SPACES OF CLOSED SUBGROUPS OF LOCALLY COMPACT GROUPS

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Abstract. The set $\mathcal{C}(G)$ of closed subgroups of a locally compact group G has a natural topology which makes it a compact space. This topology has been defined in various contexts by Vietoris, Chabauty, Fell, Thurston, Gromov, Grigorchuk, and many others.

The purpose of the talk was to describe the space $\mathcal{C}(G)$ first for a few elementary examples, then for G the complex plane, in which case $\mathcal{C}(G)$ is a 4–sphere (a result of Hubbard and Pourezza), and finally for the 3–dimensional Heisenberg group H, in which case $\mathcal{C}(H)$ is a 6–dimensional singular space recently investigated by Martin Bridson, Victor Kleptsyn and the author [BrHK].

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I. Mahler (1946)

Let n be a positive integer. A *lattice* in $\mathbf{R}^n$ is a subgroup of $\mathbf{R}^n$ generated by a basis. Two lattices $L, L' \subset \mathbf{R}^n$ ar *close to each other* if there exist basis $\{e_1, \ldots, e_n\} \subset L$, $\{e'_1, \ldots, e'_n\} \subset L'$ with e_j and e'_j close to each other for each j; this defines a topology on the space $\mathcal{L}(\mathbf{R}^n)$ of lattices in $\mathbf{R}^n$. It coincides with the natural topology on $\mathcal{L}(\mathbf{R}^n)$ viewed as the homogeneous space $GL_n(\mathbf{R})/GL_n(\mathbf{Z})$. It is easy to check that the covolume function $L \longmapsto \mathrm{vol}(\mathbf{R}^n/L)$ and the minimal norm $L \longmapsto \min L \doteqdot \min_{x \in L, x \neq 0} \|x\|^2$ are continuous functions $\mathcal{L}(\mathbf{R}^n) \longrightarrow \mathbf{R}^*_+$.

1. Mahler's Criterion [Mahl–46]**.** *For a subset $\mathcal{M}$ of $\mathcal{L}(\mathbf{R}^n)$, the two following properties are equivalent:*
(i) $\mathcal{M}$ is relatively compact;
(ii) there exist $C, c > 0$ such that $\mathrm{vol}(\mathbf{R}^n/L) \le C$ and $\min L \ge c$ for all $L \in \mathcal{M}$.

For proofs of this criterion, see for example [Bore–69, corollaire 1.9] or [Ragh–72, Corollary 10.9].

An immediate use of the criterion is the proof that, in any dimension $n \ge 1$, the space $\mathcal{L}^{\mathrm{unimod}}(\mathbf{R}^n)$ of lattices of covolume 1 contains a lattice $L_{\max}$ such that $\min L_{\max} = \sup\{\min L \mid L \in \mathcal{L}^{\mathrm{unimod}}(\mathbf{R}^n)\}$; equivalently, there exists a lattice $L_{\max}$ with a maximal

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density for the associated ball packing. (It is a much more difficult problem to identify such a lattice $L_{\max}$, and this problem is open unless $n \le 8$ or $n = 24$.)

Let us quote one other application of Mahler's Criterion. It is one of the ingredients of the proof of the following fact, special case for $SL_n(\mathbf{R})$ of a more general result conjectured by Godement and proved independently by Borel–Harish Chandra and by Mostow–Tamagawa [Ragh–72, Theorem 10.18]: *an arithmetic lattice in $SL_n(\mathbf{R})$ is cocompact if and only if it does not contain any unipotent matrix.*

II. Chabauty (1950) and Fell (1962)

For a topological space X, let 2^X denote the set of closed subsets of X. For a compact subset K and a nonempty open subset U of X, set

$$\mathcal{O}_K \;=\; \{F \in 2^X \mid F \cap K = \emptyset\} \quad \text{ and } \quad \mathcal{O}'_U \;=\; \{F \in 2^X \mid F \cap U \neq \emptyset\}.$$

The finite intersections $\mathcal{O}_{K_1} \cap \cdots \cap \mathcal{O}_{K_m} \cap \mathcal{O}'_{U_1} \cap \cdots \cap \mathcal{O}'_{U_n}$, $m, n \ge 0$, constitute a basis of the *Chabauty topology* on 2^X; observe that $\mathcal{O}_{K_1} \cap \cdots \cap \mathcal{O}_{K_m} = \mathcal{O}_{K_1 \cup \cdots \cup K_m}$.

