

ON THE SPECTRUM OF THE SUM OF GENERATORS FOR A FINITELY GENERATED GROUP

BY

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ABSTRACT

Let Γ be a finitely generated group. In the group algebra $\mathbb{C}[\Gamma]$, form the average h of a finite set S of generators of Γ . Given a unitary representation π of Γ , we relate spectral properties of the operator $\pi(h)$ to properties of Γ and π .

For the universal representation π_{un} of Γ , we prove in particular the following results. First, the spectrum $\text{Sp}(\pi_{un}(h))$ contains the complex number z of modulus one iff $\text{Sp}(\pi_{un}(h))$ is invariant under multiplication by z , iff there exists a character $\chi: \Gamma \rightarrow \mathbb{T}$ such that $\chi(S) = \{z\}$. Second, for $S^{-1} = S$, the group Γ has Kazhdan's property (T) if and only if 1 is isolated in $\text{Sp}(\pi_{un}(h))$; in this case, the distance between 1 and other points of the spectrum gives a lower bound on the Kazhdan constants. Numerous examples illustrate the results.

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Introduction

Let Γ be a finitely generated group, and let S be a finite generating set. Assume for the moment that $1 \notin S$ and that S is symmetric ($S^{-1} = S$). On the Cayley graph $\mathcal{G}(\Gamma, S)$, consider the nearest neighbour isotropic random walk: the transition probability $M_{\gamma, \gamma'}$ between two vertices γ, γ' of the graph is $|S|^{-1}$ if they are nearest neighbours (namely here if $\gamma\gamma'^{-1} \in S$) and zero otherwise. The matrix $M = (M_{\gamma, \gamma'})_{\gamma, \gamma' \in \Gamma}$ acts naturally as a bounded linear operator on the Hilbert space $l^2(\Gamma)$. In two pioneering papers [Ke1], [Ke2], Kesten started to investigate the relations between properties of the group Γ and spectral properties of transition operators such as M . He showed for example that

$$\frac{2\sqrt{|S|-1}}{|S|} \leq \|M\| \leq 1$$

with equality on the right if and only if Γ is amenable (a condition independent of the choice of S), and with equality on the left if and only if Γ is a free group on a set S_+ such that $S = S_+ \cup (S_+)^{-1}$.

Our purpose here is to modify Kesten's starting point in two ways:

- (1) We consider a finite generating set S which need not be symmetric.
- (2) We consider an arbitrary unitary representation π of Γ on a Hilbert space \mathcal{H}_π rather than only the left regular representation λ of Γ on $l^2(\Gamma)$.

Denote by $\mathbb{C}[\Gamma]$ the complex group algebra. To any representation π of Γ as above is associated a $*$ -representation of $\mathbb{C}[\Gamma]$ on the same Hilbert space \mathcal{H}_π , also denoted by π , and defined by $\pi(f) = \sum_{\gamma \in \Gamma} f(\gamma)\pi(\gamma)$ for all $f \in \mathbb{C}[\Gamma]$.

Given S as in (1), we set

$$h = \frac{1}{|S|} \sum_{s \in S} s \in \mathbb{C}[\Gamma].$$

In case S is symmetric and $1 \notin S$, observe that $\lambda(h)$ is precisely the operator M appearing in Kesten's papers. In the setting of (1) and (2) above, our programme is to investigate the relations between properties of Γ and π on one hand, and spectral properties of $\pi(h)$ on the other hand.

There are several interesting examples of such π , besides the left regular representation λ . One is the **universal representation** π_{un} , which is the direct sum of all cyclic representations of Γ (up to unitary equivalence). Other are characters $\Gamma \rightarrow \{z \in \mathbb{C}: |z| = 1\}$, including the **trivial representation** χ_1 defined by $\chi_1(\gamma) = 1$ for all $\gamma \in \Gamma$.

To state our first result, we need the following terminology. Given a representation π of Γ on a Hilbert space \mathcal{H} and a finite generating set S as above, we define the Kazhdan constant

$$\kappa(\pi, S) = \inf_{\xi \in \mathbb{S}^1(\mathcal{H})} \max_{s \in S} \|\pi(s)\xi - \xi\|$$

where we denote by $\mathbb{S}^1(\mathcal{H})$ the unit sphere $\{\xi \in \mathcal{H}: \|\xi\| = 1\}$. This constant depends on S , but the fact that it is zero or not does not; indeed $\kappa(\pi, S) = 0$ if and only if χ_1 is weakly contained in π . The latter means that $\kappa(\pi, F) = 0$ for any finite subset F of Γ . We denote by π_{\perp}^1 the restriction of π to the orthogonal complement in \mathcal{H} of the space of vectors fixed by Γ . In Section A, we shall prove.

PROPOSITION I: *Let π be a representation of Γ on a Hilbert space \mathcal{H} .*

- (1) $1 \in \text{Sp } \pi(h)$ if and only if χ_1 is weakly contained in π .
- (2) 1 is an eigenvalue of $\text{Sp } \pi(h)$ if and only if χ_1 is contained in π .
- (3) If $1 \in \text{Sp } \pi(h)$ and $1 \notin \text{Sp } \pi_{\perp}^1(h)$, then 1 is an isolated point of both $\text{Sp } \pi(h)$ and $\text{Sp } \pi(\frac{1}{2}(h + h^*))$.
- (4) If $\kappa(\pi, S) > 0$, then $\text{Sp } \pi(h)$ is disjoint from the open disc

$$\{z \in \mathbb{C}: |z - 1| < \kappa(\pi, S)^2 / (2|S|)\}.$$

Assume moreover that $S^{-1} = S$.

- (5) If 1 is isolated in $\text{Sp } \pi(h)$, then $1 \notin \text{Sp } \pi_{\perp}^1(h)$.
- (6) If $\text{Sp } \pi(h) \subset [-1, 1 - \varepsilon]$ for some $\varepsilon > 0$ then $\kappa(\pi, S) \geq \sqrt{2\varepsilon}$.

We show by an example the assumption $S^{-1} = S$ cannot be removed from (5). Actually we construct an infinite dimensional irreducible representation π of the free group \mathbb{F}_2 on a two generator set $S = \{a, b\}$ such that $\text{Sp } \pi(h) = \{1\}$. Viewing \mathbb{F}_2 as a normal subgroup of index 12 in $\text{SL}(2, \mathbb{Z})$, inducing and decomposing, we obtain an infinite dimensional irreducible representation of $\text{SL}(2, \mathbb{Z})$ which weakly contains χ_1 . This answers, almost fortuitously, a question of Bekka in [Bek]; see the end of Section A.

In Sections B and C, we deal with the peripheral spectrum of $\pi(h)$, that is with the intersection of $\text{Sp } \pi(h)$ with the unit circle \mathbb{T} . We prove, for $z \in \mathbb{T}$, results analogous to those of Proposition I for $+1$. We also study rotational symmetries of spectra. Writing $\text{Sp } h$ rather than $\text{Sp } \pi_{un}(h)$, we give the following characterization (see Propositions 3 and 5).

PROPOSITION II: Let Γ , S and h be as above and let \mathcal{G} be the Cayley graph of Γ with respect to $S \cup S^{-1}$. The peripheral spectrum of h is a closed subgroup of \mathbb{T} . Moreover, for $|z| = 1$, the following are equivalent:

- (i) $z \in \text{Sp } h$,
- (ii) $\text{Sp } h$ is invariant under multiplication by z ,
- (iii) there is a character $\chi: \Gamma \rightarrow \mathbb{T}$ such that $\chi(S) = \{z\}$.

Under conditions (i) to (iii), it is also true that $\text{Sp } \lambda(h)$ is invariant under multiplication by z . If $z = -1$, conditions (i) to (iii) are also equivalent to

- (iv) the graph \mathcal{G} is bicolorable.

Section D is about Kazhdan's property (T). Given a group Γ and a set of generators S we introduce the Kazhdan constants

$$\kappa(\Gamma, S) = \inf \kappa(\pi, S)$$

where the infimum is taken over all representations π of Γ in a separable Hilbert space which have no nonzero fixed vector, and

$$\hat{\kappa}(\Gamma, S) = \inf \{ \kappa(\pi, S) : \pi \in \hat{\Gamma}, \pi \neq \chi_1 \}$$

where $\hat{\Gamma}$ denotes the **unitary dual** of Γ , namely the set of all (equivalence classes of) irreducible representations of Γ . Here again, the fact that any of these constants are zero or not does not depend on S . Indeed $\kappa(\Gamma, S) > 0$ if and only if Γ had Kazhdan's Property (T): see [Kaz],[HaV]. It is also known that $\hat{\kappa}(\Gamma, S) > 0$ if and only if $\kappa(\Gamma, S) > 0$ [DeK: Lemme 1]. More precisely, one has obviously $\hat{\kappa}(\Gamma, S) \geq \kappa(\Gamma, S)$, possibly with strict inequality, and one may show [BaH] that $\kappa(\Gamma, S) \geq (2|S|)^{-\frac{1}{2}} \hat{\kappa}(\Gamma, S)$. The following result improves a previous result of the third author [Val].

PROPOSITION III: (1) Assume that Γ has Property (T), and set

$$\varepsilon = \frac{\hat{\kappa}(\Gamma, S)^2}{2|S|}.$$

For any z in the peripheral spectrum of h (in particular for $z = 1$), the set

$$\text{Sp } h \cap \{w \in \mathbb{C} : 0 < |w - z| < \varepsilon\}$$

is empty. Moreover, the cardinality of the peripheral spectrum is at most equal to $\pi/\text{Arcsin}(\varepsilon/2)$.

(2) We assume that $S^{-1} = S$ and we choose a real number $\varepsilon > 0$. Suppose moreover that

either $\text{Sp } h \subset [-1, 1 - \varepsilon] \cup \{1\}$

or $-1 \in \text{Sp } h \subset \{-1\} \cup [-1 + \varepsilon, 1]$.

