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Mapping class groups and outer automorphism groups of free groups are C^* -simple

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Abstract

We prove that the reduced C^* -algebras of centerless mapping class groups and outer automorphism groups of free groups are simple, as are the irreducible pure subgroups of mapping class groups and the analogous subgroups of outer automorphism groups of free groups.

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0. Introduction

A group Γ is C^* -simple if its *reduced C^* -algebra* $C_\lambda^*(\Gamma)$ is simple as a complex algebra (i.e. has no proper two-sided ideals). The main purpose of this note is to explain why mapping class groups of surfaces of finite type and outer automorphism groups of free groups are C^* -simple.

There is a fascinating analogy between lattices in semi-simple Lie groups, on the one hand, and mapping class groups and outer automorphism groups of free groups, on the other. The results recorded here resonate well with this analogy, as we shall explain in a moment. First, though, we remind the reader that the C^* -simplicity of a group Γ may be regarded as a property of the unitary representation theory of the

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group. By definition, $C_\lambda^*(\Gamma)$ is the norm closure of the image of the complex group algebra $\mathbb{C}[\Gamma]$ under the left-regular representation $\lambda_\Gamma : \mathbb{C}[\Gamma] \rightarrow \mathcal{L}(\ell^2(\Gamma))$ defined for $\gamma \in \Gamma$ by $(\lambda_\Gamma(\gamma)\xi)(x) = \xi(\gamma^{-1}x)$ for all $x \in \Gamma$ and $\xi \in \ell^2(\Gamma)$. A group Γ is C^* -simple if and only if any unitary representation π of Γ which is weakly contained in λ_Γ is weakly equivalent to λ_Γ (see Theorem 3.4.4 and Proposition 18.1.4 in [DC*-69]). Examples of C^* -simple groups include non-abelian free groups [Pow-75], non-trivial free products [PaS-79], torsion-free, non-elementary hyperbolic groups (see [Har-85, Har-88]), and Zariski-dense subgroups in centerless, connected, semisimple real Lie groups with no compact factors [BCH2-94]. (In particular $\mathrm{PSL}(n, \mathbb{Z})$ is C^* -simple [BCH1-94].)

Apart from the usual low-genus exceptions, mapping class groups and outer automorphism groups of free groups do not lie in any of the above classes of groups. But whenever one has an interesting property of groups with such a list of examples, the analogy between lattices and these important groups demands attention. Previous experience encourages us with many examples where the analogy goes through, and even leads us to expect that the shape of proof used in the classical setting might thrive in the transplanted environment. This is the case, for example, with theorems concerning aspects of rigidity [BrV-00, BrV-01, FaM-98, Iva-97] and homological stability [Ha-85, Hat-95]. It is also the case for analogues of the Tits Alternative [BFH-00, BFH2, BFH3, BLM-83, Iva-84] [McP-85]. See also [Bes-02, Iva-92, Vog-02].

As well as providing us with encouragement, these works also warn us that the tools required to adapt proofs from the classical setting to that of mapping class groups and outer automorphism groups of free groups are often non-trivial and may require significant innovation. Fortunately, though, in the present setting, the necessary tools can be readily gleaned from work of previous authors, as we shall now explain.

Although mapping class groups and outer automorphism groups of free groups are more often thought of in analogy with higher rank lattices (with $\mathrm{SL}(n, \mathbb{Z})$ springing most readily to mind), it is widely recognized that there are a number of important respects in which they exhibit rank-one phenomena (see [FLM-01]). The argument that we shall present here falls into the latter category. Indeed this is what makes our project straightforward: the argument used in [BCH-94] (following [HaJ-81]) to establish the C^* -simplicity of rank one lattices (also hyperbolic groups and non-trivial free products) rests on an elementary lemma, the input of which is dynamical information reminiscent of the classical ping-pong lemma, and the output of which allows one to follow Powers' proof of the C^* -simplicity of non-abelian free groups. In the case of rank one lattices (more generally Zariski-dense subgroups) the space on which one studies the dynamics is the boundary of the symmetric space. At the heart of the analogy between lattices and mapping class groups lies the fact that the Teichmüller space plays the rôle of the symmetric space; for outer automorphism groups of free groups, the rôle of the symmetric space is played by Culler and Vogtmann's Outer Space [CuV-86].

