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Uniform Growth in Groups of Exponential Growth

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Abstract. This is an exposition of examples and classes of finitely-generated groups which have uniform exponential growth. The main examples are non-Abelian free groups, semi-direct products of free Abelian groups with automorphisms having an eigenvalue of modulus distinct from 1, and Golod–Shafarevich infinite finitely-generated p-groups. The classes include groups which virtually have non-Abelian free quotients, nonelementary hyperbolic groups, appropriate free products with amalgamation, HNN-extensions and one-relator groups, as well as soluble groups of exponential growth. Several open problems are formulated.

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1. Introduction

Given a group Γ generated by a finite set S, we have the *word length* ℓ_S: $\Gamma \longrightarrow \mathbb{N}$, for which $\ell_S(\gamma)$ is the smallest integer n such that there exist $s_1, \ldots, s_n \in S \cup S^{-1}$ with $\gamma = s_1 \ldots s_n$, the *growth function* $\beta(\Gamma, S; -)$: $\mathbb{N} \longrightarrow \mathbb{N}$, for which $\beta(\Gamma, S; k)$ is the number of $\gamma \in \Gamma$ with $\ell_S(\gamma) \leqslant k$, and the *exponential growth rate* $\omega(\Gamma, S) = \lim_{k\to\infty} \sqrt[k]{\beta(\Gamma, S; k)}$ (the limit exists by submultiplicativity: $\beta(k_1 + k_2) \leq \beta(k_1)\beta(k_2)$). It is easy to check that $\omega(\Gamma, S) > 1$ if and only if $\omega(\Gamma, S') > 1$ for *any* finite set S' of generators of Γ. We define the *uniform exponential growth rate* of a finitely-generated group Γ by

$$\omega(\Gamma) = \inf\{\omega(\Gamma, S) \mid S \text{ is a finite set of generators of } \Gamma\}.$$

The group is of *exponential growth* if $\omega(\Gamma, S) > 1$ for some (hence, for any) finite generating set S, and of *uniform exponential growth* if $\omega(\Gamma) > 1$.

Exponential growth rates and uniform exponential growth rates are of interest in differential geometry and dynamical system theory, for example because of the following results.

Recall that the *volume entropy* of a compact Riemannian manifold M is defined by

$$h(M) = \lim_{R\to\infty} \frac{1}{R} \log \operatorname{vol} B(\tilde{m}, R),$$

where $\tilde{m}$ denotes some point in the universal covering of M and $\operatorname{vol} B(\tilde{m}, R)$ the Riemannian volume of the ball of radius R centred at $\tilde{m}$; the limit exists and is independent of the choice of $\tilde{m}$.

1.1. PROPOSITION (Švarc, Milnor, Dinaburg, Manning). (i) *With the previous notation and for M of diameter d, we have*

$$\ln(\omega(\pi_1(M))) \leqslant 2d\,h(M).$$

(ii) *If $h_{\mathrm{top}}(M)$ denotes the topological entropy of the geodesic flow on M, we have*

$$\ln(\omega(\pi_1(M))) \leqslant 2d\,h_{\mathrm{top}}(M).$$

On the proofs. Claim (i) is due independently to [Šva-55] and [Mi-68a]; see also Theorem 5.16 in [GLP-81]. Claim (ii), from [Din-71], follows, since $h(M) \leqslant h_{\mathrm{top}}(M)$ by [Man-79]; see also [Pat-99]. □

The exponential growth rate of a group may also give bounds for the entropy of various actions of the group; for these notions, see, e.g., [Gro-77], [GLW-88], and [Fri-96].

Though the subject has not been investigated properly, exponential growth rates are apparently also relevant in the theory of unitary representations. See, e.g., Corollary 3.2 in [Haa-79], which shows the following 'phase transition': for a free group F_n, $n \geqslant 2$, a free set S_n of generators, a real number $\lambda > 0$, and the function of positive type $\phi_\lambda\colon F_n \longrightarrow \mathbb{R}$ defined by $\phi_\lambda(\gamma) = \exp(-\lambda \ell_{S_n}(\gamma))$, the three following conditions are equivalent:

ϕ_λ is weakly associated with the regular representation of F_n,
$\gamma \mapsto \phi_\lambda(\gamma)(1 + \ell_{S_n}(\gamma))^2$ belongs to $\ell^2(F_n)$,
$\lambda \geqslant \frac{1}{2}\log(\omega(F_n))$

(with $\omega(F_n) = 2n - 1$, see below). This carries over to other groups, including Gromov-hyperbolic Coxeter groups, as observed in Section 2 of [JoVa-91].

A finitely generated group Γ which is nonamenable is necessarily of exponential growth [AdS-57]. It is an open question to know whether there exists for each $k \geqslant 2$ a constant $c_k < 2k - 1$ such that, if Γ has a generating subset S of k elements for which $\omega(\Gamma, S) \geqslant c_k$, then Γ is nonamenable. (If $\omega(\Gamma, S) = 2k - 1$, then it is known that Γ is free on S; hence Γ is nonamenable. See Example 2.1.)

M. Gromov has asked the following question (Remark 5.12 of [GLP-81]), which is still open [GLP-99].

1.2. MAIN OPEN PROBLEM (Gromov). *Do there exist finitely-generated groups of exponential growth which are not of uniform exponential growth?*

It is straightforward to check that exponential growth for finitely-generated groups is an invariant of quasi-isometry. If we assume a positive answer to Problem 1.2, it is natural to ask if uniform exponential growth is invariant by quasi-isometry. Here is another naturally related question.

1.3. OPEN PROBLEM. *What are the finitely-generated groups* Γ *which* realize their uniform exponential growth rate, *namely which are such that they contain a finite generating set S such that* $\omega(\Gamma, S) = \omega(\Gamma)$*?*

In particular, is it true that word-hyperbolic groups realize their uniform exponential growth rate?

