H"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& & & H' & & & \\
& & & C' \arrow[u, "\varphi'", "\simeq"'] & & & \\
& T_2 \arrow[urr, "j_2'"] & & & & T_1 \arrow[ull, "j_1'"'] & \\
H' \arrow[ur, "i_2'"] \arrow[uuurrr, bend left, "id_{H'}"] & & & H \arrow[ull, "i_2"'] \arrow[urr, "i_1"] & & & H' \arrow[ul, "i_1'"'] \arrow[uuulll, bend right, "id_{H'}"']
\end{tikzcd}.$$

This implies that the following diagram has exact rows, and is commutative:

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r, "{(-1,1)}"] \arrow[d, "\varphi' j_1' i_1"'] & H \oplus H \arrow[r, "\binom{1}{1}"] \arrow[d, "i_1 \oplus i_2"] & H \arrow[r] \arrow[d, "\varphi^{-1}", "\simeq"'] & 0 \\
0 \arrow[r] & H' \arrow[r, "{(-i_1', i_2')}"'] & T_1 \oplus T_2 \arrow[r, "\binom{j_1}{j_2}"'] & C \arrow[r] & 0\,.
\end{tikzcd}$$

Using the universal property of the pushouts $T_2 \circ T_1$ and $T_1 \circ T_2$, one can check that the maps $\varphi' j_1' i_1 = \varphi' j_2' i_2 \colon H \to H'$ and $\varphi j_1 i_1' = \varphi j_2 i_2' \colon H' \to H$ are inverse of each other, and therefore isomorphisms. By the five-lemma applied to the diagram above, i_1 and i_2 are also isomorphisms. Exchanging the roles of T_1 and T_2 leads to the same conclusion for i_1' and i_2', so both these cospans are invertible.

Now, let $G\colon \mathit{core}(\mathbf{L}_\Lambda) \to \mathbf{U}_\Lambda$ be defined by assigning to the invertible cospan $H \xrightarrow{i} T \xleftarrow{i'} H'$ the Λ-isomorphism $i'^{-1}i \colon H \to H'$. First note that isomorphic cospans are mapped to the same isomorphism. Next, observe that for any two invertible cospans $H \xrightarrow{i_1} T_1 \xleftarrow{i_1'} H'$ and $H' \xrightarrow{i_2'} T_2 \xleftarrow{i_2} H''$, we have

$$G(T_2 \circ T_1) = (j_2 i_2)^{-1}(j_1 i_1) = i_2^{-1} j_2^{-1} j_1 i_1 = i_2^{-1} i_2' i_1'^{-1} i_1 = G(T_2) \circ G(T_1)\,.$$

(Here, we used the fact that since $j_2 i_2$ and i_2 are isomorphisms, so is j_2.) We now check that if a cospan $H \xrightarrow{i} T \xleftarrow{i'} H'$ is Lagrangian, then $i'^{-1}i$ is unitary. Indeed,

$$F(T) = \overline{N_T} = \overline{\ker \binom{-i}{i'}} = \ker \binom{-i}{i'} = \ker \binom{-i'^{-1}i}{id} = \Gamma_{i'^{-1}i}$$

is a Lagrangian subspace of $(-H) \oplus H'$. Therefore, for any $x \in H$ and $y \in H'$, we have

$$0 = (-\lambda \oplus \lambda')(x \oplus i'^{-1}(i(x)), y \oplus i'^{-1}(i(y))) = -\lambda(x,y) + \lambda'(i'^{-1}(i(x)), i'^{-1}(i(y)))\,,$$

so $i'^{-1}i$ is indeed unitary. The equality $F(T) = \Gamma_{i'^{-1}i}$ displayed above also shows the commutativity of the diagram in the statement.

It only remains to check that G is a fully-faithful and essentially surjective functor. Given any unitary isomorphism $f\colon H \to H'$, the cospan $H \xrightarrow{f} H' \xleftarrow{id} H'$ is invertible, Lagrangian, and is mapped to f by G. Furthermore, if $H \xrightarrow{i_1} T_1 \xleftarrow{i'_1} H'$ and $H \xrightarrow{i_2} T_2 \xleftarrow{i'_2} H'$ are invertible cospans with $i_1'^{-1}i_1 = i_2'^{-1}i_2$, then the map $i_2 i_1^{-1} = i'_2 i_1'^{-1}\colon T_1 \to T_2$ defines an isomorphism between these two cospans. Finally, since *core*$(\mathbf{L}_\Lambda)$ and $\mathbf{U}_\Lambda$ have the same objects, G is trivially essentially surjective. □

Since Λ is an integral domain, we can consider its quotient field $Q = Q(\Lambda)$. We shall call a cospan $H \xrightarrow{i} T \xleftarrow{i'} H'$ *rationally invertible* if $i_Q := i \otimes Q$ and $i'_Q := i' \otimes Q$ are Q-isomorphisms. The next proposition can be checked by the same arguments as Proposition 14.2.5. We therefore leave the proof to the reader.

Proposition 14.2.6. *Hermitian Λ-modules, as objects, and isomorphism classes of rationally invertible Lagrangian cospans, as morphisms, form a category core$(\mathbf{L}_\Lambda)^0$. Furthermore, the map assigning to such a cospan $H \xrightarrow{i} T \xleftarrow{i'} H'$ the Q-isomorphism $i_Q'^{-1}i_Q$ gives rise to a full functor core$(\mathbf{L}_\Lambda)^0 \to \mathbf{U}^0_\Lambda$ which fits in the following commutative diagram:*

$$\begin{tikzcd} core(\mathbf{L}_\Lambda)^0 \arrow[r] \arrow[d] & \mathbf{L}_\Lambda \arrow[d, "F"] \\ \mathbf{U}^0_\Lambda \arrow[r, "\Gamma^0"] & \mathbf{Lagr}_\Lambda\,. \end{tikzcd}$$

Summarizing this section, we have six categories which all have Hermitian Λ-modules as objects. They fit in the following commutative diagram

$$\begin{tikzcd} core(\mathbf{L}_\Lambda) \arrow[r] \arrow[d, "\simeq"'] & core(\mathbf{L}_\Lambda)^0 \arrow[r] \arrow[d] & \mathbf{L}_\Lambda \arrow[d, "F"] \\ \mathbf{U}_\Lambda \arrow[r, "-\otimes Q"] \arrow[rr, bend right, "\Gamma"'] & \mathbf{U}^0_\Lambda \arrow[r, "\Gamma^0"] & \mathbf{Lagr}_\Lambda\,, \end{tikzcd} \tag{14.1}$$

where the horizontal arrows are embeddings of categories, the left-most vertical arrow is an equivalence of categories, and the two remaining ones are full functors.

14.3 Bicategories

The aim of this section is to explain how cospans and tangles can be endowed with the structure of bicategories. In Subsection 14.3.1, we begin by recalling the notions of bicategory and weak 2-functor, before defining the bicategory of Lagrangian cospans in Subsection 14.3.2. Finally Subsection 14.3.3 explains how the category of tangles can be turned into a bicategory.

14.3.1 2-categories and 2-functors

Following the original work of Bénabou [14], it is a traditional practice to use the term "2-category" for what Kapranov and Voevodsky call a "strict 2-category" [91]. As it turns out, the categories that appear in our work are not of this type, but have a richer structure: that of some type of weak 2-category known as a *bicategory*. We now recall the definition of this structure, following [14].

A *bicategory* $\mathscr{C}$ consists of the following data:

(i) A set $\mathrm{Ob}\mathscr{C}$ whose elements are called *objects*.

(ii) For each pair of objects (X, Y), a category $\mathscr{C}(X, Y)$ whose objects are called 1-*morphisms* and denoted by $f\colon X \to Y$ or by $X \xrightarrow{f} Y$, whose morphisms are called 2-*morphisms* and denoted by $\alpha\colon f \Rightarrow g$, or by $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}$, and whose composition is called *vertical composition* and denoted by

$$\left(\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}, \begin{tikzcd} X \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta"] & Y \end{tikzcd}\right) \mapsto \begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta\star\alpha"] & Y \end{tikzcd}.$$

We shall denote the identity morphism for f by $Id_f\colon f \Rightarrow f$.

(iii) For each object X, an *identity* 1-morphism $I_X\colon X \to X$.

(iv) For each triple of objects (X, Y, Z), a functor $\mathscr{C}(X, Y) \times \mathscr{C}(Y, Z) \to \mathscr{C}(X, Z)$ denoted by

$$\left(\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}, \begin{tikzcd} Y \arrow[r, bend left, "k"] \arrow[r, bend right, "\ell"'] \arrow[r, phantom, "\Downarrow\beta"] & Z \end{tikzcd}\right) \mapsto \begin{tikzcd} X \arrow[r, bend left, "k\circ f"] \arrow[r, bend right, "\ell\circ g"'] \arrow[r, phantom, "\Downarrow\beta\bullet\alpha"] & Z \end{tikzcd},$$

and called the *horizontal composition* functor.

Note that the functoriality of this composition boils down to the identity

$$Id_f \bullet Id_g = Id_{g\circ f}$$

for any composable 1-morphisms f and g, and to the *interchange law*

$$(\delta \bullet \beta) \star (\gamma \bullet \alpha) = (\delta \star \gamma) \bullet (\beta \star \alpha),$$

for each composable 2-morphisms α, β, γ and δ. This last condition is best understood by saying that the following 2-morphism is well-defined, i.e. independent of the order of the compositions:

$$\begin{tikzcd} X \arrow[r, bend left=60, phantom, "\Downarrow\alpha"] \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, bend right=60, phantom, "\Downarrow\beta"'] & Y \arrow[r, bend left=60, phantom, "\Downarrow\gamma"] \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, bend right=60, phantom, "\Downarrow\delta"'] & Z \end{tikzcd}.$$

If this horizontal composition is associative (both on the 1-morphisms and 2-morphisms) and admits I_X as a two-sided unit, then we are in the presence of a (strict) 2-category. As mentioned above, a bicategory has a richer structure: the horizontal composition is associative and unital only up to natural isomorphisms, which are part of the structure. To be more precise, a bicategory also contains the following data:

(v) For any triple of composable 1-morphisms f, g, h, an invertible 2-morphism

$$a = a_{fgh} \colon (h \circ g) \circ f \Rightarrow h \circ (g \circ f)$$

which is natural in f, g and h, and called the *associativity isomorphism*.

(vi) For any 1-morphism $X \xrightarrow{f} Y$, two invertible 2-morphisms $\ell = \ell_f \colon I_Y \circ f \Rightarrow f$ and $r = r_f \colon f \circ I_X \Rightarrow f$ which are natural in f.

These natural isomorphisms must satisfy the following two coherence axioms. Given four composable 1-morphisms e, f, g, h, there are two natural ways to pass from $((h \circ g) \circ f) \circ e$ to $h \circ (g \circ (f \circ e))$ using the associativity isomorphisms, one in two steps, the other one in three. The *associativity coherence axiom* requires that these two compositions coincide. Finally, given any 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, the *identity coherence axiom* requires the composition $(g \circ I_Y) \circ f \xRightarrow{a} g \circ (I_Y \circ f) \xRightarrow{Id_g \bullet \ell_f} g \circ f$ to coincide with $r_g \bullet Id_f$.

Let us now recall the definition of a weak 2-functor, also known as a *pseudofunctor* [85] and originally called a *homomorphism of bicategories* by Bénabou [14].

Given two bicategories $\mathscr{C}$ and $\mathscr{D}$, a *weak* 2-*functor* $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ consists of the following data:

(i) A map $\mathscr{F} : \mathrm{Ob}\mathscr{C} \to \mathrm{Ob}\mathscr{D}$.

(ii) For each pair of objects (X, Y) in $\mathscr{C}$, a functor $\mathscr{C}(X, Y) \to \mathscr{D}(\mathscr{F}(X), \mathscr{F}(Y))$ denoted by

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \alpha"] & Y \end{tikzcd} \;\mapsto\; \begin{tikzcd} \mathscr{F}(X) \arrow[r, bend left, "\mathscr{F}(f)"] \arrow[r, bend right, "\mathscr{F}(g)"'] \arrow[r, phantom, "\Downarrow \mathscr{F}(\alpha)"] & \mathscr{F}(Y) \end{tikzcd} .$$

Note that this functoriality is equivalent to the identities

$$\mathscr{F}(\beta \star \alpha) = \mathscr{F}(\beta) \star \mathscr{F}(\alpha) \quad \text{and} \quad \mathscr{F}(Id_f) = Id_{\mathscr{F}(f)} .$$

If we also have the identities $\mathscr{F}(I_X) = I_{\mathscr{F}(X)}$, $\mathscr{F}(g \circ f) = \mathscr{F}(g) \circ \mathscr{F}(f)$ and $\mathscr{F}(\beta \bullet \alpha) = \mathscr{F}(\beta) \bullet \mathscr{F}(\alpha)$, then we are in the presence of a (strict) 2-functor. Our functor has a finer structure: once again, these identities hold up to isomorphisms of functors, which are part of the data as follows.

(iii) For each object X of $\mathscr{C}$, an invertible 2-morphism $\varphi_X \colon I_{\mathscr{F}(X)} \Rightarrow \mathscr{F}(I_X)$ in $\mathscr{D}$.

(iv) For each $X \xrightarrow{f} Y \xrightarrow{g} Z$, an invertible 2-morphism $\varphi = \varphi_{fg} \colon \mathscr{F}(g) \circ \mathscr{F}(f) \Rightarrow \mathscr{F}(g \circ f)$ in $\mathscr{D}$ such that for any

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow \alpha"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow \beta"] & Y \end{tikzcd} ,$$

we have the following equality of 2-morphisms in $\mathscr{D}$:

$$\varphi_{f'g'} \star (\mathscr{F}(\beta) \bullet \mathscr{F}(\alpha)) = \mathscr{F}(\beta \bullet \alpha) \star \varphi_{fg} : \mathscr{F}(g) \circ \mathscr{F}(f) \Rightarrow \mathscr{F}(g' \circ f') .$$

Finally, these isomorphisms of functors are required to satisfy two coherence axioms.

(1) For any composable 1-morphisms f, g, h of $\mathscr{C}$, the composition

$$(\mathscr{F}(h) \circ \mathscr{F}(g)) \circ \mathscr{F}(f) \overset{a}{\Rightarrow} \mathscr{F}(h) \circ (\mathscr{F}(g) \circ \mathscr{F}(f)) \overset{Id \bullet \varphi}{\Longrightarrow} \mathscr{F}(h) \circ \mathscr{F}(g \circ f) \overset{\varphi}{\Rightarrow} \mathscr{F}(h \circ (g \circ f))$$

is equal to the composition

$$(\mathscr{F}(h) \circ \mathscr{F}(g)) \circ \mathscr{F}(f) \overset{\varphi \bullet Id}{\Longrightarrow} \mathscr{F}(h \circ g) \circ \mathscr{F}(f) \overset{\varphi}{\Rightarrow} \mathscr{F}((h \circ g) \circ f) \overset{\mathscr{F}(a)}{\Longrightarrow} \mathscr{F}(h \circ (g \circ f)) .$$

(2) For any $X \xrightarrow{f} Y$, the composition

$$I_{\mathscr{F}(Y)} \circ \mathscr{F}(f) \overset{\varphi_Y \bullet Id}{\Longrightarrow} \mathscr{F}(I_Y) \circ \mathscr{F}(f) \overset{\varphi}{\Rightarrow} \mathscr{F}(I_Y \circ f) \overset{\mathscr{F}(\ell)}{\Longrightarrow} \mathscr{F}(f)$$

coincides with $\ell_{\mathscr{F}(f)}$, and similarly for r.

14.3.2 A bicategory of cospans

Our goal is now to define a bicategory of Lagrangian cospans. To do so, we will first work in the more general setting of a category with pushouts. We wish to emphasize that our resulting bicategory of cospans differs from the usual definition considered in the literature, where the 2-morphisms are usually taken to be morphisms of cospans, see e.g. [14]. On the other hand, a notion dual to the 2-morphisms that we consider was already studied by Morton [127] in another context, see also [138].

Throughout this section, $\mathbf{C}$ is a category with pushouts in which we fix a pushout for each span.

The objects of our bicategory are the objects of $\mathbf{C}$ and the 1-morphisms are the cospans in $\mathbf{C}$, where the horizontal composition is given by our choice of a fixed pushout. It remains to define the 2-morphisms, the vertical composition, the associativity and identity isomorphisms, and what is left of the horizontal composition. A *2-cospan in* $\mathbf{C}$ from $H \xrightarrow{i_1} T_1 \xleftarrow{i'_1} H'$ to $H \xrightarrow{i_2} T_2 \xleftarrow{i'_2} H'$ consists of a cospan $T_1 \xrightarrow{\alpha_1} A \xleftarrow{\alpha_2} T_2$ in $\mathbf{C}$ for which the two following squares commute

$$\begin{tikzcd} & T_1 \arrow[d, "\alpha_1"] & \\ H \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & A & H' \arrow[ul, "i'_1"'] \arrow[dl, "i'_2"] \\ & T_2 \arrow[u, "\alpha_2"'] & \end{tikzcd} .$$

Two such 2-cospans $T_1 \xrightarrow{\alpha_1} A \xleftarrow{\alpha_2} T_2$ and $T_1 \xrightarrow{\alpha'_1} A' \xleftarrow{\alpha'_2} T_2$ are said to be *isomorphic* if there is a $\mathbf{C}$-isomorphism $f\colon A \to A'$ such that $f\alpha_1 = \alpha'_1$ and $f\alpha_2 = \alpha'_2$. Abusing notation, we shall often denote the isomorphism class of such a 2-cospan by $A\colon T_1 \Rightarrow T_2$. These will be the 2-morphisms in our 2-category.

Let us now proceed with the definition of the *vertical composition* of the 2-morphisms $A\colon T_1 \Rightarrow T_2$ and $B\colon T_2 \Rightarrow T_3$. It is best explained by the diagram

$$
\begin{array}{ccc}
\begin{tikzcd}
 & T_1 \arrow[d, "\alpha_1"'] & \\
 & A & \\
H \arrow[uur, bend left, "i_1"] \arrow[r, "i_2"] & T_2 \arrow[u, "\alpha_2"] & H' \arrow[l, "i_2'"'] \arrow[uul, bend right, "i_1'"']
\end{tikzcd}
& & \\
\star & = &
\begin{tikzcd}
 & T_1 \arrow[d, "v_A\alpha_1"] & \\
H \arrow[ur, bend left, "i_1"] \arrow[dr, bend right, "i_3"'] & B \star A & H', \arrow[ul, bend right, "i_1'"'] \arrow[dl, bend left, "i_3'"] \\
 & T_3 \arrow[u, "v_B\beta_3"'] &
\end{tikzcd}
\\
\begin{tikzcd}
H \arrow[r, "i_2"] \arrow[ddr, bend right, "i_3"'] & T_2 \arrow[d, "\beta_2"'] & H' \arrow[l, "i_2'"'] \arrow[ddl, bend left, "i_3'"] \\
 & B & \\
 & T_3 \arrow[u, "\beta_3"] &
\end{tikzcd}
& &
\end{array}
$$

where $B \star A$ and v_A, v_B are given by the pushout diagram

$$
\begin{tikzcd}
 & & B \star A & & \\
 & A \arrow[ur, "v_A"] & & B \arrow[ul, "v_B"'] & \\
T_1 \arrow[ur, "\alpha_1"] & & T_2 \arrow[ul, "\alpha_2"'] \arrow[ur, "\beta_2"] & & T_3. \arrow[ul, "\beta_3"']
\end{tikzcd}
$$

One can easily check that this indeed defines a 2-cospan.

Remark 14.3.1. In the special case where $\alpha_2 = id_{T_2}$ (resp. $\beta_2 = id_{T_2}$), this vertical composition $T_1 \xrightarrow{v_A\alpha_1} B \star A \xleftarrow{v_B\beta_3} T_3$ is isomorphic to $T_1 \xrightarrow{\beta_2\alpha_1} B \xleftarrow{\beta_3} T_3$ (resp. $T_1 \xrightarrow{\alpha_1} A \xleftarrow{\alpha_2\beta_3} T_3$). This is a direct consequence of Remark 14.2.1.

On the level 2-morphisms, the *horizontal composition* of $A\colon T_1 \Rightarrow T_2$ and $B\colon T_3 \Rightarrow T_4$ is described by the diagram

$$
\begin{tikzcd}
 & T_3 \arrow[d, "\beta_3"] & \\
H' \arrow[ur, "i_3'"] \arrow[dr, "i_4'"'] & B & H'' \arrow[ul, "i_3''"'] \arrow[dl, "i_4''"] \\
 & T_4 \arrow[u, "\beta_4"'] &
\end{tikzcd}
\bullet
\begin{tikzcd}
 & T_1 \arrow[d, "\alpha_1"] & \\
H \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & A & H' \arrow[ul, "i_1'"'] \arrow[dl, "i_2'"] \\
 & T_2 \arrow[u, "\alpha_2"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & T_3 \circ T_1 \arrow[d, "h_{31}"] & \\
H \arrow[ur, bend left, "j_1 i_1"] \arrow[dr, bend right, "j_2 i_2"'] & B \bullet A & H'', \arrow[ul, bend right, "j_3 i_3''"'] \arrow[dl, bend left, "j_4 i_4''"] \\
 & T_4 \circ T_2 \arrow[u, "h_{42}"'] &
\end{tikzcd}
$$

where $j_1, \ldots, j_4$ are the maps that arise in the compositions $T_3 \circ T_1$ and $T_4 \circ T_2$ (see the diagrams below), $B \bullet A$ is given by the pushout

$$
\begin{tikzcd}
 & & B \bullet A & & \\
 & A \arrow[ur, "h_A"] & & B \arrow[ul, "h_B"'] & \\
H \arrow[ur, "\alpha_1 i_1"] & & H' \arrow[ul, "\alpha_2 i_2'"'] \arrow[ur, "\beta_3 i_3'"] & & H'', \arrow[ul, "\beta_4 i_4''"']
\end{tikzcd}
$$

and the maps h_{31} and h_{42} are obtained as follows. Since $h_A\alpha_1 i_1' = h_A\alpha_2 i_2' = h_B\beta_3 i_3'$ and $h_A\alpha_2 i_2' = h_B\beta_3 i_3' = h_B\beta_4 i_4'$, the pushout diagrams

$$
\begin{tikzcd}
 & & B \bullet A & & \\
 & & T_3 \circ T_1 \arrow[u, "h_{31}"] & & \\
A \arrow[uurr, bend left, "h_A"] & & & & B \arrow[uull, bend right, "h_B"'] \\
T_1 \arrow[u, "\alpha_1"] \arrow[uurr, "j_1"] & & & & T_3 \arrow[u, "\beta_3"'] \arrow[uull, "j_3"'] \\
 & & H' \arrow[ull, "i_1'"] \arrow[urr, "i_3'"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & B \bullet A & & \\
 & & T_4 \circ T_2 \arrow[u, "h_{42}"] & & \\
A \arrow[uurr, bend left, "h_A"] & & & & B \arrow[uull, bend right, "h_B"'] \\
T_2 \arrow[u, "\alpha_2"] \arrow[uurr, "j_2"] & & & & T_4 \arrow[u, "\beta_4"'] \arrow[uull, "j_4"'] \\
 & & H' \arrow[ull, "i_2'"] \arrow[urr, "i_4'"'] & &
\end{tikzcd}
\tag{14.2}
$$

provide maps h_{31} and h_{42} which turn $T_3 \circ T_1 \xrightarrow{h_{31}} B \bullet A \xleftarrow{h_{42}} T_4 \circ T_2$ into a 2-cospan, as one easily checks.

The proof of the following theorem can be found in [138] in the dual context of spans. It applies without change to the present setting.

Theorem 14.3.2. *Let* $\mathbf{C}$ *be a category with pushouts in which a choice of pushout is fixed for each span. Objects in* $\mathbf{C}$*, as objects, cospans in* $\mathbf{C}$*, as morphisms, and isomorphism classes of 2-cospans in* $\mathbf{C}$*, as 2-morphisms, form a bicategory.*

Note that strictly speaking, this bicategory depends on the choice of pushouts. However, another choice would give a bicategory isomorphic in an obvious sense, see e.g. [14, p.22].

The special case where $\mathbf{C}$ is the category of Λ-modules and the morphisms are Lagrangian cospans yields the following corollary.

Corollary 14.3.3. *Fix a pushout for each span of* Λ*-modules. Hermitian* Λ*-modules, as objects, Lagrangian cospans, as morphisms, and isomorphism classes of* 2*-cospans, as* 2*-morphisms, form a bicategory.*

We shall call it "the" *bicategory of Lagrangian cospans*.

14.3.3 The bicategory of tangles

One might think that tangles produce a 2-category in a straightforward way [67]: simply define the objects and 1-morphisms as in **Tangles**, and the 2-morphisms as isotopy classes of oriented surfaces in $D^2 \times [0,1] \times [0,1]$. However, the corresponding vertical composition is not well-defined: indeed, one needs to paste two surfaces along isotopic tangles, and since the space of tangles isotopic to a fixed one is not necessarily simply-connected, different choices of isotopies can lead to different surfaces.

There are a couple of ways to circumvent this difficulty. One of them is to restrict the space of tangles whose isotopy classes form the 1-morphisms, so that the corresponding space of isotopic tangles has trivial fundamental group. Such a construction was given by Kharlamov and Turaev in [98] (see also [8]): they considered the class of so-called *generic tangles*, and proved that the space of generic tangles isotopic to a fixed one (through generic tangles) is simply-connected, thus obtaining a strict 2-category. However, it is more natural in our setting to take the following alternative approach: define 1-morphisms as oriented tangles, and consider isotopies bewteen tangles as part of the "higher structure".

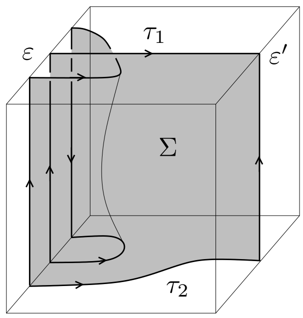

Figure 14.1: A cobordism $\Sigma \subset D^2 \times [0,1] \times [0,1]$ between two $(\varepsilon, \varepsilon')$-tangles τ_1 and τ_2, with $\varepsilon = (+1, +1, -1)$ and $\varepsilon' = (+1)$.

Let us be more precise. The objects of this bicategory are sequences ε of ± 1's, while the 1-morphisms from ε to ε' are the $(\varepsilon, \varepsilon')$-tangles in $D^2 \times [0,1]$ that are trivial near the top and bottom of the cylinder. (This is to ensure that the composition of two tangles remains a smooth 1-submanifold.)

Given two $(\varepsilon, \varepsilon')$-tangles τ_1 and τ_2, a (τ_1, τ_2)-*cobordism* is a pair consisting of the 4-ball $D^2 \times [0,1] \times [0,1]$ together with a proper oriented smooth 2-submanifold Σ whose oriented boundary is given by

$$\partial\Sigma = (\tau_2 \times \{0\}) \cup (\varepsilon' \times \{1\} \times [0,1]) \cup ((-\tau_1) \times \{1\}) \cup ((-\varepsilon) \times \{0\} \times [0,1])\,,$$

as illustrated in Figure 14.1. Note that a $(\emptyset, \emptyset)$-cobordism is nothing but a closed oriented surface embedded in the 4-ball. Two (τ_1, τ_2)-cobordisms Σ and Σ' are *isotopic* if there exists an isotopy h_t of $D^2 \times [0,1] \times [0,1]$, keeping $\partial(D^2 \times [0,1] \times [0,1])$ fixed, such that $h_1|_\Sigma \colon \Sigma \simeq \Sigma'$ is an orientation-preserving homeomorphism and $h_t(\Sigma)$ is a (τ_1, τ_2)-cobordism for all t. We shall denote by $\Sigma \colon \tau_1 \Rightarrow \tau_2$ the isotopy class of a (τ_1, τ_2)-cobordism Σ, and by Id_τ the isotopy class of the trivial (τ, τ)-cobordism $(D^2 \times [0,1], \tau) \times [0,1]$.

Fix a (τ_1, τ_2)-cobordism Σ and a (τ_2, τ_3)-cobordism Σ'. Their *vertical composition* is the (τ_1, τ_3)-cobordism $\Sigma_2 \star \Sigma_1$ obtained by gluing the two 4-balls along the cylinders containing τ_2, and shrinking the height of the resulting 4-ball $D^2 \times [0,1] \times [0,2]$ by a factor 2 (see Figure 14.2). Finally, fix $(\varepsilon, \varepsilon')$-tangles τ_1, τ_2 and $(\varepsilon', \varepsilon'')$-tangles τ_3, τ_4. Given a (τ_1, τ_2)-cobordism Σ_1 and a (τ_3, τ_4)-cobordism Σ_2, their *horizontal composition* is the $(\tau_3 \circ \tau_1, \tau_4 \circ \tau_2)$-cobordism $\Sigma_2 \bullet \Sigma_1$ obtained by gluing the two 4-balls along the cylinder $D^2 \times [0,1]$ corresponding to ε', and shrinking the length of the resulting 4-ball by a factor 2 (Figure 14.3).