Then Γ has Property (T) and $\kappa(\Gamma, S) \geq \sqrt{2\varepsilon}$.

In particular, when $S^{-1} = S$, we see that Γ has Property (T) if and only if 1 is an isolated point in $\text{Sp } h$.

In the final Section E of this paper, we consider a discrete group Γ with Property (T). It was shown by Wang and more recently reproved by Wassermann that Γ has at most finitely many (inequivalent) irreducible representations of any given dimension $m < \infty$ (see Theorems 2.5 and 2.6 of [Wan] and Corollary 2 of [Was]). We apply the preceding theory to obtain an asymptotic bound for the number $\text{Irrep}_\Gamma(m)$ of irreducible representations of Γ of degree at most m , namely:

PROPOSITION IV: *Let Γ be a discrete Kazhdan group, with a given finite symmetric generating subset S ; then*

$$\text{Irrep}_\Gamma(m) = O(e^{Am^2})$$

for some constant A depending on Γ and S .

A. The spectrum near 1

We begin by briefly discussing group C^* -algebras; although they do not play a fundamental role in this paper, they do provide a convenient framework.

Let π be a representation of the group Γ on a Hilbert space \mathcal{H} (in this paper, all representations are unitary). The norm closure of $\pi(\mathbb{C}[\Gamma])$ in the algebra of all bounded linear operators on \mathcal{H} is a C^* -algebra denoted by $C_\pi^*(\Gamma)$. For example, if π is the universal representation π_{un} of Γ , we obtain the full C^* -algebra of Γ . We shall follow common practice and denote it simply by $C^*(\Gamma)$. It has the following universal property: every representation π of Γ extends to a $*$ -representation $C^*(\Gamma) \rightarrow C_\pi^*(\Gamma)$, again denoted by π [Dix: 13.9.3]. Considering the left regular representation λ , we obtain the reduced C^* -algebra $C_\lambda^*(\Gamma)$. It follows from a standard result of Hulanicki [Ped: 7.3.9] that the canonical $*$ -homomorphism $\lambda: C^*(\Gamma) \rightarrow C_\lambda^*(\Gamma)$ is an isomorphism if and only if Γ is amenable.

From now on we assume that Γ is given together with a finite set S of generators and we consider the operator

$$h = \frac{1}{|S|} \sum_{s \in S} s \in C^*(\Gamma).$$

Clearly $\|h\| \leq 1$, and h is self-adjoint if and only if S is symmetric (i.e. $S = S^{-1}$). We denote by $\text{Sp } x$ the **spectrum** of an element x of a C^* -algebra.

PROPOSITION 1: *If $\chi: \Gamma \rightarrow \{z \in \mathbb{C}: |z| = 1\}$ is a character of Γ , then $\chi(h) \in \text{Sp } h$. In particular the spectrum of h always contains 1.*

Proof: If $x = \sum_{\gamma \in \Gamma} z_\gamma \gamma \in C^*(\Gamma)$ is an element of $\mathbb{C}[\Gamma]$, then

$$\chi(x) = \sum_{\gamma \in \Gamma} z_\gamma \chi(\gamma).$$

In particular $\chi(h - \chi(h)) = 0$ and therefore $h - \chi(h)$ is not invertible in $C^*(\Gamma)$. The second assertion follows with $\chi = \chi_1$. ■

Let \mathcal{H} be a Hilbert space. We denote by $\mathbb{S}^1(\mathcal{H})$ the unit sphere $\{\xi \in \mathcal{H}: \|\xi\| = 1\}$. Let x be an operator on \mathcal{H} . Recall that $\text{Sp } x \neq \emptyset$ if and only if $\mathcal{H} \neq \{0\}$. We say that x is a **contraction** if $\|x\| \leq 1$. In the following four lemmas, we collect standard material.

LEMMA 1: *Let x be a contraction on a Hilbert space \mathcal{H} , let $\xi \in \mathbb{S}^1(\mathcal{H})$ and let ε be a real number, $\varepsilon \geq 0$.*

(1) *If $\|x\xi - \xi\| \leq \varepsilon$, then $\|\frac{1}{2}(x + x^*)\xi - \xi\| \leq \sqrt{2\varepsilon}$.*

(2) *If $\|\frac{1}{2}(x + x^*)\xi - \xi\| \leq \varepsilon$, then $\|x\xi - \xi\| \leq \sqrt{2\varepsilon}$.*

Proof: If $\|x\xi - \xi\| \leq \varepsilon$, then $\|x\xi\| \geq \|\xi\| - \|x\xi - \xi\| \geq 1 - \varepsilon$ and

$$(1 - \varepsilon)^2 + 1 - 2\text{Re}\langle \xi | x\xi \rangle \leq \|x\xi - \xi\|^2 \leq \varepsilon^2$$

so that $\text{Re}\langle \xi | x\xi \rangle \geq 1 - \varepsilon$ and

$$1 - \langle \xi | \frac{1}{2}(x + x^*)\xi \rangle = 1 - \text{Re}\langle \xi | x\xi \rangle \leq \varepsilon.$$

Consequently

$$\begin{aligned} \|\frac{1}{2}(x + x^*)\xi - \xi\|^2 &\leq \|x\|^2 + 1 - 2\langle \xi | \frac{1}{2}(x + x^*)\xi \rangle \\ &\leq 2(1 - \langle \xi | \frac{1}{2}(x + x^*)\xi \rangle) \\ &\leq 2\varepsilon \end{aligned}$$

and (1) holds.

Similarly, under the hypothesis of (2), one has $\operatorname{Re}\langle \xi | x\xi \rangle \geq 1 - \varepsilon$ and

$$\|x\xi - \xi\|^2 \leq 2(1 - \operatorname{Re}\langle \xi | x\xi \rangle) \leq 2\varepsilon$$

so that (2) holds. ■

Remark: It is easy to see (with $\mathcal{H} = \mathbb{C}$) that the constant $\sqrt{2\varepsilon}$ in Claim (2) is best possible. However the constant in Claim (1) is not. We do not know the best constant here.

LEMMA 2: *Let x be a contraction on a Hilbert space \mathcal{H} .*

- (1) $\operatorname{Ker}(x - 1) = \operatorname{Ker}(\frac{1}{2}(x + x^*) - 1) = \operatorname{Ker}(x^* - 1)$.
- (2) $1 \in \operatorname{Sp} x$ if and only if there exists a sequence (ξ_j) of vectors in $\mathbb{S}^1(\mathcal{H})$ such that $\|x\xi_j - \xi_j\| \rightarrow 0$ as $j \rightarrow \infty$.
- (3) $1 \in \operatorname{Sp} x \Leftrightarrow 1 \in \operatorname{Sp}(\frac{1}{2}(x + x^*)) \Leftrightarrow 1 \in \operatorname{Sp} x^*$.

Proof: Claim (1) follows from Lemma 1 with $\varepsilon = 0$.

To prove the non-trivial implication in Claim (2), suppose that $1 \in \operatorname{Sp} x$. If the range of $x - 1$ is not dense in \mathcal{H} , then $\operatorname{Ker}(x^* - 1) \neq \{0\}$. But $\operatorname{Ker}(x - 1) = \operatorname{Ker}(x^* - 1)$ by (1), so that any sequence in $\operatorname{Ker}(x - 1) \cap \mathbb{S}^1(\mathcal{H})$ does the job. If the range of $x - 1$ is dense in \mathcal{H} , then $\inf\{\|(x - 1)\xi\| : \xi \in \mathbb{S}^1(\mathcal{H})\} = 0$ (otherwise $x - 1$ would be invertible), and the existence of an appropriate sequence (ξ_j) is again obvious.

Claim (3) is a straightforward consequence of Claim (2) and of Lemma 1. ■

LEMMA 3: *Consider a Hilbert space \mathcal{H} , a vector $\xi \in \mathbb{S}^1(\mathcal{H})$, a real number $\varepsilon \geq 0$, an integer $n \geq 1$, a sequence y_1, \dots, y_n of contractions on \mathcal{H} , and set*

$$x = \frac{1}{n}(y_1 + \dots + y_n).$$

If $\|x\xi - \xi\| \leq \varepsilon$, then $\|y_j\xi - \xi\| \leq \sqrt{2n\varepsilon}$ for all $j \in \{1, \dots, n\}$. In particular

$$\operatorname{Ker}(x - 1) = \bigcap_{1 \leq j \leq n} \operatorname{Ker}(y_j - 1).$$

Proof: As in the proof of Lemma 1, we compute

$$1 - \varepsilon \leq \operatorname{Re}\langle \xi | x\xi \rangle = \frac{1}{n} \sum_{k=1}^n \operatorname{Re}\langle \xi | y_k\xi \rangle.$$

As $\operatorname{Re}\langle \xi | y_k \xi \rangle \leq 1$ for all $k \in \{1, \dots, n\}$, it follows that

$$\operatorname{Re}\langle \xi | y_j \xi \rangle \geq 1 - n\varepsilon$$

and thus

$$\|y_j \xi - \xi\|^2 \leq 2n\varepsilon$$

for all $j \in \{1, \dots, n\}$. ■

LEMMA 4: Given Γ , S and π as above, $\kappa(\pi, S) = 0$ if and only if the trivial representation χ_1 of Γ is weakly contained in π .

Proof: For each integer $n \geq 0$, denote by B_n the set of those $\gamma \in \Gamma$ for which there exists a sequence s_1, \dots, s_n of generators in $S \cup S^{-1}$ such that $\gamma = s_1 \cdots s_n$.