An affirmative answer to the second question would imply that word-hyperbolic groups are Hopfian, since they are known to be *growth-tight* in the sense of [ArL]. Word hyperbolic groups which are *torsion-free* are known to be Hopfian [Sel-99].

Free groups realize their uniform exponential growth rate, as shown by Example 2.1 below.

On the other hand, A. Sambusetti has shown that $\omega(\Gamma_1 * \Gamma_2, S) > \omega(\Gamma_1 * \Gamma_2)$ whenever Γ_1 is a non-Hopfian group and Γ_2 a group not reduced to one element, for *any* finite generating set S of $\Gamma_1 * \Gamma_2$ [Sam-99]. It is an open problem to know whether the same holds for Baumslag–Solitar groups.

There is an analogous problem in differential geometry, which is to understand which Riemannian metrics (with appropriate normalisation), if any, achieve the minimum value of the volume entropy on a given compact manifold; see [BCG-95] and [BCG-96].

There are classes of groups for which exponential growth implies uniform exponential growth. They include:

free groups and related groups (Section 2),
free products with amalgamation, HNN-extensions, and one-relator groups (Section 3),
soluble groups (Section 4),
appropriate torsion groups (Section 5),
linear groups in characteristic zero (a result of Eskin–Mozes–Oh cited in Section 7).

However there are some indications that there exist groups of exponential growth which are not of uniform exponential growth, and we discuss these in Section 6. The final Section 7 contains an open problem involving Kazhdan's Property (T).

Work reported here was done partly together with co-authors, and has already partly appeared in [BuH-00], [GrH-97], [GrH], [GrH-01] and in Chapter VII of [Har-00].

2. Free Groups and Groups with Virtually Free Quotients

The following example appears as Number 5.13 in [GLP-81]:

2.1. BASIC EXAMPLE. *The free group* F_n *on* $n \geqslant 2$ *generators has uniform exponential growth, and* $\omega(F_n) = 2n - 1$.

Proof. If S_n is a free generating set of F_n, a straightforward count shows that $\beta(F_n, S_n; k) = 1 + \sum_{j=1}^{k} 2n(2n-1)^j$, and therefore that $\omega(F_n, S_n) = 2n - 1$.

Now let S be any finite generating set of F_n. The canonical image $\underline{S}$ of S in the Abelianized group $(F_n)^{\mathrm{ab}} = \mathbb{Z}^n$ generates $\mathbb{Z}^n$. Thus $\underline{S}$ contains a subset $\underline{R}$ of n elements generating a subgroup of finite index in $\mathbb{Z}^n$. Let R be a subset of S projecting bijectively onto $\underline{R}$. The subgroup $\langle R\rangle$ of F_n generated by R is free (as a subgroup of a free group), of rank at most n (because $|R| = n$) and of rank at least n (because the Abelianized group of $\langle R\rangle$ is of rank n). Hence, R is a free basis of $\langle R\rangle \approx F_n$, and it follows that $\omega(F_n, S) \geqslant \omega(\langle R\rangle, R) = \omega(F_n, S_n)$. □

Let Γ be a group with the following properties: it is non-Abelian and all its 2-generated subgroups are free. It is an obvious consequence of Example 2.1 that Γ has uniform exponential growth, indeed that $\omega(\Gamma) \geqslant 3$. It is a result of [ArO-96] that, in 'almost all' finitely-presented groups with m generators, subgroups generated by at most $m-1$ elements are free; thus, the following corollary holds.

2.2. COROLLARY (G. Arzhantseva and A. Ol'shanskii). *For $m \geqslant 3$, almost all finitely-presented groups Γ with m generators have uniform exponential growth, and satisfy $\omega(\Gamma) \geqslant 2m-3$.*

2.3. PROPOSITION. *Let Γ be a finitely generated group, let Γ' be a subgroup of finite index in Γ, and let Γ'' be a quotient of Γ'.*

(i) $\omega(\Gamma') \geqslant \omega(\Gamma'')$.
(ii) *If $\omega(\Gamma') > 1$, then $\omega(\Gamma) > 1$.*
(iii) *If Γ has a subgroup of finite index which has a non-Abelian free quotient, then Γ has uniform exponential growth.*

Proof. Claim (i) is straightforward. Claim (ii) follows from the elementary Proposition 3.3 of [ShW-92], reproduced below together with its proof. Claim (iii) follows from Example 2.1, and from Claims (i) and (ii). □

The existence of non-Abelian free subgroups in a group Γ *does not imply* any lower bound for $\omega(\Gamma)$, by Proposition 6.2 below.

2.4. PROPOSITION (Shalen-Wagreich). *Let Δ be a subgroup of finite index of a finitely-generated group Γ. Then $\omega(\Gamma)^{2[\Gamma:\Delta]-1} \geqslant \omega(\Delta)$.*

Proof. Consider first the free group F_n on a set S of n generators and a subgroup H of F_n of finite index, say d. We can identify F_n to the fundamental group of a bouquet of n circles, namely of a graph with one single vertex and n incident loops, and H to the fundamental group of a covering X of this bouquet; as the graph X has precisely d vertices, its diameter is at most $d-1$. If we choose a maximal tree of X, the free group $H = \pi_1(X)$ is generated by closed paths having all but one edge in the maximal tree and length at most $2(d-1)+1 = 2d-1$. It follows that H has a set of generators contained in the ball of radius $2d-1$ at the origin of F_n.