For example, consider the case $X = \mathbf{R}^n$, and two lattices L_0, L in this space. If K is a compact subset of $\mathbf{R}^n$ disjoint from L_0, then $L \in \mathcal{O}_K$ if and only if L is also disjoint from K; if U is an open neighbourhood of some point of L_0, then $L \in \mathcal{O}'_U$ if and only if L has also one point (at least) in U. It follows that the Chabauty topology on $2^{\mathbf{R}^n}$ induces on $\mathcal{L}(\mathbf{R}^n)$ the same topology as that considered in Section I.

This topology was defined in [Chab–50] (written in French), as a tool to show a version of Mahler's Criterion which is valid in a large class of topological groups (Proposition 3 below). Later, this topology was studied in greater detail by Fell for unrelated purposes, precisely for the study of the appropriate topology on the space of irreducible unitary representations of a locally compact group; Fell (who writes in English) does not quote Chabauty (see [Fell–62] and [Fell–64]). This *Chabauty topology,* or *Chabauty–Fell topology,* or *H–topology* (terminology of Fell), or *geometric topology* (terminology of Thurston), on 2^X should not be confused with other standard topologies on the same space, of which the study goes back to Hausdorff and Vietoris, and for which a canonical reference is [Mich–51] (however, for X compact, the Chabauty topology coincides with the F–topology of Michael). The Chabauty topology is also useful in the study of low–dimensional manifolds; see [Thurs, Definition 9.1.1], as well as [CaEG–87, Chapter I.3], where Chabauty is quoted (but Fell is not). A pleasant account of the most basic properties of this topology can be found in [Paul–07].

2. Proposition. *Let X be a topological space, and let 2^X be endowed with the Chabauty topology.*

(i) *The space 2^X is compact.*
(ii) *In case X is discrete, 2^X is homeomorphic to the product space $\{0,1\}^X$ with the compact Tychonoff product topology.*

Suppose moreover that X is a locally compact metric space.

(iii) *The Chabauty topology on 2^X is induced by the metric defined by*

$$d(F_1, F_2) \;=\; \inf \left\{ \epsilon > 0 \;\middle|\; \begin{array}{lcl} F_1 \cup \Big(X \smallsetminus B(*, 1/\epsilon) \Big) & \subset & \mathcal{V}_\epsilon \Big(F_2 \cup \Big(X \smallsetminus B(*, 1/\epsilon) \Big) \Big) \\ F_2 \cup \Big(X \smallsetminus B(*, 1/\epsilon) \Big) & \subset & \mathcal{V}_\epsilon \Big(F_1 \cup \Big(X \smallsetminus B(*, 1/\epsilon) \Big) \Big) \end{array} \right\}.$$

Here, for a subset S of X, we write $\mathcal{V}_\epsilon(S)$ for $\{x \in X \mid d(x, S) < \epsilon\}$; and $B(*, 1/\epsilon)$ is the open ball of radius $1/\epsilon$ around an arbitrarily chosen base point $* \in X$.

(iv) *A sequence* $(F_j)_{j\geq 1}$ *of closed subsets of* X *converges in* 2^X *to a closed subset* F *if and only if the two following conditions hold:*
— *for all* $x \in F$, *there exists for all* $i \geq 1$ *a point* $x_i \in F_i$ *such that* $x_i \to x$,
— *for all strictly increasing sequence* $(i_j)_{j\geq 1}$ *and for all sequences* $(x_{i_j})_{j\geq 1}$ *such that* $x_{i_j} \in F_{i_j}$ *and* $x_{i_j} \to x \in X$, *we have* $x \in F$.