Consider a real number $\varepsilon > 0$ and a vector $\xi \in \mathbb{S}^1(\mathcal{H})$ such that

$$\max_{s \in S} \|\pi(s)\xi - \xi\| \leq \kappa(\pi, S) + \varepsilon.$$

For any $\gamma = s_1 \cdots s_n \in B_n$, one has

$$\begin{aligned} \|\pi(\gamma)\xi - \xi\| &\leq \sum_{j=1}^n \|\pi(s_1 \cdots s_{j-1})(\pi(s_j)\xi - \xi)\| \\ &\leq n(\kappa(\pi, S) + \varepsilon). \end{aligned}$$

It follows that

$$\inf_{\xi \in \mathbb{S}^1(\mathcal{H})} \max_{\gamma \in B_n} \|\pi(\gamma)\xi - \xi\| \leq n\kappa(\pi, S).$$

In particular, the lemma is nothing but a reformulation of the definitions. ■

Remark: It follows from Lemma 4 that the condition $\kappa(\pi, S) = 0$ does not depend on the finite set S of generators.

Proof of Proposition I in the Introduction: Claim (1) follows from Lemmas 2(2) and 3 applied to

$$x = \pi(h) = \frac{1}{|S|} \sum_{s \in S} \pi(s),$$

and from Lemma 4.

Claim (2) follows from the last assertion of Lemma 3.

Let us denote by π_1^- the restriction of π to the space \mathcal{H}_1^- of $\pi(\Gamma)$ -fixed vectors.

Observe that $\text{Sp } \pi(h) = \text{Sp } \pi_1^{\bar{}}(h) \cup \text{Sp } \pi_1^{\perp}(h)$. Observe also that $\text{Sp } \pi_1^{\bar{}}(h)$ is either empty, if $\mathcal{H}_1^{\bar{}} = \{0\}$, or reduced to $\{1\}$, if $\mathcal{H}_1^{\bar{}} \neq \{0\}$. The first conclusion of Claim (3) follows from this. Moreover $1 \notin \text{Sp } \pi_1^{\perp}(h)$ implies that $1 \notin \text{Sp } \pi_1^{\perp}(\frac{1}{2}(h + h^*))$ by Lemma 2 (3), and the second conclusion of Claim (3) follows similarly.

Write δ for $\kappa(\pi, S)^2 / (2|S|)$ and choose $\xi \in \mathbb{S}^1(\mathcal{H})$. By the definition of $\kappa(\pi, S)$, there exists $s \in S$ such that

$$\|\pi(S)\xi - \xi\| \geq \kappa(\pi, S) = \sqrt{2|S|\delta}.$$

Lemma 3 shows that $\|\pi(h)\xi - \xi\| \geq \delta$. Choose now $w \in \mathbb{C}$ such that $|w| < \delta$. Then

$$\|\pi(h)\xi - (1 + w)\xi\| \geq \delta - |w|.$$

As π is a unitary representation, one has $\kappa(\pi, S^{-1}) = \kappa(\pi, S)$. Thus one has also

$$\|\pi(h^*)\xi - (1 + \bar{w})\xi\| \geq \delta - |w|.$$

Since these hold for all $\xi \in \mathbb{S}^1(\mathcal{H})$, the operator $\pi(h) - (1 + w)$ is invertible, and Claim (4) holds.

If $S^{-1} = S$ and 1 is isolated in $\text{Sp } \pi(h)$, we may consider the nonzero spectral projection p of the self-adjoint operator $\pi(h)$ corresponding to the isolated point 1 in the spectrum of $\pi(h)$. The restriction of $\pi(h)$ to $p\mathcal{H}$ coincides with the identity, and the restriction of $\pi(h)$ to $(1 - p)\mathcal{H}$ has its spectrum disjoint from $\{1\}$. In other words, one has $p\mathcal{H} = \mathcal{H}_1^{\bar{}}$ and $1 \notin \text{Sp } \pi_1^{\perp}(h)$.

The hypothesis of (6) and spectral theory imply that, for all $\xi \in \mathbb{S}^1(\mathcal{H})$, one has

$$\text{Re}\langle \xi | \pi(h)\xi \rangle = \langle \xi | \pi(h)\xi \rangle \leq 1 - \varepsilon.$$

Thus there exists $s \in S$ such that

$$\text{Re}\langle \xi | \pi(s)\xi \rangle \leq 1 - \varepsilon$$

and this implies that $\|\pi(s)\xi - \xi\|^2 = 2(1 - \text{Re}\langle \xi | \pi(s)\xi \rangle) \geq 2\varepsilon$. ■

Remark: In Proposition I (4), assume moreover that $S^{-1} = S$ and that $\pi(S)$ does not contain any element of order 2. Then

$$\text{Sp } \pi(h) \subset [-1, 1 - \kappa(\pi, S)^2 / |S|].$$

The point is that $2|S|$ can be replaced by $|S|$. For if $s \in S$ is chosen with $\|\pi(s)\xi - \xi\|^2 = 2 - \langle (\pi(s) + \pi(s^{-1}))\xi | \xi \rangle \geq \kappa(\pi, S)^2$ then $\langle (\pi(s) + \pi(s^{-1}))\xi | \xi \rangle \leq 2 - \kappa(\pi, S)^2$, so that $\langle \pi(h)\xi | \xi \rangle \leq 1 - \kappa(\pi, S)^2/|S|$.

SOME CONSEQUENCES OF PROPOSITION I (1):

- (a) When π is the left regular representation λ of Γ , Proposition I (1) gives Kesten's characterization of amenability [Ke2]:

$$1 \in \text{Sp } \lambda(h) \quad \text{if and only if } \Gamma \text{ is amenable .}$$

Indeed, Γ is amenable if and only if χ_1 is weakly contained in λ , by a classical result of Hulanicki [Hul]. For a generalization of Proposition I (1) see also Theorem 1 in [SoW].

- (b) For any representation π of Γ , Bekka has introduced a notion of amenability, and π is amenable if and only if χ_1 is weakly contained in $\pi \otimes \bar{\pi}$, where $\bar{\pi}$ denotes the conjugate of π . (See [Bek: Def. 1.1 and Th. 5.1]; for $\bar{\pi}$, see e.g. [Dix: 13.1.5].) As $\lambda \otimes \bar{\lambda}$ is weakly equivalent to λ [Dix: 13.11.3], it follows that λ is amenable if and only if Γ is amenable. From Proposition I (1) of Proposition 2, one has

$$1 \in \text{Sp}((\pi \otimes \bar{\pi})(h)) \quad \text{if and only if } \pi \text{ is amenable .}$$

For example, let α be the adjoint representation of Γ on $l^2(\Gamma - \{1\})$ defined by

$$(\alpha(\gamma)\xi)(\gamma_1) = \xi(\gamma^{-1}\gamma_1\gamma);$$

then Theorem 2.4 of [Bek] states that $1 \in \text{Sp}((\alpha \otimes \bar{\alpha})(h))$ if and only if Γ is **strongly inner amenable** (namely "inner amenable" in the sense of [BeH]).

Similarly, one can see that 1 is an eigenvalue of $(\pi \otimes \bar{\pi})(h)$ if and only if π has a finite dimensional subrepresentation (see the remark following Lemma 9 below).

Remark on Laplacians: Let us now explain how h is related to a "combinatorial Laplacian". Let first \mathcal{G} be a **graph** without loops or multiple edges (the graph is finite or infinite, and non-oriented). We denote by \mathcal{G}^0 the set of vertices of \mathcal{G} , by $l^2(\mathcal{G}^0)$ the space of square-summable functions from \mathcal{G}^0 to \mathbb{C} , and by $(e_v)_{v \in \mathcal{G}^0}$ the canonical orthonormal basis for the latter space. The **adjacency matrix** $A = (A_{v,w})_{v,w \in \mathcal{G}^0}$ of \mathcal{G} is defined by $A_{v,w} = 1$ if $v \neq w$ and there is an edge between v and w , and by $A_{v,w} = 0$ otherwise.

We assume from now on that \mathcal{G} has bounded degree, namely that the number of neighbours $D_v = \sum_{w \in \mathcal{G}^0} A_{v,w}$ of a vertex v is bounded by some D_{\max} . The degree operator is the bounded self-adjoint operator D on $l^2(\mathcal{G}^0)$, defined by $De_v = D_v e_v$ for all $v \in \mathcal{G}^0$. As \mathcal{G} has bounded degree, the matrix A defines a bounded self-adjoint operator on $l^2(\mathcal{G}^0)$, denoted again by A . The (combinatorial) Laplacian $\Delta_{\mathcal{G}}$ of \mathcal{G} is defined by

$$\Delta_{\mathcal{G}} = D - A.$$

In particular, let Γ be a group generated by a symmetric finite set S such that $1 \notin S$. Let $\mathcal{G} = \mathcal{G}(\Gamma, S)$ be the corresponding Cayley graph, defined by $\mathcal{G}^0 = \Gamma$ and $A_{\gamma_1, \gamma_2} = 1$ if and only if $\gamma_1 \gamma_2^{-1} \in S$. As $\lambda(h) = \frac{1}{|S|} \sum_{s \in S} \lambda(s)$, one has

$$\Delta_{\mathcal{G}} = |S| - \sum_{s \in S} \lambda(s) = |S|(1 - \lambda(h)).$$

We define more generally

$$\Delta = |S|(1 - h) \in C^*(\Gamma)$$

so that $\Delta_{\mathcal{G}} = \lambda(\Delta)$.

It is important that Δ is a positive element of $C^*(\Gamma)$. We shall express some results in terms of h and some others with Δ . Upon scaling and translating we could easily express all results for both h and Δ .

Examples:

- (1) Consider an integer $d \geq 1$, the free abelian group \mathbb{Z}^d , and the standard set $S = \{s_1^{\pm}, \dots, s_d^{\pm}\}$ of generators having $d - 1$ coordinates zero and one coordinate 1 or -1 . Then $\text{Sp } \lambda(\Delta) = [0, 4d]$, as can be viewed as follows; see also [MoW: 7.B].