Let now Γ be a group generated by a finite set S and let Δ be a subgroup of index d in Γ. By the previous argument, we can choose a set R of generators of Δ contained in the ball B of radius $2d-1$ at the origin of Γ. Consequently, we have $\omega(\Delta, R) \leqslant \omega(\Gamma, B)$. On the other hand, it is straightforward to check that $\beta(\Gamma; B, k) = \beta(\Gamma; S; k(2d-1))$ for all $k \geqslant 0$, so that* $\omega(\Gamma, B) = \omega(\Gamma, S)^{2d-1}$. Hence $\omega(\Delta) \leqslant \omega(\Gamma, S)^{2d-1}$, and the conclusion follows. □

There are many examples of groups which can be shown to have uniform exponential growth by applying Proposition 2.3. For instance, it applies to the braid groups B_n for $n \geqslant 3$, since the corresponding pure braid groups have non-abelian free quotients. It applies also to the Coxeter groups which are not virtually Abelian, by a result due independently to Gonciulea [Gon-98] and Margulis and Vinberg [MaV-00].

Proposition 2.3 provides lower bounds for $\omega(\Gamma)$. For example, if Γ_g denotes the fundamental group of an orientable closed surface of genus $g \geqslant 2$, Proposition 2.3 shows that $\omega(\Gamma_g) \geqslant 2g-1$. It is easy to improve this to $\omega(\Gamma_g) \geqslant 4g-3$ (see Proposition VII.15 of [Har-00]); however:

2.5. OPEN PROBLEM. *What is the exact value of* $\omega(\Gamma_g)$?

We wish to quote the following result of M. Koubi [Kou-98].

2.6. THEOREM (Koubi). *A nonelementary hyperbolic group has uniform exponential growth.*

For a hyperbolic group Γ, the proof consists in showing that there exists an integer $N \geqslant 1$ with the following property: for any finite set S of generators of Γ, the S-ball of radius N about the origin in Γ contains two elements which generate a non-Abelian free group.

3. Free Products with Amalgamation, HNN-Extensions, and One-Relator Groups

It is advantageous to reformulate arguments for free groups in terms of groups acting on trees. One may obtain in this way the following 'outgrowth' of Example 2.1. Recall first that a free product with amalgamation $A *_C B$, where C is a subgroup of both A and B, is *nondihedral* if the two inclusions $C < A$, $C < B$ are strict and if, moreover, the index of C is not 2 in both A and B. A HNN-extension $G *_H^\theta$, where θ is an isomorphism from some subgroup H of G onto a subgroup K of G, is *nonsemi-direct* if at least one of the inclusions $H < G$, $K < G$ is strict.

*This also shows the following fact. For any finitely generated group Γ of exponential growth and for any constant $C > 1$, there exists a finite generating set S of Γ such that $\omega(\Gamma, S) \geqslant C$.

3.1. PROPOSITION. (i) *Let $A *_C B$ be a nondihedral free product of two finitely-generated groups A and B with amalgamation along a common subgroup C. Then*

$$\omega(A *_C B) \geqslant \sqrt[4]{2}.$$

In particular, $A *_C B$ is of uniform exponential growth.

(ii) *Let $G*_H^\theta$ be a nonsemi-direct HNN-extension given by a finitely-generated group G and an isomorphism θ from a subgroup H to a subgroup $\theta(H)$ of G. Then*

$$\omega\big(G*_H^\theta\big) \geqslant \sqrt[4]{2}.$$

In particular, $G*_H^\theta$ is of uniform exponential growth.

For example, since Thompson's group F (the one which is acting on an interval– see e.g. [CFP-96]) is a nonsemi-direct HNN-extension of itself, it follows from (ii) that F has uniform exponential growth.

3.2. PROPOSITION. *Let Γ be a one-relator group of exponential growth. Then Γ is of uniform exponential growth, and more precisely $\omega(\Gamma) \geqslant \sqrt[4]{2}$.*

A one-relator group which is not of exponential growth is either cyclic, or isomorphic to the fundamental group of a 2-torus, or isomorphic to the fundamental group of the Klein bottle, and is therefore of polynomial growth. (This follows from results of Karrass and Solitar, for which we refer to [CeG-97].)

For the proofs of the two previous propositions, we refer to [BuH-00] and [GrH-01]. It is known that the constant in Proposition 3.1(i) can be replaced by $\sqrt{2}$ for the case of free products.

We can formulate the following corollary. *Let Γ be a finitely-generated group. If Γ has a generating set S such that $\omega(\Gamma, S) < \sqrt[4]{2}$, then Γ is neither a nondihedral free product with amalgamation, nor a nonsemi-direct HNN-extension, nor a one-relator group of exponential growth.*

3.3. PROBLEM. *Do Propositions* 3.1 *and* 3.2 *hold with $\sqrt{2}$ instead of $\sqrt[4]{2}$?*

The following conjecture appears with Number 5.14 in [GLP-81]; as far as I know, it is still open.

3.4. CONJECTURE (Gromov). *Let Γ be a group which has a presentation involving n generators and m relations, with $m < n$. Then $\omega(\Gamma) \geqslant 2(n - m) - 1$.*

For $m \leqslant n - 2$, it is known that Γ has a subgroup of finite index which has a non-Abelian free quotient (see [BaP-78], or pp. 82–83 of [Gro-82]), so that Γ has uniform exponential growth by Example 2.1 and Proposition 2.3. For groups Γ having n generators and $n - 1$ relations with at least one which is a proper power, it is also known that Γ has a subgroup of finite index which has a non-Abelian free quotient. (This has been conjectured in [BaP-79]; see [Gro-82] and [Stö-83] for proofs; see also [Edj-84] for some groups with n generators and n relations.)

Observe that, if Conjecture 3.4 holds, then the right-hand term in the inequality is sharp, since $\omega(F_k) = 2k - 1$.

4. Soluble Groups

The following example appears essentially as Lemma 6.2 in [Tit-81]; see also [BuH-00]. We say that a matrix $\theta \in GL(d, \mathbb{Z})$ is *irreducible* if its linear action on $\mathbb{Q}^d$ is irreducible.

