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Constructing irreducible representations of discrete groups

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Abstract. The decomposition of unitary representations of a discrete group obtained by induction from a subgroup involves commensurators. In particular Mackey has shown that quasi-regular representations are irreducible if and only if the corresponding subgroups are self-commensurizing. The purpose of this work is to describe general constructions of pairs of groups $\Gamma_0 < \Gamma$ with Γ_0 its own commensurator in Γ. These constructions are then applied to groups of isometries of hyperbolic spaces and to lattices in algebraic groups.

Keywords. Commensurator subgroups; unitary representations; quasi-regular representations; Gromov hyperbolic groups; arithmetic lattices.

1. Introduction

Let G be a separable locally compact group. The *unitary dual* $\hat{G}$ of G is the set of equivalence classes of irreducible representations of G, together with its Mackey Borel structure. In this paper, "representation" means "continuous unitary representation in a separable Hilbert space".

Let us recall the definition of this structure [Dix, 18.5]. For each $n \in \{1, 2, \ldots, \infty\}$, let $\mathrm{Irr}_n(G)$ denote the space of all irreducible representations of G in a given Hilbert space of dimension n. The set $\mathrm{Irr}_n(G)$ is endowed with the topology of the weak simple convergence on G (making the functions $\pi \mapsto \langle \pi(g)\xi | \eta \rangle$ continuous for all $g \in G$ and ξ, η in the Hilbert space of dimension n), and with the corresponding Borel structure. The dual $\hat{G}$ is the quotient of $\coprod_{1 \le n \le \infty} \mathrm{Irr}_n(G)$ by unitary equivalence, and the Mackey Borel structure on $\hat{G}$ is the quotient of the previously defined Borel structure.

In case of a countable group Γ, it follows from results of Glimm and Thoma that $\hat{\Gamma}$ is a standard Borel space if and only if Γ is virtually abelian (see [Dix], numbers 9.1, 9.5.6 and 13.11.12, or [Ped, 6.8.7]); in this case the representation theory of Γ is well understood. In all other cases there is no natural Borel coding of $\hat{\Gamma}$, i.e. $\hat{\Gamma}$ is not countably separated; for lack of a systematic procedure of constructing all irreducible representations of Γ, a natural problem is to construct large classes of irreducible representations.

Recall that two subgroups G_0 and G_1 of a group G are *commensurable* if $G_0 \cap G_1$ is of finite index in both G_0 and G_1. The *commensurator* of G_0 in G is defined to be

$$\mathrm{Com}_G(G_0) = \{g \in G \mid G_0 \text{ and } gG_0g^{-1} \text{ are commensurable}\}.$$

Let $(\Gamma_i)_{i \in I}$ be a family of pairwise non conjugate subgroups of a countable group Γ such that $\mathrm{Com}_\Gamma(\Gamma_i) = \Gamma_i$ for all $i \in I$. It follows from work of Mackey (see e.g. [Mac], and § 2