For each $\alpha \in \mathbb{R}$, define a character χ_{α} of \mathbb{Z}^d by

$$\chi_{\alpha}(s_j^{\pm}) = e^{\pm i\alpha}, \quad j = 1, \dots, d.$$

It follows from Proposition 1 that

$$\frac{1}{2d} \sum_{s \in S} \chi_{\alpha}(s) = \cos \alpha \in \text{Sp } h.$$

Thus $\text{Sp } h = [-1, 1]$ and $\text{Sp } \Delta = [0, 4d]$. As \mathbb{Z}^d is amenable, $\lambda(h)$ and h have the same spectrum.

- (2) Let Γ be the non-abelian free group on a free set S_+ of d generators, and consider the symmetric set $S = S_+ \cup S_+^{-1}$, of cardinality $N = 2d$. Then

$$\text{Sp } \lambda(\Delta) = [N - 2\sqrt{N-1}, N + 2\sqrt{N-1}].$$

Observe that the Cayley graph $\mathcal{G}(\Gamma, S)$ is the homogeneous tree of degree N , so that it is easy to give a meaning to the above formula for any integer N , even or odd; this formula is indeed a result of Kesten [Ke1]. See also [Pat: 4.31].

On the other hand the canonical map sending the generators of S to the corresponding generators of \mathbb{Z}^d in example (1) extends to a $*$ homomorphism of $C^*(\Gamma)$ onto $C^*(\mathbb{Z}^d)$. It follows that $\text{Sp}(\Delta) = [0, 4d]$.

- (3) If Γ is the non-abelian free group on d generators s_1, \dots, s_d and $S = \{s_1, \dots, s_d\}$, so that $S^{-1} \neq S$, then

$$\text{Sp } h = \{z \in \mathbb{C}: |z| \leq 1\}.$$

For if $z \in \mathbb{C}, |z| \leq 1$, there exists $z_j \in \mathbb{C}, |z_j| = 1$ ($j = 1, \dots, d$) such that

$$z = \frac{1}{d}(z_1 + \dots + z_d).$$

Let χ be the character of Γ defined by $\chi(s_j) = z_j$ for $j = 1, \dots, d$. Then $z = \chi(h) \in \text{Sp } h$ by Proposition 1.

- (4) Choose an integer $k \geq 2$, let $\Gamma = \mathbb{Z}/k\mathbb{Z} * \dots * \mathbb{Z}/k\mathbb{Z}$ (r times), identify each factor $\Gamma_j = \mathbb{Z}/k\mathbb{Z}$ of the free product to a subgroup of Γ , and define

$$S = \bigcup_{1 \leq j \leq r} (\Gamma_j - \{1\}).$$

If $r \geq 2$, then

$$\text{Sp } h = [-1/(k-1), 1]$$

by [Mlo: Prop. 3]. Note that this set is independent of r .

Let $\rho = ((k-1)(r-1))^{1/2}$ and note that $|S| = r(k-1)$. By [IoP: Th. 3] or by [KuS: Th. 1], one has

$$\text{Sp } \lambda(h) = \left\{ \begin{array}{l} \left[\frac{k-2-2\rho}{|S|}, \frac{k-2+2\rho}{|S|} \right], \quad \text{if } k \leq r, \\ \left\{ \frac{-1}{k-1} \right\} \cup \left[\frac{k-2-2\rho}{|S|}, \frac{k-2+2\rho}{|S|} \right], \quad \text{if } k > r. \end{array} \right\}$$

If Γ is the free product $\Gamma_1 * \cdots * \Gamma_r$ of r finite groups $\Gamma_1, \dots, \Gamma_r$, each of order k , and if $S = \bigcup_{1 \leq j \leq r} (\Gamma_j - \{1\})$ as above, then $\text{Sp } \lambda(h)$ remains the same, as noticed in [CS2: §4.2] (the reason being that the respective Cayley graphs are isomorphic).

Notice in this example the appearance of isolated points in $\text{Sp } \lambda(h)$. That these also occur in the case of a free product of two finite groups of distinct order was explicitly worked out in [CS1: Theorem 1].

- (5) Let Γ be a finite group and let S be a union of conjugacy classes of Γ such that $S^{-1} = S$. It is easy to write the spectra of the corresponding Δ in terms of the values on $s \in S$ of the irreducible characters of Γ . Several examples are worked out in Chapter 8 of [Lub].
- (6) Let Γ be a group generated by a finite symmetric set $S^{-1} = S$, so that $\lambda(\Delta)$ is the combinatorial Laplacian of the Cayley graph $\mathcal{G}(\Gamma, S)$. What we have already observed just after the proof of Proposition I can be reformulated as

Γ is non-amenable if and only if $\text{Sp } \lambda(\Delta) \subset [\varepsilon, 2|S|]$ for some $\varepsilon > 0$.

The last formulation carries over to other graphs, and one may estimate ε in terms of an appropriate "isoperimetric constant" of the graph; see [Ger: Th. 2] and [BMS].

- (7) Consider an integer l , the free group Γ on a set $S_+ = \{s_1, \dots, s_l\}$ of l generators, the symmetric set $S = S_+ \cup (S_+)^{-1}$, and let

$$h = \frac{1}{2l} \sum_{s \in S} s \in C^*(\Gamma)$$

be as usual. Consider also a set $\{\varphi_1, \dots, \varphi_l\}$ of isometries of the sphere \mathbb{S}^2 . There is a corresponding action of Γ on \mathbb{S}^2 whereby s_j acts as φ_j , and consequently a representation π of Γ on the Hilbert space

$$L_0^2(\mathbb{S}^2) = \left\{ f \in L^2(\mathbb{S}^2): \int_{\mathbb{S}^2} f(x) d\mu(x) = 0 \right\}$$

where μ denotes the rotation-invariant Lebesgue measure on \mathbb{S}^2 . It can be shown in this case that

$$\|\pi(h)\| \geq \frac{1}{l} \sqrt{2l-1}.$$

In case $l = (p + 1)/2$ for a prime number $p \equiv 1 \pmod{4}$, a deep theorem of Lubotzky, Phillips and Sarnak shows that one may choose the φ'_j 's such that

$$\|\pi(h)\| = \frac{2\sqrt{p}}{p+1}.$$

We refer to [CdV].

EXAMPLES FOR FREE GROUPS AND FOR $SL_2(\mathbb{Z})$.

Let us first construct an irreducible representation π of the free group \mathbb{F}_2 on two generators s_1, s_2 which shows that the hypothesis $S^{-1} = S$ cannot be removed from Claim (5) of Proposition I.

Set $\mathcal{H} = L^2[0, 1]$. Let V be the Volterra integral operator defined by

$$(V\xi)(\alpha) = \int_0^\alpha \xi(\beta) d\beta$$

for all $\xi \in L^2[0, 1]$ and for all $\alpha \in [0, 1]$. Set

$$x = (1 + V)^{-1}.$$

It is known that $\text{Sp } x = \{1\}$, that x has no eigenvector and that $\|x\| = 1$; see Number 148 and Problem 150 of [Hal]. Let $x = u|x|$ be the polar decomposition of x . As $|x|$ is a contraction, one may define two unitary operators

$$v_1 = |x| + i\sqrt{1 - |x|^2} \quad \text{and} \quad v_2 = |x| - i\sqrt{1 - |x|^2}.$$

If we set $u_1 = uv_1$ and $u_2 = uv_2$, one has

$$x = \frac{1}{2}(u_1 + u_2).$$

Define the representation π of \mathbb{F}_2 on \mathcal{H} by

$$\pi(s_j) = u_j, \quad j = 1, 2$$

so that $x = \pi(h)$. Then 1 is of course isolated in $\text{Sp } \pi(h) = \{1\}$, but $\mathcal{H} = \mathcal{H}_0^\perp$ by Proposition I(2).

Let us now show that π is irreducible. We shall use the fact (discussed in Number 151 of [Hal]) that any subspace of \mathcal{H} which is invariant under the Volterra integration operator V is one of

$$\mathcal{H}_\alpha = \{\xi \in L^2[0, 1]: \xi(\beta) = 0 \text{ for almost all } \beta \in [0, \alpha]\}$$

for some $\alpha \in [0, 1]$. Similarly, any subspace of \mathcal{H} invariant under V^* is one of

$$\mathcal{H}^\alpha = \{\xi \in L^2[0, 1]: \xi(\beta) = 0 \text{ for almost all } \beta \in [\alpha, 1]\}$$

for some $\alpha \in [0, 1]$. It follows that the C^* -algebra $C^*(V, V^*)$ generated by V and V^* acts irreducibly on \mathcal{H} . As $(1 + V)^{-1} = \frac{1}{2}(\pi(s_1) + \pi(s_2))$, the C^* -algebra $C_\pi^*(\mathbb{F}_2)$ contains $C^*(V, V^*)$, and it follows that the representation π is indeed irreducible.

Let us denote by $K(\mathcal{H})$ the algebra of all compact operators on \mathcal{H} . Since $C^*(V, V^*)$ is a subalgebra of $K(\mathcal{H})$ that acts irreducibly on \mathcal{H} , we have $C^*(V, V^*) = K(\mathcal{H})$ by [Dix: 4.1.6]. From that it follows that $C_\pi^*(\mathbb{F}_2)$ is the linear span of $K(\mathcal{H})$ and of the identity. Consequently, the only irreducible representations of \mathbb{F}_2 which are weakly contained in π are χ_1 and π itself [Dix: 4.1.10]. A fortiori χ_1 is weakly contained in $\pi \otimes \bar{\pi}$, hence π is amenable.

It was pointed out to us by M. Cowling that, already in 1951, Yoshizawa has constructed an infinite dimensional irreducible representation of \mathbb{F}_2 which is amenable. But Yoshizawa's representation has properties quite different from π , because the former weakly contains any irreducible representations of \mathbb{F}_2 [Yos: §3].