4.1. BASIC EXAMPLE. Let d be an integer, $d \geqslant 2$, and $\theta \in GL(d, \mathbb{Z})$ be an irreducible matrix which has an eigenvalue λ such that $|\lambda| > 1$. Then the corresponding semi-direct product $\Gamma = \mathbb{Z}^d \rtimes_\theta \mathbb{Z}$ has uniform exponential growth.

Remark. In particular, $\mathbb{Z}^d \rtimes_\theta \mathbb{Z}$ has exponential growth, a fact which goes back to the first paper on group growth [Šva-55].

Proof. Let S be a finite generating set of Γ. If $\pi\colon \Gamma \longrightarrow \mathbb{Z}$ denotes the canonical projection, there exists $s \in S$ such that $\pi(s) = m \neq 0$; upon changing s to s^{-1}, we may assume that $m \geqslant 1$. As Γ is not virtually Abelian, there exists $v \in S$ such that $sv \neq vs$. Set $u = svs^{-1}v^{-1}$; as $u \in \mathbb{Z}^d$, we may identify sus^{-1} with $\theta^m(u)$. Set $S_0 = \{s, u\}$ and let Γ_0 be the subgroup of Γ generated by S_0; as S_0 is inside the ball of radius 4 about e in Γ relative to the word length ℓ_S, we have $\omega(\Gamma, S)^4 \geqslant \omega(\Gamma_0, S_0)$, and it is enough to show that $\omega(\Gamma_0, S_0) \geqslant c$ for some constant $c > 1$ independent on the choices of S and S_0.

Let L be a nonzero linear form on $\mathbb{C}^d$ such that $L \circ \theta = \lambda L$. If L is of the form $(z_1, \ldots, z_n) \mapsto \sum_{1 \leqslant j \leqslant n} L_j z_j$, we have $L_j \neq 0$ for all j and, moreover, the quotients L_j/L_1 $(1 \leqslant j \leqslant n)$ are linearly independent over $\mathbb{Q}$, by the irreducibility assumption on θ. In particular $L(u) \neq 0$. Let $n \geqslant 1$ be an integer such that $|\lambda|^n \geqslant 2$. It is crucial for our argument that this n depends only on θ, but not on S and m. Observe that $|\lambda|^{mn} \geqslant 2$.

For $k \geqslant 1$ and $\epsilon_1, \ldots, \epsilon_k \in \{0, 1\}$, we have

$$\begin{aligned}
& s^n u^{\epsilon_1} s^n u^{\epsilon_2} \cdots s^n u^{\epsilon_k} s^{-kn} \\
&\quad = s^n u^{\epsilon_1} s^{-n} s^{2n} u^{\epsilon_2} s^{-2n} \cdots s^{kn} u^{\epsilon_k} s^{-kn} \\
&\quad = (\epsilon_1 s^n u s^{-n} + \epsilon_2 s^{2n} u s^{-2n} + \cdots + \epsilon_k s^{kn} u s^{-kn}, 0) \\
&\quad = (\epsilon_1 \theta^{mn}(u) + \epsilon_2 \theta^{2mn}(u) + \cdots + \epsilon_k \theta^{kmn}(u), 0) \in \Gamma.
\end{aligned}$$

The ℓ_{S_0}-word length of each of these elements is at most $k(2n+1)$. Moreover, they are all distinct because

$$L\big(\epsilon_1 \theta^{mn}(u) + \epsilon_2 \theta^{2mn}(u) + \cdots + \epsilon_k \theta^{kmn}(u)\big) = \left(\sum_{j=1}^{k} \epsilon_j \lambda^{jmn}\right) L(u)$$

and the numbers $\sum_{j=1}^{k} \epsilon_j \lambda^{jmn}$ are pairwise distinct (since $|\lambda|^{mn} \geqslant 2$). Thus the growth function of (Γ_0, S_0) satisfies $\beta_{S_0}(k(2n+1)) \geqslant 2^k$, and we have finally $\omega(\Gamma_0, S_0) \geqslant 2^{\frac{1}{2n+1}}$. □

Remark. In the situation of Example 4.1, let S_{can} denote the generating set of Γ which consists of d basic vectors in $\mathbb{Z}^d$ together with one generator of $\mathbb{Z}$, and let ρ denote the maximal modulus of the eigenvalues of θ. It is straightforward to check that

$$\beta(\Gamma, S_{\text{can}}; k) \leqslant \text{const } k\left(\rho^k\right)^d k$$

and, consequently, that $\omega(\Gamma, S_{\text{can}}) \leqslant \rho^d$.

Now let Ω_s denote the set of all numbers of the form $\rho = \rho(\theta)$, which is also the set of absolute values $\rho = |\sigma|$ of all algebraic integers σ in the half-line $[0, \infty[$ which are roots of a bi-unitary polynomial $t^d + a_{d-1}t^{d-1} + \cdots + a_1 t \pm 1$ with coefficients in $\mathbb{Z}$ and which are such that $\rho \geqslant |\tau|$ for any conjugate τ of σ; and let Ω_s^* denote the set of all numbers of the form $\rho^d = |\sigma|^d$ for $\rho = |\sigma| \in \Omega_s$ and σ of degree d. Then Ω_s is dense* in $[1, \infty[$. It would be interesting to know what is the closure of Ω_s^* in $[1, \infty[$.

Example 4.1 suggests the following general result, which was first obtained by D. Osin [Osi-a]; shortly later, it was shown independently by J. Wilson (during private conversations with the author). The case of polycyclic groups was settled independently by R. Alperin [Alp].