The *bicategory of oriented tangles* can now be defined as follows: the objects are the finite sequences of ± 1's, the 1-morphisms are given by the tangles, and the 2-morphisms are given by isotopy classes of cobordisms as described above. Finally, the associativity and identity isomorphisms

$$a \colon (\tau_3 \circ \tau_2) \circ \tau_1 \Rightarrow \tau_3 \circ (\tau_2 \circ \tau_1), \quad \ell_\tau \colon I_{\varepsilon'} \circ \tau \Rightarrow \tau, \quad r_\tau \colon \tau \circ I_\varepsilon \Rightarrow \tau$$

are given by the trace of the obvious isotopies. It is a routine check to verify that all the axioms of a bicategory are satisfied.

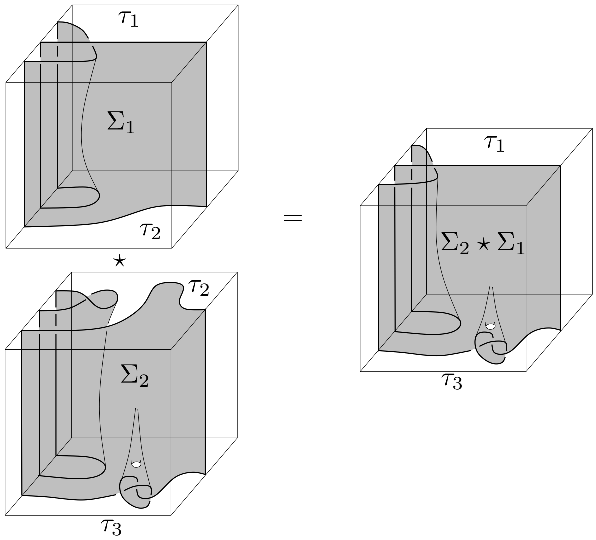

Figure 14.2: The vertical composition of a (τ_1, τ_2)-cobordism Σ_1 and a (τ_2, τ_3)-cobordism Σ_2, the (τ_1, τ_3)-cobordism $\Sigma_2 \star \Sigma_1$.

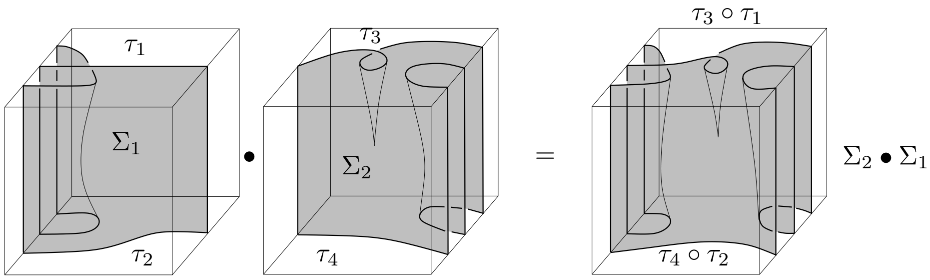

Figure 14.3: The horizontal composition of a (τ_1, τ_2)-cobordism Σ_1 and a (τ_3, τ_4)-cobordism Σ_2, the $(\tau_3 \circ \tau_1, \tau_4 \circ \tau_2)$-cobordism $\Sigma_2 \bullet \Sigma_1$.

14.4 The Burau Alexander 2-functor

Our goal is to define a weak 2-functor $\overline{\mathscr{B}}$ from the bicategory of oriented tangles to the bicategory of Lagrangian cospans where Λ is the ring $\mathbb{Z}[t^{\pm 1}]$ of Laurent polynomials in one variable with integer coefficients. We proceed in two steps: in Subsection 14.4.1, we construct a functor $\overline{\mathscr{B}}\colon \mathbf{Tangles} \to \mathbf{L}_\Lambda$, while in Subsection 14.4.2, we modifiy $\overline{\mathscr{B}}$ in order to obtain a weak 2-functor with values in the bicategory of Lagrangian cospans. In particular, these two subsections will respectively prove Theorem 14.1.1 and Theorem 14.4.2 from the introduction.

14.4.1 Proof of Theorem 14.1.1

We start by defining a functor $\overline{\mathscr{B}}\colon \mathbf{Tangles} \to \mathbf{L}_\Lambda$ on objects by following the construction of the Lagrangian functor, see Section 13.4. Denote by $\mathscr{N}(\{p_1, \ldots, p_n\})$ an open tubular neighborhood of $\{p_1, \ldots, p_n\}$ in $D^2 \subset \mathbb{R}^2$, and let S^2 be the 2-sphere obtained by the one-point compactification of $\mathbb{R}^2$. Given a sequence $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ of ± 1, recall that ℓ_ε denotes $\sum_{i=1}^n \varepsilon_i$ and that D_ε is defined as $D^2 \setminus \mathscr{N}(\{p_1, \ldots, p_n\})$ if $\ell_\varepsilon \neq 0$ and by $S^2 \setminus \mathscr{N}(\{p_1, \ldots, p_n\})$ if $\ell_\varepsilon = 0$. The natural epimomorphism $H_1(D_\varepsilon) \to \mathbb{Z}$, given by $x_j \mapsto 1$ induces an infinite cyclic covering $\widehat{D}_\varepsilon \to D_\varepsilon$ whose homology is endowed with a structure of module over $\Lambda = \mathbb{Z}[t^{\pm 1}]$.

As we saw both in Subsection 9.3.3 and in Section 13.4, the formula

$$\lambda_\varepsilon(x,y) = \sum_{k\in\mathbb{Z}} \langle t^k x, y\rangle t^{-k}$$

defines a skew-Hermitian Λ-valued pairing on $H_1(\widehat{D}_\varepsilon)$ which is non-degenerate. Therefore, following the terminology of Section 13.3, $\overline{\mathscr{B}}(\varepsilon) := (H_1(\widehat{D}_\varepsilon), \lambda_\varepsilon)$ is a free Hermitian Λ-module for any object ε of the category of oriented tangles. Note that this coincides with the definition of the Lagrangian functor $\mathscr{F}\colon \mathbf{Tangles} \to \mathbf{Lagr}_\Lambda$ at the level of objects.

Turning to morphisms, we start by recalling some notation from Section 13.4. Given an $(\varepsilon,\varepsilon')$-tangle $\tau \subset D^2 \times [0,1]$, we denote by $\mathscr{N}(\tau)$ an open tubular neighborhood of τ in $D^2\times[0,1]$. Furthermore, X_τ is defined as $(D^2\times[0,1])\setminus\mathscr{N}(\tau)$ if $\ell_\varepsilon \neq 0$ and as $(S^2\times[0,1])\setminus \mathscr{N}(\tau)$ if $\ell_\varepsilon = 0$. The homomorphism $H_1(X_\tau)\to\mathbb{Z}$ mapping each meridian of X_τ to 1 extends the previously defined homomorphisms $H_1(D_\varepsilon)\to\mathbb{Z}$ and $H_1(D_{\varepsilon'})\to\mathbb{Z}$. It determines an infinite cyclic covering $\widehat{X}_\tau \to X_\tau$ whose homology is endowed with a structure of module over Λ.

Let $i_\tau\colon H_1(\widehat{D}_\varepsilon)\to H_1(\widehat{X}_\tau)$ and $i'_\tau\colon H_1(\widehat{D}_{\varepsilon'})\to H_1(\widehat{X}_\tau)$ be the homomorphisms induced by the inclusions of $\widehat{D}_\varepsilon$ and $\widehat{D}_{\varepsilon'}$ into $\widehat{X}_\tau$. Since $\mathscr{F}(\tau) = \overline{\ker\begin{pmatrix}-i_\tau\\ i'_\tau\end{pmatrix}}$ is a Lagrangian submodule of $(-H_1(\widehat{D}_\varepsilon))\oplus H_1(\widehat{D}'_\varepsilon)$ [42, Lemma 3.3], it follows that $H_1(\widehat{D}_\varepsilon) \xrightarrow{i_\tau} H_1(\widehat{X}_\tau) \xleftarrow{i'_\tau} H_1(\widehat{D}_{\varepsilon'})$ is a Lagrangian cospan for any 1-morphism τ in the category of oriented tangles. Note that the equality above is nothing but the definition of the Lagrangian functor at the level of morphisms, see Section 13.4.

The following theorem contains the precise formulation of the statement provided in Theorem 14.1.1.

Theorem 14.4.1. *For any sequence ε of ±1's, set $\overline{\mathscr{B}}(\varepsilon) = (H_1(\widehat{D}_\varepsilon),\lambda_\varepsilon)$ and for any isotopy class τ of tangles, let $\overline{\mathscr{B}}(\tau)$ denote the isomorphism class of the Lagrangian cospan $H_1(\widehat{D}_\varepsilon) \xrightarrow{i_\tau} H_1(\widehat{X}_\tau) \xleftarrow{i'_\tau} H_1(\widehat{D}_{\varepsilon'})$. This defines a functor $\overline{\mathscr{B}}\colon \mathbf{Tangles}\to\mathbf{L}_\Lambda$ which fits in the commutative diagram*

$$\begin{tikzcd}
\mathbf{Braids} \arrow[r] \arrow[d] \arrow[dd, bend right=60, "\bar{b}"'] & \mathbf{String} \arrow[r] \arrow[d] & \mathbf{Tangles} \arrow[d, "\overline{\mathscr{B}}"] \arrow[dd, bend left=60, "\mathscr{F}"] \\
\mathit{core}(\mathbf{L}_\Lambda) \arrow[r] \arrow[d, "\simeq"'] & \mathit{core}(\mathbf{L}_\Lambda)^0 \arrow[r] \arrow[d] & \mathbf{L}_\Lambda \arrow[d, "F"] \\
\mathbf{U}_\Lambda \arrow[r, "-\otimes Q"] \arrow[rr, bend right=20, "\Gamma"'] & \mathbf{U}_\Lambda^0 \arrow[r, "\Gamma^0"] & \mathbf{Lagr}_\Lambda,
\end{tikzcd}$$

where the left-most vertical arrow is the reduced Burau functor, the horizontal arrows are the embeddings of categories described in Sections 13.3 and 13.2, and F is the full functor defined in Subsection 14.2.2, see diagram (14.1). Furthermore, if τ is an oriented link, then $\overline{\mathscr{B}}(\tau)$ is nothing but its Alexander module.

Proof. For any object ε, the cospan associated to the identity tangle id_ε is canonically isomorphic to the identity cospan $I_{\overline{\mathscr{B}}(\varepsilon)}$. Let us now check that given $\tau_1\in T(\varepsilon,\varepsilon')$ and $\tau_2\in T(\varepsilon',\varepsilon'')$, we have the equality $\overline{\mathscr{B}}(\tau_2\circ\tau_1) = \overline{\mathscr{B}}(\tau_2)\circ\overline{\mathscr{B}}(\tau_1)$. Let $H_1(\widehat{D}_\varepsilon) \overset{i_1}{\to} H_1(\widehat{X}_{\tau_1}) \overset{i'_1}{\leftarrow} H_1(\widehat{D}_{\varepsilon'})$ and $H_1(\widehat{D}_{\varepsilon'}) \overset{i'_2}{\to} H_1(\widehat{X}_{\tau_2}) \overset{i''_2}{\leftarrow} H_1(\widehat{D}_{\varepsilon''})$ be the Lagrangian cospans arising from τ_1 and τ_2.

We must show that $H_1(\widehat{D}_\varepsilon) \xrightarrow{k_1 i_1} H_1(\widehat{X}_{\tau_3\circ\tau_1}) \xleftarrow{k_2 i''_2} H_1(\widehat{D}_{\varepsilon''})$ is isomorphic to the composition $H_1(\widehat{D}_\varepsilon) \xrightarrow{j_1 i_1} H_1(\widehat{X}_{\tau_3}) \circ H_1(\widehat{X}_{\tau_1}) \xleftarrow{j_2 i''_2} H_1(\widehat{D}_{\varepsilon''})$, where k_1, k_2 are the inclusion induced maps and j_1, j_2 are maps resulting from any representative of the pushout $H_1(\widehat{X}_{\tau_3}) \circ H_1(\widehat{X}_{\tau_1})$. Observe that $\widehat{X}_{\tau_2\circ\tau_1}$ decomposes as the union of $\widehat{X}_{\tau_1}$ and $\widehat{X}_{\tau_2}$ glued along $\widehat{D}_{\varepsilon'}$. Therefore, the associated Mayer-Vietoris exact sequence

$$H_1(\widehat{D}_{\varepsilon'}) \xrightarrow{(-i'_1, i'_2)} H_1(\widehat{X}_{\tau_1}) \oplus H_1(\widehat{X}_{\tau_2}) \xrightarrow{\binom{k_1}{k_2}} H_1(\widehat{X}_{\tau_2\circ\tau_1}) \longrightarrow 0$$

together with Lemma 14.2.2 imply that $H_1(\widehat{X}_{\tau_1}) \xrightarrow{k_1} H_1(\widehat{X}_{\tau_2\circ\tau_1}) \xleftarrow{k_2} H_1(\widehat{X}_{\tau_2})$ is a representative of the pushout $H_1(\widehat{X}_{\tau_1}) \circ H_1(\widehat{X}_{\tau_2})$. The claim follows.

Then, observe that the Lagrangian functor $\mathscr{F}$ is by definition the composition of the functors $\overline{\mathscr{B}}\colon \mathbf{Tangles} \to \mathbf{L}_\Lambda$ and $F\colon \mathbf{L}_\Lambda \to \mathbf{Lagr}_\Lambda$. Also, if τ is an oriented string link, then $\overline{\mathscr{B}}(\tau)$ is a rationally invertible cospan by [103, Lemma 2.1], and thus belongs to $core(\mathbf{L}_\Lambda)^0$ by definition. If τ is an oriented braid on the other hand, then $\overline{\mathscr{B}}(\tau)$ is obviously an invertible cospan, and therefore belongs to $core(\mathbf{L}_\Lambda)$ by Proposition 14.2.5. Finally, if τ is a $(\emptyset, \emptyset)$-tangle, that is, an oriented link L, then the associated Lagrangian cospan is given by $0 \to H_1(\widehat{X}_L) \leftarrow 0$, with X_L the complement of L in the 3-ball. A straightforward Mayer-Vietoris argument shows that considering L in the 3-ball or in the 3-sphere does not change the Alexander module, and the proof is completed. □

14.4.2 Proof of Theorem 14.1.2

We are now ready to define our weak 2-functor from the bicategory of oriented tangles to the bicategory of Lagrangian cospans. Recall from Subsection 14.3.1 that we must associate a Hermitian Λ-module $\overline{\mathscr{B}}(\varepsilon)$ to each object ε, a cospan $\overline{\mathscr{B}}(\tau)$ to each tangle τ and an isomorphism class of 2-cospans to each cobordism Σ. Additionally, for each ε, we must define an invertible 2-morphism $\varphi_\varepsilon\colon I_{\overline{\mathscr{B}}(\varepsilon)} \Rightarrow \overline{\mathscr{B}}(id_\varepsilon)$ and for each pair τ_1, τ_2 of composable tangles, an invertible 2-morphism $\varphi_{\tau_1\tau_2}\colon \overline{\mathscr{B}}(\tau_2) \circ \overline{\mathscr{B}}(\tau_1) \Rightarrow \overline{\mathscr{B}}(\tau_2 \circ \tau_1)$.

Let us associate to each object ε the Hermitian Λ-module $\overline{\mathscr{B}}(\varepsilon) = (H_1(\widehat{D}_\varepsilon), \lambda_\varepsilon)$ and to each $(\varepsilon, \varepsilon')$-tangle τ the Lagrangian cospan $\overline{\mathscr{B}}(\tau)$ given by $H_1(\widehat{D}_\varepsilon) \xrightarrow{i_\tau} H_1(\widehat{X}_\tau) \xleftarrow{i'_\tau} H_1(\widehat{D}_{\varepsilon'})$. (Note that we slightly abuse notations here, as $\overline{\mathscr{B}}(\tau)$ now no longer stands for the isomorphism class of this cospan, but for the cospan itself.) As for 2-morphisms, we proceed as follows.

Fix two $(\varepsilon, \varepsilon')$-tangles τ_1, τ_2. Given a (τ_1, τ_2)-cobordism $\Sigma \subset D^2 \times [0,1] \times [0,1]$, denote by $\mathscr{N}(\Sigma)$ an open tubular neighborhood of Σ in $D^2 \times [0,1] \times [0,1]$. We shall orient the exterior

$$W_\Sigma = \begin{cases} (D^2 \times [0,1] \times [0,1]) \setminus \mathscr{N}(\Sigma) & \text{if } \ell_\varepsilon \neq 0 \\ (S^2 \times [0,1] \times [0,1]) \setminus \mathscr{N}(\Sigma) & \text{if } \ell_\varepsilon = 0 \end{cases}$$

of Σ so that the induced orientation on ∂W_Σ extends the orientation on the space $(-X_{\tau_1}) \sqcup X_{\tau_2}$. Clearly, $H_1(W_\Sigma)$ is generated by the (oriented) meridians of the connected components of Σ. The homomorphism $H_1(W_\Sigma) \to \mathbb{Z}$ obtained by mapping these meridians to 1 extends the previously defined homomorphisms $H_1(X_{\tau_1}) \to \mathbb{Z}$ and $H_1(X_{\tau_2}) \to \mathbb{Z}$. It determines an infinite cyclic covering $\widehat{W}_\Sigma \to W_\Sigma$ whose homology is endowed with a structure of module over Λ.

Denote by $H_1(\widehat{D}_\varepsilon) \xrightarrow{i_1} H_1(\widehat{X}_{\tau_1}) \xleftarrow{i'_1} H_1(\widehat{D}_{\varepsilon'})$ and $H_1(\widehat{D}_\varepsilon) \xrightarrow{i_2} H_1(\widehat{X}_{\tau}) \xleftarrow{i'_2} H_1(\widehat{D}_{\varepsilon'})$ the Lagrangian cospans arising from τ_1 and τ_2, and let $\alpha_1 \colon H_1(\widehat{X}_{\tau_1}) \to H_1(\widehat{W}_\Sigma)$ and $\alpha_2 \colon H_1(\widehat{X}_{\tau_2}) \to H_1(\widehat{W}_\Sigma)$ be the homomorphisms induced by the inclusions of $\widehat{X}_{\tau_1}$ and $\widehat{X}_{\tau_2}$ into $\widehat{W}_\Sigma$. Combining all these inclusion induced maps, the following diagram commutes

$$\begin{tikzcd}
 & H_1(\widehat{X}_{\tau_1}) \arrow[d, "\alpha_1"] & \\
H_1(\widehat{D}_\varepsilon) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & H_1(\widehat{W}_\Sigma) & H_1(\widehat{D}_{\varepsilon'}) \arrow[ul, "i'_1"'] \arrow[dl, "i'_2"] . \\
 & H_1(\widehat{X}_{\tau_2}) \arrow[u, "\alpha_2"'] &
\end{tikzcd}$$

Hence, $H_1(\widehat{X}_{\tau_1}) \xrightarrow{\alpha_1} H_1(\widehat{W}_\Sigma) \xleftarrow{\alpha_2} H_1(\widehat{X}_{\tau_2})$ is a 2-cospan, whose isomorphism class we denote by $\overline{\mathscr{B}}(\Sigma) \colon \overline{\mathscr{B}}(\tau_1) \Rightarrow \overline{\mathscr{B}}(\tau_2)$.

Given any object ε, let $\alpha_\varepsilon \colon H_1(\widehat{D}_\varepsilon) \to H_1(\widehat{X}_{id_\varepsilon})$ denote the isomorphism of Λ-modules induced by the inclusion of D_ε in $D_\varepsilon \times [0,1] = X_{id_\varepsilon}$. This isomorphism fits in the commutative diagram

$$\begin{tikzcd}
 & H_1(\widehat{D}_\varepsilon) \arrow[d, "\alpha_\varepsilon"] & \\
H_1(\widehat{D}_\varepsilon) \arrow[ur, "id"] \arrow[dr, "\alpha_\varepsilon"'] & H_1(\widehat{X}_{id_\varepsilon}) & H_1(\widehat{D}_\varepsilon) \arrow[ul, "id"'] \arrow[dl, "\alpha_\varepsilon"] . \\
 & H_1(\widehat{X}_{id_\varepsilon}) \arrow[u, "id"'] &
\end{tikzcd}$$

By Remark 14.3.1, the 2-morphism $\varphi_\varepsilon \colon I_{\overline{\mathscr{B}}(\varepsilon)} \Rightarrow \overline{\mathscr{B}}(id_\varepsilon)$ defined by this diagram is invertible, as required in the definition of a weak 2-functor.

Given an $(\varepsilon, \varepsilon')$-tangle τ_1 and an $(\varepsilon', \varepsilon'')$-tangle τ_2, the first part of the proof of Theorem 14.4.1 actually shows that there is a canonical isomorphism $\alpha_{\tau_1\tau_2} \colon H_1(\widehat{X}_{\tau_2}) \circ H_1(\widehat{X}_{\tau_1}) \to H_1(\widehat{X}_{\tau_2 \circ \tau_1})$ which fits in the commutative diagram

$$\begin{tikzcd}
 & & H_1(\widehat{X}_{\tau_2}) \circ H_1(\widehat{X}_{\tau_1}) \arrow[d, "\alpha_{\tau_1\tau_2}"] & & \\
H_1(\widehat{D}_\varepsilon) \arrow[r, "i_1"] \arrow[urr, bend left, "j_1 i_1"] \arrow[drr, bend right, "k_1 i_1"'] & H_1(\widehat{X}_{\tau_1}) \arrow[ur, "j_1"'] \arrow[dr, "k_1"] & H_1(\widehat{X}_{\tau_2 \circ \tau_1}) & H_1(\widehat{X}_{\tau_2}) \arrow[ul, "j_2"] \arrow[dl, "k_2"'] & H_1(\widehat{D}_{\varepsilon''}) , \arrow[l, "i''_2"'] \arrow[ull, bend right, "j_2 i''_2"'] \arrow[dll, bend left, "k_2 i''_2"] \\
 & & H_1(\widehat{X}_{\tau_2 \circ \tau_1}) \arrow[u, "id"'] & &
\end{tikzcd} \tag{14.3}$$

where we follow the notations of the aforementioned proof. Hence, this defines a canonical 2-morphism $\varphi_{\tau_1\tau_2} \colon \overline{\mathscr{B}}(\tau_1) \circ \overline{\mathscr{B}}(\tau_2) \Rightarrow \overline{\mathscr{B}}(\tau_2 \circ \tau_1)$, which is invertible by Remark 14.3.1.

The following theorem contains the precise formulation of the statement provided in Theorem 14.1.2.

Theorem 14.4.2. *$\overline{\mathscr{B}}$ together with the isomorphisms φ_ε and $\varphi_{\tau_1\tau_2}$ gives rise to a weak 2-functor from the bicategory of oriented tangles to the bicategory of Lagrangian cospans, whose restriction to oriented surfaces is given by the Alexander module.*

Proof. First note that isotopic cobordisms define isomorphic 2-cospans, so $\overline{\mathscr{B}}$ is well-defined at the level of 2-morphisms. Also, for any tangle τ, $\overline{\mathscr{B}}$ clearly maps the trivial concordance Id_τ to a 2-cospan canonically isomorphic to the identity 2-cospan $Id_{\overline{\mathscr{B}}(\tau)}$.

Let us now verify that $\overline{\mathscr{B}}$ preserves the vertical composition. Fix a (τ_1, τ_2)-cobordism A and a (τ_2, τ_3)-cobordism B. Let $H_1(\widehat{X}_{\tau_1}) \xrightarrow{\alpha_1} H_1(\widehat{W}_A) \xleftarrow{\alpha_2} H_1(\widehat{X}_{\tau_2})$ and $H_1(\widehat{X}_{\tau_2}) \xrightarrow{\beta_2} H_1(\widehat{W}_B) \xleftarrow{\beta_3} H_1(\widehat{X}_{\tau_3})$ be the 2-cospans arising from A and B. We need to show that $H_1(\widehat{X}_{\tau_1}) \xrightarrow{k_A\alpha_1} H_1(\widehat{W}_{B\star A}) \xleftarrow{k_B\beta_3} H_1(\widehat{X}_{\tau_3})$ is isomorphic to the vertical composition $H_1(\widehat{X}_{\tau_1}) \xrightarrow{v_A\alpha_1} H_1(\widehat{W}_B) \star H_1(\widehat{W}_A) \xleftarrow{v_B\beta_3} H_1(\widehat{X}_{\tau_3})$, where k_A, k_B are the inclusion induced maps and v_A, v_B are maps resulting from any representative of the pushout $H_1(\widehat{W}_B) \star H_1(\widehat{W}_A)$. Observe that $\widehat{W}_{B\star A}$ decomposes as the union of $\widehat{W}_B$ and $\widehat{W}_A$ glued along $\widehat{X}_{\tau_2}$. Therefore, the associated Mayer-Vietoris exact sequence

$$H_1(\widehat{X}_{\tau_2}) \xrightarrow{(-\alpha_2,\beta_2)} H_1(\widehat{W}_A) \oplus H_1(\widehat{W}_B) \xrightarrow{\binom{k_A}{k_B}} H_1(\widehat{W}_{B\star A}) \longrightarrow 0$$

together with Lemma 14.2.2 imply that $H_1(\widehat{W}_A) \xrightarrow{k_A} H_1(\widehat{W}_{B\star A}) \xleftarrow{k_B} H_1(\widehat{W}_B)$ is a representative for the pushout $H_1(\widehat{W}_A) \star H_1(\widehat{W}_B)$. Consequently, these two cospans are canonically isomorphic and the claim follows.

Given tangles and cobordisms as illustrated below

$$\begin{tikzcd} \varepsilon \arrow[r, bend left, "\tau_1"] \arrow[r, bend right, "\tau_2"'] \arrow[r, phantom, "\Downarrow A"] & \varepsilon' \arrow[r, bend left, "\tau_3"] \arrow[r, bend right, "\tau_4"'] \arrow[r, phantom, "\Downarrow B"] & \varepsilon'' \end{tikzcd},$$

our next goal is to prove the equality

$$\varphi_{\tau_2\tau_4} \star (\overline{\mathscr{B}}(B) \bullet \overline{\mathscr{B}}(A)) = \overline{\mathscr{B}}(B \bullet A) \star \varphi_{\tau_1\tau_3} \tag{14.4}$$

up to isomorphism of 2-cospans. Since the 2-morphism $\varphi_{\tau_1\tau_3}$ is represented by the 2-cospan $H_1(\widehat{X}_{\tau_3}) \circ H_1(\widehat{X}_{\tau_1}) \xrightarrow{\alpha_{\tau_1\tau_3}} H_1(\widehat{X}_{\tau_3\circ\tau_1}) \xleftarrow{id} H_1(\widehat{X}_{\tau_3\circ\tau_1})$, Remark 14.3.1 implies that the right hand side of equation (14.4) is represented by the 2-cospan

$$H_1(\widehat{X}_{\tau_3}) \circ H_1(\widehat{X}_{\tau_1}) \xrightarrow{k_{31}\alpha_{\tau_1\tau_3}} H_1(\widehat{W}_{B\bullet A}) \xleftarrow{k_{42}} H_1(\widehat{X}_{\tau_4\circ\tau_2}),$$

where k_{31} and k_{42} are induced by the inclusion maps. A similar argument shows that the left hand side of equation (14.4) is represented by the 2-cospan

$$H_1(\widehat{X}_{\tau_3}) \circ H_1(\widehat{X}_{\tau_1}) \xrightarrow{h_{31}} H_1(\widehat{W}_B) \bullet H_1(\widehat{W}_A) \xleftarrow{h_{42}\alpha^{-1}_{\tau_2\tau_4}} H_1(\widehat{X}_{\tau_4\circ\tau_2}),$$

where this time, the maps h_{31} and h_{42} are the ones which arise from the definition of horizontal composition. It now remains to find an isomorphism f of Λ-modules which fits in the following commutative diagram:

$$\begin{tikzcd} & H_1(\widehat{X}_{\tau_3}) \circ H_1(\widehat{X}_{\tau_1}) \arrow[dl, "h_{31}"'] \arrow[dr, "k_{31}\alpha_{\tau_1\tau_3}"] & \\ H_1(\widehat{W}_B) \bullet H_1(\widehat{W}_A) \arrow[rr, "f"] & & H_1(\widehat{W}_{B\bullet A}) \\ & H_1(\widehat{X}_{\tau_4\circ\tau_2}) \arrow[ul, "h_{42}\alpha^{-1}_{\tau_2\tau_4}"] \arrow[ur, "k_{42}"'] & \end{tikzcd}.$$

In order to construct f, first observe that the following diagram commutes

$$\begin{array}{ccccc} H_1(\widehat{X}_{\tau_2}) & \xleftarrow{\;i_2'\;} & H_1(\widehat{D}_{\varepsilon'}) & \xrightarrow{\;i_3'\;} & H_1(\widehat{X}_{\tau_3}) \\ {\scriptstyle\alpha_2}\big\downarrow & & \big\downarrow{\scriptstyle\cong} & & \big\downarrow{\scriptstyle\beta_3} \\ H_1(\widehat{W}_A) & \longleftarrow & H_1(\widehat{D}_{\varepsilon'}\times[0,1]) & \longrightarrow & H_1(\widehat{W}_B), \end{array}$$

where all the maps are induced by inclusions. Hence, identifying $H_1(\widehat{D}_{\varepsilon'}\times[0,1])$ with $H_1(\widehat{D}_{\varepsilon'})$, the first map in the Mayer-Vietoris exact sequence

$$H_1(\widehat{D}_{\varepsilon'}) \longrightarrow H_1(\widehat{W}_A)\oplus H_1(\widehat{W}_B) \longrightarrow H_1(\widehat{W}_{B\bullet A}) \longrightarrow 0$$

is given by $(-\alpha_2 i_2', \beta_3 i_3')$. It now follows from Lemma 14.2.2 that the cospan of inclusion induced maps $H_1(\widehat{W}_A) \overset{k_A}{\to} H_1(\widehat{W}_{B\bullet A}) \overset{k_B}{\leftarrow} H_1(\widehat{W}_B)$ is a representative of the pushout $H_1(\widehat{W}_A) \overset{h_A}{\to} H_1(\widehat{W}_B)\bullet H_1(\widehat{W}_A) \overset{h_B}{\leftarrow} H_1(\widehat{W}_B)$. Invoking the corresponding universal property, this produces a Λ-module isomorphism $f\colon H_1(\widehat{W}_B)\bullet H_1(\widehat{W}_A) \to H_1(\widehat{W}_{B\bullet A})$ with $fh_A = k_A$ and $fh_B = k_B$. Using successively the definition of $\alpha_{\tau_1\tau_3}$ (see diagram (14.3) for the relevant notations), the commutativity of inclusion induced maps, and the equalities above, one gets

$$f^{-1}k_{31}\alpha_{\tau_1\tau_3}j_3 = f^{-1}k_{31}k_3 = f^{-1}k_B\beta_3 = h_B\beta_3\,.$$

The equality $f^{-1}k_{31}\alpha_{\tau_1\tau_3}j_1 = h_A\alpha_1$ is proved similarly. Hence, the universal property of diagram (14.2) implies that $h_{31} = f^{-1}k_{31}\alpha_{\tau_1\tau_3}$. The equality $h_{42} = f^{-1}k_{42}\alpha_{\tau_2\tau_4}$ can be dealt with in the same way, and equation (14.4) is proved.