(v) *If* $X' = X \sqcup \{\omega\}$ *is the one–point compactification of* X, *then* $F \longmapsto F \sqcup \{\omega\}$ *is a homeomorphism from* 2^X *to the subspace of* $2^{X'}$ *of closed sets containing* $\{\omega\}$. *Moreover, the subset* $\big\{\{x\} \in 2^X \mid x \in X\big\} \cup \emptyset$ *of* 2^X *and its image in* $2^{X'}$ *are homeomorphic to* X'.

Suppose moreover that $X = G$ *is a locally compact group, and let* $\mathcal{C}(G)$ *be the subset of* 2^G *of closed subgroups.*

(vi) *The subspace* $\mathcal{C}(G)$ *of* 2^G *is closed, and therefore compact.*

(vii) *In* $\mathcal{C}(G)$, *a basis of neighbourhoods of a closed subgroup* C *is given by*

$$\mathcal{N}_{K,U}(C) \;=\; \{D \in \mathcal{C}(G) \mid D \cap K \subset CU \quad \text{and} \quad C \cap K \subset DU\}.$$

Comments. (i) The proof appears in [Fell–62, Theorem 1]. In general, 2^X need not be Hausdorff, even if X is metrisable; thus in a french–like terminology, 2^X is quasi–compact. However, if X is locally compact and possibly non–Hausdorff, then 2^X is a compact Hausdorff space. In the relevant context and in terms of the geometric topology (defined by the conditions of (iv)), the proof of (i) appears also in [Thurs, Proposition 9.1.6], where it is established that $\mathcal{C}(G)$ is compact for a Lie group G.

(ii) For a positive integer k, let F_k denote the free group on k generators. The space $\mathcal{N}(F_k)$ of normal subgroups of F_k is closed in $2^{F_k} = \{0,1\}^{F_k}$, and therefore compact. This space can be naturally identifed with the *space of marked groups on k generators*, namely of groups Γ given together with a generating set $\{s_1, \ldots, s_k\}$, or equivalently together with a quotient homomorphism $F_k \longrightarrow \Gamma$. This space has been intensively studied in recent years: see among others [Grom–81, final remarks], [Grig–84], [Cham–00], [ChGu–05], and [CoGP–07].

(iii) and (iv) See for example [Paul–07], Proposition 1.8, Page 60.

(v) See [Fell–62, Page 475], or [Bour–63], chapitre VIII, § 5, exercice 1.

(vi) See [Fell–62, Page 474].

From now on, $\mathcal{C}(G)$ will denote the compact space of closed subgroups of a locally compact group G, furnished with the Chabauty topology. It has several subspaces of interest, including:

the space $\mathcal{D}(G)$ of discrete subgroups of G,
the space $\mathcal{L}(G)$ of lattices of G,
the space $\mathcal{A}(G)$ of closed abelian subgroups of G,
the space $\mathcal{N}(G)$ of closed normal subgroups of G.

(Recall that a *lattice* in G is a discrete subgroup Λ such that G/Λ has a G-invariant probability measure.)

3. Proposition (Chabauty's Mahler's Criterion). *Let* G *be a unimodular*[1] *locally compact group satisfying some extra technical conditions, for example let* G *be a connected unimodular Lie group, and let* M *be a subset of* $\mathcal{L}(G)$. *Then* $\mathcal{M}$ *is relatively compact if and only if*

(i) *there exists a constant* $C > 0$ *such that* $\mathrm{vol}(G/\Lambda) \leq C$ *for all* $\Lambda \in \mathcal{M}$,

(ii) *there exists a neighbourhood* U *of* e *in* G *such that* $\Lambda \cap U = \{e\}$ *for all* $\Lambda \in \mathcal{M}$.

[1]Recall that, if G was not unimodular, it would not contain any lattice at all; in other terms, one would have $\mathcal{L}(G) = \emptyset$. Even if G is unimodular, it may happen that $\mathcal{L}(G) = \emptyset$; this happens for example with nilpotent Lie groups of which the Lie algebra $\mathfrak{g}$ has no rational form; a *rational form* of a Lie algebra $\mathfrak{g}$ is a Lie algebra $\mathfrak{g}_0$ over $\mathbf{Q}$ such that $\mathfrak{g}_0 \otimes_{\mathbf{Q}} \mathbf{R}$ and $\mathfrak{g}$ are isomorphic real Lie algebras.