4.2. PROPOSITION (Osin). *A finitely-generated group which is soluble and of exponential growth is of uniform exponential growth.*

Remark. It is known from [Mi-68b] and [Wol-68] that a finitely-generated soluble group either is of exponential growth or contains a nilpotent subgroup of finite index (and is therefore of polynomial growth). It is also known from [Ros-74] that a finitely-generated soluble group is of exponential growth if and only if it contains a free sub-semi-group on two generators.

On the proof. Osin first establishes the proposition for polycyclic groups of exponential growth. Then he uses the following lemma, of independent interest: if Δ is a 2-generated Abelian-by-(infinite cyclic)** such that $\omega(\Delta) < \sqrt[5]{2}$, then Δ is polycyclic.

Wilson first establishes the proposition for *primitive metabelian groups*. These fall into two types, those of prime characteristic and those of characteristic zero. Typical of the former are the wreath products $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}$ with p prime. The ones of characteristic zero are of the form $A \rtimes B$ where A is torsion-free Abelian with $\dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q}) < \infty$, where B is free Abelian of finite rank, $B \neq 1$, and where B acts

*It is straightforward that 1 is a limit point of Ω_s. Indeed, Ω_s contains the largest real root ρ_n of $t^n - t - 1$, which is inside $]1, 1 + (1/n - 1)[$. On the other hand, as $\lim_{n\to\infty}(1 + (\ln 2/n))^n = e^{\ln 2} = 2$, we have $\lim_{n\to\infty}(\rho_n)^n = 2$. Also, if $\rho_1, \rho_2 \in \Omega_s$, then $\rho_1\rho_2 \in \Omega_s$. Compare with Proposition 5.2 of [Lin-84].

**This means here that there exists a short exact sequence $1 \longrightarrow A \longrightarrow \Delta \longrightarrow \mathbb{Z} \longrightarrow 1$ with A abelian.

faithfully on A, with any subgroup of B of finite index acting irreducibly on $A \otimes_{\mathbb{Z}} \mathbb{Q}$; examples include $\mathbb{Z}[1/2] \rtimes \mathbb{Z}$, where a generator of $\mathbb{Z}$ acts on $\mathbb{Z}[1/2]$ by multiplication by 2. Wilson uses then the fact that any finitely-generated soluble group which is not virtually nilpotent has a subgroup of finite index which maps epimorphically to a primitive metabelian group (similar facts have been known for a long time see, for example, [RoW-84]). □

Proposition 4.2 carries over to elementary amenable groups [Osi-b].

5. Torsion Groups

5.1. BASIC EXAMPLE. *The Golod–Shafarevich finitely-generated infinite p-groups defined below have uniform exponential growth.*

Comments. This class of examples appears as an observation in Section 3 of [BaG-99], with a proof which reduces essentially to quoting earlier results, from [GoS-64], [Gol-64], and [Gri-89]. We expose these below.

Step one: Definition and growth of a graded algebra. Consider an integer $d \geqslant 2$, a field $\mathbb{K}$, and the corresponding noncommutative polynomial algebra in d indeterminates, which is naturally a graded algebra

$$\mathbb{K}\{x_1, \dots, x_d\} = \bigoplus_{k=0}^{\infty} \mathbb{K}\{x_1, \dots, x_d\}_k$$

with $\mathbb{K}\{x_1, \dots, x_d\}_k$ the space of homogeneous elements of degree k. (This can also be described as the tensor algebra of $\mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_d$, with its basis of noncommutative monomials in $x_1, \dots, x_d$.) For each $k \geqslant 2$, consider a finite subset R_k, say of size r_k, in $\mathbb{K}\{x_1, \dots, x_d\}_k$; all these elements generate a graded ideal $\mathcal{I}$ of $\mathbb{K}\{x_1, \dots, x_d\}$. Let

$$A = \mathbb{K}\{x_1, \dots, x_d\}/\mathcal{I} = \bigoplus_{k=0}^{\infty} A_k$$

be the quotient graded algebra.

Consider also the Hilbert–Poincaré series $H_A(t) = \sum_{k \geqslant 0} (\dim_{\mathbb{K}} A_k) t^k$ of this algebra and set $H_R(t) = \sum_{k \geqslant 2} r_k t^k$. By Step 2 below, we have the *Golod–Shafarevich inequality*

$$H_A(t)(1 - \mathrm{d}t + H_R(t)) \geqslant 1 \tag{GS}$$

which means that the left-hand side is a series of which the coefficient of the constant term is 1 and all other coefficients are nonnegative.

The standard consequence of (GS) is that, under appropriate bounds on the r_k's, the series $H_A(t)$ is not a polynomial, and in particular the dimension of A as a vector space over $\mathbb{K}$ is infinite. For the purpose of the present exposition, let us assume here that $r_k \in \{0, 1\}$ for all $k \geqslant 2$ and that $r_k = 0$ for $k \leqslant 7$. For $t = \frac{3}{4}$, we have

$$1 - \mathrm{d}t + H_R(t) \leqslant 1 - 2\frac{3}{4} + \frac{\left(\frac{3}{4}\right)^8}{1 - \frac{3}{4}} = -\frac{1}{2} + \frac{3^8}{4^7} < 0.$$

If $\frac{3}{4}$ were in the open disc of convergence of the series $H_A(t)$, and since this series has nonnegative coefficients, this would imply $H_A\left(\frac{3}{4}\right)\left(1 - d\frac{3}{4} + H_R\left(\frac{3}{4}\right)\right) < 0$, in contradiction with (GS). Thus

$$\limsup_{k\to\infty} \sqrt[k]{\dim_{\mathbb{K}}(A_k)} \geqslant \frac{4}{3},$$

so that A has exponential growth; in particular, A is infinite-dimensional.