Given an $(\varepsilon,\varepsilon')$-tangle τ, we must now show that the 2-morphism $\overline{\mathscr{B}}(r_\tau)\star\varphi_{I_\varepsilon\tau}\star(I_{\overline{\mathscr{B}}(\tau)}\bullet\varphi_\varepsilon)$ coincides with $r_{\overline{\mathscr{B}}(\tau)}\colon \overline{\mathscr{B}}(\tau)\circ I_{\overline{\mathscr{B}}(\varepsilon)} \Rightarrow \overline{\mathscr{B}}(\tau)$. First observe that by Remark 14.2.1, one can choose representatives of the pushouts so that for any cospan $H\to T\leftarrow H'$, one has $T\circ I_H = T$. In particular, we only need to prove that, for this choice of pushouts,

$$\overline{\mathscr{B}}(r_\tau)\star\varphi_{I_\varepsilon\tau}\star(I_{\overline{\mathscr{B}}(\tau)}\bullet\varphi_\varepsilon) = I_{\overline{\mathscr{B}}(\tau)}. \tag{14.5}$$

As a first step, using the definition of the horizontal composition and Remark 14.2.1, we deduce that $I_{\overline{\mathscr{B}}(\tau)}\bullet\varphi_\varepsilon$ is represented by the 2-cospan

$$H_1(\widehat{X}_\tau) \xrightarrow{\;id\;} H_1(\widehat{X}_\tau) \xleftarrow{\;h\;} H_1(\widehat{X}_\tau)\circ H_1(\widehat{X}_{id_\varepsilon})\,,$$

where h is the unique morphism which fits in the following commutative diagram (recall diagram (14.2)):

$$\begin{tikzcd} & & H_1(\widehat{X}_\tau) & & \\ & & H_1(\widehat{X}_\tau)\circ H_1(\widehat{X}_{id_\varepsilon}) \arrow[u, "h"] & & \\ H_1(\widehat{D}_\varepsilon) \arrow[uurr, bend left, "i"] & & & & H_1(\widehat{X}_\tau) \arrow[uull, bend right, "id"'] \\ H_1(\widehat{X}_{id_\varepsilon}) \arrow[u, "\alpha_\varepsilon^{-1}"] \arrow[uurr, "j_1"] & & & & H_1(\widehat{X}_\tau). \arrow[u, "id"'] \arrow[uull, "j_2"'] \\ & & H_1(\widehat{D}_\varepsilon) \arrow[ull, "\alpha_\varepsilon"] \arrow[urr, "i"'] & & \end{tikzcd}$$

A short computation using Remark 14.2.1 then shows that the left hand side of equation (14.5) is represented by the 2-cospan

$$H_1(\widehat{X}_\tau) \xrightarrow{\alpha_{id_\varepsilon\tau}h^{-1}} H_1(\widehat{X}_{\tau\circ id_\varepsilon}) \xleftarrow{r^{-1}} H_1(\widehat{X}_\tau)\,,$$

where $r\colon H_1(\widehat{X}_{\tau\circ id_\varepsilon}) \to H_1(\widehat{X}_\tau)$ is the isomorphism induced by the obvious isotopy from $\tau\circ id_\varepsilon$ to τ. We now claim that r induces a 2-cospan isomorphism from $I_{\overline{\mathscr{B}}(\tau)}$ to this cospan. To prove this claim, we only need to show the equality $r\alpha_{id_\varepsilon\tau}h^{-1} = id_{H_1(\widehat{X}_\tau)}$, i.e. to check that $r\alpha_{id_\varepsilon\tau}$ satisfies the defining property of h displayed above. Since $\alpha_{id_\varepsilon\tau}j_1$ and $\alpha_{id_\varepsilon\tau}j_2$ are the inclusion induced homomorphisms (recall diagram (14.3)), this follows from the functoriality of homology. The proof of the equality $\overline{\mathscr{B}}(\ell_\tau)\star\varphi_{\tau I_{\varepsilon'}}\star(\varphi_{\varepsilon'}\bullet I_{\overline{\mathscr{B}}(\tau)}) = \ell_{\overline{\mathscr{B}}(\tau)}$ is dealt with in the same way.

Finally, the axiom involving the associativity isomorphisms is left to the reader: although the proof is tedious, it involves no other ideas than the ones presented up to now. Therefore we have proved that $\overline{\mathscr{B}}$ is a weak 2-functor and we turn to the last statement of the theorem. If Σ is a $(\emptyset,\emptyset)$-cobordism, that is, a closed oriented surface in the 4-ball, then the associated 2-cospan is given by

$$\begin{tikzcd} & 0 \arrow[d] & \\ 0 \arrow[ur] \arrow[dr] & H_1(\widehat{W}_\Sigma) & 0\,, \arrow[ul] \arrow[dl] \\ & 0 \arrow[u] & \end{tikzcd}$$

with W_Σ the exterior of Σ in the 4-ball. This is nothing but the Alexander module of Σ. $\square$

14.5 Further remarks

Recall from Chapter 9 that apart from the reduced Burau representation $\overline{\mathscr{B}}_t\colon B_n \to GL_{n-1}(\Lambda)$, we also have the unreduced Burau $\mathscr{B}_t\colon B_n \to GL_n(\Lambda)$ as well as the (un)reduced colored Gassner representations in the multivariable case. It is therefore natural to ask whether these variations of the Burau representation can also be extended to weak 2-functors. This is indeed the case, and is the subject of this slightly informal last section.

More precisely, we start in Subsection 14.5.1 by explaining how $\mathscr{B}_t$ can be extended to a functor $\mathscr{B}$ on tangles. This functor is no longer Lagrangian ($\mathscr{B}_t$ is not unitary) but it is monoidal and behaves well with respect to traces. In Subsection 14.5.2, we indicate how to extend it to a weak 2-functor. Finally, in Subsection 14.5.3, we briefly explain how all of these constructions can be extended to multivariable versions, defined on the category of colored tangles.

14.5.1 Extending the unreduced Burau representation to a monoidal functor

Given an integral domain Λ, let $\mathbf{C}_\Lambda$ denote the category with finitely generated Λ-modules as objects, and isomorphism classes of cospans as morphisms, composed by pushouts. Also, let $\mathbf{GL}_\Lambda$ denote the groupoid with the same objects as $\mathbf{C}_\Lambda$ and Λ-isomorphisms as morphisms. As in Section 13.3, one can check that the map assigning to an invertible cospan $H \xrightarrow{i} T \xleftarrow{i'}$

H' the Λ-isomorphism $i'^{-1}i\colon H \to H'$ defines an equivalence of categories $\mathit{core}(\mathbf{C}_\Lambda) \xrightarrow{\simeq} \mathbf{GL}_\Lambda$. Note that the direct sum endows these categories with a monoidal structure, with the trivial Λ-module $H = 0$ being the identity object. Given an endomorphism of $\mathbf{C}_\Lambda$, i.e. a cospan of the form $H \xrightarrow{i} T \xleftarrow{i'} H$, define the *trace* of T as the coequalizer

$$\begin{tikzcd} H \arrow[r, shift left, "i"] \arrow[r, shift right, "i'"'] & T \arrow[r, "j"] & \mathrm{tr}(T)\,. \end{tikzcd}$$

Viewing $\mathrm{tr}(T)$ as the isomorphism class of the cospan $0 \to \mathrm{tr}(T) \leftarrow 0$, the trace actually defines a map $\mathrm{tr}\colon \mathrm{End}(H) \to \mathrm{End}(0)$. It is an amusing exercise to check that it satisfies the following properties, as it should, see e.g. [152, p. 22].

i. If T_1 is a cospan from H to H' and T_2 from H' to H, then $\mathrm{tr}(T_1 \circ T_2) = \mathrm{tr}(T_2 \circ T_1)$.

ii. If T_1 and T_2 are two endomorphisms, then $\mathrm{tr}(T_1 \oplus T_2) = \mathrm{tr}(T_1) \circ \mathrm{tr}(T_2)$.

iii. If T is an endomorphism of 0, then $\mathrm{tr}(T) = T$.

These additional structures are also present in the category of tangles. Indeed, the juxtaposition endows **Tangles** with a monoidal structure, with the empty set $\varepsilon = \emptyset$ being the identity object. Furthermore, the closure of a tangle defines a natural trace function $\mathrm{End}(\varepsilon) \to \mathrm{End}(\emptyset)$. In this context, the unreduced Burau representation can be understood as a monoidal functor $b\colon \mathbf{Braids} \to \mathbf{GL}_\Lambda$, where $\Lambda = \mathbb{Z}[t^{\pm 1}]$.

We now sketch the construction of a monoidal functor $\mathscr{B}\colon \mathbf{Tangles} \to \mathbf{C}_\Lambda$ extending b, and behaving well with respect to traces. We shall follow the notation of Section 14.4, apart from the fact that all exteriors will be considered in the unit disc D^2, and not the sphere S^2 even when ℓ_ε vanishes. Let x_0 be the point $(-1, 0)$ in D^2. For any sequence ε of ± 1's, set $\mathscr{B}(\varepsilon) = H_1(\widehat{D}_\varepsilon, \widehat{x_0})$ and for any isotopy class τ of tangles, let $\mathscr{B}(\tau)$ denote the isomorphism class of the cospan $H_1(\widehat{D}_\varepsilon, \widehat{x_0}) \xrightarrow{i_\tau} H_1(\widehat{X}_\tau, \widehat{x_0 \times I}) \xleftarrow{i'_\tau} H_1(\widehat{D}_{\varepsilon'}, \widehat{x_0})$, where $\widehat{Y}$ stands for the inverse image of a subspace $Y \subset X_\tau$ by the infinite cyclic covering map $\widehat{X_\tau} \to X_\tau$. Following almost *verbatim* the proof of Theorem 14.4.1, one checks that this defines a functor $\mathscr{B}\colon \mathbf{Tangles} \to \mathbf{C}_\Lambda$ which fits in the commutative diagram

$$\begin{tikzcd} & \mathbf{Braids} \arrow[dl, bend right, "b"'] \arrow[r] \arrow[d] & \mathbf{Tangles} \arrow[d, "\mathscr{B}"] \\ \mathbf{GL}_\Lambda & \mathit{core}(\mathbf{C}_\Lambda) \arrow[l, "\simeq"'] \arrow[r] & \mathbf{C}_\Lambda\,. \end{tikzcd}$$

Furthermore, an additional application of Mayer-Vietoris shows that this functor is monoidal. (The basepoint x_0 is chosen so that the juxtaposition of tangles can be realized in a natural way by gluing discs along intervals, with x_0 a common endpoint of these intervals.) Finally, if τ is an $(\varepsilon, \varepsilon)$-tangle, then $\mathrm{tr}(\mathscr{B}(\tau))$ is nothing but the relative Alexander module of the oriented link in $D^2 \times I$ (or equivalently, in S^3) obtained by the closure of τ.

14.5.2 $\mathscr{B}$ as a monoidal weak 2-functor

One can modify $\mathbf{C}_\Lambda$ to obtain a bicategory in the exact same way as we did for $\mathbf{L}_\Lambda$, with 2-morphisms given by isomorphism classes of 2-cospans (recall Subsection 14.3.2). Furthermore, the direct sum endows this bicategory with a monoidal structure.

Also, the juxtaposition endows the bicategory of tangles with a monoidal structure. Here again, some care is needed, as different conventions such as the ones in [98] and [8] will lead to different monoidal bicategories. We will not go into these details, but only mention that our construction is robust enough to be valid in these different settings.

Let us sketch how the functor $\mathscr{B}$ can be extended to a weak 2-functor, following the notation of Subsection 14.4.2. Given a (τ_1, τ_2)-cobordism Σ, let us denote by $\mathscr{B}(\Sigma)\colon \mathscr{B}(\tau_1) \Rightarrow \mathscr{B}(\tau_2)$ the isomorphism class of the 2-cospan

$$H_1(\widehat{X}_{\tau_1}, \widehat{x_0 \times I}) \xrightarrow{\alpha_1} H_1(\widehat{W}_\Sigma, x_0 \widehat{\times I \times} I) \xleftarrow{\alpha_2} H_1(\widehat{X}_{\tau_2}, \widehat{x_0 \times I})\,.$$

One can check that this defines a weak 2-functor, that is monoidal in a sense that, once again, we shall not discuss in detail here.

14.5.3 Multivariable versions

Let μ be a positive integer. Recall that a *μ-colored tangle* consists of an oriented tangle τ together with a surjective map assigning to each component of τ an integer in $\{1, \ldots, \mu\}$. As explained in Section 13.2, μ-colored tangles naturally form a category $\mathbf{Tangles}_\mu$, with the $\mu = 1$ case being nothing but **Tangles**. Obviously, assigning a color to the cobordisms and proceeding as in subsection 14.4.2, one obtains a bicategory of μ-colored tangles.

All the results of the present chapter extend to this multivariable setting in a straightforward way, that we now very briefly summarize. The coloring of points, tangles and cobordisms induces homomorphisms from the homology of the corresponding exterior onto $\mathbb{Z}^\mu$, thus defining free abelian covers whose homology is a module over the ring of multivariable Laurent polynomials $\mathbb{Z}[\mathbb{Z}^\mu] = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}] =: \Lambda_\mu$. This allows one to construct a weak 2-functor from the bicategory of μ-colored tangles to the bicategory of Lagrangian cospans over Λ_μ, which extends the colored Gassner representation of μ-colored braids, and whose restriction to μ-colored links and surfaces is nothing but the multivariable Alexander module. The results of subsections 14.5.1 and 14.5.2 can be extended in the same way.

Part III

Non-additivity of classical link invariants

Chapter 15

Introduction and statement of the results

Consider an arbitrary link invariant $\mathscr{I}$ taking values in an abelian group. Precomposing this invariant with the braid closure defines maps $\alpha \mapsto \mathscr{I}(\widehat{\alpha})$ from the braid groups B_n to this abelian group, and one might wonder whether these maps are group homomorphisms. In other words, one can ask whether

$$\mathscr{I}(\widehat{\alpha\beta}) - \mathscr{I}(\widehat{\alpha}) - \mathscr{I}(\widehat{\beta})$$

vanishes for all $\alpha, \beta \in B_n$. This question has an easy answer: the only invariant with this property is the trivial one. However, one can ask the more refined question of "evaluating" the homomorphism defect displayed above. This can yield interesting consequences, both from the theoretical viewpoint, if this defect is expressed in terms of *a priori* unrelated objects, and from the practical viewpoint, as it reduces the computation of the invariant to the computation of this defect (together with the value of $\mathscr{I}$ on the closure of the standard generators of the braid group, i.e. unlinks).

This program was carried out by Gambaudo and Ghys ([78], recently republished in [80]) in the case of the Levine-Tristram signature

$$\mathrm{sign}(L)\colon S^1 \to \mathbb{Z}, \quad \omega \mapsto \mathrm{sign}_\omega(L)\,.$$

The great success of Gambaudo and Ghys was to express the homomorphism defect of this signature in terms of another classical object, the reduced Burau representation

$$\overline{\mathscr{B}}_t\colon B_n \to GL_{n-1}(\mathbb{Z}[t^{\pm 1}])\,.$$

More precisely, recall from Subsection 9.3.3 that this representation is unitary with respect to some skew-Hermitian form. Therefore, given two braids $\alpha, \beta \in B_n$ and a root of unity ω, one can consider the *Meyer cocycle* of the two unitary matrices $\overline{\mathscr{B}}_\omega(\alpha)$ and $\overline{\mathscr{B}}_\omega(\beta)$. Denoting by $\mathrm{sign}_\omega(L)$ the Levine-Tristram signature of an oriented link L, the main theorem of [78] is the equality

$$\mathrm{sign}_\omega(\widehat{\alpha\beta}) - \mathrm{sign}_\omega(\widehat{\alpha}) - \mathrm{sign}_\omega(\widehat{\beta}) = -\mathit{Meyer}(\overline{\mathscr{B}}_\omega(\alpha), \overline{\mathscr{B}}_\omega(\beta)) \tag{15.1}$$

for all $\alpha, \beta \in B_n$ and $\omega \in S^1$ of order coprime to n. (These authors actually work with the braid group on infinitely many strands B_∞, and obtain an equality valid for any ω of finite order; however, their proof does yield the finer result stated above.) Let us mention that this equality not only relates two very much studied objects in knot theory, but also gives a very efficient algorithm for the computation of the signature, as the Meyer cocycle is easy to calculate (and the signature of unlinks vanishes).

Recall from Section 3.4 that the Levine-Tristram signature admits a generalization, the *multivariable signature* which associates to a μ-colored link L a map

$$\operatorname{sign}(L)\colon \mathbb{T}^\mu \to \mathbb{Z}, \quad \omega = (\omega_1, \dots, \omega_\mu) \mapsto \operatorname{sign}_\omega(L)$$

on the μ-dimensional torus $\mathbb{T}^\mu$ (we now write $\operatorname{sign}_\omega(L)$ instead of $\sigma_L(\omega)$). We also know from Section 9.4 that the reduced Burau representation has a multivariable extension, called the reduced colored Gassner representation, which is unitary, and it is natural to wonder if (15.1) holds in this multivariable setting.

Also, braids are but a very special kind of *tangles*, whose definition was given in Section 13.2. Recall that oriented tangles no longer form groups, but are the morphisms of a category. Furthermore, the tangles that are endomorphims of a given object of this category can not only be composed, but also closed up to give oriented links, just like braids. Therefore, it makes sense to ask the same question as above, i.e. try to evaluate the defect of additivity of the signature on tangles. As we recalled in Chapter 13, the reduced Burau representation admits an extension to tangles, due to Cimasoni-Turaev [42], in the form of the Lagrangian functor $\mathscr{F}$. It extends the reduced Burau representation in the sense that if the tangle is a braid α, then $\mathscr{F}(\alpha)$ is the graph of the unitary automorphism $\overline{\mathscr{B}}_t(\alpha)$. One cannot consider the Meyer cocycle of (pairs of) objects in this Lagrangian category, but it makes sense to consider the *Maslov index* of three objects in this category, evaluated at some $t = \omega \in S^1$, see Subsection 16.2. Therefore, one can ask whether the additivity defect of the signature of tangles is related to the Maslov index of the image by $\mathscr{F}$ of these tangles, evaluated at $t = \omega$.

In the third part of this thesis (which is based on joint work with David Cimasoni [39]), we answer both these questions simultaneously. The precise statement will be given in Theorem 16.4.1 and Theorem 17.3.2 below, but in a nutshell, it can be phrased as follows.

Theorem 15.0.1. *Given an object c of the category of μ-colored tangles and two endomorphisms τ_1, τ_2 of this object, the equality*

$$\operatorname{sign}_\omega(\widehat{\tau_1\tau_2}) - \operatorname{sign}_\omega(\widehat{\tau}_1) - \operatorname{sign}_\omega(\widehat{\tau}_2) = \mathit{Maslov}(\mathscr{F}_\omega(\overline{\tau}_1), \mathscr{F}_\omega(id_c), \mathscr{F}_\omega(\tau_2))$$

holds for all $\omega = (\omega_1, \dots, \omega_\mu)$ in an open dense subset of $\mathbb{T}^\mu$, where $\overline{\tau}$ denotes the horizontal reflection of the tangle τ, and $\mathscr{F}_\omega$ is the evaluation at $t = \omega$ of the multivariable extension of the Lagrangian functor $\mathscr{F}$.

In the case of colored braids, this functor gives back the graph of the reduced colored Gassner representation, the horizontal reflection of a braid is its inverse, and the Maslov index of the graphs of unitary automorphisms γ_1^{-1}, id and γ_2 is one possible definition of the Meyer cocycle of γ_1 and γ_2. Therefore, in the case of μ-colored braids, our theorem is exactly the expected multivariable extension of (15.1).

We would like to point out that, although our demonstration roughly follows the same lines as the original proof of Gambaudo and Ghys, several clarifications are made along the way. Actually, the paper [78] contains a very detailed proof in the case $\omega = -1$, but only a brief description of the necessary modifications needed for the case of ω a root of unity. Therefore, it is our hope that this work will be of use not only to those interested in the full generality of our main result (Theorem 16.4.1), but also to those merely curious about oriented tangles (Corollary 16.4.3), colored braids (Corollary 16.4.4), or a new algorithm for the computation of multivariable signatures (Remark 16.4.6).

Note that the final part of this thesis has a somewhat special status for reasons which we now outline. Firstly, it assumes familiarity with Part I and part II of this thesis, especially with Sections 3.4 and 13.4. The second reason is more relevant:

Remark 15.0.2. This chapter is based on the paper [39] which was written before the machinery of Chapters 8 and 11 was developed. A posteriori, we have little doubt that Theorem 15.0.1 and its proof could be improved upon using the aforementioned technology, but this shall not be attempted.

Ironically, the awkwardness of some of the arguments presented here may finish convincing the reader of the usefulness of local coefficients (although the more geometrically minded reader might enjoy the use of branched covers and the lack of homological algebra). Be that as it may, these arguments led us to develop the machinery of the previous chapters of this thesis and so it only seems fair to leave them untouched.

Elaborating on the first pararagraph of this remark, the interested reader might attempt to generalize Theorem 16.4.1 as follows. Replace the use of the "generalized eigenspaces" of Subsections 17.1.1 and 17.1.2 by $\mathbb{C}^\omega$-twisted homology, substitute the "ω-signatures" of Subsection 17.1.2 with the twisted signatures of Section 7.2, define the isotropic functor using $\mathbb{C}^\omega$-coefficients instead of branched covers and, finally, adapt the arguments of Section 17.3 by making use of the universal coefficient spectral sequences of Subsection 7.5.2. The proof would then require the interpretation of the multivariable signature given in Theorem 8.1.1. In the case of braids, the result could then be understood as relating signatures to reduced $\mathbb{C}^\omega$-twisted Burau maps, as described in Chapter 11.

Most importantly, the paper [39] is dedicated to the memory of Ruty Ben-Zion and so I have absolutely no desire to rewrite it.

Chapter 16

Definitions, statement of the theorem, and examples

The first aim of this section is to give precise definitions of the objects which appeared in the introduction: isotropic categories, the Maslov index and Meyer cocycle, and the isotropic functor are introduced in Sections 16.1, 16.2 and 16.3. Our main result is then stated in subsection 16.4, where several corollaries and examples are also given.

16.1 The isotropic and Lagrangian categories

In this paragraph, we introduce the category $\mathbf{Isotr}_\Lambda$ of isotropic relations over a ring Λ. This is a slight modification of the category $\mathbf{Lagr}_\Lambda$ of Lagrangian relations reviewed in Section 13.3.

Fix an integral domain Λ endowed with a ring involution $a \mapsto \overline{a}$. In this chapter a *Hermitian Λ-module H* will be a finitely generated Λ-module endowed with a *possibly degenerate* skew-Hermitian form λ.

Remark 16.1.1. In Chapters 13 and 14, following [42, 43], skew-Hermitian forms were assumed to be non-degenerate. In the current chapter, we drop this assumption.

The same module H with the opposite form $-\lambda$ will be denoted by $-H$. Given a submodule V of a Hermitian Λ-module H, its *annihilator* is the submodule

$$\mathrm{Ann}(V) = \{x \in H \mid \lambda(v, x) = 0 \text{ for all } v \in V\}\,.$$

A submodule V of a Hermitian module H is *isotropic* if $V \subset \mathrm{Ann}(V)$ or, equivalently, if λ vanishes identically on V. A submodule of a Hermitian module is *Lagrangian* if it is equal to its annihilator. If H_1 and H_2 are Hermitian Λ-modules, an *isotropic relation* $N\colon H_1 \Rightarrow H_2$ is an isotropic submodule of $(-H_1) \oplus H_2$. For instance, given a Hermitian Λ-module H, the *diagonal relation* $\Delta_H = \{h \oplus h \in H \oplus H\}$ is an isotropic relation $H \Rightarrow H$. Given two isotropic relations $N_1\colon H_1 \Rightarrow H_2$ and $N_2\colon H_2 \Rightarrow H_3$, their *composition* is defined as $N_2 \circ N_1 := N_2N_1\colon H_1 \Rightarrow H_3$ where N_2N_1 denotes the following submodule of $(-H_1) \oplus H_3$:

$$N_2N_1 = \{h_1 \oplus h_3 \mid h_1 \oplus h_2 \in N_1 \text{ and } h_2 \oplus h_3 \in N_2 \text{ for a certain } h_2 \in H_2\}\,.$$

Isotropic relations should be understood as a generalization of unitary isomorphisms: if γ is a unitary isomorphism, then Γ_γ is an isotropic submodule of $(-H_1) \oplus H_2$, that is, an isotropic relation $H_1 \Rightarrow H_2$. We will denote by $\widetilde{\mathbf{U}}_\Lambda$ the category of Hermitian Λ-modules and unitary isomorphisms. The proof of the following proposition is straightforward.

Proposition 16.1.2. *Hermitian Λ-modules, as objects, and isotropic relations as morphisms, form a category $\mathbf{Isotr}_\Lambda$. Furthermore the map $\gamma \mapsto \Gamma_\gamma$ defines an embedding of categories $\Gamma\colon \widetilde{\mathbf{U}}_\Lambda \hookrightarrow \mathbf{Isotr}_\Lambda$.*

We now relate the isotropic category to the Lagrangian category $\mathbf{Lagr}_\Lambda$ which was reviewed in Section 13.3. First, we recall that the composition in $\mathbf{Lagr}_\Lambda$ is defined slightly differently than in $\mathbf{Isotr}_\Lambda$. Given of a submodule A of a Hermitian Λ-module H, we set

$$\overline{A} = \{x \in H \mid ax \in A \text{ for a non-zero } a \in \Lambda\}\,.$$

If H_1 and H_2 are non-degenerate Hermitian Λ-modules, a *Lagrangian relation* $N\colon H_1 \Rightarrow H_2$ is a Lagrangian submodule of $(-H_1) \oplus H_2$. Given two Lagrangian relations $N_1\colon H_1 \Rightarrow H_2$ and $N_2\colon H_2 \Rightarrow H_3$, their *composition* is defined as $N_2 \circ N_1 := \overline{N_2 N_1}\colon H_1 \Rightarrow H_3$. As we saw in Theorem 13.3.1, non-degenerate Hermitian Λ-modules, as objects, and Lagrangian relations, as morphisms, form a category $\mathbf{Lagr}_\Lambda$, and the map $\gamma \mapsto \Gamma_\gamma$ defines an embedding of categories $\Gamma\colon \mathbf{U}_\Lambda \hookrightarrow \mathbf{Lagr}_\Lambda$. Here $\mathbf{U}_\Lambda$ denotes the category of *non-degenerate* Hermitian Λ-modules and unitary isomorphisms.

Summarizing the content of this paragraph. For any integral domain Λ endowed with a ring involution, we have the diagram

$$\begin{tikzcd} \mathbf{U}_\Lambda \arrow[r, "\Gamma"] \arrow[d] & \mathbf{Lagr}_\Lambda \arrow[d, dashed] \\ \widetilde{\mathbf{U}}_\Lambda \arrow[r, "\Gamma"] & \mathbf{Isotr}_\Lambda, \end{tikzcd}$$

where the horizontal arrows are the embeddings of categories given by the graph, and the vertical arrows denote the natural embeddings of categories. The right hand side arrow is dashed because the composition in $\mathbf{Lagr}_\Lambda$ is not always defined in the same way as in $\mathbf{Isotr}_\Lambda$. However, if Λ is a field, then $\overline{A}$ coincides with A for every subspace $A \subset H$, and this arrow does represent a functor.