The Chabauty topology provides some natural compactifications. More precisely, let S be a space of which the points are in natural correspondance with closed subgroups of a given locally compact group G, in such a way that the corresponding injection $\varphi : S \longrightarrow \mathcal{C}(G)$ is continuous. Then $\overline{\varphi(S)}$ is a compactification of S. Examples to which this applies are:

- → Riemannian symmetric spaces of the non–compact type S, for which each point of S corresponds to a maximal compact subgroup of the isometry group S;
- → Bruhat–Tits buildings;
- → the space of complete Riemannian manifolds of dimension n and of constant sectional curvature -1, given together with a base point and an orthonormal basis of the tangent space at this base point; this is a space which can be identified with a space of discrete subgroups of the isometry group of the hyperbolic space H^n.

III. First examples

If $G = \mathbf{R}$, the space $\mathcal{C}(\mathbf{R})$ is homeomorphic to a compact interval $[0, \infty]$. The points 0, λ (with $0 < \lambda < \infty$), and ∞ correspond respectively to the subgroups $\{0\}$, $\frac{1}{\lambda}\mathbf{Z}$, and $\mathbf{R}$.

The space $\mathcal{C}(\mathbf{Z})$ is homeomorphic to the subspace $\left\{\frac{1}{n}\right\}_{n\geq 1} \cup \{0\}$ of $[0,1]$, with $\frac{1}{n}$ corresponding to $n\mathbf{Z}$ and 0 to $\{0\}$. [Exercise: the spaces $\mathcal{C}(\mathbf{R}/\mathbf{Z})$ and $\mathcal{C}(\mathbf{Z})$ are homeomorphic.]

Even if the list of easily understandable spaces $\mathcal{C}(G)$ could be slightly extended (exercise: look at $SO(3)$ and at the affine group of the real line), it is essentially a very short list. This was our main motivation to understand two more cases : the additive group of $\mathbf{C}$ ($= \mathbf{R}^2$), see [HuPo–79] and Section IV below, and the Heisenberg group H, see [BrHK] and Sections V to VII.

In $\mathbf{R}^n$, $n \geq 1$, any closed subgroup is isomorphic to one of $\mathbf{R}^a \oplus \mathbf{Z}^b$, where a, b are non–negative integers such that $0 \leq a + b \leq n$. For a given pair (a, b), the subspace[2] $\mathcal{C}_{\mathbf{R}^a\oplus\mathbf{Z}^b}(\mathbf{R}^n)$ of $\mathcal{C}(\mathbf{R}^n)$ is a homogeneous space of $GL_n(\mathbf{R})$, for example $\mathcal{C}_{\mathbf{Z}^n}(\mathbf{R}^n) = \mathcal{L}(\mathbf{R}^n) = GL_n(\mathbf{R})/GL_n(\mathbf{Z})$, as already observed in Section I. But the way these different "strata" are glued to each other to form $\mathcal{C}(\mathbf{R}^n)$ is complicated, and we do not know of any helpful description if $n \geq 3$.

To describe the case $n = 2$, it will be convenient to think of $\mathbf{C}$ rather than of $\mathbf{R}^2$.

IV. Hubbard and Pourezza (1979)

We will describe $\mathcal{C}(\mathbf{C})$ in three steps: first the space $\mathcal{C}_{\rm nl}(\mathbf{C}) = \mathcal{C}_{\{0\},\mathbf{Z},\mathbf{R},\mathbf{R}\oplus\mathbf{Z},\mathbf{C}}(\mathbf{C})$ of closed subgroups of $\mathbf{C}$ which are *not* lattices (easy step), then the space $\mathcal{L}(\mathbf{C})$ of lattices (classical step in complex analysis), and finally the way these two parts are glued to each other to form a 4–sphere (the contribution of Hubbard and Pourezza).

IV.1. The space $\mathcal{C}_{\rm nl}(\mathbf{C})$ is a 2–sphere.