Step two: Proof of (GS). For the ease of the reader, we repeat here the proof as it appears in § 26 of [KaM-85]. For each $k \geqslant 2$, let $\mathcal{I}_k$ denote the corresponding homogeneous component of $\mathcal{I}$; choose a linear complement A_k^* of $\mathcal{I}_k$ in $\mathbb{K}\{x_1, \ldots, x_d\}_k$ and set $a_k = \dim_{\mathbb{K}}(A_k) = \dim_{\mathbb{K}}(A_k^*)$. Counting dimensions we have $d^k = \dim_{\mathbb{K}}(\mathcal{I}_k) + a_k$. By definition, $\mathcal{I}_k$ is linearly spanned by elements of the form $\xi = \mu\rho\nu$, with homogenous elements $\mu, \nu \in \mathbb{K}\{x_1, \ldots, x_d\}$ and $\rho \in \bigcup_{j \geqslant 2} R_j$. If ν is of degree at least 1, we have $\xi = (\mu\rho)\nu \in \mathcal{I}_{k-1}\mathbb{K}\{x_1, \ldots, x_d\}_1$. If ν is of degree zero, there is no loss of generality in assuming that $\xi = \mu\rho$, say with μ of degree $k - j$ and ρ of degree j. Write $\mu = \mu_1 + \mu_2$ with $\mu_1 \in \mathcal{I}_{k-j}$ and $\mu_2 \in A_{k-j}^*$. Then $\mu_1\rho \in \mathcal{I}_{k-1}\mathbb{K}\{x_1, \ldots, x_d\}_1$ and $\mu_2\rho \in A_{k-j}^* R_j$. We have shown that

$$\mathcal{I}_k \subset \mathcal{I}_{k-1}\mathbb{K}\{x_1, \ldots, x_d\}_1 + \sum_{j=2}^{k} A_{k-j}^* R_j.$$

It follows that

$$d^k - a_k \leqslant \left(d^{k-1} - a_{k-1}\right)d + \sum_{j=2}^{k} a_{k-j} r_j,$$

and consequently that

$$a_k - a_{k-1}d + \sum_{j=2}^{k} a_{k-j} r_j \geqslant 0$$

for all $k \geqslant 2$. This is precisely (GS).

Step three: Definition of a finitely-generated p-group. Assume now that the field $\mathbb{K}$ is finite or infinite countable, so that the algebra $\mathbb{K}\{x_1, \ldots, x_d\}$ is countable; let $(t_i)_{i \geqslant 1}$ be an enumeration of the elements of $\bigoplus_{k \geqslant 1} \mathbb{K}\{x_1, \ldots, x_d\}_k$. Choose first an integer $M_1 \geqslant 8$, and let $\tilde{R}_1$ be the set of homogeneous components of $(t_1)^{M_1}$. We have

$$\tilde{R}_1 = R_{M_1} \cup R_{M_1+1} \cup \cdots \cup R_{N_1}$$

for some integer $N_1 \geqslant M_1$, where each R_j is either empty or a singleton subset of $\mathbb{K}\{x_1, \ldots, x_d\}_j$. Choose next an integer $M_2 > N_1$ and let

$$\tilde{R}_2 = R_{M_2} \cup R_{M_2+1} \cup \cdots \cup R_{N_2}$$

be the set of homogeneous components of $(t_2)^{M_2}$. Iterating this, we obtain an infinite set $R = \bigcup_{i \geqslant 1} R_i$ of homogeneous elements, at most one for each degree (and none of degree $\leqslant 7$). Choose now for $\mathcal{I}$ the ideal generated by R, and let A be the corresponding quotient. By construction, any element in $A_+ = \bigoplus_{k \geqslant 1} A_k$ is nilpotent. (This is how Golod and Shafarevich solved the 'Kurosh problem', showing that there exist $\mathbb{K}$-algebras which are not nilpotent⋆, but which are 'nil-algebras', namely in which every element is nilpotent.)

Observe that any element in A of the form $1 - x$ with $x \in A_+$ is invertible, with inverse $\sum_{k \geqslant 0} x^k$ (which is a finite sum since x is nilpotent). In particular, we can consider the group Γ of invertible elements of A generated by $S_{can} = \{1 - x_1, \ldots, 1 - x_d\}$.

Assume moreover that $\mathbb{K}$ is a field of characteristic $p \geqslant 2$, and that all integers $N_1, N_2, \ldots$ appearing above are positive powers of p. For any $x \in A_+$, we have now

$$(1 - x)^{p^n} = 1 + (-1)^p x^{p^n} = 1$$

for n large enough, so that Γ is a p-group generated by the finite set S_{can}.

Denote by $\mathbb{K}[\Gamma]$ the group $\mathbb{K}$-algebra of Γ, by $\epsilon\colon \mathbb{K}[\Gamma] \longrightarrow \mathbb{K}$ the augmentation mapping, by $\Delta = \mathrm{Ker}(\epsilon)$ the principal ideal, by

$$\mathbb{K}[\Gamma] = \Delta^0 \supset \Delta = \Delta^1 \supset \Delta^2 \supset \cdots$$

the filtration by powers of the ideal Δ, and by

$$\mathrm{Grad}\,\mathbb{K}[\Gamma] = \bigoplus_{k=0}^{\infty} (\Delta^k / \Delta^{k+1})$$

the corresponding graded algebra.

As the algebra A is generated as a unital algebra by $\{x_1, \ldots, x_d\}$, it is also generated by $\{1 - x_1, \ldots, 1 - x_d\}$, and therefore by Γ. By the universal property⋆⋆ of group algebras, the embedding of Γ in A provides a morphism of algebras from $\mathbb{K}[\Gamma]$ onto A. This morphism maps Δ onto A_+, and therefore Δ^k onto $\bigoplus_{l \geqslant k} A_l$ for each $k \geqslant 1$. As A is graded, we have moreover a morphism of algebras from $\mathrm{Grad}\,\mathbb{K}[\Gamma]$ onto A. This and Step one show that

$$\limsup_{k \to \infty} \sqrt[k]{\dim_{\mathbb{K}} (\Delta^k / \Delta^{k+1})} \geqslant \limsup_{k \to \infty} \sqrt[k]{\dim_{\mathbb{K}} (A_k)} \geqslant \frac{4}{3};$$

in particular, the finitely-generated p-group Γ is infinite. (This is how Golod and Shafarevich solved the general Burnside problem.)