Consider now the group $\Gamma = \text{SL}(2, \mathbb{Z})$, together with the subgroup Γ_0 generated by

$$s_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

It is a classical fact that Γ_0 is free on $\{s_1, s_2\}$, and that it is a normal subgroup of index 12 in Γ . Let σ be the representation induced by π from Γ_0 to Γ . The following result, which is certainly well-known, shows that σ decomposes into finitely many irreducible representations of Γ . We owe the proof to D. Poguntke.

LEMMA 5: *Let Γ_0 be a normal subgroup of finite index of a group Γ , let π be an infinite dimensional irreducible representation of Γ_0 and let σ be the representation induced by π from Γ_0 to Γ . Then there exists a finite number of infinite dimensional irreducible representations $\sigma_1, \dots, \sigma_n$ of Γ such that $\sigma = \bigoplus_{1 \leq j \leq n} \sigma_j$.*

Proof: Let $p: E \rightarrow \Gamma/\Gamma_0$ be the bundle associated with the principal bundle $\Gamma \rightarrow \Gamma/\Gamma_0$ and with π , and let \mathcal{K} be the Hilbert space of sections of p , so that σ acts naturally on \mathcal{K} . For each $x \in \Gamma/\Gamma_0$, let \mathcal{K}_x be the subspace of sections with support in $\{x\}$. As Γ_0 is normal in Γ , each \mathcal{K}_x is $\sigma(\Gamma_0)$ -invariant and $\mathcal{K} =$

$\bigoplus_{x \in \Gamma/\Gamma_0} \mathcal{K}_x$ is a decomposition of \mathcal{K} into infinite dimensional irreducible $(\sigma|_{\Gamma_0})$ -subspaces. It follows that the commutant $\sigma(\Gamma_0)'$ of $\sigma(\Gamma_0)$ in the algebra of all bounded operators on \mathcal{K} is a subalgebra of the algebra of N -by- N matrices, where $N = [\Gamma: \Gamma_0]$; moreover any non-zero projection of $\sigma(\Gamma_0)'$ is infinite dimensional.

A fortiori, the commutant $\sigma(\Gamma)'$ of $\sigma(\Gamma)$ is a finite dimensional algebra. Let $\{p_1, \dots, p_n\}$ be a set of pairwise commuting minimal projections in $\sigma(\Gamma)'$ such that $\sum_{1 \leq j \leq n} p_j = 1$. For each $j \in \{1, \dots, n\}$, the image of p_j is an infinite dimensional subspace of \mathcal{K} which is invariant under and irreducible with respect to $\sigma(\Gamma)$, and one may define σ_j as the corresponding subrepresentation of σ .

■

Let us consider again $\Gamma = SL(2, \mathbb{Z}) \supset \Gamma_0 = \langle s_1, s_2 \rangle$, the representation π of Γ_0 on $L^2([0, 1])$ constructed above and the corresponding $\sigma = \text{Ind}_{\Gamma_0}^{\Gamma}(\pi) = \bigoplus_{1 \leq j \leq n} \sigma_j$. Since the trivial representation of Γ_0 is weakly contained in π , the trivial representation of Γ is weakly contained in σ , and thus also in σ_j for some $j \in \{1, \dots, n\}$. A fortiori, the trivial representation of Γ is weakly contained in $\sigma_j \otimes \bar{\sigma}_j$, meaning that σ_j is amenable. We have proved:

PROPOSITION 2: *There exists an infinite dimensional irreducible representation of $SL(2, \mathbb{Z})$ which is amenable.*

This answers a question of M. Bekka [Bek: Section 5] that was motivated by the fact that, because of property (T), no such representation exists for $SL(n, \mathbb{Z})$ when $n \geq 3$.

B. The peripheral spectrum

We consider again a finitely generated group Γ , a finite set S of generators of Γ and

$$h = \frac{1}{|S|} \sum_{s \in S} s \in C^*(\Gamma).$$

Let π be a unitary representation of Γ on a Hilbert space \mathcal{H} . In this section, we study the peripheral spectrum of $\pi(h)$ by means of the techniques of Section A.

Given any multiplicative character χ of Γ (i.e. a homomorphism χ from Γ to the circle group \mathbb{T}), we define

$$\kappa_{\chi}(\pi, S) = \inf_{\xi \in S^1(\mathcal{H})} \max_{s \in S} \|\pi(s)\xi - \chi(s)\xi\|$$

Clearly, if $\chi = \chi_1$, then $\kappa_\chi(\pi, S) = \kappa(\pi, S)$; also $\kappa_\chi(\pi, S) = \kappa(\bar{\chi}\pi, S)$ for any character χ , where $\bar{\chi}$ is the conjugate character and the product $\bar{\chi}\pi$ is the pointwise product. We say that χ is weakly contained in π if $\kappa_\chi(\pi, S) = 0$.

LEMMA 6: Let π be a representation of Γ on a Hilbert space \mathcal{H} , and let z be a complex number. One has $z \in \text{Sp } \pi(h) \cap \mathbb{T}$ if and only if there exists a character χ of Γ such that $\chi(S) = \{z\}$, and χ is weakly contained in π .

Proof: The "if" part is clear. For the "only if" part, Lemmas 2 and 3 applied to $\bar{z}\pi(h)$ show that there exists a sequence (ξ_j) of vectors in $S^1(\mathcal{H})$ such that

$$(*) \quad \lim_{j \rightarrow \infty} \|\pi(s)\xi_j - z\xi_j\| = 0$$

for all $s \in S$. Remark the following: given $\gamma, \gamma' \in \Gamma$ such that there exists $k, k' \in \mathbb{Z}$ with

$$\begin{aligned} \lim_{j \rightarrow \infty} \|\pi(\gamma)\xi_j - z^k \xi_j\| &= 0, \\ \lim_{j \rightarrow \infty} \|\pi(\gamma')\xi_j - z^{k'} \xi_j\| &= 0, \end{aligned}$$

then

$$\lim_{j \rightarrow \infty} \|\pi(\gamma\gamma')\xi_j - z^{k+k'} \xi_j\| = 0.$$

This remark shows that the constant function $\chi: S \rightarrow \mathbb{T}$ of constant value z extends unambiguously to a character χ of Γ . Finally, it follows from (*) that χ is weakly contained in π . ■

Lemma 6 has the following consequence for the spectra of h in the universal and regular representations.

LEMMA 7: For any z in the peripheral spectrum of h , one has $\text{Sp } h = z \cdot \text{Sp } h$ and $\text{Sp } \lambda(h) = z \cdot \text{Sp } \lambda(h)$.

Proof: Let χ be a character of Γ such that $\chi(S) = \{z\}$, as in Lemma 6. Then $\chi\pi_{un}$ is unitarily equivalent to π_{un} , so $(\chi\pi_{un})(h) = z\pi_{un}(h)$ has the same spectrum as $\pi_{un}(h)$. Observe now that the preceding argument is valid for any representation π of Γ such that $\chi\pi$ is unitarily equivalent to π for any character χ of Γ . In particular it holds for $\pi = \lambda$ (see [Dix: 13.11.3]).

Definition: Given an integer $n \geq 2$, we say that the set S of generators of Γ is n -coloring (bicoloring for $n=2$) if there exists z , a primitive n -th root of 1, and $\chi_{S,z}$, a character of Γ mapping S onto $\{z\}$.

Notice that if $\chi_{S,z}$ exists for one primitive n -th root z , then it exists for all. Below, we shall write $\kappa_z(\pi, S)$ for $\kappa_{\chi_{S,z}}(\pi, S)$.

PROPOSITION 3: Let $n \geq 2$ be an integer.

- (1) The peripheral spectrum of h is a closed subgroup of \mathbb{T} . It contains the set of n -th roots of 1 if and only if S is n -coloring. It coincides with \mathbb{T} if and only if there exists a homomorphism $\alpha: \Gamma \rightarrow \mathbb{Z}$ such that $\alpha(S) = \{1\}$.
- (2) If S is n -coloring, then the Cayley graph $\mathcal{G}(\Gamma, S \cup S^{-1})$ is n -colorable. The converse is true if $n = 2$, namely:

$$S \text{ is bicoloring} \Leftrightarrow -1 \in \text{Sp } h \Leftrightarrow \mathcal{G}(\Gamma, S \cup S^{-1}) \text{ is bicolorable.}$$

- (3) The peripheral spectrum of $\lambda(h)$ is either empty or a closed subgroup of \mathbb{T} . Moreover, the following are equivalent:
 - (a) Γ is not amenable;
 - (b) $1 \notin \text{Sp } \lambda(h)$;
 - (c) $\text{Sp } \lambda(h) \cap \mathbb{T} = \emptyset$.

Proof:

- (1) The first assertion follows from Lemma 7, the second from Lemma 6. The “if” part of the third assertion is easy; for the “only if” part, select some irrational real number θ , and set $z = e^{2\pi i\theta}$. By Lemma 6, there exists a character χ of Γ such that $\chi(S) = \{z\}$. Identifying the subgroup generated by z with \mathbb{Z} , we see that we can take χ for α .
- (2) Recall from [Gra:Chapter VII] that a graph \mathcal{G} is n -colorable if there exists a partition $\mathcal{G}^\circ = \mathcal{G}_1^\circ \amalg \cdots \amalg \mathcal{G}_n^\circ$ of its vertices, such that no edge of \mathcal{G} has both its extremities in the same class. From that definition, both assertions are straightforward.
- (3) The first assertion follows from Lemma 7. The second follows from the fact that a group Γ is amenable if and only if its regular representation λ weakly contains *some* finite-dimensional representation [Fel: Theorem 3]. ■

Definition: Fix an integer $n \geq 2$, and z , a primitive n -th root of 1. Let S be a n -coloring finite set of generators of a group Γ , and let $\chi_{S,z}$ be the corresponding character. Given a representation π of Γ on a Hilbert space \mathcal{H} , we set

$$\mathcal{H}_{S,z}^\pi = \{\xi \in \mathcal{H}: \pi(\gamma)\xi = \chi_{S,z}(\gamma)\xi \text{ for all } \gamma \in \Gamma\}$$

and we denote by $\mathcal{H}_{S,z}^\perp$ the orthogonal complement of $\mathcal{H}_{S,z}^\perp$ in \mathcal{H} . The two corresponding subrepresentations of π are denoted by $\pi_{S,z}^\perp$ and $\pi_{S,z}^\perp$.