The mention of ping-pong in this setting brings to mind the construction of free groups in the proof of the Tits alternative [Har-83, Tits-72]. The Tits alternative has

been established for both mapping class groups and outer automorphism groups of free groups. In the case of outer automorphism groups of free groups, this is recent work that represents the culmination of a long-term project by Bestvina, Feighn and Handel developing “train-track technology” [BeH-92] to produce refined topological representatives of free group automorphisms. This project was motivated by Thurston’s work on train-track representatives of surface automorphisms, which is closely related to his work on the boundary of Teichmüller space. It is the action of the mapping class group on this boundary that provides us with the ping-pong behaviour required to establish C^* -simplicity for the mapping class group. In the case of the (outer) automorphism groups of free groups, we appeal to the work of Bestvina–Feighn–Handel [BFH-97] and a refinement of Levitt and Lustig [LeL-03] concerning the dynamics of automorphisms on the boundary of Outer Space.

Our account of the action of $\text{Out}(F_n)$ on the boundary of Outer Space owes a great deal to the insights of Karen Vogtmann. We are most grateful to her for sharing these and other insights during conversations in Geneva in the summer of 2002.

1. Powers’ criterion for C^* -simplicity

We define a group Γ to be a *Powers group* if

for any finite subset F in $\Gamma \setminus \{e\}$ and for any integer $N \geq 1$, there exists a partition $\Gamma = C \sqcup D$ and elements $\gamma_1, \dots, \gamma_N$ in Γ such that $fC \cap C = \emptyset$ for all $f \in F$, and $\gamma_j D \cap \gamma_k D = \emptyset$ for all $j \neq k$ in $\{1, \dots, N\}$.

The terminology is in honour of [Pow-75]. The first thing to be said about this definition is that it implies C^* -simplicity; the argument is essentially that of the original paper [Pow-75] and is repeated here as an appendix for the reader’s convenience. The other important thing to note is that there is a simple dynamical criterion that enables one to show that many interesting groups are Powers groups. In order to describe this criterion we need the following vocabulary.

A homeomorphism γ of a Hausdorff space Ω is said to be *hyperbolic* if it has two fixed points $s_\gamma, r_\gamma \in \Omega$ and exhibits *north–south dynamics*: for any pair of neighbourhoods S of s_γ and R of r_γ , there exists $n_0 \in \mathbb{N}$ such that $\gamma^n(\Omega \setminus S) \subset R$ and $\gamma^{-n}(\Omega \setminus R) \subset S$ for all $n \geq n_0$. The points s_γ and r_γ are called the *source* and the *range* of γ , respectively.

Two hyperbolic homeomorphisms of Ω are *transverse* if they have no common fixed point.

Proposition 1.1. *Let Γ be a group acting by homeomorphisms on a Hausdorff space Ω . Assume that the following two conditions hold.*

- (i) Γ contains two transverse hyperbolic homeomorphisms of Ω .
- (ii) For any finite subset F of $\Gamma \setminus \{e\}$, there exists a point $t \in \Omega$ fixed by some hyperbolic homeomorphism of Γ such that $ft \neq t$ for all $f \in F$.

Then Γ is a Powers group, and in particular $C_\lambda^*(\Gamma)$ is a simple C^* -algebra.

Proof. Consider $F \subset \Gamma \setminus \{e\}$ and $N \geq 1$ as in the definition of a Powers group. By hypothesis, there exist hyperbolic homeomorphisms $\gamma, \gamma', \gamma'' \in \Gamma$ and a neighbourhood C_Ω of the range r of γ such that γ', γ'' are transverse and such that $fC_\Omega \cap C_\Omega = \emptyset$ for all $f \in F$. Let $\gamma_1, \dots, \gamma_N$ be pairwise transverse conjugates of γ' by appropriate powers of γ'' . Upon conjugating $\gamma_1, \dots, \gamma_N$ by a large power of γ , we may assume that, for each $j \in \{1, \dots, N\}$, both the source s_j and the range r_j of γ_j are in C_Ω . We choose neighbourhoods S_j of s_j and R_j of r_j in such a way that $S_1, R_1, \dots, S_N, R_N$ are pairwise disjoint and inside C_Ω . Upon replacing now each of $\gamma_1, \dots, \gamma_N$ by a large enough power of itself, we may furthermore assume that $\gamma_j(\Omega \setminus C_\Omega) \subset R_j$, and in particular that the $\gamma_j(\Omega \setminus C_\Omega)$ are pairwise disjoint subsets of Ω .