16.2 The Maslov index and the Meyer cocycle

The Maslov index associates an integer to three isotropic subspaces of a symplectic vector space, while the Meyer cocycle associates an integer to two symplectic automorphisms. The aim of this paragraph is to review these constructions in the spirit of [152, Chapter IV.3], adapting them to the setting of Hermitian complex vector spaces.

Fix a finite dimensional Hermitian complex vector space (H, λ), and let L_1, L_2 and L_3 be three isotropic subspaces of H. Consider the Hermitian form f defined on $(L_1 + L_2) \cap L_3$ as follows: for $a, b \in (L_1 + L_2) \cap L_3$, write $a = a_1 + a_2$ with $a_i \in L_i$, and set $f(a, b) = \lambda(a_2, b)$. One easily checks that f is a well-defined Hermitian form.

Definition 47. The signature of f is called the *Maslov index* of L_1, L_2 and L_3. It will be denoted by $\mathit{Maslov}(L_1, L_2, L_3)$.

It should be noted that two other definitions occur in the literature. In [156], the Maslov index is defined by considering the same Hermitian form but on the quotient $\frac{(L_1+L_2)\cap L_3}{(L_1\cap L_3)+(L_2\cap L_3)}$, see also [135]. In [78], the authors consider the space

$$V = \{v_1 \oplus v_2 \oplus v_3 \in L_1 \oplus L_2 \oplus L_3 \mid v_1 + v_2 + v_3 = 0\}$$

and the Hermitian form defined by sending elements $a_1 \oplus a_2 \oplus a_3, b_1 \oplus b_2 \oplus b_3 \in V$ to $\lambda(a_2, b_1)$. These definitions are equivalent to ours. This can be seen by noting that if f is a Hermitian form on H and A is a subspace contained in $\mathrm{Ann}(H)$, then f descends to a Hermitian form on H/A whose signature remains unchanged.

We record the following easy lemma for further use.

Lemma 16.2.1. *The Maslov index satisfies the following properties.*

(i) *If L_1, L_2, L_3 (resp. L_1', L_2', L_3') are isotropic subspaces of H (resp. H') then $L_1 \oplus L_1', L_2 \oplus L_2', L_3 \oplus L_3'$ are isotropic subspaces of $H \oplus H'$, and*

$$Maslov(L_1 \oplus L_1', L_2 \oplus L_2', L_3 \oplus L_3') = Maslov(L_1, L_2, L_3) + Maslov(L_1', L_2', L_3')\,.$$

(ii) *For any isotropic subspaces $L_1, L_2 \subset H$, $Maslov(L_1, L_2, L_2)$ vanishes.*

(iii) *If L_1, L_2, L_3 are isotropic subspaces of H and ψ is a unitary automorphism of H, then $\psi(L_1), \psi(L_2), \psi(L_3)$ are isotropic subspaces of H, and*

$$Maslov(\psi(L_1), \psi(L_2), \psi(L_3)) = Maslov(L_1, L_2, L_3)\,.$$

As we briefly mentioned in Section 7.2, the Maslov index plays a crucial role in the additivity of signatures of 4-manifolds, see Section 17.1.2 for details. Furthermore, note that the state of the literature is quite confusing with regards to the term "Maslov index". A helpful survey can be found in [27]. Regarding this issue, we shall simply mention that the Maslov index we are studying first appeared in a 1969 paper by Wall [156]; it must not be confused with the Maslov index used in [3, 119].

Let us now introduce the second object of this paragraph.

Definition 48. The *Meyer cocycle* of two unitary automorphisms γ_1, γ_2 of H is the integer

$$Meyer(\gamma_1, \gamma_2) = -Maslov(\Gamma_{\gamma_1^{-1}}, \Gamma_{id}, \Gamma_{\gamma_2}).$$

As for the Maslov index, some equivalent definitions appear in the literature. The Meyer cocyle was originally defined by Meyer in [120, 121] by considering the space

$$M_{\gamma_1,\gamma_2} = \{(v_1, v_2) \mid (\gamma_1^{-1} - id)v_1 = (id - \gamma_2)v_2\}$$

and taking the signature of the bilinear form B on M_{γ_1,γ_2} obtained by setting

$$B(v, w) = \lambda(v_1 + v_2, \gamma_1^{-1}(w_1) - w_1)$$

for $v = (v_1, v_2)$ and $w = (w_1, w_2) \in M_{\gamma_1,\gamma_2}$. On the other hand, for computational purposes, the most practical definition of the Meyer cocycle is given in [78]: the authors consider the space

$$E_{\gamma_1,\gamma_2} = \mathrm{im}(\gamma_1^{-1} - id) \cap \mathrm{im}(id - \gamma_2)$$

and take the signature of the Hermitian form obtained by setting $b(e,e') = \lambda(x_1+x_2,e')$ for $e = \gamma_1^{-1}(x_1) - x_1 = x_2 - \gamma_2(x_2) \in E_{\gamma_1,\gamma_2}$. It can be checked that these definitions are equivalent to the one we gave in terms of the Maslov index. Finally, note that despite our algebraic treatment of this cocycle, Meyer's original goal was to compute signatures of certain surface bundles over surfaces.

Let us show how to use the latter definition on a couple of explicit examples.

Example 16.2.2. Let ω be any complex number of modulus 1, and consider the one-dimensional complex vector space $H = \mathbb{C}$ endowed with the skew-Hermitian form given by the matrix

$$\lambda(\omega) = (\omega - \overline{\omega}) \ .$$

The automorphism γ of H given by multiplication by $-\omega$ is unitary with respect to the matrix $\lambda(\omega)$. Since $(\gamma^{-1} - id)(-\omega) = (id-\gamma)(1) = 1+\omega =: e$, we get $E_{\gamma,\gamma} = \mathbb{C}e$ and

$$b(e,e) = (1-\omega)(\omega-\overline{\omega})(1+\overline{\omega}) = \|\omega - \overline{\omega}\|^2.$$

This leads to

$$Meyer(\gamma,\gamma) = \begin{cases} 1 & \text{if } \omega \neq \pm 1; \\ 0 & \text{if } \omega = \pm 1. \end{cases} \tag{16.1}$$

Example 16.2.3. Let ω be a complex number of modulus 1, and consider the two-dimensional complex vector space H endowed with the skew-Hermitian form given by the matrix

$$\lambda(\omega) = \begin{pmatrix} \omega - \overline{\omega} & 1-\omega \\ -1+\overline{\omega} & \omega - \overline{\omega} \end{pmatrix} .$$

The automorphism γ_1 of H given by the matrix

$$\gamma_1 = \begin{pmatrix} -\omega & 1 \\ 0 & 1 \end{pmatrix}$$

is unitary with respect to $\lambda(\omega)$, so $\gamma_2 := \gamma_1^2$ is unitary as well. If $\omega = 1$, $\lambda(\omega)$ vanishes. For $\omega \neq 1$, an immediate computation yields $E_{\gamma_1,\gamma_2} = \mathbb{C}e$, with

$$e := \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (\gamma_1^{-1} - id) \begin{pmatrix} 0 \\ \omega \end{pmatrix} = (id - \gamma_2) \begin{pmatrix} 0 \\ (\omega-1)^{-1} \end{pmatrix}$$

This leads to

$$b(e,e) = \begin{pmatrix} 0 & \omega + (\omega-1)^{-1} \end{pmatrix} \begin{pmatrix} \omega - \overline{\omega} & 1-\omega \\ -1+\overline{\omega} & \omega - \overline{\omega} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 - 2\,\mathrm{Re}(\omega) \, ,$$

so

$$Meyer(\gamma_1,\gamma_2) = \mathrm{sgn}(1 - 2\,\mathrm{Re}(\omega)). \tag{16.2}$$

Example 16.2.4. Fix $(\omega_1,\omega_2) \in \mathbb{T}^2$, and consider the one-dimensional complex vector space endowed with the skew-Hermitian form given by the matrix

$$\lambda(\omega_1,\omega_2) = ((\omega_1 - \overline{\omega}_1) + (\omega_2 - \overline{\omega_2}) - (\omega_1\omega_2 - \overline{\omega}_1\overline{\omega}_2)) \ .$$

The automorphism γ given by multiplication by $\omega_1\omega_2$ is unitary with respect to $\lambda(\omega_1,\omega_2)$. Since $(\gamma^{-1} - id)(\omega_1\omega_2) = (id-\gamma)(1) = 1 - \omega_1\omega_2 =: e$, we get $E_{\gamma,\gamma} = \mathbb{C}e$, and

$$\begin{aligned} Meyer(\gamma,\gamma) &= \mathrm{sgn}(b(e,e)) = \mathrm{sgn}((\omega_1\omega_2 + 1)\lambda(\omega_1,\omega_2)(1 - \overline{\omega}_1\overline{\omega}_2)) \\ &= \mathrm{sgn}\left(\mathrm{Re}\left[(1-\omega_1)(1-\omega_2)(1-\omega_1\omega_2)\right]\right) \ . \end{aligned}$$

Let us conclude this paragraph with one last observation. Note that the definition of the Meyer cocycle depends on whether one chooses $\lambda \oplus -\lambda$ or $-\lambda \oplus \lambda$ as the skew-Hermitian form on $H \oplus H$. We chose the latter, which explains the presence of the minus sign in our definition, a sign which does not appear in [78].

16.3 The isotropic functor

In order to state our main result, we need to extend the evaluation of the reduced colored Gassner representation at $t = \omega \in \mathbb{T}^\mu$ from braids to tangles. This requires the definition, for each ω, of a modified functor $\mathscr{F}_\omega$, which is constructed in the same manner as the Lagrangian functor $\mathscr{F}$, but using finite abelian branched covers instead of free abelian ones. In this section, we only state the main properties of $\mathscr{F}_\omega$, postponing to Subsection 17.1.3 its construction and the proof of these statements.

We start by recalling some properties of the Lagrangian functor from Section 13.4. The Lagrangian functor $\mathscr{F} \colon \mathbf{Tangles}_\mu \to \mathbf{Lagr}_{\Lambda_\mu}$ is an extension of the reduced colored Gassner representations in the following sense. If α is an n-stranded μ-colored braid, then $\mathscr{F}(\alpha)$ is the graph of the reduced colored Gassner representation. In other words, the Lagrangian functor fits in the commutative diagram

$$\begin{tikzcd} \mathbf{Braids}_\mu \arrow[r] \arrow[d] & \mathbf{Tangles}_\mu \arrow[d, "\mathscr{F}"] \\ \mathbf{U}_{\Lambda_\mu} \arrow[r, "\Gamma"] & \mathbf{Lagr}_{\Lambda_\mu}, \end{tikzcd}$$

where the horizontal arrows are the embeddings of categories described in Sections 13.2 and 13.3. In order to evaluate this functor, for each torsion element ω in $\mathbb{T}^\mu$, Theorem 17.1.10 shall provide a functor $\mathscr{F}_\omega \colon \mathbf{Tangles}_\mu \to \mathbf{Isotr}_\mathbb{C}$ which fits in the commutative diagram

$$\begin{tikzcd} \mathbf{Braids}_\mu \arrow[r] \arrow[d] & \mathbf{Tangles}_\mu \arrow[d, "\mathscr{F}_\omega"] \\ \widetilde{\mathbf{U}}_\mathbb{C} \arrow[r, "\Gamma"] & \mathbf{Isotr}_\mathbb{C}\,. \end{tikzcd}$$

In general, the object $\mathscr{F}_\omega(c)$ of $\mathbf{Isotr}_\mathbb{C}$ associated to a sequence c of $\pm 1, \dots, \pm\mu$ is a complex vector space endowed with a skew-Hermitian form which can be degenerate. However, let us assume that each coordinate ω_i of ω is of order $k_i > 1$ with these k_i's pairwise coprime. If the sequence c is such that for all $i = 1, \dots, \mu$, $\ell(c)_i := \sum_{j; c_j = \pm i} \operatorname{sgn}(c_j)$ does not vanish and is coprime to k_i, then $\mathscr{F}_\omega(c)$ is non-degenerate. To be more precise, let us denote by $\mathbf{Tangles}_\mu^\omega$ (resp. $\mathbf{Braids}_\mu^\omega$) the full subcategory of $\mathbf{Tangles}_\mu$ (resp. $\mathbf{Braids}_\mu$) given by sequences fulfilling the condition above. Then, Proposition 17.1.12 will show that the restriction of $\mathscr{F}_\omega$ to $\mathbf{Tangles}_\mu^\omega$ defines a functor which fits in the commutative diagram

$$\begin{tikzcd} \mathbf{Braids}_\mu^\omega \arrow[r] \arrow[d] & \mathbf{Tangles}_\mu^\omega \arrow[d, "\mathscr{F}_\omega"] \\ \mathbf{U}_\mathbb{C} \arrow[r, "\Gamma"] & \mathbf{Lagr}_\mathbb{C}. \end{tikzcd}$$

Finally, one might wonder how the functors $\mathscr{F}$ and $\mathscr{F}_\omega$ are related. As far as objects are concerned, the answer is very simple: for any $c\colon \{1,\dots,n\} \to \{\pm 1,\dots,\pm\mu\}$, a matrix for the skew-Hermitian form of the complex vector space $\mathscr{F}_\omega(c)$ can be obtained by evaluating at $t=\omega$ a matrix for the skew-Hermitian form λ_c of the (localized) module $\mathscr{F}(c)$, see Proposition 17.3.1. Here are some concrete examples of these statements.

Example 16.3.1. In the case $\mu = 1$, a matrix for the skew-Hermitian complex form given by $\mathscr{F}_\omega(c)$ is obtained by evaluating the matrix described in Example 13.4.1 at $t=\omega$. The result is :

$$\begin{pmatrix} \frac{1}{2}(\varepsilon_1+\varepsilon_2)(\omega-\overline{\omega}) & 1-\omega^{\varepsilon_2} & 0 & \dots & 0 \\ \overline{\omega}^{\varepsilon_2}-1 & \frac{1}{2}(\varepsilon_2+\varepsilon_3)(\omega-\overline{\omega}) & & \ddots & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & \ddots & & & 1-\omega^{\varepsilon_n} \\ 0 & \dots & 0 & \overline{\omega}^{\varepsilon_n}-1 & \frac{1}{2}(\varepsilon_{n-1}+\varepsilon_n)(\omega-\overline{\omega}) \end{pmatrix}. \tag{16.3}$$

Example 16.3.2. In the case $n=\mu=2$ and $c=(1,2)$, a matrix for the skew-Hermitian complex form associated to $\mathscr{F}_\omega(c)$ is obtained by evaluating the matrix described in Example 9.4.8 at $t_1=\omega_1$ and $t_2=\omega_2$. The result is

$$\lambda_c(\omega_1,\omega_2) = \left((\omega_1-\overline{\omega}_1)+(\omega_2-\overline{\omega}_2)-(\omega_1\omega_2-\overline{\omega}_1\overline{\omega}_2)\right) .$$

For morphisms, the relation between $\mathscr{F}$ and $\mathscr{F}_\omega$ is trickier. Roughly speaking, $\mathscr{F}_\omega$ is the evaluation of $\mathscr{F}$ at $t=\omega$ "whenever that makes sense". The precise statement is slightly technical, but the following special case will suffice for the purpose of the present discussion, see Subsection 17.3.1 for details. Let us say that a tangle is *topologically trivial* if its exterior is homeomorphic to the exterior of a trivial braid. It turns out that if a (c,c')-tangle τ is topologically trivial, then working over the localized ring Λ_S, one obtains that $\mathscr{F}(\tau)$ is a free submodule of the free Λ_S-module $(-\mathscr{F}(c))\oplus\mathscr{F}(c')$, so this inclusion can be encoded by a matrix $M(t)$ with coefficients in Λ_S. In this case, $\mathscr{F}_\omega(\tau)$ is equal to the complex subspace of $(-\mathscr{F}_\omega(c))\oplus\mathscr{F}_\omega(c')$ encoded by the matrix $M(\omega)$.

Remark 16.3.3. From now on, the (localized) reduced colored Gassner representation will be denoted by $\overline{\mathscr{B}}_t$ instead of $\overline{\mathscr{B}}_{\psi_c}$ and $\overline{\mathscr{B}}^{\mathrm{loc}}_{\psi_c}$. This new notation is more convenient regarding evaluations at roots of unity.

Finally, note that since a braid α is topologically trivial, $\mathscr{F}_\omega(\alpha)$ is nothing but the graph of the unitary automorphism given by the evaluation $\overline{\mathscr{B}}_\omega(\alpha)$ of a matrix $\overline{\mathscr{B}}_t(\alpha)$ of the reduced colored Gassner representation at $t=\omega$.

16.4 Statement of the result and examples

We are finally ready to state our main result in a precise way, and to illustrate it with examples.

Let $\mathbb{T}^\mu_{cP}$ denote the dense subset of the μ-dimensional torus $\mathbb{T}^\mu$ composed of the elements $\omega=(\omega_1,\dots,\omega_\mu)$ such that the orders $k_1,\dots,k_\mu$ of $\omega_1,\dots,\omega_\mu$ are greater than 1 and pairwise coprime (this set must not be confused with the set $\mathbb{T}^\mu_P$ which appears in Section 3.4

and in Chapter 8). Recall that for a coloring $c\colon \{1,\dots,n\} \to \{\pm 1,\dots,\pm\mu\}$, we defined $\ell(c)$ as the element in $\mathbb{Z}^\mu$ whose i^{th} coordinate is given by $\ell(c)_i = \sum_{j;c_j=\pm i} \operatorname{sgn}(c_j)$. Given a coloring c with $\ell(c)_i \neq 0$ for all i, let $\mathbb{T}^\mu_c$ denote the dense subset of $\mathbb{T}^\mu$ given by the elements ω such that for all i, k_i is coprime to $\ell(c)_i$.

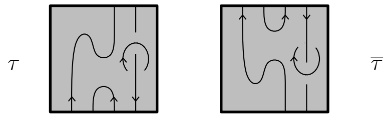

Figure 16.1: A tangle τ and its reflexion $\overline{\tau}$.

Finally, given a (c,c')-tangle τ, let us denote by $\overline{\tau}$ the (c',c)-tangle obtained from τ by a reflection with respect to the horizontal disk $D^2 \times \{1/2\}$, see Figure 16.1.

Theorem 16.4.1. *For any c such that $\ell(c)$ is nowhere zero and for any (c,c)-colored tangles τ_1 and τ_2, the equality*

$$\operatorname{sign}_\omega(\widehat{\tau_1\tau_2}) - \operatorname{sign}_\omega(\widehat{\tau}_1) - \operatorname{sign}_\omega(\widehat{\tau}_2) = Maslov(\mathscr{F}_\omega(\overline{\tau}_1), \Delta, \mathscr{F}_\omega(\tau_2))$$

holds for all ω in the dense subset $\mathbb{T}^\mu_c \cap \mathbb{T}^\mu_{cP}$ of the torus $\mathbb{T}^\mu$.

Remark 16.4.2. Note that the conditions of $\ell(c)$ being nowhere zero and ω belonging to $\mathbb{T}^\mu_c$ are not restrictive. To see this, first note that the category $\mathbf{Tangles}_\mu$ is endowed with a monoidal structure given by the juxtaposition $\tau_1 \sqcup \tau_2$ of colored tangles, see Figure 16.2. Next, assume that we want to evaluate the integer

$$\delta_\omega(\tau_1,\tau_2) := \operatorname{sign}_\omega(\widehat{\tau_1\tau_2}) - \operatorname{sign}_\omega(\widehat{\tau}_1) - \operatorname{sign}_\omega(\widehat{\tau}_2)$$

for a given $\omega \in \mathbb{T}^\mu_{cP}$ and (c,c)-tangles τ_1,τ_2. Then, one can always find a sequence c' such that $\ell(c\sqcup c')$ is nowhere zero and ω belongs to $\mathbb{T}^\mu_{c\sqcup c'}$. Since the signature of any colored unlink vanishes, we have the equality

$$\delta_\omega(\tau_1,\tau_2) = \delta_\omega(\tau_1 \sqcup id_{c'}, \tau_2 \sqcup id_{c'})$$

for any c' and any ω; note that here we implicitly used the following easy fact: given any two closable μ-colored tangles τ_1 and τ_2, the equality $\operatorname{sign}_\omega(\widehat{\tau_1 \sqcup \tau_2}) = \operatorname{sign}_\omega(\widehat{\tau}_1) + \operatorname{sign}_\omega(\widehat{\tau}_2)$ holds for all $\omega \in \mathbb{T}^\mu$. Therefore, Theorem 16.4.1 allows us to compute this defect for any pair of colored tangles and any $\omega \in \mathbb{T}^\mu_{cP}$.

The special case $\mu = 1$, which corresponds to oriented tangles and the Levine-Tristram signature, takes the following form.

Corollary 16.4.3. *For any sequence ε of ± 1's whose sum $\ell(\varepsilon)$ does not vanish, and for any $(\varepsilon,\varepsilon)$-tangles τ_1 and τ_2, the equality*

$$\operatorname{sign}_\omega(\widehat{\tau_1\tau_2}) - \operatorname{sign}_\omega(\widehat{\tau}_1) - \operatorname{sign}_\omega(\widehat{\tau}_2) = Maslov(\mathscr{F}_\omega(\overline{\tau}_1), \Delta, \mathscr{F}_\omega(\tau_2))$$

holds for all $\omega \in S^1 \setminus \{1\}$ whose order is coprime to $\ell(\varepsilon)$.

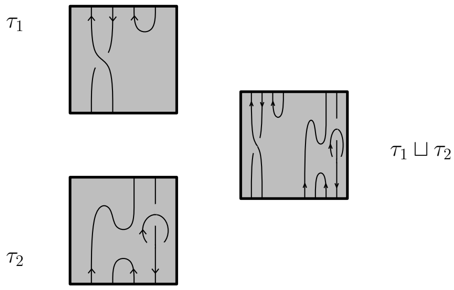

Figure 16.2: On the left, two tangles τ_1, τ_2; on the right their disjoint union $\tau_1 \sqcup \tau_2$.

Note that by Remark 16.4.2, this actually allows us to compute the additivity defect of the Levine-Tristram signature evaluated at any root of unity.

Let us now come back to the general multivariable case, but specialized to colored braids. As explained in Section 16.3, if α is a colored braid, then $\mathscr{F}_\omega(\alpha) = \Gamma_{\overline{\mathscr{B}}_\omega(\alpha)}$, the graph of the reduced colored Gassner representation evaluated at ω. Moreover, the horizontal reflection $\overline{\alpha}$ of α is nothing but its inverse. Finally, as stated in Subsection 16.2, the Meyer cocycle and Maslov index are related by the equality

$$Maslov(\Gamma_{\gamma_1^{-1}}, \Delta, \Gamma_{\gamma_2}) = Maslov(\Gamma_{\gamma_1^{-1}}, \Gamma_{id}, \Gamma_{\gamma_2}) = -Meyer(\gamma_1, \gamma_2).$$

Consequently, we obtain the following corollary.

Corollary 16.4.4. *For any c such that $\ell(c)$ is nowhere zero and for any two colored braids $\alpha, \beta \in B_c$, the equality*

$$\operatorname{sign}_\omega(\widehat{\alpha\beta}) - \operatorname{sign}_\omega(\widehat{\alpha}) - \operatorname{sign}_\omega(\widehat{\beta}) = -Meyer(\overline{\mathscr{B}}_\omega(\alpha), \overline{\mathscr{B}}_\omega(\beta))$$

holds for all ω in $\mathbb{T}_c^\mu \cap \mathbb{T}_{cP}^\mu$.

Obviously, Remark 16.4.2 applies to this particular case, so we can compute the multivariable signature evaluated at any $\omega \in \mathbb{T}_{cP}^\mu$. However, more can be said in this case.

Remark 16.4.5. Observe that both sides of the equality in Corollary 16.4.4 are defined for all $\omega \in \mathbb{T}^\mu$. Moreover, it is known that the multivariable signature $\operatorname{sign}(L)$ is constant on the connected components of $\mathbb{T}_*^\mu \setminus V_L$, where V_L denotes the intersection of the torus with the algebraic variety defined by the "colored" Alexander ideal of L, see Proposition 3.4.3. Therefore, the left-hand side of this equality is constant on the connected components of $\mathbb{T}^\mu \setminus V$, where V is some algebraic variety defined by the Alexander ideals of the closures of $\alpha\beta$, α, and β. A similar but so far, less precise statement can be proved for the right-hand side of this equality: it is constant on the connected components of $\mathbb{T}^\mu \setminus V'$, where V' is some algebraic variety. Since Corollary 16.4.4 establishes the equality of these functions on a dense subset of the torus, we can conclude that they coincide on the *open* dense subset $\mathbb{T}^\mu \setminus (V \cup V')$.

Remark 16.4.6. Let $\alpha_\pm \in B_c$ be an arbitrary colored braid, and let $\alpha_\mp \in B_c$ be obtained from $\alpha_\pm$ by a crossing change. Up to conjugation in the group B_c, we have $\alpha_\mp = \sigma_i^{\mp 2}\alpha_\pm$ for some i. Therefore, Corollary 16.4.4 gives

$$\operatorname{sign}_\omega(\widehat{\alpha_\mp}) - \operatorname{sign}_\omega(\widehat{\alpha_\pm}) = \operatorname{sign}_\omega(\widehat{\sigma_i^{\mp 2}}) - Meyer(\mathscr{B}_\omega(\alpha_\pm), \mathscr{B}_\omega(\sigma_i^{\mp 2}))\,.$$

Note that $\operatorname{sign}_\omega(\widehat{\sigma_i^{\mp 2}}) = \pm 1$ if both strands involved in the crossing have the same color, and it vanishes otherwise, see Example 3.4.2. Furthermore, the explicit (sparse) form of $\mathscr{B}_\omega(\sigma_i^{\mp 2})$ implies that this Meyer cocycle is in $\{-1, 0, 1\}$. This gives an explicit formula relating the multivariable signature of two links related by a crossing change.

As a first consequence, we obtain a new proof of the well-known fact that half of the signature provides a lower bound for the unlinking number of a link, see e.g. [78, Proposition 5.3] for the univariate case, and [41, Section 5] for the general case. Furthermore, since any colored link can be realized as the closure of a colored braid, and any colored braid can be transformed by crossing changes into a braid whose closure is a trivial link, this provides a new algorithm for the computation of the multivariable signature of any colored link.

Remark 16.4.7. Recall that given a coloring $c\colon \{1, \ldots, n\} \to \{\pm 1, \ldots, \pm \mu\}$, the corresponding reduced colored Gassner representation evaluated at $t = \omega$ is a homomorphism $\overline{\mathscr{B}}_\omega$ from the associated colored braid group B_c to the group of unitary automorphisms of a Hermitian complex vector space of dimension $n - 1$. Since the Meyer cocycle evaluated on this group is the signature of a Hermitian form on a space of dimension at most $2(n - 1)$, Corollary 16.4.4 implies the inequality

$$\left| \operatorname{sign}_\omega(\widehat{\alpha\beta}) - \operatorname{sign}_\omega(\widehat{\alpha}) - \operatorname{sign}_\omega(\widehat{\beta}) \right| \leq 2(n-1)$$

for all $\alpha, \beta \in B_c$ and ω in $\mathbb{T}^\mu_c \cap \mathbb{T}^\mu_{cP}$. In particular, for any such ω, the map $B_c \to \mathbb{R}$, $\alpha \mapsto \operatorname{sign}_\omega(\widehat{\alpha})$ is a *quasimorphism*.

Note that even the intersection of Corollaries 16.4.3 and 16.4.4, i.e. the case of oriented braids and the Levine-Tristram signature, is slightly more general than the main theorem of Gambaudo and Ghys stated in the introduction, as we allow the strands of the braids to be oriented in different directions.

With all these positive results, one might wonder whether the equation of Corollary 16.4.4 does not hold true for all $\omega \in \mathbb{T}^\mu$. This is *not* the case, even for the classical signature, as demonstrated by the following simple example.

Example 16.4.8. Consider the classical case $\mu = 1$ and $c = (1, 1)$, and let $\alpha = \beta$ be the standard (positive) generator of the braid group B_2. Since $\widehat{\alpha\beta}$ is the positive Hopf link $\mathscr{H}$ and $\widehat{\alpha}$ the unknot, Example 3.4.2 leads to

$$\operatorname{sign}_\omega(\widehat{\alpha\beta}) - \operatorname{sign}_\omega(\widehat{\alpha}) - \operatorname{sign}_\omega(\widehat{\beta}) = \operatorname{sign}_\omega(\mathscr{H}) = -1$$

for all $\omega \in S^1 \setminus \{1\}$. On the other hand, we know by (9.3) that a matrix for the reduced Burau representation evaluated at α is equal to $\overline{\mathscr{B}}_t(\alpha) = (-t)$, which is unitary with respect to the form $(t - t^{-1})$, recall (13.1). By (16.1), we have

$$\mathit{Meyer}(\overline{\mathscr{B}}_\omega(\alpha), \overline{\mathscr{B}}_\omega(\beta)) = \begin{cases} 1 & \text{if } \omega \neq \pm 1; \\ 0 & \text{if } \omega = \pm 1. \end{cases}$$

Therefore, we see that the equality in Corollary 16.4.4 is satisfied for all $\omega \in S^1 \setminus \{-1\}$, but *not* for $\omega = -1$, whose order is not coprime to $n = 2$. This shows that, even in the most basic case of the classical (Murasugi) signature, one does need n to be odd for this equality to hold.