The following result, analogous to Proposition I, can easily be rephrased for any complex number z of modulus 1. For notational convenience, we chose to stick to roots of 1.

PROPOSITION 4: Fix an integer $n \geq 2$, and z a primitive n -th root of 1. Let π be a representation of Γ on a Hilbert space \mathcal{H} .

- (1) $z \in \text{Sp } \pi(h)$ if and only if S is n -coloring and $\chi_{S,z}$ is weakly contained in π .
 - (2) z is an eigenvalue of $\pi(h)$ if and only if S is n -coloring and $\chi_{S,z}$ is contained in π (that is $\mathcal{H}_{S,z}^\perp \neq \{0\}$).
 - (3) If S is n -coloring, if $z \in \text{Sp } \pi(h)$ and if $z \notin \text{Sp } \pi_{S,z}^\perp(h)$, then z is an isolated point of $\text{Sp } \pi(h)$.
 - (4) If $\kappa_z(\pi, S) > 0$ then $\text{Sp } \pi(h)$ is disjoint from the open ball $\{w \in \mathbb{C}: |w - z| < \kappa_z(\pi, S)^2 / (2|S|)\}$.
- Assume moreover that $S = S^{-1}$.
- (5) If S is bicoloring and if -1 is isolated in $\text{Sp } \pi(h)$, then $-1 \notin \text{Sp } \pi_{S,-1}^\perp(h)$.
 - (6) If $\text{Sp } \pi(h) \subset [-1 + \varepsilon, 1]$ for some $\varepsilon > 0$, then $\kappa_{-1}(\pi, S) \geq \sqrt{2\varepsilon}$.

Proof: (1) is just a special case of Lemma 6. We leave the details of (2) to (5) to the reader (see Proposition I). For (6), observe that

$$\text{Re}\langle \xi | \pi(s)\xi \rangle \geq -1 + \varepsilon$$

implies

$$\|\pi(s)\xi + \xi\|^2 = 2(1 + \text{Re}\langle \xi | \pi(s)\xi \rangle) \geq 2\varepsilon. \quad \blacksquare$$

A CONSEQUENCE OF PROPOSITIONS I(1) AND 4(1).

Recall that a group Γ is amenable if and only if its regular representation λ weakly contains some finite dimensional representation [Fel: Theorem 3]. Consequently, for Γ and S as usual, the following are equivalent.

- (i) Γ is not amenable.
- (ii) $\text{Sp } \lambda(h)$ is disjoint from $\{1\}$.
- (iii) $\text{Sp } \lambda(h)$ is disjoint from $\{1, -1\}$.

There is a straightforward analogue of Proposition 4 involving a complex number $z \in \mathbb{T}$ and a character $\chi_{S,z}: \Gamma \rightarrow \mathbb{T}$ such that $\chi_{S,z}(S) = \{z\}$. It follows that (i), (ii), (iii) above are equivalent to

(iv) $\text{Sp } \lambda(h)$ is disjoint from \mathbb{T} .

C. Symmetric spectra

Let \mathcal{G} now be an oriented finite graph. One defines naturally an (in general not symmetric) adjacency matrix $A = (A_{v,w})_{v,w \in \mathcal{G}^0}$ for \mathcal{G} . Suppose that \mathcal{G} is connected by oriented paths, namely that the matrix A is irreducible in the sense of Perron–Frobenius theory. It is known that the spectral radius ρ of A is a simple eigenvalue of A , and moreover that the following are equivalent:

- (i) \mathcal{G} is bicolorable,
 - (ii) $-\rho \in \text{Sp } A$,
 - (iii) $\mu \in \text{Sp } A \Leftrightarrow -\mu \in \text{Sp } A$.
- (See e.g. Chapter XIII of [Gan].)

The purpose of this section is to start to investigate how this could carry over to the situation of the previous sections.

Given a self-adjoint operator x on a Hilbert space \mathcal{H} , we set

$$\begin{aligned} \min \text{Sp } x &= \min\{\alpha : \alpha \in \text{Sp } x\} = \min\{\langle \xi | x \xi \rangle : \xi \in \mathbb{S}^1(\mathcal{H})\}, \\ \max \text{Sp } x &= \max\{\alpha : \alpha \in \text{Sp } x\} = \max\{\langle \xi | x \xi \rangle : \xi \in \mathbb{S}^1(\mathcal{H})\}. \end{aligned}$$

Let again Γ , S and h be as in Section A.

PROPOSITION 5: *We assume that Γ is not reduced to one element.*

- (1) *If the set S is bicoloring, then the spectrum of $\lambda(h)$ is symmetric with respect to 0.*
- (2) *The following are equivalent:*
 - (i) *S in bicoloring.*
 - (ii) *The spectrum of h is symmetric with respect to 0.*
 - (iii) *$-1 \in \text{Sp } h$.*

Suppose moreover that $S^{-1} = S$.

- (3) *One has*

$$\max \text{Sp } h \geq \max \text{Sp } \lambda(h) \geq \frac{1}{|S|} > 0$$

and

$$\min \text{Sp } h \leq \min \text{Sp } \lambda(h) \leq 0.$$

- (4) *If $1 \notin S$, then one has moreover*

$$\min \text{Sp } h \leq \min \text{Sp } \lambda(h) \leq -\frac{1}{|S|} < 0.$$

(5) (Due to Kesten.) One has

$$|\min \text{Sp } \lambda(h)| \leq \max \text{Sp } \lambda(h).$$

(6) (Due to Kesten.) If $S^{-1} = S$ one has

$$\max \text{Sp } \lambda(h) \geq 2 \frac{\sqrt{|S| - 1}}{|S|}.$$

If $|S| > 2$ then one has equality if and only if Γ is free on a set S_+ of generators such that $S = S_+ \cup (S_+)^{-1}$.

Proof:

(1) This is a consequence of Lemma 7.

(2) This follows from Lemma 7 and Proposition 3 (2).

As usual, we denote by $(\delta_\gamma)_{\gamma \in \Gamma}$ the canonical basis of $l^2(\Gamma)$. Suppose first that Γ is finite and $\Gamma = S$. Then $\lambda(h)$ is the projection from $l^2(\Gamma)$ onto the line $\mathbb{C} \sum_{\gamma \in \Gamma} \delta_\gamma$, so that $\text{Sp } \lambda(h) = \{0, 1\}$.

To prove (3) and (4), we may assume from now on that $\Gamma \neq S$. As S generates Γ , and as $S^{-1} = S$, it follows that S is not a subgroup of Γ , so that there exist two (not necessarily distinct) elements $s, t \in S$ such that $st \notin S$. We define the vectors

$$\eta_+ = \frac{1}{\sqrt{2}}(\delta_t + \delta_{st})$$

and

$$\eta_- = \frac{1}{\sqrt{2}}(\delta_t - \delta_{st})$$

in $\mathbb{S}^1(l^2(\Gamma))$. We have

$$\langle \lambda(h)\eta_+ | \eta_+ \rangle = \frac{1}{2} \{ \langle \lambda(h)\delta_t | \delta_t \rangle + \langle \lambda(h)\delta_t | \delta_{st} \rangle + \langle \lambda(h)\delta_{st} | \delta_t \rangle + \langle \lambda(h)\delta_{st} | \delta_{st} \rangle \}.$$

The right hand side is equal to

$$\frac{1}{2} \left\{ 0 + \frac{1}{|S|} + \frac{1}{|S|} + 0 \right\} = 1/|S|,$$

if $1 \notin S$ and to $2/|S|$ if $1 \in S$. Thus

$$\max \text{Sp } \lambda(h) \geq 1/|S|$$

in all cases.

Similarly $\langle \lambda(h)\eta_- | \eta_- \rangle$ is equal to $-1/|S|$ if $1 \notin S$ and to 0 if $1 \in S$, so that $\min \text{Sp } \lambda(h)$ is bounded above by $-1/|S|$ if $1 \notin S$ and by 0 in all cases. This completes the proof of Claims (3) and (4).

Let us now show (5), which is due to Kesten [Ke1, Formula 2.11]. Let $\{E_t\}_{t \in \mathbb{R}}$ be the spectral resolution of $\lambda(h)$ (see e.g. [RSN: no. 107]), and let σ be the probability Borel measure defined on \mathbb{R} by

$$\sigma(B) = \int_B d\langle E_t \delta_1 | \delta_1 \rangle$$

for all Borel subsets $B \subset \mathbb{R}$, where $\delta_1 \in l^2(\Gamma)$ denotes the Dirac function of support $\{1\} \subset \Gamma$.