⋆An element a in an algebra A is *nilpotent* if there exists an integer $n > 0$ such that $a^n = 0$. An algebra A is nilpotent if there exists an integer $n > 0$ such that $a_1 a_2 \ldots a_n = 0$ for any n-uple $a_1, a_2, \ldots, a_n$ of elements of A.

⋆⋆Let Γ be a group and $\mathbb{K}$ be a field. For any $\mathbb{K}$-algebra B and morphism of monoids $f\colon \Gamma \longrightarrow B$, there exists a unique morphism of $\mathbb{K}$-algebras $F\colon \mathbb{K}[\Gamma] \longrightarrow B$ which extends f; see, e.g., §V.1 in [Lan-65].

Step four: Lower bound for $\omega(\Gamma)$ (this is Lemma 8 of [Gri-89]). Consider first a group Γ and the filtration $\mathbb{K}[\Gamma] \supset \Delta \supset \Delta^2 \supset \cdots$ of its group $\mathbb{K}$-algebra, as above. For $s, s' \in \Gamma$, we have

$$1 - ss' = (1-s) + (1-s') - (1-s)(1-s') \Longrightarrow$$
$$1 - ss' \equiv (1-s) + (1-s') (\mathrm{mod}\, \Delta^2)$$
$$1 - s^{-1} = -(1-s) + (1-s)(1-s^{-1}) \Longrightarrow 1 - s^{-1} \equiv -(1-s) (\mathrm{mod}\, \Delta^2)$$

and therefore

$$1 - s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_k^{\epsilon_k} \equiv \sum_{1 \leqslant j \leqslant k} \epsilon_j (1 - s_j) \quad (\mathrm{mod}\, \Delta^2) \tag{$*$}$$

for $k \geqslant 1$, $s_1, \ldots, s_k \in \Gamma$, and $\epsilon_1, \ldots, \epsilon_k \in \{-1, 1\}$. By definition, the power Δ^k of Δ is linearly generated by elements of the form

$$x_0 (1 - \gamma_1) x_1 (1 - \gamma_2) \cdots (1 - \gamma_k) x_k$$

with $x_0, x_1, \ldots, x_k \in \mathbb{K}[\Gamma]$ and $\gamma_1, \ldots, \gamma_k \in \Gamma$.

Assume now that Γ has a finite generating set S. By ($*$), and since $x \equiv \epsilon(x) (\mathrm{mod}\, \Delta)$ for all $x \in \mathbb{K}[\Gamma]$, the quotient Δ^k / Δ^{k+1} is linearly generated by elements of the form

$$(1 - s_1)(1 - s_2) \cdots (1 - s_k)$$

with $s_1, \ldots, s_k \in S$. Denote by $\beta(S^* \subset \Gamma, S; k)$ the number of elements of word-length with respect to S at most k in the sub-semi-group of Γ generated by S. We have

$$\beta(\Gamma, S; k) \geqslant \beta(S^* \subset \Gamma, S; k) \geqslant \dim_{\mathbb{K}} (\Delta^k / \Delta^{k+1})$$

for all $k \geqslant 0$. Consequently

$$\omega(\Gamma, S) = \lim_{k \to \infty} \sqrt[k]{\beta(\Gamma, S; k)} \geqslant \limsup_{k \to \infty} \sqrt[k]{\dim_{\mathbb{K}} (\Delta^k / \Delta^{k+1})}. \tag{$**$}$$

In the situation of the previous steps, ($**$) implies that $\omega(\Gamma, S) \geqslant \frac{4}{3}$, and this ends the proof of Example 5.1.

[If S_{can} is as in Step 3, I guess that $\omega(\Gamma) = \omega(\Gamma, S_{\mathrm{can}})$.] □

It should be both possible and interesting to have other constructions of finitely-generated torsion groups of uniform exponential growth.

I do not know if the infinite Burnside groups have uniform exponential growth.

5.2. OPEN PROBLEM. *Show that the groups of Example* 5.1 *are not amenable.*

Remark. For the case of no relations, i.e. of $\mathcal{I} = \{0\}$, we have

$$\limsup_{k \to \infty} \sqrt[k]{\dim_{\mathbb{K}} \mathbb{K}\{x_1, \ldots, x_d\}_k} = \limsup_{k \to \infty} \sqrt[k]{d^k} = d.$$

In the group of invertible elements of the algebra $\mathbb{K}\{x_1, \ldots, x_d\}$, the set $\{1 - x_1, \ldots, 1 - x_d\}$ generates a group which is free of rank d. Indeed, for any word

$$w = (1 - x_{i_1})^{k_1}(1 - x_{i_2})^{k_2} \cdots (1 - x_{i_n})^{k_n}$$

with $i_1, \ldots, i_n \in \{1, \ldots, d\}$ and $i_{j+1} \neq i_j$ when $j \in \{1, \ldots, n-1\}$, and $k_1, \ldots, k_n \in \mathbb{Z}\setminus\{0\}$, we have $w \neq 1$ since w contains the monomial $k_1 \cdots k_n x_{i_1} \cdots x_{i_n}$. Step three above shows that

$$\limsup_{k\to\infty} \sqrt[k]{\dim_{\mathbb{K}}(\Delta^k/\Delta^{k+1})} \geqslant d$$

and step four that $\omega(F_d) \geqslant d$. Compare with Example 2.1.