In $\mathcal{C}(\mathbf{C})$, the subset of closed subgroups isomorphic to $\mathbf{R}$ is the real projective line $\mathbf{P}^1$, namely a circle. Each infinite cyclic subgroup of $\mathbf{C}$ is contained in a unique real line, and the subgroup $\{0\}$ is contained in all of them. It follows that the space $\mathcal{C}_{\{0\},\mathbf{Z},\mathbf{R}}(\mathbf{C})$ is a cone over $\mathbf{P}^1$, namely a closed disc. We rather think of it as the closed lower hemisphere of a 2–sphere $\mathbf{S}^2$.

[2]Let $G, A, \dots, B$ be topological groups. We denote by $\mathcal{C}_{A,\dots,B}(G)$ the subspace of $\mathcal{C}(G)$ of closed subgroups of G topologically isomorphic to one of $A, \dots, B$.

A closed subgroup C of $\mathbf{C}$ isomorphic to $\mathbf{R} \oplus \mathbf{Z}$ is uniquely determined by its connected component C^o, isomorphic to $\mathbf{R}$, and by the minimal norm $\min_{z \in C, z \notin C^o} |z|$. When this minimal norm is very large [respectively very small], C is "near" a closed subgroup of $\mathbf{C}$ isomorphic to $\mathbf{R}$ [respectively is "near" $\mathbf{C}$]. It follows that the space $\mathcal{C}_{\mathbf{R},\mathbf{R}\oplus\mathbf{Z},\mathbf{C}}(\mathbf{C})$ is also a cone over $\mathbf{P}^1$, which can be identified to the closed upper hemisphere of $\mathbf{S}^2$.

Consequently, $\mathcal{C}_{\text{nl}}(\mathbf{C})$ is homeomorphic to $\mathbf{S}^2$, with

$\{0\}$ corresponding to the South Pole,
$\mathcal{C}_{\mathbf{Z}}(\mathbf{C})$ to the complement of the South Pole in the open lower hemisphere,
$\mathcal{C}_{\mathbf{R}}(\mathbf{C})$ to the equator,
$\mathcal{C}_{\mathbf{R}\oplus\mathbf{Z}}(\mathbf{C})$ to the complement of the North Pole in the open upper hemisphere,
$\mathbf{C}$ to the North Pole.

IV.2. The space $\mathcal{L}(\mathbf{C})$ is the product of an open interval with the complement of a trefoil knot in $\mathbf{S}^3$.

For a lattice L in $\mathbf{C}$, set as usual

$$g_2(L) = 60 \sum_{z \in L, z \neq 0} z^{-4}, \qquad g_3(L) = 140 \sum_{z \in L, z \neq 0} z^{-6}, \qquad \Delta(L) = g_2(L)^3 - 27 g_3(L)^3.$$

Let Σ be the complex curve in $\mathbf{C}^2$ of equation $a^3 = 27b^2$. It is a classical fact that we have a homeomorphism

$$g : \begin{cases} \mathcal{L}(\mathbf{C}) & \longrightarrow & \mathbf{C}^2 \smallsetminus \Sigma \\ L & \longmapsto & (g_2(L), g_3(L)) \end{cases}$$

(see for example [SaZy–65, § VIII.13]). By the same formulas, g can be extended to a homeomorphism

$$g' : \mathcal{L}_{\{0\},\mathbf{Z},\mathbf{Z}^2}(\mathbf{C}) \longrightarrow \mathbf{C}^2.$$

Observe that Σ, viewed as a real surface in $\mathbf{C}^2 \approx \mathbf{R}^4$, is smooth outside the origin, and that its intersection with the unit sphere $\mathbf{S}^3$ of equation $|a|^2 + |b|^2 = 1$ is a *trefoil knot*

$$T = \left\{ (a,b) \in \mathbf{S}^3 \mid a^3 = b^2 \right\}.$$

(It follows that the origin is indeed a singular point of Σ.)

The multiplicative group $\mathbf{C}^*$ acts on $\mathcal{C}(\mathbf{C})$ and on $\mathbf{C}^2$ by

$$(s, C) \longmapsto \sqrt{s} C \quad \text{and} \quad \big(s, (a,b)\big) \longmapsto (s^{-2}a, s^{-3}b)$$

(observe that $\sqrt{s}C$ is well defined, because $-C = C$). Moreover, the homeomorphism g' is $\mathbf{C}^*$–equivariant.