Let us first observe that the spectrum $\text{Sp } \lambda(h)$ coincides with the support $\text{Supp } \sigma$. Indeed, let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} \alpha \in \text{Sp } \lambda(h) &\Leftrightarrow \text{for all } \varepsilon > 0 \text{ one has } E([\alpha - \varepsilon, \alpha + \varepsilon]) \neq 0, \text{ by definition of} \\ &\text{the projection-valued measure } E \text{ associated to } \{E_t\}_{t \in \mathbb{R}} \\ &\Leftrightarrow \text{for all } \varepsilon > 0 \text{ there exists } \gamma \in \Gamma \text{ such that} \\ &\langle E([\alpha - \varepsilon, \alpha + \varepsilon]) \delta_\gamma | \delta_\gamma \rangle \neq 0, \text{ because a non-zero projection has} \\ &\text{some non-zero diagonal matrix element} \\ &\Leftrightarrow \text{for all } \varepsilon > 0 \text{ one has } \langle E([\alpha - \varepsilon, \alpha + \varepsilon]) \delta_1 | \delta_1 \rangle \neq 0, \text{ because} \\ &\lambda(h) \text{ and } E([\alpha - \varepsilon, \alpha + \varepsilon]) \text{ commute with the right regular} \\ &\text{representation of } \Gamma \\ &\Leftrightarrow \alpha \in \text{Supp } \sigma, \text{ by definition of } \sigma. \end{aligned}$$

Let us also observe that all moments of σ are non-negative. Indeed, for each $n \geq 0$ the n -th moment of σ is

$$\mu_n = \int_{\mathbb{R}} t^n d\sigma(t) = \langle \lambda(h)^n \delta_1 | \delta_1 \rangle.$$

As $|S|\lambda(h)$ is the adjacency matrix of the Cayley graph $\mathcal{G}(\Gamma, S)$, the number $\langle |S|^n \lambda(h)^n \delta_1 | \delta_1 \rangle$ is also the number of closed paths of length n in $\mathcal{G}(\Gamma, S)$ starting at 1, so that $\mu_n \geq 0$. (Otherwise said: μ_n is the probability that the appropriate random walk starting at 1 goes back to 1 after n steps, so that $\mu_n \geq 0$.)

Claim (5) is now a consequence of the following standard lemma, from measure theory. For claim (6) we refer to [Ke1]. ■

LEMMA 8: Let σ be a probability Borel measure with compact support on the real line. Set

$$\begin{aligned} -m &= \min\{t \in \mathbb{R}: t \in \text{Supp } \sigma\}, \\ M &= \max\{t \in \mathbb{R}: t \in \text{Supp } \sigma\}. \end{aligned}$$

Assume that one has

$$\begin{aligned} -m &< 0 < M, \\ \mu_n &= \int_{\mathbb{R}} t^n d\sigma(t) \geq 0 \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Then one has

$$\limsup_{n \rightarrow \infty} (\mu_n)^{1/n} = M = \max(m, M).$$

Proof: Set $F(t) = \int_{-\infty}^t d\sigma(t)$ for all $t \in \mathbb{R}$.

We claim first that

$$\limsup_{n \rightarrow \infty} (\mu_n)^{1/n} \geq \max(m, M).$$

For each integer $n \geq 0$, one has

$$\mu_n \leq \int_{\mathbb{R}} |t|^n d\sigma(t) \leq \max(m^n, M^n)[F(M+1) - F(m-1)].$$

As the square bracket is strictly positive by definition of m and M , this implies the claim.

Let us now consider the moments of even order. For each $n \geq 0$ and for each small enough real number $\varepsilon > 0$, one has

$$\begin{aligned} \mu_n &\geq \max \left\{ \int_{-m-\varepsilon}^{-m+\varepsilon} t^{2n} d\sigma(t), \int_{M-\varepsilon}^{M+\varepsilon} t^{2n} d\sigma(t) \right\} \\ &\geq \max \left\{ (m-\varepsilon)^{2n} [F(-m+\varepsilon) - F(-m-\varepsilon)], \right. \\ &\quad \left. (M-\varepsilon)^{2n} [F(M+\varepsilon) - F(M-\varepsilon)] \right\}. \end{aligned}$$

As the square brackets are strictly positive, this implies

$$\limsup_{n \rightarrow \infty} (\mu_{2n})^{1/2n} \geq \max\{m-\varepsilon, M-\varepsilon\}.$$

As this holds for all small enough ε , the left-hand side is also larger than $\max\{m, M\}$, so that one has

$$\limsup_{n \rightarrow \infty} (\mu_n)^{1/n} = \max(m, M).$$

Let us finally consider the moments of odd order. For each $n \geq 0$ and $\varepsilon > 0$ as above, one has

$$\mu_{2n+1} = \int_{-m-\varepsilon}^0 t^{2n+1} d\sigma(t) + \int_0^{M+\varepsilon} t^{2n+1} d\sigma(t) \geq 0$$

so that

$$\begin{aligned} (M + \varepsilon)^{2n+1} [F(M + \varepsilon) - F(0)] &\geq \int_0^{M+\varepsilon} t^{2n+1} d\sigma(t) \\ &\geq \int_{-m-\varepsilon}^{-m+\varepsilon} |t|^{2n+1} d\sigma(t) \geq (m - \varepsilon)^{2n+1} [F(-m + \varepsilon) - F(-m - \varepsilon)]. \end{aligned}$$

This implies again $M + \varepsilon \geq m - \varepsilon$, so that one has finally $M \geq m$. ■

Remarks: (1) Suppose that S is symmetric. If S is bicoloring, then

$$-\min \operatorname{Sp} \lambda(h) = \max \operatorname{Sp} \lambda(h)$$

by Proposition 5 (1). Conversely, answering a question in a preliminary version of our paper, D.I. Cartwright has shown [Car] that the equality above implies that S is bicoloring.

D. A characterization of discrete Kazhdan groups

We now prove Proposition III from the introduction.

- (1) The first assertion follows from Proposition I (4) for $z = 1$, and from Proposition 4 (4) for each other z on the peripheral spectrum of h ; indeed, the sum of all (equivalence classes of) irreducible representations of Γ defines a faithful $*$ -representation of $C^*(\Gamma)$ [Ped: 4.3.7]. To check the second assertion we notice that, by the first assertion, the distance between two distinct elements in the peripheral spectrum of h is at least ε . Since the angle subtended by an interval of length ε inscribed in \mathbb{T} is $2\operatorname{Arcsin} \frac{\varepsilon}{2}$, we see that the peripheral spectrum has at most $\pi / \operatorname{Arcsin} \frac{\varepsilon}{2}$ points.
- (2) If $\operatorname{Sp} h \subset [-1, 1 - \varepsilon] \cup \{1\}$, the conclusion follows from Proposition I (6); if $-1 \in \operatorname{Sp} h \subset \{-1\} \cup [-1 + \varepsilon, 1]$, then S is bicoloring by Proposition 3 (2), and the conclusion follows from Proposition 4 (6), and the fact that $\kappa_{-1}(\pi, S) = \kappa(\chi_{S, -1}\pi, S)$.

QUESTIONS:

- (1) Are Kazhdan groups characterized by the fact that 1 is an isolated point of $\text{Sp } h$ in the case when S is not symmetric?
- (2) Assume that Γ has property (T). Consider a symmetric set of generators $S^{-1} = S$. Let

$$\varepsilon_- = 1 + \min\{\alpha \in \text{Sp } h: \alpha > -1\},$$

$$\varepsilon_+ = 1 - \max\{\alpha \in \text{Sp } h: \alpha < 1\},$$

so that

$$\{-1 + \varepsilon_-, 1 - \varepsilon_+\} \subset \text{Sp } h \subset \{-1\} \cup [-1 + \varepsilon_-, 1 - \varepsilon_+] \cup \{1\}.$$

If S is bicoloring, then Proposition 5 (2) shows that $\varepsilon_- = \varepsilon_+$. How do ε_- and ε_+ compare in general?

Remarks:

- (1) It has been shown in [Val] that a locally compact group G is a Kazhdan group if and only if there is a (necessarily unique) projection p_G in $C^*(G)$ which is annihilated by every irreducible representation of G distinct from χ_1 . If Γ is a countable Kazhdan group, S is a symmetric finite generating subset of Γ , and h is as usual, then 1 is isolated in $\text{Sp } h$ by Proposition III (1). It also follows from Proposition I (6) that the spectral projection of h corresponding to 1 is the projection p_Γ mentioned above.
- (2) If S is symmetric then everything in Proposition III (2) can be reformulated in terms of the Laplacian $\Delta = |S|(1 - h)$. For example, if Γ is a Kazhdan group then $\text{Sp } \Delta \subset \{0\} \cup [\frac{1}{2}\hat{\kappa}(\Gamma, S)^2, 2|S|]$. Conversely if $\text{Sp } \Delta \subset \{0\} \cup [|S|\varepsilon, 2|S|]$ for some $\varepsilon > 0$, then Γ is a Kazhdan group and $\kappa(\Gamma, S) \geq \sqrt{2\varepsilon}$.

■

Denote by $\mu_1(\mathcal{G})$ the smallest positive eigenvalue of the Laplacian of a finite graph \mathcal{G} .

COROLLARY TO PROPOSITION III: Let Γ be a discrete Kazhdan group with a finite set $S = S^{-1}$ of generators. Let $\hat{\kappa} = \hat{\kappa}(\Gamma, S)$. Let φ be a homomorphism of Γ onto a finite group Γ_0 whose restriction to $S \cup \{1\}$ is injective. Consider the Cayley graph $\mathcal{G} = \mathcal{G}(\Gamma_0, \varphi(S))$. Then $\mu_1(\mathcal{G}) \geq \hat{\kappa}^2/2$.