6. Speculations

We reproduce now some of the speculations of [GrM-97] which indicate a possible strategy to construct finitely-generated groups of exponential growth, not of uniform exponential growth.

Let Γ be a group generated by a finite generating set S. Assume that Γ is non-Hopfian, namely that there exists an endomorphism ϕ of Γ which is onto and such that $N_1 = \phi^{-1}(1) \neq \{1\}$. Set $N_n = \phi^{-n}(1)$ for each $n \geqslant 1$ and $N_\infty = \bigcup_{n\geqslant 1} N_n$. Denote by Γ_n the quotient Γ/N_n. We denote by the same letter S and its canonical images in Γ_n for $n \in \{1, 2, \ldots, \infty\}$. In a sense which can be made precise, the group Γ_∞ is a limit of the groups Γ_n as $n \to \infty$. It follows that the exponential growth rates $\omega(\Gamma_n, S)$ are decreasing and that

$$\omega(\Gamma_\infty, S) = \lim_{n\to\infty} \omega(\Gamma_n, S)$$

(Theorem 1 of [GrM-97]). Moreover, as a consequence of Gromov's result on groups of polynomial growth being virtually nilpotent, if $\omega(\Gamma_\infty, S) = 1$, then Γ_∞ is of intermediate growth. As Γ_n is isomorphic to Γ for all $n \geqslant 1$, this shows the following proposition.

6.1. PROPOSITION. *Let Γ be a finitely-generated non-Hopfian group of exponential growth such that, with the notation above, $\omega(\Gamma_\infty, S) = 1$. Then Γ is not of uniform exponential growth.*

In [GrM-97], there are investigations⋆ of groups of the form Γ_∞, but there are so far no known examples such that $\omega(\Gamma_\infty, S_\infty) = 1$.

Other investigations ended (so far) with the following result.

⋆The construction of Γ_∞ makes sense for any non-Hopfian group Γ and surjective endomorphism ϕ of Γ with non-trivial kernel. It is known that, if Γ is torsion free, then so is Γ_∞. For example, if Γ is the Baumslag-Solitar group $\langle b, t \mid t^{-1}b^2t = b^3\rangle$ and if ϕ is defined as usual by $\phi(t) = t$ and $\phi(b) = b^2$, then Γ_∞ is known to be the group of matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a \in (2/3)^{\mathbb{Z}}$ and $b \in \mathbb{Z}(\frac{1}{6})$. See [GrM-97].

6.2. PROPOSITION [GrHa]. *There is a finitely-generated group Γ_{-1} of uniform exponential growth and a sequence of quotients*

$$\Gamma_{-1} \longrightarrow \Gamma_0 \longrightarrow \cdots \longrightarrow \Gamma_n \longrightarrow \cdots \longrightarrow \Gamma_\infty$$

such that, if S denotes both an appropriate finite symmetric generating set in Γ_{-1} and its canonical images in each of the Γ_n's, we have $\omega(\Gamma_n, S) \geqslant \omega(\Gamma_n) > 1$ for each n with $-1 \leqslant n < \infty$ and

$$\lim_{n\to\infty} \omega(\Gamma_n, S) = \omega(\Gamma_\infty, S) = 1.$$

Moreover Γ_n is commensurable to a direct product of 2^n copies of a non-Abelian finitely-generated free group for all $n \geqslant 0$.

7. A Recent Result and an Open Problem

The main problem of our subject is clearly Problem 1.2. Since the Haifa Conference where the present report was first presented, the following important result has been proved [EMO].

7.1. THEOREM (Eskin–Mozes–Oh). *Let Γ be a finitely generated group which is linear in characteristic zero, namely which is isomorphic to a subgroup of* $\mathrm{GL}(n, \mathbb{K})$ *for some integer n and some field $\mathbb{K}$ of characteristic zero. Then Γ is either virtually nilpotent* (*and thus of polynomial growth*) *or of uniform exponential growth.*

The proof requires an important number of prerequisites, including Osin's result on uniform exponential growth for solvable groups [Osi-a].

7.2. OPEN PROBLEM. *Does there exist an infinite group with Kazhdan Property* (T) *which is not of uniform exponential growth?*

Here is a possibly related result of Y. Shalom. Let Γ be a finitely-generated group of which the regular representation λ_Γ is 'uniformly isolated from the unit representation', i.e. such that there exists a constant $\epsilon > 0$ with the following property: for *any* generating set S of Γ, and for any unit vector $\xi \in \ell^2(\Gamma)$, the inequality

$$\sup_{s\in S} \|\lambda_\Gamma(s)\xi - \xi\| \geqslant \epsilon$$

holds. Then Γ has uniform exponential growth (Proposition 8.3 of [Sha-00]).

If constants measuring exponential growth can have uniform bounds in terms of the generating sets (see the examples discussed in this paper), let us finally report that other constants show the opposite behaviour. For example, T. Gelander and A. Zuk have shown that, in many cases, Kazhdan constants depend on the chosen generating sets in a more important way [GeZ].

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Added in Proof. John Wilson has announced a positive solution to the main open problem (1.2 above). More precisely, there exist finitely generated groups with trivial Abelianization which are isomorphic to their permutational wreath product with the alternating group on 31 letters. let $\Gamma \approx \Gamma \wr A_{31}$ be any group of this kind; on the one hand, there exists a sequence $(S_n = \{x_n, y_n\})_{n \geq 1}$ of generating sets of Γ, with $x_n^2 = y_n^3 = 1$ for all $n \geq 1$, such that the limit of the corresponding exponential growth rates is 1, in symbols $\lim_{n\to\infty}\omega(\Gamma, S_n) = 1$; on the other hand, for an appropriate choice of Γ, there exist non-Abelian free subgroups in Γ, so that in particular Γ is of exponential growth.

Moreover, Wilson has also announced a proof of Conjucture 3.4.