Let us restrict these actions to the subgroup $\mathbf{R}^*_+$ of $\mathbf{C}^*$. The resulting action of $\mathbf{R}^*_+$ on $\mathcal{L}(\mathbf{C})$ is free, and its orbits are transverse to the subset $\mathcal{L}^{\text{unimod}}(\mathbf{C})$ of unimodular lattices in $\mathbf{C}$. The resulting action of $\mathbf{R}^*_+$ on $\mathbf{C}^2 \smallsetminus \Sigma$ is also free, and its orbits are transverse to the sphere $\mathbf{S}^3$. And the homeomorphism g is $\mathbf{R}^*_+$–equivariant. [The restriction of the action to the subgroup of complex numbers of modulus one is also interesting, producing on $\mathcal{L}^{\text{unimod}}(\mathbf{C})$ the structure of a Seifert manifold, but we will not expand this here.]

However, note that the transversals $\mathcal{L}^{\text{unimod}}(\mathbf{C})$ and $\mathbf{S}^3 \smallsetminus T$ *do not* correspond to each other by the homeomorphism g. Compare with the first comment following Theorem 4 below.

IV.3. The space $\mathcal{C}(\mathbf{C})$ is a 4–sphere.

The covolume is usually defined on the set of lattices, but there is no difficulty to see it as a continuous mapping $\mathcal{C}_{\{0\},\mathbf{Z},\mathbf{Z}^2}(\mathbf{C}) \longrightarrow [0,\infty]$, cyclic subgroups being of infinite covolume. We can therefore consider the set $\mathcal{C}^{\mathrm{covol}\geq 1}(\mathbf{C})$ of closed subgroups of $\mathbf{C}$ of covolume at least 1; it is a subspace of $\mathcal{C}_{\{0\},\mathbf{Z},\mathbf{Z}^2}(\mathbf{C})$ which contains $\mathcal{C}_{\{0\},\mathbf{Z}}(\mathbf{C})$.

Denote by B^4 the open unit ball $\{(a,b) \in \mathbf{C}^2 \mid |a|^2+|b|^2 < 1\}$, by $\overline{B^4} = B^4 \sqcup \mathbf{S}^3$ its closure, and by γ the restriction to $\overline{\mathbf{B}}^4$ of the inverse of the homeomorphism g' defined above. We modify γ to obtain a mapping

$$f \,:\, \overline{\mathbf{B}}^4 \quad\longrightarrow\quad \mathcal{C}^{\mathrm{covol}\geq 1}(\mathbf{C}) \cup \mathcal{C}_{\mathbf{R}}(\mathbf{C})$$

defined as follows:

— $f(0,0) = \{0\}$,
— if $(a,b) \in \mathbf{S}^3 \smallsetminus T$, then $f(a,b) = \frac{1}{\sqrt{\mathrm{covol}(\gamma(a,b))}}\gamma(a,b)$,
— if $(a,b) \in B^4 \smallsetminus \{(0,0)\}$ is in the $\mathbf{R}^*_+$–orbit of a point $(a_1,b_1) \in \big(\mathbf{S}^3 \smallsetminus T\big)$, then $f(a,b) = \frac{1}{\sqrt{h(a,b)}}\gamma(a_1,b_1)$ for an appropriate factor $h(a,b) \in]0, \mathrm{covol}(\gamma(a_1,b_1))[$,
— if $(a,b) \in T$ and $\gamma(a,b) = \mathbf{Z}\omega$, then $f(a,b) = \mathbf{R}\omega$,
— if $(a,b) \in B^4 \smallsetminus \{(0,0)\}$ is in the $\mathbf{R}^*_+$–orbit of a point $(a_1,b_1) \in T$, then $f(a,b) = \frac{1}{\sqrt{h(a,b)}}\gamma(a_1,b_1)$ for an appropriate factor $h(a,b) \in]0,\infty[$.

For a precise definition of the function h, see [HuPo–79] and [BrHK]. It can be shown that f is a homeomorphism.