Example 16.4.9. Let us now consider the case $\mu = 1$, $n = 2$, and $\varepsilon = (+1, -1)$. By (16.3), the associated skew-Hermitian form $\lambda_c(\omega)$ vanishes for all ω. Hence, all the Meyer cocycles computed in this setting will vanish as well. On the other hand, one can of course form oriented links with non-vanishing signature by closing up braids in B_c. (The Hopf link is the simplest example.) This shows that the assumption $\ell(c) \neq 0$ is necessary for our result to hold, even in the simple case of oriented braids.

At this point, it is necessary to make a brief remark regarding conventions.

Remark 16.4.10. Since we are heavily using the formalism of Cimasoni-Turaev [42, 43], we choose to use their conventions, some of which differ from those of Chapters 9 and 11, see Remarks 13.4.3 and 13.4.4. In particular, from now on, the generator σ_i of B_n involves a positive crossing change. Note that this was already implicit in Example 16.4.8: we claimed that the positive Hopf link could be obtained as the closure of σ_1^2.

We conclude this section with some more examples and remind the reader that the reduced colored Gassner representation is now denoted by $\overline{\mathscr{B}}_t$ instead of $\overline{\mathscr{B}}_{\psi_c}$.

Example 16.4.11. Let us compute the Levine-Tristram signature of the positive trefoil knot T without using any Seifert surface. Since we want to make sure we get the correct value at $\omega = -1$, consider the standard generator σ_1 in B_3 (and not B_2). Applying Corollary 16.4.4 to $\alpha = \sigma_1$ and $\beta = \sigma_1^2$, and using the fact that $\widehat{\alpha\beta} = T$, $\widehat{\beta} = \mathscr{H}$ whose signature is -1, and $\widehat{\alpha}$ is the unknot whose signature vanishes, we get

$$\operatorname{sign}_\omega(T) + 1 = -\mathit{Meyer}(\gamma, \gamma^2)\,, \quad \text{where} \quad \gamma = \overline{\mathscr{B}}_\omega(\sigma_1) = \begin{pmatrix} -\omega & 1 \\ 0 & 1 \end{pmatrix}$$

by (9.3). Using (16.3), the relevant form is given by the matrix

$$\begin{pmatrix} \omega - \overline{\omega} & 1 - \omega \\ -1 + \overline{\omega} & \omega - \overline{\omega} \end{pmatrix},$$

so (16.2) gives

$$\operatorname{sign}_\omega(T) = -1 + \operatorname{sgn}(2\operatorname{Re}(\omega) - 1).$$

This turns out to be the correct value at all $\omega \in S^1$, even at the roots of unity of order divisible by 3.

Example 16.4.12. Consider the 2-colored link L illustrated in Figure 3.2. Clearly, it is the closure of the square of the standard generator A_{12} of the pure braid group P_2. Applying Corollary 16.4.4 to $\alpha = \beta = A_{12} \in P_2$, and using the fact that $\widehat{A_{12}}$ is the 2-colored Hopf link whose 2-variable signature vanishes, we get

$$\operatorname{sign}_{(\omega_1,\omega_2)}(L) = -\mathit{Meyer}(\gamma, \gamma)\,, \quad \text{where} \quad \gamma = \overline{\mathscr{B}}_{(\omega_1,\omega_2)}(A_{12}) = (\omega_1\omega_2)$$

by Example 9.5.4. Using Examples 16.3.2 and 16.2.4, we obtain

$$\operatorname{sign}_{(\omega_1,\omega_2)}(L) = -\operatorname{sgn}\left(\operatorname{Re}\left[(1-\omega_1)(1-\omega_2)(1-\omega_1\omega_2)\right]\right).$$

Comparing this with Example 3.4.1, it remains to see when

$$\operatorname{Re}\left[(1-\omega_1)(1-\omega_2)\right] \quad \text{and} \quad \operatorname{Re}\left[(1-\omega_1)(1-\omega_2)(1-\omega_1\omega_2)\right]$$

have the same sign. Writing $\omega_1 = e^{i\theta_1}$ and $\omega_2 = e^{i\theta_2}$, one easily checks that this is the case if and only if $\omega_1\omega_2 \neq 1$. In particular, this holds when the orders of the roots of unity ω_1, ω_2 are coprime, as predicted by Corollary 16.4.4. However, this example shows that the hypothesis $\omega \in \mathbb{T}^{\mu}_{cP}$ is necessary for this result to hold.

Chapter 17

Proof of Theorem 16.4.1 and its corollaries

This chapter is organized as follows. Section 17.1 introduce the tools needed to prove Theorem 16.4.1, while the core of the proof can be found in Section 17.2. Finally Section 17.3 relates the Lagrangian functor to our isotropic functor.

17.1 Algebraic and topological preliminaries

The aim of this section is to introduce the tools needed to prove our main result. In Subsection 17.1.1, we deal with generalized eigenspaces while in Subsection 17.1.2, we review signatures of 4-manifolds. Then, building on this, we define and study the isotropic functor in Subsection 17.1.3.

17.1.1 Generalized eigenspaces

Let $k_1, \dots, k_\mu$ be positive integers, and let G denote the finite abelian group $C_{k_1} \times \cdots \times C_{k_\mu}$. In all this paragraph, we fix a $\mathbb{C}$-algebra homomorphism

$$\chi \colon \mathbb{C}[G] \to \mathbb{C}\,.$$

Note that such a homomorphism is simply given by a character of G, or equivalently, by the choice for $i = 1, \dots, \mu$ of an element $\omega_i \in S^1$ whose order divides k_i. In other words, it is given by an element $\omega = (\omega_1, \dots, \omega_\mu)$ of $\mathbb{T}^\mu$. Note also that such a χ automatically preserves the involutions given by $\sum z_g g \mapsto \sum \overline{z_g} g^{-1}$ on $\mathbb{C}[G]$ and by the complex conjugation on $\mathbb{C}$.

Terminology 17.1.1. Given a $\mathbb{C}[G]$-module H, the *generalized eigenspace* associated to the character $\chi \colon \mathbb{C}[G] \to \mathbb{C}$ is the complex vector space

$$H_\chi = \{x \in H \mid gx = \chi(g)x \text{ for all } g \in G\}\,.$$

Since H_χ is completely determined by the element $\omega \in \mathbb{T}^\mu$ corresponding to χ, we shall often write H_ω instead of H_χ.

Denote by c_χ the element of $\mathbb{C}[G]$ defined by

$$c_\chi = \frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}g\,.$$

One can easily check that for any $g \in G$, one has $g\,c_\chi = \chi(g)c_\chi$, which implies that $c_\chi c_\chi = c_\chi$. The additional equality $\chi(c_\chi) = 1$ is also easy to check. These properties are useful to give an alternative characterization of generalized eigenspaces.

Lemma 17.1.2. *For any $\mathbb{C}[G]$-module H, the generalized eigenspace H_χ is equal to $c_\chi H$.*

Proof. If $c_\chi x$ is an arbitrary element of $c_\chi H$, then $g\,c_\chi x = \chi(g)c_\chi x$ so $c_\chi x$ belongs to H_χ. Conversely, if x is an element of H_χ, then

$$c_\chi x = \frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}gx = \frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\chi(g)x = x\,,$$

so $x = c_\chi x$ lies in $c_\chi H$. □

If H and H' are $\mathbb{C}[G]$-modules, then any $\mathbb{C}[G]$-linear map $f\colon H \to H'$ restricts to a map $f_\chi\colon H_\chi \to H'_\chi$ on generalized eigenspaces, thus defining a functor from the category of $\mathbb{C}[G]$-modules to the category of complex vector spaces. Let us analyse some further properties of this functor.

Proposition 17.1.3. *1. The functor $H \mapsto H_\chi$ preserves exact sequences.*

2. If V is a submodule of H satisfying $c_\chi V = 0$, then $(H/V)_\chi = H_\chi$.

3. If λ is a non-degenerate form on H and G acts on H by unitary isomorphisms, then the restriction of λ to H_χ is also non-degenerate.

Proof. For the first assertion, consider $\mathbb{C}[G]$-linear maps $f\colon H \to H'$ and $g\colon H' \to H''$ such that $\ker(g) = \mathrm{im}(f)$; we must show that $\ker(g_\chi) = \mathrm{im}(f_\chi)$. One inclusion follows directly from the functoriality $(g\circ f)_\chi = g_\chi \circ f_\chi$. For the other one, fix $c_\chi x \in \ker(g_\chi) = H'_\chi \cap \ker(g)$. By exactness, there exists $y \in H$ such that $f(y) = c_\chi x$. As $f_\chi(c_\chi y) = c_\chi f(y) = c_\chi c_\chi x = c_\chi x$, the equality is proved. The second statement follows from the first one together with Lemma 17.1.2 and the hypothesis $c_\chi V = 0$:

$$(H/V)_\chi = H_\chi/V_\chi = H_\chi/c_\chi V = H_\chi\,.$$

Finally, let $c_\chi x$ be an element of $c_\chi H = H_\chi$ such that $\lambda(c_\chi x, c_\chi y) = 0$ for every $c_\chi y \in H_\chi$. Using the fact that the elements of G act by isometries, together with the equality $c_\chi c_\chi = c_\chi$, one obtains

$$0 = \lambda(c_\chi x, c_\chi y) = \lambda(c_\chi c_\chi x, y) = \lambda(c_\chi x, y)$$

for every $y \in H$. As λ is non-degenerate on H, this forces $c_\chi x = 0$. □

As the reader might have guessed, another description of these generalized eigenspaces can be given using tensor products. Indeed, the homomorphism $\chi\colon \mathbb{C}[G] \to \mathbb{C}$ endows the field $\mathbb{C}$ with a structure of module over $\mathbb{C}[G]$. To emphasize this action, we shall denote this $\mathbb{C}[G]$-module by $\mathbb{C}_\chi$. Given a Hermitian $\mathbb{C}[G]$-module (H,λ), one can therefore consider the complex vector space $H\otimes_{\mathbb{C}[G]}\mathbb{C}_\chi$ endowed with the skew-Hermitian form $\lambda^\chi(x\otimes u, y\otimes v) = u\overline{v}\chi(\lambda(x,y))$.

Proposition 17.1.4. *Given any Hermitian $\mathbb{C}[G]$-module (H,λ), the map $\Phi_H\colon H\otimes_{\mathbb{C}[G]}\mathbb{C}_\chi\to H_\chi$ defined by $\Phi_H(x\otimes u)=uc_\chi x$ is an isomorphism of complex vector spaces, unitary with respect to the forms λ^χ and $\chi\circ\lambda$. Furthermore, $f_\chi\circ\Phi_H=\Phi_{H'}\circ(f\otimes id_{\mathbb{C}_\chi})$ for any $\mathbb{C}[G]$-linear map $f\colon H\to H'$.*

Proof. The map Φ_H is surjective thanks to Lemma 17.1.2, while its injectivity follows from the equation

$$x\otimes 1=x\otimes\chi(c_\chi)=c_\chi x\otimes 1=\Phi(x\otimes 1)\otimes 1\,.$$

The equality $\chi(c_\chi)=1$ easily implies that Φ_H is unitary. Finally, the last statement follows from the definitions. □

Note that if H if a free $\mathbb{C}[G]$-module of rank n, then $H_\chi=H\otimes_{\mathbb{C}[G]}\mathbb{C}_\chi$ is a complex vector space of dimension n. By standard properties of the tensor product, we also have the following result, that we record here for further use.

Lemma 17.1.5. *Let $f\colon H\to H'$ be a $\mathbb{C}[G]$-linear map between free $\mathbb{C}[G]$-modules and let λ be a skew-Hermitian form on H. Fix bases $v_1,\dots,v_m$ for H and $w_1,\dots,w_n$ for H'. Then, the matrix for $f\otimes id_{\mathbb{C}_\chi}$ (resp. λ^χ) with respect to the bases $v_1\otimes 1,\dots,v_m\otimes 1$ and $w_1\otimes 1,\dots,w_n\otimes 1$ is equal to the componentwise evaluation by χ of the matrix for f (resp. λ).*

By Proposition 17.1.4, the same result holds with $f\otimes id_{\mathbb{C}_\chi}$, λ^χ and $v_i\otimes 1$ replaced by f_χ,$\chi\circ\lambda$ and $c_\chi v_i$, respectively. Finally, note that all the results of this paragraph still hold if we consider Hermitian forms instead of skew-Hermitian ones.

17.1.2 Signatures of 4-manifolds

The aim of this paragraph is to review the signatures associated to 4-manifolds endowed with the action of a finite abelian group. In particular, we shall recall the celebrated *Novikov-Wall theorem* on the non-additivity of these signatures.

Let M be a compact oriented $2n$-dimensional manifold endowed with the action of a finite abelian group G. The homology of M with complex coefficients is endowed with a structure of module over $\mathbb{C}[G]$. In particular, if $\chi\colon\mathbb{C}[G]\to\mathbb{C}$ is a $\mathbb{C}$-algebra homomorphism, one may consider the generalized eigenspace

$$H_n(M)_\chi:=\{x\in H_n(M;\mathbb{C})\mid \chi(g)x=gx \text{ for all } g\in G\}\,.$$

This complex vector space comes equipped with a $(-1)^n$-Hermitian form given by the restriction of the intersection form $\langle\ ,\ \rangle$ of $H_n(M;\mathbb{C})$. On the other hand, for any $x,y\in H_n(M;\mathbb{C})$, one can define

$$\lambda(x,y)=\sum_{g\in G}\langle gx,y\rangle g^{-1}\,,$$

and $\chi\circ\lambda$ gives another complex valued $(-1)^n$-Hermitian form on $H_n(M)_\chi$. It turns out that these two forms are closely related.

Proposition 17.1.6. *On the space $H_n(M)_\chi$, the intersection form $\langle\ ,\ \rangle$ and the pairing $\chi\circ\lambda$ coincide up to a positive multiplicative constant.*

Proof. Fix arbitrary elements $c_\chi x, c_\chi y \in H_n(M)_\chi = c_\chi H_n(M;\mathbb{C})$. Using the fact that χ is a ring homomorphism, the definition of c_χ and the equality $c_\chi c_\chi = c_\chi$, we get

$$\begin{aligned}\chi(\lambda(c_\chi x, c_\chi y)) &= \sum_{g\in G}\langle g c_\chi x, c_\chi y\rangle\overline{\chi(g)} = \langle\sum_{g\in G}\overline{\chi(g)}g c_\chi x, c_\chi y\rangle \\ &= |G|\langle c_\chi c_\chi x, c_\chi y\rangle = |G|\langle c_\chi x, c_\chi y\rangle\,,\end{aligned}$$

and the proposition is proved. □

Proposition 17.1.6, Proposition 17.1.4 and Lemma 17.1.5 immediately yield the following corollary.

Corollary 17.1.7. *Up to a positive multiplicative constant, a matrix of the restriction of the intersection form to $H_n(M)_\chi$ is given by a matrix of the form λ evaluated componentwise by χ.*

Note that when n is even, these two forms are Hermitian and therefore have a well-defined (identical) signature. It is called the *χ-signature* of M, and will be denoted by $\sigma_\chi(M)$. Since it is completely determined by an element ω of $\mathbb{T}^\mu$, we shall sometimes write $\sigma_\omega(M)$ instead of $\sigma_\chi(M)$, and call it the *ω-signature* of M.

The (non-)additivity of this signature is well-understood thanks to a famous theorem of C.T.C Wall [156]. (Strictly speaking, Wall only stated and proved his result for ordinary signatures of manifolds; however, he did mention in [156, p.274] that it extends to G-manifolds and G-signatures.) This result holds for any even n, but we shall restrict ourselves to low-dimensional manifolds. To state this theorem, we need the following well-known consequence of Poincaré duality.

Lemma 17.1.8. *Let X be a compact oriented 3-dimensional manifold-with-boundary endowed with the action of a finite abelian group G. Then, its boundary $\Sigma = \partial X$ inherits an orientation and a G-action from X, and the kernel of the map induced by the inclusion of Σ in X is a Lagrangian subspace of $H_1(\Sigma)_\chi$ with respect to the intersection form.*

Let M be an oriented compact 4-manifold endowed with the action of a finite abelian group G and let X_0 be an oriented compact 3-manifold properly embedded into M, so that X_0 intersects ∂M along $\partial X_0 = X_0 \cap \partial M$. Assume that X_0 splits M into two manifolds M_1 and M_2. For $i = 1, 2$, denote by X_i the compact 3-manifold $\partial M_i \setminus \mathrm{Int}(X_0)$. Orient X_1 and X_2 so that $\partial M_1 = X_0 \cup (-X_1)$ and $\partial M_2 = (-X_0) \cup X_2$. Note that the orientations of X_0, X_1 and X_2 induce the same orientation on the surface $\Sigma = \partial X_0 = \partial X_1 = \partial X_2$. By Lemma 17.1.8, we know that given any $\mathbb{C}$-algebra homomorphism $\chi\colon \mathbb{C}[G] \to \mathbb{C}$, the subspace $(L_i)_\chi = \ker(H_1(\Sigma)_\chi \to H_1(X_i)_\chi)$ is Lagrangian in $H_1(\Sigma)_\chi$ for $i = 0, 1, 2$.

Theorem 17.1.9 (Wall [156])**.** *Under the conditions above, the χ-signature of M is given by*

$$\sigma_\chi(M) = \sigma_\chi(M_1) + \sigma_\chi(M_2) + \mathit{Maslov}((L_1)_\chi, (L_0)_\chi, (L_2)_\chi).$$

As in [78], this result will be one of the main tools in the proof of our result. We will often refer to it as the *Novikov-Wall theorem.*

17.1.3 The isotropic functor

Given a fixed torsion element ω in $\mathbb{T}^\mu$, the aim of this subsection is now to define a functor $\mathscr{F}_\omega \colon \mathbf{Tangles}_\mu \to \mathbf{Isotr}_\mathbb{C}$ using branched coverings. We will then give a sufficient condition for this functor to take its values in the Lagrangian category $\mathbf{Lagr}_\mathbb{C}$.

Given a positive integer n, recall that p_j denotes the point $((2j-n-1)/n, 0)$ in the closed unit disk D^2, for $j = 1, \dots, n$. Let $\mathscr{N}(\{p_1, \dots, p_n\})$ be an open tubular neighborhood of $\{p_1, \dots, p_n\}$. Given a map $c \colon \{1, \dots, n\} \to \{\pm 1, \dots, \pm\mu\}$, we shall denote by D_c the compact surface

$$D_c = D^2 \setminus \mathscr{N}(\{p_1, \dots, p_n\})$$

endowed with the counterclockwise orientation and a basepoint z. The same space with the clockwise orientation will be denoted by $-D_c$. The fundamental group $\pi_1(D_c, z)$ is freely generated by $\{x_1, \dots, x_n\}$, where x_j is a simple loop turning once around p_j counterclockwise if $\operatorname{sgn}(c_j) = 1$, clockwise if $\operatorname{sgn}(c_j) = -1$.

Fix a torsion element $\omega = (\omega_1, \dots, \omega_\mu) \in \mathbb{T}^\mu$, let k_i be the order of ω_i and G be the finite abelian group $C_{k_1} \times \cdots \times C_{k_\mu}$. Also, let C_∞^μ be the (multiplicative) free abelian group with basis $t_1, \dots, t_\mu$. Composing the coloring induced homomorphism $H_1(D_c) \to C_\infty^\mu,\ x_j \mapsto t_{|c_j|}$ with the canonical projection $C_\infty^\mu \to G$ yields a regular G-covering $\overline{D}_c \to D^2$ branched along the punctures. The homology group $H_1(\overline{D}_c; \mathbb{C})$ is endowed with a structure of module over $\mathbb{C}[G]$. Let $\langle\ ,\ \rangle_c \colon H_1(\overline{D}_c; \mathbb{C}) \times H_1(\overline{D}_c; \mathbb{C}) \to \mathbb{C}$ be the (skew-Hermitian) intersection form obtained by lifting the orientation of D_c to $\overline{D}_c$. Restricting this form to the generalized eigenspace $H_1(\overline{D}_c)_\omega$ (recall Subsection 17.1.1) turns the latter into a Hermitian complex vector space.

Given a colored tangle $\tau \in T_\mu(c, c')$ with m components, denote by $\mathscr{N}(\tau)$ an open tubular neighborhood of τ, and let

$$X_\tau = (D^2 \times [0,1]) \setminus \mathscr{N}(\tau)$$

be the exterior of τ. We shall orient X_τ so that the induced orientation on ∂X_τ extends the orientation on $(-D_c) \sqcup D_{c'}$. The long exact sequence of the pair $(D^2 \times [0,1], X_\tau)$, excision and duality yield $H_1(X_\tau) = \bigoplus_{j=1}^m \mathbb{Z} m_j$, where m_j is a meridian linking once the j^{th} component of τ.

Composing the coloring induced homomorphism $H_1(X_\tau) \to C_\infty^\mu,\ m_j \mapsto t_{|c_j|}$ with the canonical projection $C_\infty^\mu \to G$ yields a regular G-covering $p \colon \overline{X}_\tau \to D^2 \times [0,1]$ branched along τ. Let $\hat{\imath}_\tau \colon H_1(\overline{D}_c)_\omega \to H_1(\overline{X}_\tau)_\omega$ and $\hat{\imath}'_\tau \colon H_1(\overline{D}_{c'})_\omega \to H_1(\overline{X}_\tau)_\omega$ be the homomorphisms induced by the inclusions of $\overline{D}_c$ and $\overline{D}_{c'}$ in $\overline{X}_\tau$. Finally, let $\hat{\jmath}_\tau$ be the homomorphism

$$\hat{\jmath}_\tau \colon H_1(\overline{D}_c)_\omega \oplus H_1(\overline{D}_{c'})_\omega \to H_1(\overline{X}_\tau)_\omega$$

given by $\hat{\jmath}_\tau(x, x') = \hat{\imath}'_\tau(x') - \hat{\imath}_\tau(x)$.

Theorem 17.1.10. *Fix a torsion element ω in $\mathbb{T}^\mu$. Let $\mathscr{F}_\omega$ assign to each map $c \colon \{1, \dots, n\} \to \{\pm 1, \dots, \pm\mu\}$ the pair $(H_1(\overline{D}_c)_\omega, \langle\ ,\ \rangle_c)$ and to each tangle $\tau \in T_\mu(c, c')$ the subspace* $\ker(\hat{\jmath}_\tau)$ *of $H_1(\overline{D}_c)_\omega \oplus H_1(\overline{D}_{c'})_\omega$. Then $\mathscr{F}_\omega$ is a functor* $\mathbf{Tangles}_\mu \to \mathbf{Isotr}_\mathbb{C}$ *which fits in the commutative diagram*

$$\begin{tikzcd} \mathbf{Braids}_\mu \arrow[r] \arrow[d] & \mathbf{Tangles}_\mu \arrow[d, "\mathscr{F}_\omega"] \\ \widetilde{\mathbf{U}}_\mathbb{C} \arrow[r, "\Gamma"] & \mathbf{Isotr}_\mathbb{C}\,, \end{tikzcd}$$

where the horizontal arrows are the embeddings of categories described in Sections 13.2 and 16.1.

Proof. By the discussion above, for any object c of $\mathbf{Tangles}_\mu$, $\mathscr{F}_\omega(c)$ is a Hermitian complex vector space, i.e. an object of $\mathbf{Isotr}_\mathbb{C}$.

Now, fix a μ-colored tangle $\tau \in T_\mu(c, c')$ and let us check that $\mathscr{F}_\omega(\tau)$ is an isotropic subspace of $(-H_1(\overline{D}_c)_\omega) \oplus H_1(\overline{D}_c)_\omega$. During this discussion, we shall denote by λ (respectively λ') the skew-Hermitian intersection form on $H_1(\overline{D}_c)_\omega$ (respectively $H_1(\overline{D}_{c'})_\omega$). Recall that we oriented X_τ so that the orientation of ∂X_τ extends the one of $(-D_c) \sqcup D_{c'}$. Consequently, the composition of the form Ω on $H_1(\partial \overline{X}_\tau)_\omega$ with the homomorphism induced by the inclusion $(-\overline{D}_c) \sqcup \overline{D}_{c'} \subset \partial \overline{X}_\tau$ is equal to $(-\lambda) \oplus \lambda'$. Observe that the map $\hat{\jmath}_\tau$ is given by the composition

$$H_1(\overline{D}_c)_\omega \oplus H_1(\overline{D}_{c'})_\omega \xrightarrow{\psi} H_1(\overline{D}_c)_\omega \oplus H_1(\overline{D}_{c'})_\omega \xrightarrow{i} H_1(\partial \overline{X}_\tau)_\omega \xrightarrow{\varphi} H_1(\overline{X}_\tau)_\omega \,,$$

where $\psi = (-id) \oplus id$ while i and φ are the inclusion induced maps. Writing $L := \ker(\varphi \circ i)$, we find $\ker(\hat{\jmath}_\tau) = \psi(L)$ and consequently $\mathrm{Ann}(\ker(\hat{\jmath}_\tau)) = \mathrm{Ann}(\psi(L)) = \psi(\mathrm{Ann}(L))$. Let us now check that L is isotropic. Given $x, y \in L$, the elements $i(x)$ and $i(y)$ belong to $\ker(\varphi)$ which is known to be Lagrangian by Lemma 17.1.8. As the form Ω "restricts to" $(-\lambda) \oplus \lambda'$ on $H_1((-\overline{D}_c) \sqcup \overline{D}_{c'})_\omega$, we get

$$((-\lambda) \oplus \lambda')(x, y) = \Omega(i(x), i(y)) = 0$$

as desired. Combining these observations, it follows that

$$\ker(\hat{\jmath}_\tau) = \psi(L) \subset \psi(\mathrm{Ann}(L)) = \mathrm{Ann}(\ker(\hat{\jmath}_\tau)) \,,$$

which shows that $\mathscr{F}_\omega(\tau) = \ker(\hat{\jmath}_\tau)$ is isotropic.

The proof of the functoriality follows by restricting the arguments given in [42, Lemma 3.4] to generalized eigenspaces. (Recall that the first point of Proposition 17.1.3 ensures that exactness is preserved.) Finally, the restriction of this functor to braids can be analysed by a straightforward adaptation of the proof of [42, Proposition 5.1]. □

Our next goal is to find a sufficient condition for the functor $\mathscr{F}_\omega$ to take its values in the Lagrangian category $\mathbf{Lagr}_\mathbb{C}$. Given $\omega \in \mathbb{T}^\mu$, let $\mathbf{Tangles}^\omega_\mu$ be the full subcategory of $\mathbf{Tangles}_\mu$ whose objects are maps $c\colon \{1, \dots, n\} \to \{\pm 1, \dots, \pm\mu\}$ such that $\ell(c)$ is nowhere zero and such that ω belongs to $\mathbb{T}^\mu_c$ (recall Subsection 16.4). We shall denote by $T^\omega_\mu(c, c')$ the set of morphisms between two objects c and c' of this category.

Lemma 17.1.11. *Fix ω in $\mathbb{T}^\mu_{cP}$. If c is an object of $\mathbf{Tangles}^\omega_\mu$, then the surface $\overline{D}_c$ has one boundary component, and the restriction of the skew-Hermitian intersection form to $H_1(\overline{D}_c)_\omega$ is non-degenerate.*

Proof. As the branch set of the covering is contained in the interior of the disk D^2, the composition $\pi_1(\partial D^2) \to \pi_1(D_c) \to G$ induces a regular G-covering. Consequently, the boundary of the total space $\overline{D}_c$ has one component if and only if this homomorphism is surjective. In other words, the connectedness of $\partial \overline{D}_c$ is equivalent to the image of the generator of $\pi_1(\partial D^2)$ spanning G. The image of the generator of $\pi_1(\partial D^2)$ in $\pi_1(D_c)$ goes once around each of the punctures, so it is sent by the above composition to the class of $\ell(c)$ in G. Using the Chinese remainder theorem, this element generates G if and only if the $k_i > 1$'s are pairwise coprime (i.e. ω belongs to $\mathbb{T}^{\mu}_{cP}$), and $\ell(c)_i$ and k_i are coprime for each i (i.e. c is an object of $\mathbf{Tangles}^{\omega}_{\mu}$). In this case, $\overline{D}_c$ has one boundary component, so the intersection form on $H_1(\overline{D}_c;\mathbb{C})$ is non-degenerate. The last claim follows from the third part of Proposition 17.1.3. $\square$

Lemma 17.1.11 gives a sufficient condition for the functor $\mathscr{F}_\omega$ to take its values in the Lagrangian category $\mathbf{Lagr}_{\mathbb{C}}$.