Proof: If λ_0 is the regular representation of Γ_0 on $l^2(\Gamma_0)$, then $\lambda_0 \circ \varphi(\Delta)$ is the combinatorial Laplacian of \mathcal{G} . The conclusion follows from the above Remark,

since

$$\text{Sp } \lambda_0 \circ \varphi(\Delta) \subset \text{Sp } \Delta. \quad \blacksquare$$

Remarks:

- (1) $\mu_1(\mathcal{G})$ provides qualitative information about the graph. For example, graphs with large μ_1 tend to have large connectivity and small diameter. See [Bie] for a survey of results in this area. The above Corollary is a slight improvement on Lemma 2.3 of [AIM], since the inequality $\hat{\kappa}(\Gamma, S) \geq \kappa(\Gamma, S)$ may be strict, even for finite groups. Also $\hat{\kappa}(\Gamma, S)$ is easier to calculate.
- (2) M. Burger has obtained inequalities related to some Kazhdan constants for $\text{SL}(3, \mathbb{Z})$ [Bur]. These can be combined with the Corollary to give a lower bound for μ_1 when

$$\Gamma = \Gamma_0 = \text{SL}(3, \mathbb{Z}/N\mathbb{Z}) \quad (N \geq 2)$$

and S is the set of all matrices of the form

$$\begin{bmatrix} 1 & 2 & j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & j \\ 0 & 0 & 1 \end{bmatrix}, \quad j = -1, 0, 1$$

and their inverses. The result is that for the corresponding Cayley graph \mathcal{G} ,

$$\mu_1(\mathcal{G}) \geq (1 - n^{-1/2})^2 / 32$$

where n is the product of the distinct prime factors of N .

E. Finite dimensional irreducible representations of a Kazhdan group

The aim of this section is to prove Proposition IV from the introduction.

We first need the following simple result.

LEMMA 9: *Let Γ be a finitely generated group with finite generating set S . If π_1 and π_2 are irreducible representations of Γ and 1 is an eigenvalue of the operator $(\pi_1 \otimes \bar{\pi}_2)(h)$, then π_1 and π_2 are equivalent finite dimensional representations.*

Proof: Let $\mathcal{H}_1, \mathcal{H}_2$ be the representation spaces of π_1, π_2 respectively. Using the usual identification of $\mathcal{H}_1 \otimes \bar{\mathcal{H}}_2$ with the space of Hilbert-Schmidt operators from \mathcal{H}_2 into \mathcal{H}_1 , we write $\pi_1 \otimes \bar{\pi}_2$ in the form

$$(\pi_1 \otimes \bar{\pi}_2)(\gamma)\xi = \pi_1(\gamma)\xi\pi_2(\gamma)^{-1}$$

where $\xi: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ is Hilbert–Schmidt.

Choose ξ with Hilbert–Schmidt norm $\|\xi\|_2 = 1$ such that $(\pi_1 \otimes \bar{\pi}_2)(h)\xi = \xi$.

That is

$$\frac{1}{|S|} \sum_{s \in S} (\pi_1 \otimes \bar{\pi}_2)(s)\xi = \xi.$$

By Lemma 3, we have $(\pi_1 \otimes \bar{\pi}_2)(s)\xi = \xi$ for all $s \in S$. It follows that $\pi_1(\gamma)\xi = \xi\pi_2(\gamma)$, for all $\gamma \in \Gamma$. Since π_1, π_2 are irreducible, $\pi_1 \simeq \pi_2$ and ξ is an isomorphism. Since ξ is a compact operator, π_1, π_2 are necessarily finite dimensional.

■

Remark: A similar argument shows that a (possibly reducible) unitary representation π of Γ has a finite dimensional subrepresentation if and only if 1 is an eigenvalue of $(\pi \otimes \bar{\pi})(h)$. ■

Now let $\mathcal{H} = \mathbb{C}^m$ be a fixed finite dimensional Hilbert space. Denote by $\|x\|_2 = \text{tr}(x^*x)^{1/2}$ the Hilbert–Schmidt norm of an operator x on \mathcal{H} . The next result is corollary 2 of [Was].

LEMMA 10: *Let Γ be a discrete Kazhdan group, let S be a finite set of generators of Γ and set*

$$\varepsilon = \frac{1}{2|S|} \hat{\kappa}(\Gamma, S)^2.$$

If π_1, π_2 are irreducible representations of Γ on $\mathcal{H} = \mathbb{C}^m$ such that $\|\pi_1(s) - \pi_2(s)\|_2 < \varepsilon\sqrt{m}$, for all $s \in S$, then π_1 is equivalent to π_2 .

Proof: We may assume that $S = S^{-1}$, since

$$\|\pi_1(s^{-1}) - \pi_2(s^{-1})\|_2 = \|\pi_1(s) - \pi_2(s)\|_2.$$

If $I: \mathcal{H} \rightarrow \mathcal{H}$ is the identity map, then

$$\begin{aligned} \|I - (\pi_1 \otimes \bar{\pi}_2)(h)(I)\|_2 &= \left\| I - \frac{1}{|S|} \sum_{s \in S} \pi_1(s)\pi_2(s)^{-1} \right\|_2 \\ &\leq \frac{1}{|S|} \sum_{s \in S} \|I - \pi_1(s)\pi_2(s)^{-1}\|_2 \\ &= \frac{1}{|S|} \sum_{s \in S} \|\pi_2(s) - \pi_1(s)\|_2 \\ &< \varepsilon\sqrt{m}. \end{aligned}$$

Let $\eta = m^{-1/2}I$, so that $\|\eta\|_2 = 1$. Then

$$1 - \langle \eta | (\pi_1 \otimes \bar{\pi}_2)(h) \eta \rangle = \langle \eta | \eta - (\pi_1 \otimes \bar{\pi}_2)(h) \eta \rangle < \varepsilon.$$

Therefore $\text{Sp}(\pi_1 \otimes \bar{\pi}_2)(h)$ meets $(1 - \varepsilon, 1]$. It follows from Proposition III (1) that 1 is an eigenvalue of $(\pi_1 \otimes \bar{\pi}_2)(h)$ and so $\pi_1 \simeq \pi_2$, by the preceding Lemma.

■

Proof of Proposition IV from the Introduction: We continue with the notation of Lemma 10.

Let $n = |S|$ and $S = \{s_1, \dots, s_n\}$. Define a norm on $B(\mathcal{H})^n$ by

$$\|(x_1, \dots, x_n)\| = \max_{1 \leq j \leq n} \|x_j\|_2.$$

Lemma 10 says that $\pi_1 \simeq \pi_2$ whenever π_1, π_2 are irreducible representations of Γ on $\mathcal{H} = \mathbb{C}^m$ satisfying

$$\|(\pi_1(s_1), \dots, \pi_1(s_n)) - (\pi_2(s_1), \dots, \pi_2(s_n))\| < \varepsilon\sqrt{m}.$$

Let k be the number of balls of Hilbert–Schmidt radius $\varepsilon\sqrt{m}/2$ which are required to cover the unitary group $U(m)$. Then $U(m)^n$ is covered by k^n balls of radius $\varepsilon\sqrt{m}/2$, so there are at most k^n inequivalent irreducible representations of Γ on \mathbb{C}^m . It remains to estimate k .

Now $U(m) \subset \{x \in B(\mathbb{C}^m) : \|x\|_2 = \sqrt{m}\}$. Using matrix entries as coordinates, and identifying \mathbb{C} with \mathbb{R}^2 , we have, for the usual Euclidean norm,

$$U(m) \subset \{x \in \mathbb{R}^{2m^2} : \|x\| = \sqrt{m}\}.$$

It is therefore enough to find the number of balls of radius $\varepsilon\sqrt{m}/2$ which are required to cover a sphere of radius \sqrt{m} in \mathbb{R}^{2m^2} . This is the same as covering a sphere of radius 1 with balls of radius $\varepsilon/2$. A sharp asymptotic bound for this is given by the Corollary in [Wyn]. There it is shown that $k = e^{\alpha m^2}$ balls are enough to do the job, where $\alpha > 0$ is constant. In fact, for sufficiently large m , we need only choose $\alpha > -2 \log \sin \theta$, where θ is the half-angle subtended at 0 by a ball of radius $\varepsilon/2$ with centre on the sphere of radius one. Letting $A = A(\Gamma, S)$ denote the constant αn of the preceding argument, we obtain

$$\text{Irrep}_\Gamma(m) = O(e^{Am^2}).$$

Remark: There exists a discrete Kazhdan group with no nontrivial finite dimensional representations. This follows from [Gro: Chapter 5]. The result is that any lattice in $\mathrm{Sp}(1, q)$, $q \geq 2$, has uncountably many infinite quotients which are simple and torsion. Such a quotient provides the desired example. For if there is a non-trivial finite dimensional representation then it is faithful (by simplicity), so the group is linear. However a linear non-amenable group has to contain a copy of the free group on two generators by Tits' theorem [Tit]. This contradicts the fact that the group is a torsion group.

Let Γ be the group $\mathrm{SL}(n, \mathbb{Z})$, with $n \geq 3$. Steinberg [Ste] has shown that any finite dimensional unitary representation of Γ factorizes through $\mathrm{SL}(n, \mathbb{Z}/k\mathbb{Z})$ for some integer k . Using this and information on the characters of $\mathrm{SL}(n, \mathbb{Z}/p\mathbb{Z})$ for prime p 's, it should be possible to improve the bound of Proposition IV for $\mathrm{SL}(n, \mathbb{Z})$, and also to obtain lower asymptotic bounds for $\mathrm{Irrep}_{\Gamma}(m)$. ■

In the line of our work, there are numerous questions which remain open. Let us mention here the following ones:

When is $\mathrm{Sp}(\lambda(h))$ connected?

When does $\lambda(h)$ have eigenvalues besides 1 and -1 ?

When is the spectral measure of $\lambda(h)$ absolutely continuous with respect to the Lebesgue measure on $[-1, 1]$?

Finally we remark that Proposition I of the present paper has been used in [HRV] to obtain results on exactness for group C^* -algebras.

Note added in proof: In a remarkable piece of work, D. I. Cartwright, W. Mlotkowski and T. Steger, *Property (T) and \tilde{A}_2 groups*, preprint, University of Sydney, 1992, the authors show by combinatorial methods that certain groups associated with buildings have property (T). Moreover they calculate the spectrum of h and obtain the exact values of the Kazhdan constants, using the estimates of Proposition I (6).

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