Choose $\omega \in \Omega$; let $C = \{\gamma \in \Gamma \mid \gamma\omega \in C_\Omega\}$ and $D = \{\gamma \in \Gamma \mid \gamma\omega \notin C_\Omega\}$. Then $fC \cap C = \emptyset$, since $fC_\Omega \cap C_\Omega = \emptyset$, for all $f \in F$, and $\gamma_j D \cap \gamma_k D = \emptyset$, since $R_j \cap R_k = \emptyset$, for all $j \neq k$. \square

The following perturbation of Proposition 1.1 is well-adapted to the examples in which we are interested. We write $\text{Stab}(x)$ to denote the stabilizer in Γ of a point $x \in \Omega$.

Corollary 1.2. *Let Γ be a group acting by homeomorphisms on a Hausdorff space Ω . Assume that the following two conditions hold.*

- (i) Γ contains a hyperbolic homeomorphism γ_0 with source s_0 and range r_0 .
- (ii) There exists a non-trivial element $\gamma_1 \in \Gamma$ such that for each integer $i \neq 0$, the set

$$\gamma_1^{-i}(\text{Stab}(r_0) \cup \text{Stab}(s_0))\gamma_1^i \cap (\text{Stab}(r_0) \cup \text{Stab}(s_0))$$

is just $\{e\}$.

Then Γ is a Powers group, and in particular $C_\lambda^*(\Gamma)$ is a simple C^* -algebra.

Proof. The hyperbolic homeomorphisms γ_0 and $\gamma_1^{-1}\gamma_0\gamma_1$ are transverse. Indeed for any positive integer n , the hyperbolic homeomorphisms $\gamma_i := \gamma_1^{-i}\gamma_0\gamma_1^i$, $i = 1, \dots, n$ are pairwise transverse and no element of Γ fixes a source or range of more than one of them, by condition (ii). We write s_i and r_i to denote the source and range of γ_i .

Given a finite set F , of cardinality m say, we choose n so that $2n > m$. Since each element of Γ fixes at most two of the $2n$ points $s_1, r_1, \dots, s_n, r_n$, at least one point on this list is moved by every element of F . \square

2. Mapping class groups and outer automorphism groups of free groups satisfy the Powers' criterion

Let Mod_S denote the group of isotopy classes of orientation-preserving homeomorphisms of a compact orientable surface S (which may have non-empty boundary). For an introduction to the properties of such groups, see Ivanov's

excellent survey [Iva-02]. To avoid the well-known quirks associated with small examples¹ we assume that S is neither a sphere with ≤ 4 boundary circles, nor a torus with ≤ 2 boundary circles, nor a closed surface of genus 2. In all of the remaining cases, the centre of Mod_S is trivial and the group acts effectively on the Thurston boundary of the associated Teichmüller space. There is a natural identification of the Thurston boundary with the space of projective measured foliations² \mathcal{PMF}_S on S (the basic reference for this material is [FLP-79], while Ivanov's monograph [Iva-92] and survey [Iva-02] provide an excellent overview). \mathcal{PMF}_S is a topological sphere; in particular it is Hausdorff. We shall apply the considerations of Section 2 to the action of Mod_S on this space.

A *pseudo-Anosov* in Mod_S is an element that acts as a hyperbolic homeomorphism of \mathcal{PMF}_S . It is well-known that Mod_S contains transverse pairs of pseudo-Anosov classes—see, for example, Lemma 2.5 in [McP-89] or Corollary 7.15 in [Iva-92].