Proposition 17.1.12. *Fix an element ω in $\mathbb{T}^{\mu}_{cP}$ (i.e. assume that the component ω_i of ω is of order $k_i > 1$ with these k_i's pairwise coprime). Then the restriction of $\mathscr{F}_\omega$ to $\mathbf{Tangles}^{\omega}_{\mu}$ defines a functor which fits in the commutative diagram*

$$\begin{tikzcd} \mathbf{Braids}^{\omega}_{\mu} \arrow[r] \arrow[d] & \mathbf{Tangles}^{\omega}_{\mu} \arrow[d, "\mathscr{F}_\omega"] \\ \mathbf{U}_{\mathbb{C}} \arrow[r, "\Gamma"] & \mathbf{Lagr}_{\mathbb{C}}. \end{tikzcd}$$

Proof. Let $\tau \in T^{\omega}_{\mu}(c,c')$ be a colored tangle. Applying the same notation and reasoning as in the proof of Theorem 17.1.10, we only need to show that $\ker(\varphi\circ i)$ is Lagrangian. Since $\ker(\varphi)$ is Lagrangian by Lemma 17.1.8, we are left with the proof that i is an isomorphism. To check this claim, note that

$$\partial\overline{X}_\tau = (\overline{D}_c \sqcup \overline{D}_{c'}) \cup p^{-1}(S^1\times[0,1])\,,$$

where $p^{-1}(S^1\times[0,1])$ consists of a certain number of disjoint cylinders. As ω belongs to $\mathbb{T}^{\mu}_{cP}$ and τ to $T^{\omega}_{\mu}(c,c')$, Lemma 17.1.11 implies that both $\overline{D}_c$ and $\overline{D}_{c'}$ have a single boundary component. Consequently, $p^{-1}(S^1\times[0,1])$ consists of a single cylinder, so the inclusion $\overline{D}_c\sqcup\overline{D}_{c'} \hookrightarrow \overline{X}_\tau$ induces an isomorphism on the first homology groups, as claimed. $\square$

Remark 17.1.13. Fix an element ω in $\mathbb{T}^{\mu}_{cP}$ and a tangle $\tau\in T^{\omega}_{\mu}(c,c')$. The proof of the previous proposition shows that

$$\mathscr{F}_\omega(\tau) = \psi(\ker(j))\,,$$

where ψ is the unitary automorphism of $H_1(\overline{D}_c)_\omega \oplus H_1(\overline{D}_{c'})_\omega$ given by $\psi = (-id)\oplus id$ while j is the inclusion induced homomorphism

$$j\colon H_1(\overline{D}_c)_\omega \oplus H_1(\overline{D}_{c'})_\omega \cong H_1(\partial\overline{X}_\tau)_\omega \to H_1(\overline{X}_\tau)_\omega\,.$$

This characterization will help us later on for using the Novikov-Wall theorem.

We conclude this section with one last property of this functor, namely the fact that it behaves well with respect to juxtaposition of colored tangles.

Proposition 17.1.14. *For any $\omega\in\mathbb{T}^{\mu}_{cP}$, $\tau_1\in T^{\omega}_{\mu}(c_1,c_1')$ and $\tau_2\in T^{\omega}_{\mu}(c_2,c_2')$, we have*

1. *$H_1(\overline{D}_{c_1\sqcup c_2})_\omega \cong H_1(\overline{D}_{c_1})_\omega \oplus H_1(\overline{D}_{c_2})_\omega \oplus A$ as Hermitian complex vector spaces,*
2. *$\mathscr{F}_\omega(\tau_1\sqcup\tau_2) \cong \mathscr{F}_\omega(\tau_1)\oplus\mathscr{F}_\omega(\tau_2)\oplus\Delta_A$,*

where A is some subspace of $H_1(\overline{D}_{c_1\sqcup c_2})_\omega$ and $\Delta_A = \{x\oplus x \mid x\in A\}$ the associated diagonal.

Proof. The space $D_{c_1 \sqcup c_2}$ can be obtained by gluing D_{c_1} and D_{c_2} along an interval I in their boundary. As intervals are contractible, this decomposition lifts to $\overline{D}_{c_1 \sqcup c_2} = \overline{D}_{c_1} \cup_{I \times G} \overline{D}_{c_2}$. Applying the same line of reasoning to the tangle exteriors X_{τ_1} and X_{τ_2} and using the corresponding Mayer-Vietoris exact sequence leads to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_1(\overline{D}_{c_1})_\omega \oplus H_1(\overline{D}_{c_2})_\omega \arrow[r] \arrow[d, "\hat{\imath}_{\tau_1} \oplus \hat{\imath}_{\tau_2}"] & H_1(\overline{D}_{c_1 \sqcup c_2})_\omega \arrow[r, "\partial"] \arrow[d, "\hat{\imath}_{\tau_1 \sqcup \tau_2}"] & \operatorname{im}(\partial) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_1(\overline{X}_{\tau_1})_\omega \oplus H_1(\overline{X}_{\tau_2})_\omega \arrow[r] & H_1(\overline{X}_{\tau_1 \sqcup \tau_2})_\omega \arrow[r, "\partial"] & \operatorname{im}(\partial) \arrow[r] & 0.
\end{tikzcd}$$

Splitting these short exact sequences of vector spaces and writing $A = \operatorname{im}(\partial)$, one gets the decompositions

$$H_1(\overline{D}_{c_1 \sqcup c_2})_\omega \cong H_1(\overline{D}_{c_1})_\omega \oplus H_1(\overline{D}_{c_2})_\omega \oplus A$$

and

$$H_1(\overline{X}_{\tau_1 \sqcup \tau_2})_\omega \cong H_1(\overline{X}_{\tau_1})_\omega \oplus H_1(\overline{X}_{\tau_2})_\omega \oplus A\,.$$

At the level of maps, one obtains $\hat{\imath}_{\tau_1 \sqcup \tau_2} = \hat{\imath}_{\tau_1} \oplus \hat{\imath}_{\tau_2} \oplus id_A$, so the vector space $\mathscr{F}_\omega(\tau_1 \sqcup \tau_2)$ is isomorphic to $\ker(\hat{\jmath}_{\tau_1}) \oplus \ker(\hat{\jmath}_{\tau_2}) \oplus \Delta_A$ as claimed. To conclude the proof, we still need to check that the first decomposition displayed above is orthogonal with respect to the intersection forms. As ω belongs to $\mathbb{T}^\mu_{cP}$ and the tangles are morphisms of $\mathbf{Tangles}^\omega_\mu$, Lemma 17.1.11 implies that $\overline{D}_{c_1}$ and $\overline{D}_{c_2}$ are compact surfaces with one boundary component. It follows that the section of the exact sequence above can be chosen so that the corresponding decomposition is orthogonal. □

17.2 Proof of Theorem 16.4.1

This section is devoted to the proof of our main result, a proof which extends (and hopefully, at times, also clarifies) the one of [78]. Let us very briefly outline the underlying strategy. We will build 4-manifolds whose ω-signatures are equal to the terms appearing in the theorem. Gluing these manifolds together yields a manifold whose ω-signature is equal to

$$\operatorname{sign}_\omega(\widehat{\tau_1\tau_2}) - \operatorname{sign}_\omega(\widehat{\tau_1}) - \operatorname{sign}_\omega(\widehat{\tau_2}) - \mathit{Maslov}(\mathscr{F}_\omega(\overline{\tau}_1), \Delta, \mathscr{F}_\omega(\tau_2))\,.$$

It will then only remain to show that this signature vanishes.

Actually, it is sufficient to show all these equalities up to a uniformly bounded constant, thanks to a reduction of our main theorem to a looser statement. This is the subject of the first subsection.

17.2.1 A reduction

If n and m are two integers depending on some tangles, we shall write $n \simeq m$ if $|n-m|$ is bounded by a constant that is independent of the tangles. The aim of this paragraph is to prove the following proposition which will spare us the trouble of keeping track of (most of) the Novikov-Wall defects.

Proposition 17.2.1. *To prove Theorem 16.4.1, it is enough to show that for any c such that $\ell(c)$ is nowhere zero and for any (c,c)-tangles τ_1 and τ_2, we have*

$$\operatorname{sign}_\omega(\widehat{\tau_1\tau_2}) - \operatorname{sign}_\omega(\widehat{\tau_1}) - \operatorname{sign}_\omega(\widehat{\tau_2}) \simeq Maslov(\mathscr{F}_\omega(\overline{\tau}_1), \Delta, \mathscr{F}_\omega(\tau_2))$$

for all ω in $\mathbb{T}^\mu_c \cap \mathbb{T}^\mu_{cP}$.

As we mentioned in Remark 16.4.2, the signature defect (i.e. the left-hand side of the equation displayed above) is additive with respect to the disjoint union of tangles. Furthermore, Lemma 16.2.1 and Proposition 17.1.14 immediately imply that the right-hand side of this equation, that we shall denote by $M_\omega(\tau_1,\tau_2)$, satisfies

$$M_\omega(\tau_1 \sqcup \tau_1', \tau_2 \sqcup \tau_2') = M_\omega(\tau_1,\tau_2) + M_\omega(\tau_1',\tau_2')$$

if ω belongs to $\mathbb{T}^\mu_{cP}$ and the tangles $\tau_1,\tau_2,\tau_1',\tau_2'$ are morphisms of the category $\mathbf{Tangles}^\omega_\mu$. This will be the key ingredient in the proof of the reduction. We will also need the following easy lemma.

Lemma 17.2.2. *Given two coprime integers $\ell \neq 0$ and $k > 0$, there exist two positive integers m and n such that $m+n+\ell$, $2\ell+m$ and n are positive and coprime to k.*

Proof. Set $m = \lambda k - \ell$ and $n = \lambda k + \ell$ for any integer $\lambda > 0$ such that m and n are positive. $\square$

Proposition 17.2.1 will be an easy consequence of the following statement.

Lemma 17.2.3. *Let ω be an element of $\mathbb{T}^\mu_{cP}$. For any $\tau_1,\tau_2 \in T^\omega_\mu(c,c)$, there exists a coloring c' that is an object of the category $\mathbf{Tangles}^\omega_\mu$ and (c',c')-colored tangles τ_1',τ_2' such that $M_\omega(\tau_1',\tau_2') = 2M_\omega(\tau_1,\tau_2)$.*

Proof. Since c is an object of $\mathbf{Tangles}^\omega_\mu$, we can apply Lemma 17.2.2 to $\ell_i = \ell(c)_i \neq 0$ and $k_i > 0$, thus producing positive integers m_i and n_i for $i = 1,\dots,\mu$. Set

$$\tau_1' := \tau_1 \sqcup \tau_1 \sqcup \textstyle\bigsqcup_{i=1}^\mu m_i \quad \text{and} \quad \tau_2' := \tau_2 \sqcup \tau_2 \sqcup \textstyle\bigsqcup_{i=1}^\mu m_i\,,$$

where m_i denotes the trivial tangle with m_i (upward oriented) strands of color i, and let c' be the corresponding coloring. The fact that c' is an object of $\mathbf{Tangles}^\omega_\mu$ follows from the second point of Lemma 17.2.2. By the second and third points of this same lemma, we have

$$M_\omega(\tau_1',\tau_2') = M_\omega\left(\tau_1 \sqcup \tau_1 \sqcup \textstyle\bigsqcup_{i=1}^\mu (m_i \sqcup n_i), \tau_2 \sqcup \tau_2 \sqcup \textstyle\bigsqcup_{i=1}^\mu (m_i \sqcup n_i)\right)\,,$$

which splits as

$$M_\omega(\tau_1,\tau_2) + M_\omega\left(\tau_1 \sqcup \textstyle\bigsqcup_{i=1}^\mu (m_i \sqcup n_i), \tau_2 \sqcup \textstyle\bigsqcup_{i=1}^\mu (m_i \sqcup n_i)\right)$$

by the first point of Lemma 17.2.2 and the fact that τ is a morphism of $\mathbf{Tangles}^\omega_\mu$. Finally, adding the trivial tangle id_c and using twice more a combination of the first part of Lemma 17.2.2 and the fact that τ is a morphism $\mathbf{Tangles}^\omega_\mu$, we obtain

$$M_\omega(\tau_1',\tau_2') = 2M_\omega(\tau_1,\tau_2) + M_\omega\left(\textstyle\bigsqcup_{i=1}^\mu (m_i \sqcup n_i \sqcup \ell_i), \textstyle\bigsqcup_{i=1}^\mu (m_i \sqcup n_i \sqcup \ell_i)\right) = 2M_\omega(\tau_1,\tau_2)\,,$$

which concludes the proof. $\square$

proof of Proposition 17.2.1. Assume by contradiction that for a fixed map c, there are (c,c)-tangles τ_1, τ_2 and an element ω of $\mathbb{T}_c^\mu \cap \mathbb{T}_{cP}^\mu$ such that

$$N_\omega(\tau_1,\tau_2) := \mathrm{sign}_\omega(\widehat{\tau_1\tau_2}) - \mathrm{sign}_\omega(\widehat{\tau}_1) - \mathrm{sign}_\omega(\widehat{\tau}_2) - \mathit{Maslov}(\mathscr{F}_\omega(\overline{\tau}_1), \Delta, \mathscr{F}_\omega(\tau_2))$$

does not vanish. For any positive integer m, using inductively Lemma 17.2.3, one obtains a coloring $c(m)$ and tangles $\tau_1(m), \tau_2(m)$ such that ω belongs to $T_{c(m)} \cap \mathbb{T}_{cP}^\mu$ and

$$N_\omega(\tau_1(m), \tau_2(m)) = 2^m \, N_\omega(\tau_1, \tau_2)\,.$$

Since this quantity goes to infinity as m grows, this concludes the proof. □

Remark 17.2.4. The idea of this reduction comes from the paper [78] of Gambaudo and Ghys. Let us mention however that these authors use a simpler version of this trick, which turns out to be slightly incorrect. (See the last line of [78, p.559], where it is claimed that the reduced Burau representation evaluated at $t = -1$ is always additive under disjoint union.) To the best of our knowledge, the more involved trick given above seems to be needed even in the case $\mu = 1$.

17.2.2 The manifold $P_G(\tau_1, \tau_2)$.

Fix a map $c\colon \{1, \ldots, n\} \to \{\pm 1, \ldots, \pm\mu\}$ with $\ell(c)$ nowhere zero, an element ω in $\mathbb{T}_c^\mu \cap \mathbb{T}_{cP}^\mu$ and two (c,c)-tangles τ_1, τ_2. Denote by G the finite abelian group $C_{k_1} \times \cdots \times C_{k_\mu}$, where k_i is the order of ω_i.

Let P be the sphere with three holes more commonly known as a "pair of pants", with a fixed orientation that will be pictured as counterclockwise. Let I_1 and I_2 be closed intervals joining the inner boundary components of the pair of pants to the outer boundary component. Thicken these intervals to get rectangles $J_1 = I_1 \times [0,1]$ and $J_2 = I_2 \times [0,1]$, as illustrated in Figure 17.1.

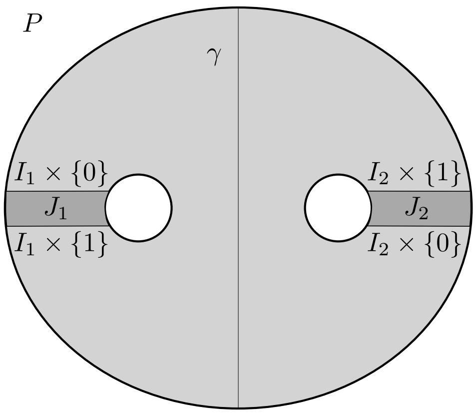

Figure 17.1: The various decompositions of the pair of pants P.

Define $R(\tau_1, \tau_2)$ as the surface in $P \times D^2$ which coincides with

1. the surface $(P \setminus (J_1 \cup J_2)) \times \{x_1, \ldots, x_n\}$ on $(P \setminus (J_1 \cup J_2)) \times D^2$,
2. the surface $I_1 \times \tau_1$, on $J_1 \times D^2 = I_1 \times ([0,1] \times D^2)$, and

3. the surface $I_2 \times \tau_2$, on $J_2 \times D^2 = I_2 \times ([0,1] \times D^2)$.

Observe that for each point $x \in I_i$ ($i = 1,2$), the surface $R(\tau_1, \tau_2)$ contains a copy of τ_i; therefore, its complement $P \times D^2 \setminus R(\tau_1, \tau_2))$ contains one copy of the tangle exterior X_{τ_i} for each point in I_i. Recall from Subsection 17.1.3 that for any (c,c)-tangle τ, there is a natural map $H_1(X_\tau) \to G$ obtained by composing the colored-induced map with the canonical projection.

Lemma 17.2.5. *There exists a homomorphism $H_1(P \times D^2 \setminus R(\tau_1, \tau_2)) \to G$ which is trivial when restricted to loops in $P \times \{x\}$ (with $x \in \partial D^2$), and whose composition with the homomorphism induced by the inclusion of any copy of X_{τ_i} into $P \times D^2 \setminus R(\tau_1, \tau_2)$ coincides with the natural map $H_1(X_{\tau_i}) \to G$ (for $i = 1,2$).*

Proof. Decompose the space $X = (P \times D^2) \setminus R(\tau_1, \tau_2)$ as the union of $A = (P \setminus (J_1 \cup J_2)) \times D_c$ and $B = ((J_1 \times D^2) \setminus (I_1 \times \tau_1)) \sqcup ((J_2 \times D^2) \setminus (I_2 \times \tau_2))$. As $P \setminus (J_1 \cup J_2)$ is contractible, A retracts onto D_c. As B is equal to $(I_1 \times X_{\tau_1}) \sqcup (I_2 \times X_{\tau_2})$, it has the homotopy type of $X_{\tau_1} \sqcup X_{\tau_2}$. Finally, $A \cap B$ has the homotopy type of four disjoint copies of the punctured disc D_c. Therefore the associated Mayer-Vietoris exact sequence has the form

$$H_1(\textstyle\bigsqcup_1^4 D_c) \to H_1(D_c) \oplus H_1(X_{\tau_1}) \oplus H_1(X_{\tau_2}) \to H_1(X) \to H_0(\textstyle\bigsqcup_1^4 D_c)\,,$$

which allows us to extend $H_1(X_{\tau_1}) \oplus H_1(X_{\tau_2}) \to G$ to the desired map $H_1(X) \to G$. $\square$

Using the homomorphism of Lemma 17.2.5, one obtains a G-covering $P_G(\tau_1, \tau_2) \to P \times D^2$ branched along $R(\tau_1, \tau_2)$. Let us start by studying its boundary, which is nothing but the lift of $\partial(P \times D^2) = \partial P \times D^2 \cup_{\partial P \times \partial D^2} P \times \partial D^2$. By definition, the surface $R(\tau_1, \tau_2)$ intersects the three components of $\partial P \times D^2$ in the closure of the three tangles τ_1, τ_2 and $\tau_1\tau_2$ in solid tori $S^1 \times D^2$. Therefore, Lemma 17.2.5 implies that $\partial P \times D^2$ lifts to

$$\overline{X}_{\widehat{\tau_1}} \sqcup \overline{X}_{\widehat{\tau_2}} \sqcup \overline{X}_{\widehat{\tau_1\tau_2}} \subset \partial P_G(\tau_1, \tau_2)\,,$$

where $X_{\widehat{\tau}}$ denotes the exterior of $\widehat{\tau}$ in $S^1 \times D^2$ and $\overline{X}_{\widehat{\tau}}$ the corresponding cover. Since ω belongs to $\mathbb{T}^\mu_c \cap \mathbb{T}^\mu_{cP}$, Lemma 17.1.11 ensures that the boundary of each of these components is a single torus. For the same reason, together with the first condition in Lemma 17.2.5, $P \times \partial D^2$ lifts to a single copy of $P \times \partial D^2 \subset \partial P_G(\tau_1, \tau_2)$. Combining these remarks, we get

$$\partial P_G(\tau_1, \tau_2) = (\overline{X}_{\widehat{\tau_1}} \sqcup \overline{X}_{\widehat{\tau_2}} \sqcup \overline{X}_{\widehat{\tau_1\tau_2}}) \cup_{\partial P \times \partial D^2} (P \times \partial D^2)\,.$$

Before applying the Novikov-Wall theorem, we must slightly modify $P_G(\tau_1, \tau_2)$, as follows. Consider the space $\widetilde{P}_G(\tau_1, \tau_2)$ given by

$$\widetilde{P}_G(\tau_1, \tau_2) = P_G(\tau_1, \tau_2) \cup_{P \times \partial D^2} (P \times D^2)\,.$$

By the discussion above, this manifold has boundary

$$\partial \widetilde{P}_G(\tau_1, \tau_2) = \overline{Y}_{\widehat{\tau_1}} \sqcup \overline{Y}_{\widehat{\tau_2}} \sqcup \overline{Y}_{\widehat{\tau_1\tau_2}}\,,$$

where $\overline{Y}_{\widehat{\tau}}$ is the *closed* 3-manifold given by $\overline{Y}_{\widehat{\tau}} = \overline{X}_{\widehat{\tau}} \cup_{S^1 \times \partial D^2} (S^1 \times D^2)$.

Proposition 17.2.6. *For any ω and τ_1, τ_2 as above, we have*

$$\sigma_\omega(P_G(\tau_1, \tau_2)) \simeq \sigma_\omega(\widetilde{P}_G(\tau_1, \tau_2)) = \mathit{Maslov}(\mathscr{F}_\omega(\overline{\tau}_1), \Delta, \mathscr{F}_\omega(\tau_2))\,.$$

Proof. Let us start by applying Novikov-Wall to $X_0 = P \times \partial D^2 \subset \widetilde{P}_G(\tau_1, \tau_2)) = M$. Note that the other corresponding spaces are given by $M_1 = P_G(\tau_1, \tau_2)$, $M_2 = P \times D^2$ whose signature vanishes as it has no degree 2 homology, while Σ consists of a union of three tori. Therefore, we immediately obtain that the difference between the ω-signatures of $P_G(\tau_1, \tau_2)$ and $\widetilde{P}_G(\tau_1, \tau_2)$ is uniformly bounded.

To show the second equality, start by cutting the pair of pants P along the path γ illustrated in Figure 17.1. This splits P into two cylinders $C_1 = I_1 \times S^1$ and $C_2 = I_2 \times S^1$; let us analyze the corresponding splitting of the manifold $\widetilde{P}_G(\tau_1, \tau_2)$. By construction, $\gamma \times D^2 \subset P \times D^2$ lifts to $\gamma \times \overline{D}_c = \overline{X}_{id_c}$. In $\widetilde{P}_G(\tau_1, \tau_2)$, the corresponding manifold is $X_0 := \overline{Y}_{id_c}$, whose boundary is given by $\Sigma := \partial X_0$ which consists of two copies of $\overline{D}_c \cup_{\partial D^2} D^2$. Similarly, the space $C_i \times D^2 \subset P \times D^2$ lifts to $I_i \times \overline{X}_{\widehat{\tau_i}} \subset P_G(\tau_1, \tau_2)$ (for $i = 1, 2$). In $\widetilde{P}_G(\tau_1, \tau_2)$, the corresponding manifold is $M_i := I_i \times \overline{Y}_{\widehat{\tau_i}}$. As these manifolds are of the form $[0,1] \times N^3$, for some 3-manifold N^3, their signature vanishes. Moreover, the manifolds $\overline{Y}_{\widehat{\tau_i}}$ being closed, we can apply the Novikov-Wall additivity theorem. Writing $X_i = \partial M_i \setminus X_0$ for $i = 1, 2$, we obtain

$$\begin{aligned}\sigma_\omega(\widetilde{P}_G(\tau_1, \tau_2)) &= \sigma_\omega(M_1) + \sigma_\omega(M_2) + Maslov((L_1)_\omega, (L_0)_\omega, (L_2)_\omega) \\ &= Maslov((L_1)_\omega, (L_0)_\omega, (L_2)_\omega)\,,\end{aligned}$$

where $(L_i)_\omega$ is the kernel of the map induced by the inclusion of Σ in X_i ($i = 0, 1, 2$). We now determine these spaces $(L_i)_\omega$.

First, one can check that

$$\partial X_0 = \partial X_1 = \partial X_2 = \Sigma = (\overline{D}_c \cup_{\partial D^2} D^2) \sqcup (\overline{D}_c \cup_{\partial D^2} D^2)\,.$$

As $\overline{D}_c$ is a compact orientable surface with one boundary component, its first homology is unaffected by capping off its boundary with a disk. Therefore, the spaces $H_1(\Sigma)$ and $H_1(\overline{D}_c) \oplus H_1(\overline{D}_c)$ are canonically isomorphic, and so are the corresponding generalized eigenspaces. An easy Mayer-Vietoris argument shows that the inclusion $\overline{X}_{id_c} \subset X_0$ induces an isomorphism on the first homology. Therefore, the map $H_1(\Sigma)_\omega \to H_1(X_0)_\omega$ can be identified with the inclusion induced map

$$j_0 \colon H_1(\overline{D}_c)_\omega \oplus H_1(\overline{D}_c)_\omega \cong H_1(\partial \overline{X}_{id_c})_\omega \to H_1(\overline{X}_{id_c})_\omega\,.$$

Similarly, the map $H_1(\Sigma)_\omega \to H_1(X_2)_\omega$ can be identified with the inclusion induced map

$$j_2 \colon H_1(\overline{D}_c)_\omega \oplus H_1(\overline{D}_c)_\omega \cong H_1(\partial \overline{X}_{\tau_2})_\omega \to H_1(\overline{X}_{\tau_2})_\omega$$

while $H_1(\Sigma)_\omega \to H_1(X_1)_\omega$ is the inclusion induced map

$$j_1 \colon H_1(\overline{D}_c)_\omega \oplus H_1(\overline{D}_c)_\omega \cong H_1(\partial \overline{X}_{\overline{\tau}_1})_\omega \to H_1(\overline{X}_{\overline{\tau}_1})_\omega\,.$$

(The appearance of the reflection of τ_1 should be clear from Figure 17.1.)

Summarizing, we have shown that

$$\sigma_\omega(\widetilde{P}_G(\tau_1, \tau_2)) = Maslov(\ker(j_1), \ker(j_0), \ker(j_2))\,.$$

Applying Remark 17.1.13 to each of the three tangles $\overline{\tau}_1$, id_c and τ_2, and the last point of Lemma 16.2.1 to the unitary involution $\psi = (-id) \oplus id$, we have

$$\begin{aligned}\sigma_\omega(\widetilde{P}_G(\tau_1, \tau_2)) &= Maslov(\psi(\mathscr{F}_\omega(\overline{\tau}_1)), \psi(\mathscr{F}_\omega(id_c)), \psi(\mathscr{F}_\omega(\tau_2))) \\ &= Maslov(\mathscr{F}_\omega(\overline{\tau}_1), \mathscr{F}_\omega(id_c), \mathscr{F}_\omega(\tau_2))\,,\end{aligned}$$

and the proof is completed. □

17.2.3 The manifold $C_G(\tau)$.

Next, we build the manifold that encodes the signature of the tangle closure. This will require some notations. Let D^4 denote the (oriented) unit 4-ball, $S^3 = \partial D^4$ its oriented boundary, and $T = S^1 \times D^2 \subset S^3$ the standardly embedded solid torus.

Closing a colored tangle $\tau \subset [0,1] \times D^2$ yields a colored link $\widehat{\tau} \subset T$. Consider a collection $S(\tau)$ of surfaces that bound $\widehat{\tau} \subset S^3$ and that are "in general position" in D^4. In other words, $S(\tau)$ consists of a collection of surfaces $F_1 \cup \cdots \cup F_\mu$ smoothly embedded in D^4, whose only intersections are transverse double points (between different surfaces), and such that for all i, F_i meets $S^3 = \partial D^4$ along the sublink of $\widehat{\tau} \subset T \subset S^3$ of color i. (Such a surface can be obtained, for example, by taking any C-complex for $\widehat{\tau}$ and by pushing it inside the 4-ball.)

Let us further assume that $S(\tau)$ meets the radius one-half sphere $\frac{1}{2}S^3$ along the closure of the trivial tangle id_c (i.e. the n-component unlink) in a way that respects the coloring c. Finally, we shall assume that the intersection of $S(\tau)$ with the closure of $D^4 \setminus \frac{1}{2}D^4$ is contained in the subspace N of $cl(D^4 \setminus \frac{1}{2}D^4) \cong [0,1] \times S^3$ given by

$$N = \{x \in D^4 \mid 1/2 \leq ||x|| \leq 1,\ x/||x|| \in T\} \cong [0,1] \times T\,.$$

One easily checks that such a surface can be obtained by pushing a C-complex for $\widehat{\tau}$ inside D^4 and isotopying it in the appropriate way.