Moreover, f can be extended to a mapping $\mathbf{C}^2 \cup \{\infty\} \longrightarrow \mathcal{C}(\mathbf{C})$ by defining for (a,b) outside $\mathbf{B}^4$

$$f(a,b) \;=\; \big(f(\sigma(a,b))\big)^{\sharp}$$

where σ denotes the inversion $(a,b) \longmapsto \frac{(a,b)}{|a|^2+|b|^2}$ of $\mathbf{C}^2$ fixing $\mathbf{S}^3$ and where the *dual* of a closed subgroup C of $\mathbf{C}$ is defined by

$$C^{\sharp} \;=\; \{z \in \mathbf{C} \;\mid\; \mathrm{Im}(\overline{z}\,c) \in \mathbf{Z} \;\text{ for all }\; c \in C\}.$$

(Observe that $C^{\sharp} = C$ if and only if C is either a unimodular lattice or a subgroup isomorphic to $\mathbf{R}$.) The one–point compactification $\mathbf{C}^2 \cup \{\infty\}$ of $\mathbf{C}^2$ can be identified to the 4–sphere $\mathbf{S}^4$, and we have:

4. Theorem [HuPo–79]**.** *The mapping $f : \mathbf{S}^4 \longrightarrow \mathcal{C}(\mathbf{C})$ is a homeomorphism.*

Comments on Theorem 4. By the homeomorphisms f, the equator $\mathbf{S}^3$ of $\mathbf{S}^4$ corresponds to the union of the set $\mathcal{L}^{\mathrm{unimod}}(\mathbf{C})$ of unimodular lattices and the set $\mathcal{C}_{\mathbf{R}}(\mathbf{C})$ of subgroups isomorphic to $\mathbf{R}$, the latter corresponding to a trefoil knot $T \subset \mathbf{S}^3$.

This cannot be seen using only the homeomorphism g' of Subsection IV.1. Indeed, whenever a point $(a,b) \in \mathbf{C}^2 \smallsetminus \Sigma$ converges towards a point $(a_{\lim}, b_{\lim}) \in \Sigma$, the corresponding $(g')^{-1}(a,b)$ has a volume which tends to infinite. When (a,b) is rescaled in such a way that the corresponding lattice is unimodular, then (a,b) escapes any compact subset of $\mathbf{C}^2$, the minimal norm of the corresponding lattice tends to 0, and the lattice itself tends inside $\mathcal{C}(\mathbf{C})$ to a subgroup isomorphic to $\mathbf{R}$.

If we view $\mathbf{S}^4$ as the suspension of its equator $\mathbf{S}^3$, the two–dimensional sphere $\mathbf{S}^2$ of Subsection IV.1, which corresponds to the complement of $\mathcal{L}(\mathbf{C})$ in $\mathcal{C}(\mathbf{C})$, corresponds to the suspension of T. A pole has a typical neighbourhood inside $\mathbf{S}^2$ which is a cone over the knot T (other points of $\mathbf{S}^2$ have typical neighbourhoods in $\mathbf{S}^2$ which are cones over unknotted closed curves inside $\mathbf{S}^3$); in particular, the embedding of this $\mathbf{S}^2$ in the total space $\mathbf{S}^4$ is not locally flat at the South and North Poles.

We know that the space $\mathcal{L}^{\mathrm{unimod}}(\mathbf{C})$ is not simply connected; indeed, it is an aspherical space with fundamental group the inverse image of $SL_2(\mathbf{Z})$ in the universal covering group of $SL_2(\mathbf{R})$. An oriented closed curve ℓ in $\mathcal{L}^{\mathrm{unimod}}(\mathbf{C})$ can be viewed as inside $\mathbf{S}^3$ and disjoint from the trefoil knot T (oriented in some way), so that there is an associated *linking number* $\mathrm{link}(\ell, T)$. An interesting particular case is that of a periodic orbit of the *geodesic flow* on $SL_2(\mathbf{R})/SL_2(\mathbf{Z})$, viewed as the unit tangent bundle of the *modular surface* $\mathbf{D}^2/PSL_2(\mathbf{Z})$, as explained in [Ghys–07].