The stabilizers of the fixed points of pseudo-Anosovs are understood (see Lemma 2.5 in [McP-89] and Lemma 5.10 in [Iva-92], for example):

Lemma 2.1. *If $\phi \in \text{Mod}_S$ is pseudo-Anosov, then the stabilizer of each of its fixed points in \mathcal{PMF}_S is virtually cyclic.*

In outline, one proves this lemma as follows. The fixed points of a pseudo-Anosov ϕ are the projective classes of its stable and unstable laminations. If $\psi \in \text{Mod}_S$ fixes one of these points $[\mu]$, then it multiplies the measure on the underlying lamination by a constant factor, $\lambda(\psi)$ say. The map $\psi \mapsto \lambda(\psi)$ is a homomorphism from the stabilizer of $[\mu]$ in Mod_S to the multiplicative group of positive reals; the image of this homomorphism is discrete (hence cyclic), the image of ϕ is non-trivial, and the kernel is finite. It follows that if γ_1 is a pseudo-Anosov such that ϕ and γ_1 do not have common powers, then ϕ and γ_1 are transverse (and hence have powers that generate a non-abelian free group). Moreover, for typical (but not all) ϕ , the stabilizer of $[\mu]$ is actually cyclic, generated by γ_0 say. Now γ_0 and γ_1 satisfy the conditions of Corollary 1.2. Thus we have:

Theorem 2.2. *Mod_S is a Powers group; in particular its reduced C^* -algebra is simple.*

A subgroup $\Gamma \subseteq \text{Mod}_S$ is called *reducible* if there is a non-empty closed 1-dimensional submanifold $C \subset S$ such that for every $f \in \Gamma$ there is a homeomorphism F in the isotopy class f with $F(C) = C$. The *pure* elements of Mod_S (the definition of which is somewhat technical) contain a torsion-free subgroup of finite index in Mod_S . Every non-trivial irreducible subgroup $\Gamma \subseteq \text{Mod}_S$ consisting of pure elements contains a pseudo-Anosov (see Theorem 5.9 of [Iva-92]). Arguing as above, it follows easily from Lemma 2.1 that if Γ is not cyclic, then it contains a pair of

¹In each of these exceptional cases Mod_S has a non-trivial centre, and hence the reduced C^* algebra is not simple. However these cases can be dealt with individually. For example, if S is a closed surface of genus 1 or 2, the quotient of the group Mod_S by its centre of order 2 is a C^* -simple group.

²Alternatively, laminations.

transverse pseudo-Anosovs $\{\gamma_0, \gamma_1\}$ satisfying the conditions of Corollary 1.2. Moreover, in this context we do not need to exclude the low-genus exceptions mentioned at the beginning of this section.

Theorem 2.3. *For every compact surface Σ , every non-cyclic, pure, irreducible subgroup of Mod_Σ is a Powers group.*

We now turn our attention to $\text{Out}(F_n)$, the group of outer automorphisms of a free group of rank n . By definition, $\text{Out}(F_n) = \text{Aut}(F_n)/\text{Inn}(F_n)$, where $\text{Aut}(F_n)$ is the group of automorphisms of F_n and $\text{Inn}(F_n)$ is the group of inner automorphisms (conjugations). In the case $n = 2$, the natural map $\text{Out}(F_2) \rightarrow \text{GL}(2, \mathbb{Z})$ is an isomorphism, hence the centre of $\text{Out}(F_2)$ has order two, and the quotient by this centre is C^* -simple (see [BekH-00]).

Henceforth we assume that $n \geq 3$. In this case it is easy to check that the centre of $\text{Out}(F_n)$ is trivial. Let \mathcal{C} denote the set of conjugacy classes of F_n . We consider the vector space $\mathbb{R}^{\mathcal{C}}$ of real-valued functions on \mathcal{C} , equipped with the product topology, and the corresponding projective space $\mathbb{P}\mathbb{R}^{\mathcal{C}}$ with the quotient topology. (Note that these spaces are Hausdorff.) The natural action of $\text{Out}(F_n)$ on \mathcal{C} induces an action on $\mathbb{P}\mathbb{R}^{\mathcal{C}}$.

Culler and Vogtmann's Outer Space X_n may be described as the space of equivalence classes of free actions of F_n by isometries on \mathbb{R} -trees. Given such an action on a tree T , we associate to each $w \in F_n$ the positive number $\|w\| = \inf\{d(wx, x) \mid x \in T\}$. The number $\|w\|$ depends only on the equivalence class of w , and the function $w \mapsto \|w\|$ completely determines the equivalence class of the action. Thus we obtain a natural equivariant injection $j: X_n \hookrightarrow \mathbb{P}\mathbb{R}^{\mathcal{C}}$. The set $\overline{j(X_n)} \setminus j(X_n)$ is called *the boundary of outer space* and is denoted $\partial_\infty(X_n)$. This space is compact. For a survey of these and related matters, see [Vog-02].