A standard computation shows that $H_1(D^4 \setminus S(\tau))$ is free abelian of rank μ. Let

$$C_G(\tau) \to N$$

be the G-cover branched over $S(\tau) \cap N$ induced by the composition of the inclusion induced homomorphism $H_1(N \setminus (S(\tau) \cap N)) \to H_1(D^4 \setminus S(\tau))$ with the canonical projection $H_1(D^4 \setminus S(\tau)) \to G$. Let us analyse its boundary. Writing $C = [0,1] \times S^1$, the boundary of $N = C \times D^2$ can be written as $\partial(C \times D^2) = (\partial C \times D^2) \cup_{\partial C \times \partial D^2} (C \times \partial D^2)$. Thanks to the conditions stated above, $\partial C \times D^2$ lifts to $\overline{X}_{\widehat{\tau}} \sqcup \overline{X}_{\widehat{id_c}}$. On the other hand, as ω is in $\mathbb{T}^\mu_c \cap \mathbb{T}^\mu_{cP}$, the space $C \times \partial D^2$ lifts to a single copy of $C \times \partial D^2$. Summarizing, we get

$$\partial C_G(\tau) = (\overline{X}_{\widehat{\tau}} \sqcup \overline{X}_{\widehat{id_c}}) \cup_{\partial C \times \partial D^2} (C \times \partial D^2)\,.$$

We are now ready to compute the ω-signature of $C_G(\tau)$.

Proposition 17.2.7. *For any ω and τ as above, $\sigma_\omega(C_G(\tau)) \simeq \operatorname{sign}_\omega(\widehat{\tau})$.*

Proof. Let $W_{\widehat{\tau}} \to D^4$ be the G-cover of D^4 branched along $S(\tau) \subset D^4$ given by the homomorphism $H_1(D^4 \setminus S(\tau)) \to G$. By Theorem 3.4.9, $\sigma_\omega(W_{\widehat{\tau}}) = \operatorname{sign}_\omega(\widehat{\tau})$, so we are left with the proof that $\sigma_\omega(W_{\widehat{\tau}}) \simeq \sigma_\omega(C_G(\tau))$. To do so, we will apply the Novikov-Wall theorem twice.

First, the space $cl(D^4 \setminus \frac{1}{2}D^4) \cong [0,1] \times S^3$ can be obtained by gluing a copy of $[0,1] \times D^2 \times \partial D^2$ to N along $[0,1] \times S^1 \times \partial D^2$. Lifting this to the covers, one gets a manifold M obtained by gluing a copy of $[0,1] \times D^2 \times \partial D^2$ to $C_G(\tau)$ along $X_0 := [0,1] \times S^1 \times \partial D^2$, with $\Sigma := \partial X_0$ consisting of two disjoint tori. As the boundary of $C_G(\tau)$ is $(\overline{X}_{\widehat{\tau}} \sqcup \overline{X}_{\widehat{id_c}}) \cup_\Sigma X_0$ while the boundary of $[0,1] \times D^2 \times \partial D^2$ is $(\{0,1\} \times D^2 \times \partial D^2) \cup_\Sigma X_0$, Novikov-Wall yields

$$\sigma_\omega(M) \simeq \sigma_\omega(C_G(\tau)) + \sigma_\omega([0,1] \times D^2 \times \partial D^2) = \sigma_\omega(C_G(\tau))\,.$$

Next, glue the ball $\frac{1}{2}D^4$ to $cl(D^4 \setminus \frac{1}{2}D^4)$ along $\frac{1}{2}S^3$ in order to obtain D^4. Lifting this to the covers, it corresponds to recovering the manifold $W_{\widehat{\tau}}$ by gluing $W_{\widehat{id_c}}$ to M along the preimage of $\frac{1}{2}S^3$. As the latter space is closed, Novikov-Wall additivity applies trivially and we get

$$\sigma_\omega(W_{\widehat{\tau}}) = \sigma_\omega(M) + \sigma_\omega(W_{\widehat{id_c}}) = \sigma_\omega(M) + \operatorname{sign}_\omega(\widehat{id_c}) = \sigma_\omega(M)\,.$$

This concludes the proof. □

17.2.4 The manifold $M_G(\tau_1, \tau_2)$.

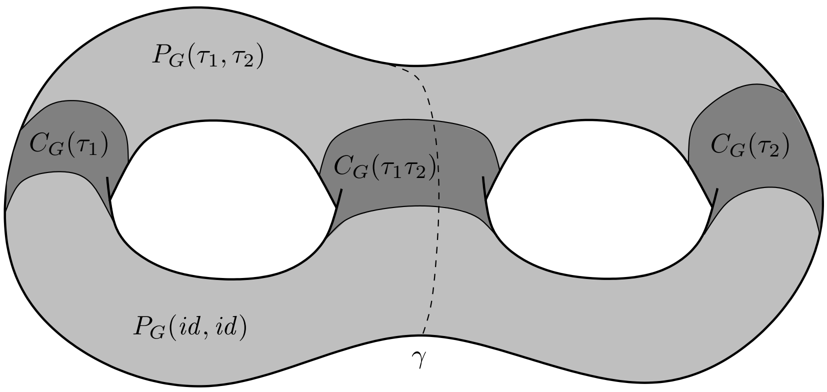

Figure 17.2: The manifold $M_G(\tau_1, \tau_2)$.

Our goal is now to glue several copies of the manifolds $C_G(\tau)$ and $P_G(\tau_1, \tau_2)$ along their boundary in order to obtain an oriented 4-manifold $M_G(\tau_1, \tau_2)$. Recall that these boundaries are given by

$$\partial C_G(\tau) = (\overline{X}_{\widehat{\tau}} \sqcup \overline{X}_{\widehat{id_c}}) \cup_{\partial C \times \partial D^2} (C \times \partial D^2)\,,$$

with $C = [0,1] \times S^1$, while

$$\partial P_G(\tau_1, \tau_2) = (\overline{X}_{\widehat{\tau_1}} \sqcup \overline{X}_{\widehat{\tau_2}} \sqcup \overline{X}_{\widehat{\tau_1\tau_2}}) \cup_{\partial P \times \partial D^2} (P \times \partial D^2)\,,$$

where P denotes the pair of pants. It therefore makes sense to define the manifold $M_G(\tau_1, \tau_2)$ by gluing $P_G(\tau_1, \tau_2)$ "on one side" of the disjoint union of $C_G(\tau_1)$, $C_G(\tau_2)$ and $C_G(\tau_1\tau_2)$, and $P_G(id_c, id_c)$ "on the other side", see Figure 17.2. More precisely, set

$$M_G(\tau_1,\tau_2) = P_G(\tau_1,\tau_2) \cup_{\overline{X}_{\widehat{\tau_1}} \sqcup \overline{X}_{\widehat{\tau_2}} \sqcup \overline{X}_{\widehat{\tau_1\tau_2}}} (C_G(\tau_1) \sqcup C_G(\tau_2) \sqcup C_G(\tau_1\tau_2)) \cup_{\overline{X}_{\widehat{id_c}} \sqcup \overline{X}_{\widehat{id_c}} \sqcup \overline{X}_{\widehat{id_c}}} P_G(id_c, id_c)\,.$$

By construction, this 4-manifold is a covering of

$$(P \times D^2) \cup_{\partial P \times D^2} (C \times D^2 \sqcup C \times D^2 \sqcup C \times D^2) \cup_{\partial P \times D^2} (P \times D^2) = \Sigma_2 \times D^2\,,$$

where Σ_2 is the closed orientable surface of genus 2 (see Figure 17.2), branched over

$$T(\tau_1,\tau_2) = R(\tau_1,\tau_2) \cup_{\widehat{\tau_1} \sqcup \widehat{\tau_2} \sqcup \widehat{\tau_1\tau_2}} (S(\tau_1) \sqcup S(\tau_2) \sqcup S(\tau_1\tau_2)) \cup_{\widehat{id_c} \sqcup \widehat{id_c} \sqcup \widehat{id_c}} R(id_c, id_c)\,.$$

Moreover, its ω-signature is precisely what we wish to bound, as shown by the following proposition.

Proposition 17.2.8. *The 4-manifold $M_G(\tau_1,\tau_2)$ can be endowed with an orientation, so that*

$$\sigma_\omega(M_G(\tau_1,\tau_2)) \simeq \operatorname{sign}(\widehat{\tau_1\tau_2}) - sign_\omega(\widehat{\tau}_1) - sign_\omega(\widehat{\tau}_2) - Maslov(\mathscr{F}_\omega(\overline{\tau}_1), \mathscr{F}_\omega(id_c), \mathscr{F}_\omega(\tau_2))\,.$$

Proof. We need to be more precise about the orientation of the 4-manifolds in play. First note that any (arbitrary but fixed) orientation on the cylinder $C = [0,1]\times S^1$ and on the unit disk D^2 defines an orientation on their product $C\times D^2$, which lifts to an orientation on the cover $C_G(\tau)$, and induces an orientation on $\overline{X}_{\widehat{\tau}} \sqcup \overline{X}_{\widehat{id_c}} \subset \partial C_G(\tau)$. However, note that these two spaces are now endowed with *opposite* orientations (with respect to a fixed orientation of the solid torus $S^1\times D^2$ which lifts to an orientation of $\overline{X}_{\widehat{\tau}}$ for any tangle τ). This can be written

$$\partial C_G(\tau) \supset \overline{X}_{\widehat{\tau}} \sqcup -\overline{X}_{\widehat{id_c}}\,.$$

By the same arguments, the fixed orientation on the pair of pants P (and on D^2) induces an orientation on $P_G(\tau_1,\tau_2)$ such that

$$\partial P_G(\tau_1,\tau_2) \supset \overline{X}_{\widehat{\tau_1}} \sqcup \overline{X}_{\widehat{\tau_2}} \sqcup -\overline{X}_{\widehat{\tau_1\tau_2}}\,.$$

Therefore, for the manifold $M_G(\tau_1,\tau_2)$ to be oriented, we need to paste *positively oriented* copies of $P_G(\tau_1,\tau_2)$ and $C_G(\tau_1\tau_2)$ together with *negatively oriented* copies of $C_G(\tau_1)$, $C_G(\tau_2)$ and $P_G(id_c,id_c)$ (or the opposite). It only remains to apply the Novikov-Wall theorem a couple of times, as follows.

Let M be the manifold obtained by gluing $P_G(\tau_1,\tau_2)$ to $-C_G(\tau_1)\sqcup -C_G(\tau_2)\sqcup C_G(\tau_1\tau_2)$ along the 3-manifold $X_0 := \overline{X}_{\widehat{\tau_1}} \sqcup \overline{X}_{\widehat{\tau_2}} \sqcup -\overline{X}_{\widehat{\tau_1\tau_2}}$ whose boundary Σ consists of 3 disjoint tori. By Novikov-Wall, Proposition 17.2.6 and Proposition 17.2.7, we have

$$\begin{aligned}\sigma_\omega(M) &\simeq \sigma_\omega(P_G(\tau_1,\tau_2)) + \sigma_\omega(C_G(\tau_1\tau_2)) - \sigma_\omega(C_G(\tau_1)) - \sigma_\omega(C_G(\tau_2))\\ &\simeq Maslov(\mathscr{F}_\omega(\overline{\tau}_1), \Delta, \mathscr{F}_\omega(\tau_2)) + \operatorname{sign}(\widehat{\tau_1\tau_2}) - \operatorname{sign}_\omega(\widehat{\tau}_1) - \operatorname{sign}_\omega(\widehat{\tau}_2).\end{aligned}$$

Applying the exact same line of reasoning to the gluing of $P_G(id_c,id_c)$, the result follows from Proposition 17.2.6 as

$$\sigma_\omega(P_G(id_c,id_c)) \simeq Maslov(\Delta,\Delta,\Delta) = 0\,,$$

by the second point of Lemma 16.2.1. □

To prove Theorem 16.4.1, it only remains to show that the ω-signature of $M_G(\tau_1,\tau_2)$ vanishes up to an uniformly bounded additive constant. This requires a small lemma.

Lemma 17.2.9. *For well-chosen surfaces $S(\tau_1)$, $S(\tau_2)$ and $S(\tau_1\tau_2)$ in the construction above, the branched covering $M_G(\tau_1,\tau_2)\to\Sigma_2\times D^2$ satisfies the following property: there exists a curve γ in the genus 2 surface Σ_2 such $\gamma\times D^2$ intersects the branch set $T(\tau_1,\tau_2)\subset\Sigma_2\times D^2$ in the n disjoint circles $\gamma\times\{x_1,\dots,x_n\}$.*

Proof. Fix C-complexes $S(\tau_1)$ and $S(\tau_2)$ for the links $\widehat{\tau}_1$ and $\widehat{\tau}_2$, and build a C-complex $S(\tau_1\tau_2)$ for $\widehat{\tau_1\tau_2}$ by connecting $S(\tau_1)$ and $S(\tau_2)$ along disjoint bands far from the tangles. Let us use the same notation for the surfaces obtained by pushing these C-complex inside D^4 in such a way that they intersect $[0,1]\times\{s_1\}\times D^2$ along $[0,1]\times\{s_1\}\times\{p_1,\dots,p_n\}$, where s_1 is some point on the circle S^1 far from the tangle.

The strategy is to build the curve γ from four intervals $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ alternatively contained in the pairs of pants and in the central cylinder, as illustrated in Figure 17.2. For the branch set $R(\tau_1, \tau_2)$ (resp. $R(id_c, id_c)$), one can simply pick an interval γ_1 (resp. γ_3) in the pair of pants as illustrated in Figure 17.1. For the branch set $S(\tau_1\tau_2)$, one must find two intervals satisfying the same property as above. Using the way we pushed the C-complex $S(\tau_1\tau_2)$ into the 4-ball, we can set $\gamma_2 = [0,1] \times \{s_1\}$. Finally, using the way we built $S(\tau_1\tau_2)$ from $S(\tau_1)$ and $S(\tau_2)$, there exists a second point $s_2 \neq s_1$ such that the C-complex intersects $[0,1] \times \{s_2\} \times D^2$ along $[0,1] \times \{s_2\} \times \{p_1, \dots, p_n\}$. Set $\gamma_3 = [0,1] \times \{s_2\}$. Gluing these intervals $\gamma_1, \dots, \gamma_4$ together produces the required curve γ. □

We may now conclude.

Proposition 17.2.10. *The ω-signature of $M_G(\tau_1, \tau_2)$ vanishes up to an uniformly bounded constant.*

Proof. Cutting the genus 2 surface Σ_2 along the curve γ provided by Lemma 17.2.9 yields a decomposition $\Sigma_2 = \Sigma_1 \cup_\gamma \Sigma_1$, where Σ_1 denotes the genus 1 surface with one boundary component. The induced decomposition of $\Sigma_2 \times D^2$ lifts to

$$M_G(\tau_1, \tau_2) = Q_G(\tau_1) \cup_{X_0} Q_G(\tau_2)\,,$$

with $X_0 = \gamma \times \overline{D}_c$ thanks to the way we chose γ. Applying the Novikov-Wall theorem to this decomposition, we get

$$\sigma_\omega(M_G(\tau_1, \tau_2)) \simeq \sigma_\omega(Q_G(\tau_1)) + \sigma_\omega(Q_G(\tau_2))\,.$$

It remains to show that $\sigma_\omega(Q_G(\tau)) \simeq 0$ for any tangle τ. Using the second point of Lemma 16.2.1 together with Proposition 17.2.8 and the equation displayed above for $(\tau_1, \tau_2) = (\tau, id_c)$, we have

$$\begin{aligned} 0 &= \sigma_\omega(\widehat{\tau}) - \sigma_\omega(\widehat{\tau}) - \sigma_\omega(\widehat{id_c}) - Maslov(\mathscr{F}_\omega(\overline{\tau}), \Delta, \Delta) \\ &\simeq \sigma_\omega(M_G(\tau, id_c)) \simeq \sigma_\omega(Q_G(\tau)) + \sigma_\omega(Q_G(id_c)) \end{aligned}$$

independently of the tangle τ. Taking $\tau = id_c$ yields the result. □

17.3 The isotropic functor as an evaluation

In this section, we relate the Lagrangian functor to our isotropic functor. In particular, we will see in Theorem 17.3.2 below why the isotropic functor is, in some sense, an evaluation of the Lagrangian functor. Recall that these results were used in Chapter 16 to compute several examples, and to prove Corollary 16.4.4.

17.3.1 Statement of the results

Under some mild assumptions on the colored tangle, we shall show how the Lagrangian functor is related to the isotropic functor. In particular, it will follow that if α is a braid, then $\mathscr{F}_\omega(\alpha)$ can be understood by evaluating a matrix of the reduced colored Gassner representation at $t = \omega$. The proofs of these slightly technical statements will be given in the next paragraph.

Fix an element ω in $\mathbb{T}^\mu$ and assume that the component ω_i of ω is of finite order $k_i > 1$. Let G be the finite abelian group $C_{k_1} \times \cdots \times C_{k_\mu}$. For $i = 1, \dots, \mu$ let t_i be a generator of C_{k_i} and let χ_ω be the character of G sending the generator t_i to the root of unity $\omega_i \in \mathbb{C}\setminus\{1\}$. As usual, we shall simply denote by H_ω the associated generalized eigenspaces (recall Subsection 17.1.1). Note that χ_ω induces a ring homomorphism $\Lambda_\mu \to \mathbb{C}$ which endows $\mathbb{C}$ with the structure of a module over Λ_μ, that will be emphasized by the notation $\mathbb{C}_\omega$. Note also that since $\omega_i \neq 1$, this homomorphism factors through the localized ring $\Lambda_S = \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}, (t_1-1)^{-1}, \dots, (t_\mu - 1)^{-1}]$. As customary, we shall write H_S for the localization $H \otimes_{\Lambda_\mu} \Lambda_S$ of a Λ_μ-module H. Note the identity $H \otimes_{\Lambda_\mu} \mathbb{C}_\omega = H_S \otimes_{\Lambda_S} \mathbb{C}_\omega$.

Proposition 17.3.1. *For any $c\colon \{1,\dots,n\} \to \{\pm 1, \dots, \pm\mu\}$, there is a natural isomorphism of complex vector spaces*

$$\Phi_c\colon H_1(\widehat{D}_c) \otimes_{\Lambda_\mu} \mathbb{C}_\omega \longrightarrow H_1(\overline{D}_c)_\omega\,.$$

Furthermore, if $\lambda_c(t)$ is the matrix for the form of $\mathscr{F}(c)_S$ with respect to some Λ_S-basis $\{v_j\}_j$ of $H_1(\widehat{D}_c)_S$, then the matrix for the form $\langle\ ,\ \rangle_c$ of $\mathscr{F}_\omega(c)$ with respect to the basis $\{\Phi_c(v_j \otimes 1)\}_j$ of $H_1(\overline{D}_c)_\omega$ is given by the componentwise evaluation of $\lambda_c(t)$ at $t = \omega$, up to a positive multiplicative constant.

Next, we introduce some terminology. A free submodule N of $H \oplus H'$ is determined by a matrix of the inclusion $N \subset H \oplus H'$ with respect to a basis of N. Following [43], we will say that $N \subset H \oplus H'$ is *encoded* by this matrix. For instance, the graph of a linear map $\gamma\colon H \to H'$ is encoded by the matrix $(I \quad M_\varphi)^T$, where M_φ is a matrix for φ and I the identity matrix.

Theorem 17.3.2. *Assume that ω is in $\mathbb{T}^\mu_{cP}$ and let $\tau \in T^\omega_\mu(c,c')$ be such that the Λ_μ-module $\ker(j_\tau)$ is Lagrangian, and its localization is a free Λ_S-module. Then, with respect to the isomorphisms Φ_c and $\Phi_{c'}$ of Proposition 17.3.1, an encoding matrix for the complex vector space $\mathscr{F}_\omega(\tau) = \ker(\hat{\jmath}_\tau)$ can be obtained by evaluating an encoding matrix for $\mathscr{F}(\tau)_S = \ker(j_\tau)_S$ at $t = \omega$.*

Recall that a tangle is *topologically trivial* if its exterior is homeomorphic to the exterior of a trivial braid. It is easy to check that in such a case, the condition in Theorem 17.3.2 above is always satisfied, see [43, Section 4]. In particular, braids are topologically trivial, so we have the following corollary.

Corollary 17.3.3. *For all $\alpha \in B_c$ with $\ell(c)$ nowhere zero and $\omega \in \mathbb{T}^\mu_c \cap \mathbb{T}^\mu_{cP}$, $\mathscr{F}_\omega(\alpha) = \Gamma_{\overline{\mathscr{B}_\omega(\alpha)}}$.*

17.3.2 Proofs of Proposition 17.3.1 and Theorem 17.3.2

In order to prove these results, the first step consists of understanding the relation between free abelian coverings and finite branched coverings. This is the subject of the following two lemmata. For punctured disks D_c and tangle exteriors X_τ, we shall denote by D_c^* and X_τ^* the respective unbranched finite abelian coverings.

Lemma 17.3.4. *For any c and τ as above, we have natural isomorphisms*

$$H_1(\overline{D}_c)_\omega \cong H_1(D_c^*)_\omega \quad \textit{and} \quad H_1(\overline{X}_\tau)_\omega \cong H_1(X_\tau^*)_\omega\,.$$

Proof. Recall that the first homology group of the punctured disk D_c is freely generated by the loops $x_1, \dots, x_n$, where x_j is a simple loop turning once around the puncture p_j. Let $\widetilde{x}_j$ be a lift of the loop x_j to D_c^* for $j = 1, \dots, n$. By definition, the branched covering $\overline{D}_c \to D^2$

is obtained from the unbranched covering $D_c^* \to D_c$ by gluing n disks to D_c (in order to recover D^2) and lifting these gluings to the covering. Applying the Mayer-Vietoris exact sequence to this decomposition of $\overline{D}_c$ shows that the inclusion induced homomorphism defines an isomorphism $H_1(D_c^*)/V \cong H_1(\overline{D}_c)$, where V is the $\mathbb{Z}[G]$-submodule generated by the loops $(1+t_i+\dots+t_i^{k_i-1})\widetilde{x}_j$ for $j=1,\dots,n$ and i stands for c_j. Since $\omega_i \neq 1$ is a k_i^{th}-root of unity, $\chi(1+t_i+\dots+t_i^{k_i-1})$ vanishes; this implies that $c_\chi V = 0$, and the conclusion follows from the second point of Proposition 17.1.3. The case of the tangle exterior can be treated in the same way. □

Lemma 17.3.5. *There are natural isomorphisms*

$$H_1(\overline{D}_c)_\omega \cong H_1(\widehat{D}_c) \otimes_{\Lambda_\mu} \mathbb{C}_\omega \quad \textit{and} \quad H_1(\overline{X}_\tau)_\omega \cong H_1(\widehat{X}_\tau) \otimes_{\Lambda_\mu} \mathbb{C}_\omega\,.$$

Proof. Both statements will be proved by using standard cut and paste arguments. Let $I_1,\dots,I_n$ be disjoint intervals in the disk D^2 such that for $j=1,\dots,n$ the interval I_j joins the j^{th} puncture x_j to the boundary ∂D^2, and let $N_j = I_j \times [-1,1]$ be a bicollar neighborhood of I_j in D^2. Set $N = \bigcup_{j=1}^n (N_j \cap D_c)$, $Y = D_c \setminus \bigcup_{j=1}^n \operatorname{Int}(N_j)$, $R = N \cap Y$ and let $\widehat{p}\colon \widehat{D}_c \to D_c$ be the free abelian covering map. The decomposition $D_c = N \cup Y$ leads to the Mayer-Vietoris exact sequence of Λ_μ-modules

$$H_1(\widehat{R}) \to H_1(\widehat{N}) \oplus H_1(\widehat{Y}) \to H_1(\widehat{D}_c) \to H_0(\widehat{R}) \to H_0(\widehat{N}) \oplus H_0(\widehat{Y})\,,$$

where $\widehat{R}$, $\widehat{N}$ and $\widehat{Y}$ stand for $\widehat{p}^{-1}(R)$, $\widehat{p}^{-1}(N)$ and $\widehat{p}^{-1}(Y)$, respectively. Writing $\overline{p}\colon D_c^* \to D_c$ for the finite abelian covering map and repeating the same argument yields the Mayer-Vietoris exact sequence of $\mathbb{Z}[G]$-modules

$$H_1(R^*) \to H_1(N^*) \oplus H_1(Y^*) \to H_1(D_c^*) \to H_0(R^*) \to H_0(N^*) \oplus H_0(Y^*)\,,$$

where R^*, N^* and Y^* stand for $\overline{p}^{-1}(R)$, $\overline{p}^{-1}(N)$ and $\overline{p}^{-1}(Y)$, respectively. In the free abelian case, the map $H_0(\widehat{R}) \to H_0(\widehat{N}) \oplus H_0(\widehat{Y})$ is injective while in the finite abelian case, the kernel V of the corresponding map $H_0(R^*) \to H_0(N^*) \oplus H_0(Y^*)$ is freely generated by the n loops $\{(1+t_i+\dots+t_i^{k_i-1})\widetilde{x}_j\}_{j=1}^n$, where i stands for c_j. It follows that the first homology groups of these two coverings are related by

$$H_1(D_c^*) \cong \Big(H_1(\widehat{D}_c) \otimes_{\Lambda_\mu} \mathbb{Z}[G]\Big) \oplus V\,.$$

Since ω_i is a k_i^{th} root of unity different from 1, we have $V_\omega \cong V \otimes_{\mathbb{Z}[G]} \mathbb{C}_\omega = 0$. Therefore, using Lemma 17.3.4, Proposition 17.1.4 and the isomorphism displayed above, one obtains

$$H_1(\overline{D}_c)_\omega \cong H_1(D_c^*)_\omega \cong H_1(D_c^*) \otimes_{\mathbb{Z}[G]} \mathbb{C}_\omega \cong H_1(\widehat{D}_c) \otimes_{\Lambda_\mu} \mathbb{C}_\omega\,.$$

Let us now deal with the tangle exterior. As $\ell(c) = \ell(c')$, one can always obtain a colored link L from the tangle τ by joining the punctures with disjoint colored strands contained in the boundary of the cylinder $D^2 \times [0,1]$. Pick a C-complex F for the colored link L, which can be assumed to be contained in the cylinder. By [41, Section 3], it is possible to recover the free abelian covering of the link exterior by cutting it along S. Consequently, if we denote by $F_1,\dots,F_\mu$ the components of F, let $N_i = F_i \times [-1,1]$ be a bicollar neighborhood of $F_i \subset D^2 \times [0,1]$, and set $N = \bigcup_{i=1}^\mu (N_i \cap X_\tau)$, $Y = X_\tau \setminus \bigcup_{i=1}^\mu \operatorname{Int}(N_i)$, and $R = N \cap Y$, we can then follow the exact same steps as above. □

Proposition 17.3.1 now follows readily.

proof of Proposition 17.3.1. The isomorphism Φ_c is given by Lemma 17.3.5. Note that this isomorphism is natural, in the sense that it is given by the composition of several inclusion induced isomorphisms. In particular, it preserves the intersection numbers, so Proposition 17.3.1 follows from Corollary 17.1.7 applied to $M = \overline{D}_c$. One needs to work over the ring Λ_S to ensure that $H_1(\widehat{D}_c)_S$ is free, but this is not an issue, as the homomorphism $\Lambda_\mu \to \mathbb{C}$ mapping t_i to $\omega_i \neq 1$ factors through Λ_S. □

The proof of Theorem 17.3.2 will rely on one last intermediate statement.

Lemma 17.3.6. *Assume that ω is in $\mathbb{T}^\mu_{cP}$ and let $\tau \in T^\omega_\mu(c, c')$ be such that the Λ_μ-module $\ker(j_\tau)$ is Lagrangian, and its localization is a free Λ_S-module. Then, via the isomorphisms of Lemma 17.3.5, we have*

$$\mathscr{F}_\omega(\tau) = \ker(j_\tau) \otimes_{\Lambda_\mu} \mathbb{C}_\omega\,.$$

Proof. By definition, the isomorphims of Lemma 17.3.5 allow us to identify $\mathscr{F}_\omega(\tau)$ with the kernel of the map

$$j_\tau \otimes id_{\mathbb{C}_\omega} \colon (H_1(\widehat{D}_c) \oplus H_1(\widehat{D}_c)) \otimes_{\Lambda_\mu} \mathbb{C}_\omega \to H_1(\widehat{X}_\tau) \otimes_{\Lambda_\mu} \mathbb{C}_\omega\,.$$

To prove the assertion, it is therefore enough to show the equality

$$\ker(j_\tau \otimes id_{\mathbb{C}_\omega}) = \ker(j_\tau) \otimes_{\Lambda_\mu} \mathbb{C}_\omega\,.$$

The inclusion from right to left is straightforward. To get equality, we will argue that both spaces have the same (complex) dimension.