An element $\gamma \in \text{Out}(F_n)$ is an *iwip* (irreducible with irreducible powers) if no proper free factor of F_n is mapped to a conjugate of itself by a non-zero power of any representative $\tilde{\gamma} \in \text{Aut}(F_n)$ of γ . Bestvina, Feighn and Handel show that iwip outer automorphisms of free groups behave in close analogy with pseudo-Anosov automorphisms of surfaces. In particular, in [BFH-97] they define a set \mathcal{SL} of "stable laminations" on which $\text{Out}(F_n)$ acts. Each iwip $\phi \in \text{Out}(F_n)$ has two fixed points $A_\phi^+, A_\phi^- \in \mathcal{SL}$. Theorem 2.14 of [BFH-97], whose proof is closely analogous to that of Lemma 2.1 sketched above, amounts to the following statement:

Lemma 2.4. *If $\phi \in \text{Out}(F_n)$ is an iwip then the stabilizers of the fixed points of ϕ in \mathcal{SL} are virtually cyclic.*

And as in the case of the mapping class group, one can choose ϕ so that these stabilizers are actually cyclic. This information about stabilizers of the fixed points of ϕ can be transferred to the action of $\text{Out}(F_n)$ on $\partial_\infty(X_n)$ by virtue of Corollary 3.6 of [BFH-97]:

Lemma 2.5. *There is an $\text{Out}(F_n)$ -equivariant injection $\mathcal{I}\mathcal{L} \hookrightarrow \partial_\infty(X_n)$.*

As in the proof of Theorem 2.2 we will be in a position to apply Corollary 1.2 once we know that the action of an iwip $\phi \in \text{Out}(F_n)$ on $\partial_\infty(X_n)$ has north–south dynamics. And this was proved by Levitt and Lustig [LeL-03]. Thus we have:

Theorem 2.6. *If $n \geq 3$, then $\text{Out}(F_n)$ is a Powers group; in particular its reduced C^* -algebra is simple.*

If $\Gamma \subset \text{Out}(F_n)$ is torsion-free, the stabilizers in Γ of the fixed points of iwips are cyclic, so the arguments above establish:

Theorem 2.7. *If $\Gamma \subseteq \text{Out}(F_n)$ is torsion-free, non-cyclic and contains an iwip, then it is a Powers group; in particular its reduced C^* -algebra is simple.*

Let Γ be a group, A a unital C^* -algebra, and α an action of Γ on A such that A contains no non-trivial $\alpha(\Gamma)$ -invariant two-sided ideals. In general, the reduced crossed product $A \rtimes_{\alpha,r} \Gamma$ need not be simple. However, it will always be simple if Γ is a Powers group [HaS-86]. Thus we have the following corollary.

Corollary 2.8. *Let Γ be a group as in Theorems 2.2, 2.3, 2.6 or 2.7. If Γ acts on a unital C^* -algebra A and leaves no non-trivial 2-sided ideals invariant, then the corresponding reduced crossed product is a simple C^* -algebra.*

This corollary applies, for example, to the reduced crossed product $\mathcal{C}(L_\Gamma) \rtimes_r \Gamma$ associated to the action of Γ on the algebra of continuous functions on the limit set of Γ in \mathcal{PMF}_S or $\partial_\infty(X_n)$.

Remarks on minimality and compactness: In previous papers, Condition (ii) of Proposition 1.1 is replaced by other conditions involving minimality for the action of Γ on Ω and a compactness assumption on Ω . As we have seen, compactness is not necessary. Minimality can always be obtained, if desired, by replacing Ω with the closure of the set of fixed points of hyperbolic elements. We remark that the action of Mod_S on \mathcal{PMF}_S is minimal for all $g \geq 1$ (see e.g. Section VII of exposé 6 in [FLP-79]), but the action of $\text{Out}(F_n)$ on $\partial_\infty(X_n)$ is not: the ideal points of the simplicial spine $K_n \subset X_n$ defined in [CuV-86] form a closed invariant subset, for example.

The limitations of the method: It would be interesting to understand more precisely the limitations of the elementary method used here to establish the C^* -simplicity of groups. We used Corollary 1.2 in a rather weak form: in our examples the stabilizers of the endpoints of our hyperbolic homeomorphisms were virtually cyclic, whereas Corollary 1.2 would allow large malnormal³ stabilizers, for example. Since the centralizer of a hyperbolic element must stabilize its fixed points, a natural challenge arises in the case of $F_n \times F_n$, where all centralizers contain a copy of \mathbb{Z}^2 but are not

³A subgroup H of a group G is *malnormal* if $gHg^{-1} \cap H = \{e\}$ for any $g \in G$ such that $g \notin H$.

malnormal. Does this group admit a faithful action satisfying the conditions of Proposition 1.1? (Note that the direct product of two non-abelian free groups is C^* -simple since a spatial tensor product of simple C^* -algebras is a simple C^* -algebra [Tak-64].)

Other challenging examples are the groups $PSL(n, \mathbb{Z})$ for $n \geq 3$: we know that these groups are C^* -simple [BCH-94], but we do not know whether they are Powers groups.

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Appendix. On the sufficiency of Powers’ criterion

For the convenience of the reader, we include a proof of the result of Powers quoted at the beginning of Section 1. Powers’ original proof was formulated only for non-abelian free groups [Pow-75], but the following adaptation is entirely straightforward.

Theorem A.1. *Powers groups are C^* -simple.*

Proof (following Powers [Pow-75]). Consider a Powers group Γ , its left-regular representation λ_Γ , a non-zero ideal \mathcal{I} of its reduced C^* -algebra, and an element $U \neq 0$ in \mathcal{I} . We want to show that \mathcal{I} contains an element Z such that $\|Z - 1\| < 1$, and in particular such that Z is invertible (with inverse $\sum_{n=0}^\infty (1 - Z)^n$).

Upon replacing U by a scalar multiple of U^*U , we may assume that $U = 1 + X$ and $X = \sum_{x \in \Gamma, x \neq e} z_x \lambda_\Gamma(x)$, with $z_x \in \mathbb{C}$. Choose ε, δ with $0 < \varepsilon < \delta \leq 1$ (here, we could set $\delta = 1$, but we will use the freeness in choosing δ in the next proof). Then there exists a finite subset F of $\Gamma \setminus \{e\}$ such that, if

$$X' = \sum_{f \in F} z_f \lambda_\Gamma(f),$$

then $\|X' - X\| < \varepsilon$. We choose F symmetric, so that X' is selfadjoint. Set $U' = 1 + X'$, so that $\|U' - U\| < \varepsilon$. Choose an integer N so large that $\frac{2}{\sqrt{N}} \|X'\| < \delta - \varepsilon$.

Let now $\Gamma = C \sqcup D$ and $\gamma_1, \dots, \gamma_N$ be as in the definition of a Powers group. Set

$$V = \frac{1}{N} \sum_{j=1}^N \lambda_\Gamma(\gamma_j) U \lambda_\Gamma(\gamma_j^{-1}), \quad V' = \frac{1}{N} \sum_{j=1}^N \lambda_\Gamma(\gamma_j) U' \lambda_\Gamma(\gamma_j^{-1}),$$

$$Y = \frac{1}{N} \sum_{j=1}^N \lambda_\Gamma(\gamma_j) X \lambda_\Gamma(\gamma_j^{-1}), \quad Y' = \frac{1}{N} \sum_{j=1}^N \lambda_\Gamma(\gamma_j) X' \lambda_\Gamma(\gamma_j^{-1}).$$

Note that $V = 1 + Y \in \mathcal{I}$ and $V' = 1 + Y'$. We show below that $\|Y'\| < \delta - \varepsilon$. This implies that $\|Y\| \leq \|Y'\| + \|Y - Y'\| \leq \|Y'\| + \|X - X'\| < \delta \leq 1$. As \mathcal{I} contains the invertible element $V = 1 + Y$, the C^* -algebra $C_\lambda^*(\Gamma)$ is indeed simple.