Assume that D_c and $D_{c'}$ are punctured n and n' times, respectively. Then, we know that $H_1(\widehat{D}_c)$ and $H_1(\widehat{D}_{c'})$ are Λ_μ-modules of respective rank $(n-1)$ and $(n'-1)$. Since ω belongs to $\mathbb{T}^\mu_{cP}$ and τ to $T^\omega_\mu(c, c')$, the subspace $\mathscr{F}_\omega(\tau)$ is Lagrangian by Proposition 17.1.12. As the form on $H_1(\overline{D}_c)_\omega \oplus H_1(\overline{D}_{c'})_\omega$ is non-degenerate (Lemma 17.1.11), the dimension of $\ker(j_\tau \otimes id_{\mathbb{C}_\omega}) \cong \mathscr{F}_\omega(\tau)$ is half that of $(H_1(\widehat{D}_c) \otimes_{\Lambda_\mu} \mathbb{C}_\omega) \oplus (H_1(\widehat{D}_c) \otimes_{\Lambda_\mu} \mathbb{C}_\omega)$, that is,

$$\dim(\ker(j_\tau \otimes id_{\mathbb{C}_\omega})) = ((n-1) + (n'-1))/2\,.$$

On the other hand, $\ker(j_\tau)$ is a Lagrangian submodule of a non-degenerate Hermitian Λ_μ-module, and its localization is free over Λ_S; hence, the dimension of $\ker(j_\tau) \otimes_{\Lambda_\mu} \mathbb{C}_\omega = \ker(j_\tau)_S \otimes_{\Lambda_S} \mathbb{C}_\omega$ is also equal to $((n-1) + (n'-1))/2$. □

We can finally prove Theorem 17.3.2.

proof of Theorem 17.3.2: By standard properties of the tensor product (recall Lemma 17.1.5), the componentwise evaluation by χ_ω of a matrix for the inclusion map i of $\ker(j_\tau)_S$ inside $H_1(\widehat{D}_c)_S \oplus H_1(\widehat{D}_{c'})_S$ yields a matrix for

$$i \otimes id_{\mathbb{C}_\omega} \colon \ker(j_\tau) \otimes_{\Lambda_\mu} \mathbb{C}_\omega \to (H_1(\widehat{D}_c) \oplus H_1(\widehat{D}_{c'})) \otimes_{\Lambda_\mu} \mathbb{C}_\omega\,.$$

By Lemma 17.3.6, this map can be identified with the inclusion of $\mathscr{F}_\omega(\tau)$ into $H_1(\overline{D}_c)_\omega \oplus H_1(\overline{D}_c)_\omega$, and the proof is completed. □

Résumé de la thèse en français

Cette thèse se concentre sur la topologie de basse dimension et plus spécifiquement sur la théorie des noeuds. Elle se décompose en trois parties. La première est consacrée aux invariants d'entrelacs colorés, la seconde concerne la représentation de Burau du groupe de tresses et ses généralisations. La troisième partie s'appuie sur les deux premières et étudie la non-additivité des signatures. Résumons brièvement ces trois parties en autant de paragraphes.

Un noeud consiste en un cercle plongé dans la sphère de dimension trois. Les noeuds sont étudiés en assignant une quantité algébrique (appelée "invariant de noeud") à chaque noeud de façon à ce que des noeuds isotopes aient des invariants identiques. Les invariants dits classiques comprennent le polynôme d'Alexander, la signature de Levine-Tristram et la forme de Blanchfield. Dans le cas des entrelacs (i.e. des unions disjointes de noeuds), des invariants multivariés similaires existent, mais ils s'avèrent plus difficiles à étudier. La première partie de cette thèse étudie ce type d'invariants: nous utilisons la signature multivariée de Cimasoni-Florens afin de fournir de nouvelles bornes inférieures sur le *splitting number*, nous donnons la première formule explicite de la forme de Blanchfield d'un entrelacs quelconque, nous améliorons la compréhension 4-dimensionelle de la signature multivariée et, en particulier, nous montrons qu'elle est invariante par *1-solvable cobordism.*

De façon sommaire, une tresse à n brins consiste en n intervalles monotones dans le cylindre. L'ensemble des classes d'isotopie de tresses à n brins forme un groupe appelé le groupe de tresses. La représentation de Burau (réduite) du groupe de tresses est connue pour sa relation avec le polynôme d'Alexander. Nous généralisons cette relation au contexte des polynômes d'Alexander tordus ainsi qu'aux invariants L^2. Nos résultats sont obtenus en définissant des applications de Burau tordues et L^2. Les tresses peuvent aussi être comprises comme cas particuliers des enchevêtrements, qui sont un type de 1-sous-variétés du cylindre. Les enchevêtrements ne forment plus un groupe mais sont les morphismes d'une catégorie. La représentation de Burau admet alors une généralisation aux enchevêtrements sous la forme du foncteur Lagrangien de Cimasoni-Turaev. En utilisant la théorie des *cospans*, nous montrons comment ce foncteur peut être promu en un 2-foncteur sur la bicatégorie des enchevêtrements.

La dernière partie de cette thèse concerne la non-additivité des invariants de signature. Soit $\mathscr{I}$ un invariant d'entrelacs à valeurs dans un groupe abélien. En précomposant cet invariant avec la clôture $\alpha \mapsto \widehat{\alpha}$ de tresses, nous obtenons des applications des groupes de tresses vers ce groupe abélien. Par conséquent, il est naturel d'essayer d'évaluer le défaut d'additivité $\mathscr{I}(\widehat{\alpha\beta}) - \mathscr{I}(\widehat{\alpha}) - \mathscr{I}(\widehat{\beta})$. Cette tâche a été accomplie par Gambaudo et Ghys qui ont exprimé le défaut d'additivité de la signature de Levine-Tristram en termes de la représentation de Burau. Nous étendons ce théorème aux enchevêtrement colorés. Le résultat relie le défaut d'additivité de la signature multivariée au foncteur Lagrangien et à l'indice de Maslov.

Bibliography

[1] J. W. Alexander. Topological invariants of knots and links. *Trans. Amer. Math. Soc.*, 30(2):275–306, 1928.

[2] James Alexander. A lemma on a system of knotted curves. *Proc. Natl. Acad. Sci. USA.*, 9:93–95, 1923.

[3] V. I. Arnol'd. On a characteristic class entering into conditions of quantization. *Funkcional. Anal. i Priložen.*, 1:1–14, 1967.

[4] Emil Artin. Theorie der Zöpfe. *Abh. Math. Sem. Univ. Hamburg*, 4(1):47–72, 1925.

[5] Matthias Aschenbrenner, Stefan Friedl, and Henry Wilton. *3-manifold groups.* EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2015.

[6] M. F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. In *Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974)*, pages 43–72. Astérisque, No. 32–33. Soc. Math. France, Paris, 1976.

[7] Michael Atiyah, Vijay Patodi, and Isadore Singer. Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69, 1975.

[8] John C. Baez and Laurel Langford. 2-tangles. *Lett. Math. Phys.*, 43(2):187–197, 1998.

[9] Jean Barge, Jean Lannes, François Latour, and Pierre Vogel. Λ-sphères. *Ann. Sci. École Norm. Sup. (4)*, 7:463–505 (1975), 1974.

[10] Joshua Batson and Cotton Seed. A link-splitting spectral sequence in Khovanov homology. *Duke Math. J.*, 164(5):801–841, 2015.

[11] Fathi Ben Aribi. *A study of properties and computation techniques of the L^2-Alexander invariant in knot theory.* PhD thesis, Université Paris Diderot, 2015.

[12] Fathi Ben Aribi. The L^2-Alexander invariant detects the unknot. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, XV(5):683–708, 2016.

[13] Fathi Ben Aribi and Anthony Conway. L^2-Burau maps and L^2-Alexander torsions. *Osaka J. Math.*, (to appear), 2017.

[14] Jean Bénabou. *Reports of the Midwest Category Seminar*, chapter Introduction to bicategories, pages 1–77. Springer Berlin Heidelberg, Berlin, Heidelberg, 1967.

[15] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators*, volume 298 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[16] Stephen Bigelow. The Burau representation is not faithful for $n = 5$. *Geom. Topol.*, 3:397–404, 1999.

[17] Joan S. Birman. *Braids, links, and mapping class groups*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 82.

[18] Richard C. Blanchfield. Intersection theory of manifolds with operators with applications to knot theory. *Ann. of Math. (2)*, 65:340–356, 1957.

[19] Hans U. Boden, Emily Dies, Anne Isabel Gaudreau, Adam Gerlings, Eric Harper, and Andrew J. Nicas. Alexander invariants for virtual knots. *J. Knot Theory Ramifications*, 24(3):1550009 (62 pages), 2015.

[20] Maciej Borodzik and Stefan Friedl. The unknotting number and classical invariants, I. *Algebr. Geom. Topol.*, 15(1):85–135, 2015.

[21] Maciej Borodzik, Stefan Friedl, and Mark Powell. Blanchfield forms and gordian distance. *Journal of the Mathematical Society of Japan.*

[22] Maciej Borodzik and Eugene Gorsky. Immersed concordances of links and Heegaard Floer homology. January 2016. http://arxiv.org/abs/1601.07507.

[23] Glen E. Bredon. *Topology and geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1993.

[24] William Browder. *Surgery on simply-connected manifolds*. Springer-Verlag, New York-Heidelberg, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65.

[25] Werner Burau. Über Zopfgruppen und gleichsinnig verdrillte Verkettungen. *Abh. Math. Sem. Univ. Hamburg*, 11(1):179–186, 1935.

[26] Gerhard Burde, Heiner Zieschang, and Michael Heusener. *Knots*, volume 5 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, extended edition, 2014.

[27] Sylvain E. Cappell, Ronnie Lee, and Edward Y. Miller. On the Maslov index. *Comm. Pure Appl. Math.*, 47(2):121–186, 1994.

[28] A. J. Casson and C. McA. Gordon. On slice knots in dimension three. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2*, Proc. Sympos. Pure Math., XXXII, pages 39–53. Amer. Math. Soc., Providence, R.I., 1978.

[29] A. J. Casson and C. McA. Gordon. Cobordism of classical knots. In *À la recherche de la topologie perdue*, volume 62 of *Progr. Math.*, pages 181–199. Birkhäuser Boston, Boston, MA, 1986. With an appendix by P. M. Gilmer.

[30] Jae Choon Cha. Symmetric Whitney tower cobordism for bordered 3-manifolds and links. *Trans. Amer. Math. Soc.*, 366(6):3241–3273, 2014.

[31] Jae Choon Cha and Stefan Friedl. Twisted torsion invariants and link concordance. *Forum Math.*, 25(3):471–504, 2013.

[32] Jae Choon Cha, Stefan Friedl, and Mark Powell. Splitting numbers of links. 2013. arXiv:1308.5638.

[33] Jae Choon Cha and Taehee Kim. Covering link calculus and iterated Bing doubles. *Geom. Topol.*, 12(4):2173–2201, 2008.

[34] Jae Choon Cha and Charles Livingston. Linkinfo. 2015. Linkinfo.

[35] T. A. Chapman. Topological invariance of Whitehead torsion. *Amer. J. Math.*, 96:488–497, 1974.

[36] David Cimasoni. A geometric construction of the Conway potential function. *Comment. Math. Helv.*, 79(1):124–146, 2004.

[37] David Cimasoni. The Conway function of a splice. *Proc. Edinb. Math. Soc. (2)*, 48(1):61–73, 2005.

[38] David Cimasoni and Anthony Conway. A Burau-Alexander 2-functor on tangles. *Fund.Math.*, (to appear), 2017.

[39] David Cimasoni and Anthony Conway. Coloured tangles and signatures. *Math. Proc. Cambridge Philos. Soc.*, (to appear), 2017.

[40] David Cimasoni, Anthony Conway, and Kleopatra Zacharova. Splitting numbers and signatures. *Proc. Amer. Math. Soc.*, 144(12):5443–5455, 2016.

[41] David Cimasoni and Vincent Florens. Generalized Seifert surfaces and signatures of colored links. *Trans. Amer. Math. Soc.*, 360(3):1223–1264 (electronic), 2008.

[42] David Cimasoni and Vladimir Turaev. A Lagrangian representation of tangles. *Topology*, 44(4):747–767, 2005.

[43] David Cimasoni and Vladimir Turaev. A Lagrangian representation of tangles. II. *Fund. Math.*, 190:11–27, 2006.

[44] Tim Cochran, Shelly Harvey, and Constance Leidy. Link concordance and generalized doubling operators. *Algebr. Geom. Topol.*, 8(3):1593–1646, 2008.

[45] Tim D. Cochran and Kent E. Orr. Not all links are concordant to boundary links. *Ann. of Math. (2)*, 138(3):519–554, 1993.

[46] Tim D. Cochran and Kent E. Orr. Homology boundary links and Blanchfield forms: concordance classification and new tangle-theoretic constructions. *Topology*, 33(3):397–427, 1994.

[47] Tim D. Cochran, Kent E. Orr, and Peter Teichner. Knot concordance, Whitney towers and L^2-signatures. *Ann. of Math. (2)*, 157(2):433–519, 2003.

[48] Tim D. Cochran, Kent E. Orr, and Peter Teichner. Structure in the classical knot concordance group. *Comment. Math. Helv.*, 79(1):105–123, 2004.

[49] P. E. Conner and E. E. Floyd. *Differentiable periodic maps.* Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Band 33. Academic Press Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1964.

[50] Anthony Conway. Burau maps and twisted Alexander polynomials. *Proc. Edinburgh Math. Soc.*, (to appear), 2017.

[51] Anthony Conway. An explicit computation for the Blanchfield pairing of arbitrary links. June 2017. http://arxiv.org/pdf/1706.00226.pdf.

[52] Anthony Conway, Stefan Friedl, and Enrico Toffoli. The Blanchfield pairing of colored links. *Indiana Univ. Math. J.*, (to appear), 2017.

[53] Anthony Conway, Matthias Nagel, and Enrico Toffoli. Multivariable signatures, genus bounds and 1-solvable cobordisms. March 2017. http://arxiv.org/pdf/1703.07540v1.pdf.

[54] D. Cooper. The universal abelian cover of a link. In *Low-dimensional topology (Bangor, 1979)*, volume 48 of *London Math. Soc. Lecture Note Ser.*, pages 51–66. Cambridge Univ. Press, Cambridge-New York, 1982.

[55] Daryl Cooper. Signatures of surfaces with applications to knot and link cobordism. 1982. University of Warwick.

[56] Peter R. Cromwell. *Knots and links.* Cambridge University Press, Cambridge, 2004.

[57] R. H. Crowell and D. Strauss. On the elementary ideals of link modules. *Trans. Amer. Math. Soc.*, 142:93–109, 1969.

[58] Richard H. Crowell and Ralph H. Fox. *Introduction to knot theory.* Based upon lectures given at Haverford College under the Philips Lecture Program. Ginn and Co., Boston, Mass., 1963.

[59] Marc Culler, Nathan M. Dunfield, and Jeffrey R. Weeks. SnapPy, a computer program for studying the topology of 3-manifolds.

[60] James F. Davis and Paul Kirk. *Lecture notes in algebraic topology*, volume 35 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2001.

[61] Alex Degtyarev, Vincent Florens, and Ana Lecuona. The signature of a splice. *ArXiv e-prints*, 2014.

[62] Jérôme Dubois, Stefan Friedl, and Wolfgang Lück. The L^2-Alexander torsion is symmetric. *Algebr. Geom. Topol.*, 15(6):3599–3612, 2015.

[63] Jérôme Dubois, Stefan Friedl, and Wolfgang Lück. The L^2-Alexander torsion of 3-manifolds. *J. Topol.*, 9(3):889–926, 2016.

[64] Jérôme Dubois and Christian Wegner. L^2-Alexander invariant for torus knots. *C. R. Math. Acad. Sci. Paris*, 348(21-22):1185–1189, 2010.

[65] David Eisenbud and Walter Neumann. *Three-dimensional link theory and invariants of plane curve singularities*, volume 110 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 1985.

[66] Dieter Erle. Quadratische Formen als Invarianten von Einbettungen der Kodimension 2. *Topology*, 8:99–114, 1969.

[67] John E. Fischer, Jr. 2-categories and 2-knots. *Duke Math. J.*, 75(2):493–526, 1994.

[68] V. Florens. Signatures of colored links with application to real algebraic curves. *J. Knot Theory Ramifications*, 14(7):883–918, 2005.

[69] Ralph H. Fox. Free differential calculus. II. The isomorphism problem of groups. *Ann. of Math. (2)*, 59:196–210, 1954.

[70] Ralph H. Fox and John W. Milnor. Singularities of 2-spheres in 4-space and cobordism of knots. *Osaka J. Math.*, 3:257–267, 1966.

[71] Stefan Friedl. Eta invariants as sliceness obstructions and their relation to Casson-Gordon invariants. *Algebr. Geom. Topol.*, 4:893–934, 2004.

[72] Stefan Friedl. L^2-eta-invariants and their approximation by unitary eta-invariants. *Math. Proc. Cambridge Philos. Soc.*, 138(2):327–338, 2005.

[73] Stefan Friedl and Taehee Kim. The Thurston norm, fibered manifolds and twisted Alexander polynomials. *Topology*, 45(6):929–953, 2006.

[74] Stefan Friedl, Taehee Kim, and Takahiro Kitayama. Poincaré duality and degrees of twisted Alexander polynomials. *Indiana Univ. Math. J.*, 61(1):147–192, 2012.

[75] Stefan Friedl, Constance Leidy, Matthias Nagel, and Mark Powell. Twisted Blanchfield pairings and decompositions of 3-manifolds. *Homology, homotopy and applications.*, (to appear), 2017.

[76] Stefan Friedl and Mark Powell. A calculation of Blanchfield pairings of 3-manifolds and knots. 2015. arXiv:1512.04603.

[77] Stefan Friedl and Stefano Vidussi. A survey of twisted Alexander polynomials. In *The mathematics of knots*, volume 1 of *Contrib. Math. Comput. Sci.*, pages 45–94. Springer, Heidelberg, 2011.

[78] Jean-Marc Gambaudo and Étienne Ghys. Braids and signatures. *Bull. Soc. Math. France*, 133(4):541–579, 2005.

[79] Stavros Garoufalidis. Signatures of links and finite type invariants of cyclic branched covers. In *Tel Aviv Topology Conference: Rothenberg Festschrift (1998)*, volume 231 of *Contemp. Math.*, pages 87–97. Amer. Math. Soc., Providence, RI, 1999.

[80] Étienne Ghys and Andrew Ranicki, editors. *Six papers on signatures, braids and Seifert surfaces*, volume 30 of *Ensaios Matemáticos [Mathematical Surveys].* Sociedade Brasileira de Matemática, Rio de Janeiro, 2016.

[81] Patrick M. Gilmer and Charles Livingston. Signature jumps and Alexander polynomials for links. *Proc. Amer. Math. Soc.*, 144(12):5407–5417, 2016.

[82] Robert E. Gompf and András I. Stipsicz. *4-manifolds and Kirby calculus*, volume 20 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1999.

[83] C. McA. Gordon. Some aspects of classical knot theory. In *Knot theory (Proc. Sem., Plans-sur-Bex, 1977)*, volume 685 of *Lecture Notes in Math.*, pages 1–60. Springer, Berlin, 1978.

[84] Marco Grandis. Collared cospans, cohomotopy and TQFT (cospans in algebraic topology. II). *Theory Appl. Categ.*, 18:No. 19, 602–630, 2007.

[85] Alexander Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*, volume 224 of *Lecture notes in mathematics*. Springer-Verlag, 1971.

[86] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[87] Jonathan Hillman. *Algebraic invariants of links*, volume 52 of *Series on Knots and Everything*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second edition, 2012.

[88] Jonathan A. Hillman. *Alexander ideals of links*, volume 895 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1981.

[89] Jonathan A. Hillman, Charles Livingston, and Swatee Naik. Twisted Alexander polynomials of periodic knots. *Algebr. Geom. Topol.*, 6:145–169 (electronic), 2006.

[90] Friedrich Hirzebruch, Walter D. Neumann, and Sebastian S. Koh. *Differentiable manifolds and quadratic forms*. Marcel Dekker, Inc., New York, 1971. Appendix II by W. Scharlau, Lecture Notes in Pure and Applied Mathematics, Vol. 4.

[91] M. M. Kapranov and V. A. Voevodsky. 2-categories and Zamolodchikov tetrahedra equations. In *Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991)*, volume 56 of *Proc. Sympos. Pure Math.*, pages 177–259. Amer. Math. Soc., Providence, RI, 1994.

[92] Christian Kassel and Vladimir Turaev. *Braid groups*, volume 247 of *Graduate Texts in Mathematics*. Springer, New York, 2008. With the graphical assistance of Olivier Dodane.

[93] Louis H. Kauffman. *On knots*, volume 115 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1987.

[94] Louis H. Kauffman and Laurence R. Taylor. Signature of links. *Trans. Amer. Math. Soc.*, 216:351–365, 1976.

[95] Akio Kawauchi. *A survey of knot theory*. Birkhäuser Verlag, Basel, 1996. Translated and revised from the 1990 Japanese original by the author.

[96] C. Kearton. Blanchfield duality and simple knots. *Trans. Amer. Math. Soc.*, 202:141–160, 1975.

[97] Cherry Kearton. Blanchfield duality and simple knots. *Trans. Amer. Math. Soc.*, 202:141–160, 1975.

[98] V. M. Kharlamov and V. G. Turaev. On the definition of the 2-category of 2-knots. In *Mathematics in St. Petersburg*, volume 174 of *Amer. Math. Soc. Transl. Ser. 2*, pages 205–221. Amer. Math. Soc., Providence, RI, 1996.

[99] Min Hoon Kim. Whitney towers, gropes and Casson-Gordon style invariants of links. *Algebr. Geom. Topol.*, 15(3):1813–1845, 2015.

[100] Paul Kirk and Matthias Lesch. On the rho invariant for manifolds with boundary. *Algebr. Geom. Topol.*, 3:623–675 (electronic), 2003.

[101] Paul Kirk and Matthias Lesch. The η-invariant, Maslov index, and spectral flow for Dirac-type operators on manifolds with boundary. *Forum Math.*, 16(4):553–629, 2004.

[102] Paul Kirk and Charles Livingston. Twisted Alexander invariants, Reidemeister torsion, and Casson-Gordon invariants. *Topology*, 38(3):635–661, 1999.

[103] Paul Kirk, Charles Livingston, and Zhenghan Wang. The Gassner representation for string links. *Commun. Contemp. Math.*, 3(1):87–136, 2001.

[104] Teruaki Kitano. Twisted Alexander polynomial and Reidemeister torsion. *Pacific J. Math.*, 174(2):431–442, 1996.

[105] Ki Hyoung Ko. Seifert matrices and boundary link cobordisms. *Trans. Amer. Math. Soc.*, 299(2):657–681, 1987.

[106] Peter Kohn. Unlinking two component links. *Osaka J. Math.*, 30(4):741–752, 1993.

[107] Constance Leidy. Higher-order linking forms for knots. *Comment. Math. Helv.*, 81(4):755–781, 2006.

[108] J. Levine. Knot cobordism groups in codimension two. *Comment. Math. Helv.*, 44:229–244, 1969.

[109] J. P. Levine. Link invariants via the eta invariant. *Comment. Math. Helv.*, 69(1):82–119, 1994.

[110] Jerome Levine. Knot modules. I. *Trans. Amer. Math. Soc.*, 229:1–50, 1977.

[111] Weiping Li and Weiping Zhang. An L^2-Alexander invariant for knots. *Commun. Contemp. Math.*, 8(2):167–187, 2006.

[112] W. B. Raymond Lickorish. *An introduction to knot theory*, volume 175 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997.

[113] Xiao Song Lin. Representations of knot groups and twisted Alexander polynomials. *Acta Math. Sin. (Engl. Ser.)*, 17(3):361–380, 2001.

[114] Richard Litherland. Cobordism of satellite knots. Four-manifold theory, Proc. AMS-IMS-SIAM Joint Summer Res. Conf., Durham/N.H. 1982, Contemp. Math. 35, 327-362 (1984)., 1984.

[115] Charles Livingston. A survey of classical knot concordance. In *Handbook of knot theory*, pages 319–347. Elsevier B. V., Amsterdam, 2005.

[116] D. D. Long and M. Paton. The Burau representation is not faithful for $n \geq 6$. *Topology*, 32(2):439–447, 1993.

[117] Wolfgang Lück. *L^2-invariants: theory and applications to geometry and K-theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.

[118] Wolfgang Lück. Twisting L^2-invariants with finite-dimensional representations. *arXiv preprint arXiv:1510.00057*, 2015.

[119] V.P. Maslov. Theory of perturbations and asymptotic mathods (in R ussian). *MGU*, 1965.

[120] Werner Meyer. Die Signatur von lokalen Koeffizientensystemen und Faserbündeln. *Bonn. Math. Schr.*, (53):viii+59, 1972.

[121] Werner Meyer. Die Signatur von Flächenbündeln. *Math. Ann.*, 201:239–264, 1973.

[122] J. Milnor. Whitehead torsion. *Bull. Amer. Math. Soc.*, 72:358–426, 1966.

[123] John Milnor. A duality theorem for Reidemeister torsion. *Ann. of Math. (2)*, 76:137–147, 1962.

[124] John W. Milnor. Infinite cyclic coverings. In *Conference on the Topology of Manifolds (Michigan State Univ., E. Lansing, Mich., 1967)*, pages 115–133. Prindle, Weber & Schmidt, Boston, Mass., 1968.

[125] John Moody. The faithfulness question for the Burau representation. *Proc. Amer. Math. Soc.*, 119(2):671–679, 1993.

[126] H. R. Morton. The multivariable Alexander polynomial for a closed braid. In *Low-dimensional topology (Funchal, 1998)*, volume 233 of *Contemp. Math.*, pages 167–172. Amer. Math. Soc., Providence, RI, 1999.

[127] Jeffrey Colin Morton. Two-vector spaces and groupoids. *Appl. Categ. Structures*, 19(4):659–707, 2011.

[128] Kunio Murasugi. On a certain numerical invariant of link types. *Trans. Amer. Math. Soc.*, 117:387–422, 1965.

[129] Matthias Nagel and Mark Powell. Concordance invariance of Levine-Tristram signatures of links. *Doc. Math.*, 22:25–43, 2017.

[130] Walter D. Neumann. Signature related invariants of manifolds. I. Monodromy and γ-invariants. *Topology*, 18(2):147–172, 1979.

[131] Walter D. Neumann. A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves. *Trans. Amer. Math. Soc.*, 268(2):299–344, 1981.

[132] Liviu I. Nicolaescu. *The Reidemeister torsion of 3-manifolds*, volume 30 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2003.

[133] Mark Powell. Twisted Blanchfield pairings and symmetric chain complexes. 2016. arXiv:1605.06800.

[134] Mark Powell. The four-genus of a link, Levine-Tristram signatures and satellites. *J. Knot Theory Ramifications*, 26(2):1740008, 28, 2017.

[135] Pierre Py. Indice de Maslov et théorème de Novikov-Wall. *Bol. Soc. Mat. Mexicana (3)*, 11(2):303–331, 2005.

[136] Andrew Ranicki. *Algebraic and geometric surgery.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2002. Oxford Science Publications.

[137] Andrew Ranicki and Desmond Sheiham. Blanchfield and Seifert algebra in high-dimensional boundary link theory. I. Algebraic K-theory. *Geom. Topol.*, 10:1761–1853, 2006.

[138] Franciscus Rebro. Building the bicategory $\mathrm{Span}_2(\mathscr{C})$. January 2015. http://arxiv.org/pdf/1501.00792v1.pdf.

[139] Kurt Reidemeister. Überdeckungen von Komplexen. *J. Reine Angew. Math.*, 173:164–173, 1935.

[140] Dale Rolfsen. *Knots and links. 2nd print. with corr.* Houston, TX: Publish or Perish, 2nd print. with corr. edition, 1990.

[141] H. Seifert. Über das Geschlecht von Knoten. *Math. Ann.*, 110(1):571–592, 1935.

[142] Jean-Pierre Serre. *Trees.* Springer-Verlag, Berlin-New York, 1980. Translated from the French by John Stillwell.

[143] Yaichi Shinohara. On the signature of knots and links. *Trans. Amer. Math. Soc.*, 156:273–285, 1971.

[144] Craig C. Squier. The Burau representation is unitary. *Proc. Amer. Math. Soc.*, 90(2):199–202, 1984.

[145] Richard G. Swan. Projective modules over Laurent polynomial rings. *Trans. Amer. Math. Soc.*, 237:111–120, 1978.

[146] Guillermo Torres. On the Alexander polynomial. *Ann. of Math. (2)*, 57:57–89, 1953.

[147] A. G. Tristram. Some cobordism invariants for links. *Proc. Cambridge Philos. Soc.*, 66:251–264, 1969.

[148] H. F. Trotter. Homology of group systems with applications to knot theory. *Ann. of Math. (2)*, 76:464–498, 1962.

[149] V. G. Turaev. Reidemeister torsion in knot theory. *Uspekhi Mat. Nauk*, 41(1(247)):97–147, 240, 1986.

[150] Vladimir Turaev. *Introduction to combinatorial torsions*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2001. Notes taken by Felix Schlenk.

[151] Vladimir Turaev. Faithful linear representations of the braid groups. *Astérisque*, (276):389–409, 2002. Séminaire Bourbaki, Vol. 1999/2000.

[152] Vladimir G. Turaev. *Quantum invariants of knots and 3-manifolds*, volume 18 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, revised edition, 2010.

[153] O. Ja. Viro. Branched coverings of manifolds with boundary, and invariants of links. I. *Izv. Akad. Nauk SSSR Ser. Mat.*, 37:1241–1258, 1973.

[154] Oleg Viro. Twisted acyclicity of a circle and signatures of a link. *J. Knot Theory Ramifications*, 18(6):729–755, 2009.

[155] Masaaki Wada. Twisted Alexander polynomial for finitely presentable groups. *Topology*, 33(2):241–256, 1994.

[156] C. T. C. Wall. Non-additivity of the signature. *Invent. Math.*, 7:269–274, 1969.

[157] C. T. C. Wall. *Surgery on compact manifolds*, volume 69 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 1999. Edited and with a foreword by A. A. Ranicki.

[158] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.