For $j \in \{1, \dots, N\}$, denote by P_j the orthogonal projection of $\ell^2(\Gamma)$ onto $\ell^2(\gamma_j D)$. We have

$$(1 - P_j)\lambda_\Gamma(\gamma_j)X'\lambda_\Gamma(\gamma_j^{-1})(1 - P_j) = 0;$$

indeed, since $fC \cap C = \emptyset$ for all $f \in F$, we have

$$\begin{aligned} (\lambda_\Gamma(\gamma_j)X'\lambda_\Gamma(\gamma_j^{-1})(1 - P_j))(\ell^2(\Gamma)) &\subset (\lambda_\Gamma(\gamma_j)X')(\ell^2(C)) \\ &\subset (\lambda_\Gamma(\gamma_j))(\ell^2(D)) = P_j(\ell^2(\Gamma)). \end{aligned}$$

It follows that

$$V' = 1 + \frac{1}{N} \sum_{j=1}^N P_j \lambda_\Gamma(\gamma_j) X' \lambda_\Gamma(\gamma_j^{-1}) + \left(\frac{1}{N} \sum_{j=1}^N P_j \lambda_\Gamma(\gamma_j) X' \lambda_\Gamma(\gamma_j^{-1})(1 - P_j) \right)^*.$$

Since the subsets $\gamma_j D$ of Γ are pairwise disjoint, the operators $X'_j \doteq P_j \lambda_\Gamma(\gamma_j) X' \lambda_\Gamma(\gamma_j^{-1})$ have pairwise orthogonal ranges in $\ell^2(\Gamma)$, and we have

$$\left\| \frac{1}{N} \sum_{j=1}^N X'_j \right\| \leq \frac{1}{\sqrt{N}} \max_{j=1}^n \|X'_j\| \leq \frac{1}{\sqrt{N}} \|X'\|.$$

Similarly

$$\left\| \left(\frac{1}{N} \sum_{j=1}^N X'_j (1 - P_j) \right)^* \right\| = \left\| \frac{1}{N} \sum_{j=1}^N X'_j (1 - P_j) \right\| \leq \frac{1}{\sqrt{N}} \|X'\|.$$

Consequently

$$\|Y'\| = \|V' - 1\| \leq \frac{2}{\sqrt{N}} \|X'\| < \delta - \varepsilon.$$

As already observed, this completes the proof. \square

In Section 6 of [BekL-00] the above proof is recast in the language of functions of positive type.

A linear form τ on $C_\lambda^*(\Gamma)$ is a *normalised trace* if $\tau(1) = 1$ and $\tau(U^*U) \geq 0$, $\tau(UV) = \tau(VU)$ for all $U, V \in C_\lambda^*(\Gamma)$. We have $\|\tau(X)\| \leq \|X\|$ for all $X \in C_\lambda^*(\Gamma)$ (see, e.g., Proposition 2.1.4 in [DC*-69]). The *canonical trace* is uniquely defined by

$$\tau_{\text{can}} \left(\sum_{f \in F} z_f \lambda_\Gamma(f) \right) = z_e$$

for every *finite* sum $\sum_{f \in F} z_f \lambda_\Gamma(f)$ where $z_f \in \mathbb{C}$ and $F \subset \Gamma$ contains e .

Proposition A.2. *If Γ is a Powers group, then the canonical trace is the only normalized trace on $C_\lambda^*(\Gamma)$.*

Proof. Let τ be a normalised trace on $C_\lambda^*(\Gamma)$ and let $X \in C_\lambda^*(\Gamma)$ be such that $\tau_{\text{can}}(X) = 0$. It is enough to show that $\tau(X) = 0$.

Choose $\delta > 0$. The previous proof shows that there exist $N \geq 1$ and $\gamma_1, \dots, \gamma_N \in \Gamma$ such that $\|Y\| \leq \delta$ for

$$Y = \frac{1}{N} \sum_{j=1}^N \lambda_\Gamma(\gamma_j) X \lambda_\Gamma(\gamma_j^{-1}).$$

As $\tau(Y) = \tau(X)$, we have $\|\tau(X)\| \leq \|Y\| < \delta$. As δ is arbitrary, this implies $\tau(X) = 0$. \square

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