V. On the Heisenberg group

The Heisenberg group is a 3–dimensional nilpotent Lie group, which is connected and simply connected. We use the model[3]

$$H \,=\, \mathbf{C} \times \mathbf{R} \quad \text{with} \quad (z_1, t_1)(z_2, t_2) \,=\, \Big(z_1 + z_2 \;,\; t_1 + t_2 + \frac{1}{2}\,\mathrm{Im}(z_1 \overline{z_2})\Big),$$

we denote by $p : H \longrightarrow \mathbf{C}$ the projection $(z, t) \longmapsto z$, and we identify $\mathbf{R}$ to $\{0\} \times \mathbf{R} \subset H$. The following is a collection of easily verified properties.

5. Proposition. *(i) The subset $\mathbf{R}$ of H is both the commutator subgroup and the centre $Z(H)$ of the Heisenberg group; moreover, commutators are given by*

$$[(z_1, t_1), (z_2, t_2)] \,=\, (0, \mathrm{Im}(\overline{z_1} z_2)).$$

(ii) Maximal abelian subgroups of H are of the form $\mathbf{R}z_0 \times R$, with $z_0 \in \mathbf{C}^*$.

(iii) For any $C \in \mathcal{C}(H)$ with $C \cap Z(H) \neq \emptyset$, the projection $p(C)$ is a closed subgroup of $\mathbf{C}$ (this applies in particular to non–abelian closed subgroups of H).

(iv) For any non–abelian $C \in \mathcal{C}(H)$, either $Z(H) \subset C$ or $C \in \mathcal{L}(H)$.

(v) For any lattice Λ in H, the commutator subgroup $[\Lambda, \Lambda]$ is a subgroup of finite index in the centre $Z(\Lambda)$.

(vi) The automorhism group of H is a semi–direct product $(H/Z(H)) \rtimes GL_2(\mathbf{R})$, with $H/Z(H)$ the group of inner automorphisms and with $GL_2(\mathbf{R})$ acting on H by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,,\, (x + iy, t)\right) \quad \longmapsto \quad \Big((ax + by) + i(cx + dy) \;,\; (ad - bc)^2 t\Big).$$

For any positive integer n, we denote by $\mathcal{L}_n(H)$ the subspace of lattices $\Lambda \in \mathcal{L}(H)$ with $[\Lambda, \Lambda]$ of index n in $Z(\Lambda)$. We denote by $\mathcal{L}_\infty(H)$ the subspace of $\mathcal{C}(H)$ of subgroups of the form $p^{-1}(L)$, with $L \in \mathcal{L}(\mathbf{C})$.

[3] An element $(x+iy, t)$ corresponds to a matrix $\begin{pmatrix} 1 & x & t + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ in the matrix picture $\begin{pmatrix} 1 & \mathbf{R} & \mathbf{R} \\ 0 & 1 & \mathbf{R} \\ 0 & 0 & 1 \end{pmatrix}$ of the Heisenberg group.

6. Examples. *For any $n \geq 1$, the subgroup Λ_n of H generated by $(1,0)$, $(i,0)$, and $(0,1/n)$ is in $\mathcal{L}_n(H)$. Moreover,*

$$\Lambda_n = \mathbf{Z}[i] \times \frac{1}{n}\mathbf{Z}$$

if n is even and

$$\Lambda_n = \left\{ \left(x+iy, \frac{t}{2n} \right) \in \mathbf{Z}[i] \times \frac{1}{2n}\mathbf{Z} \;\middle|\; xy \equiv t \pmod 2 \right\}$$

if n is odd. In the matrix picture for H, we have $\Lambda_n = \begin{pmatrix} 1 & \mathbf{Z} & \frac{1}{n}\mathbf{Z} \\ 0 & 1 & \mathbf{Z} \\ 0 & 0 & 1 \end{pmatrix}$ for all n.

The subgroup of H generated by $(1,0)$, $(i,0)$, and $Z(H)$ is in $\mathcal{L}_\infty(H)$.

VI. Three subspaces of $\mathcal{C}(H)$

The space $\mathcal{C}(H)$ consists of several parts which, as it is the case for $\mathcal{C}(\mathbf{C})$, are relatively easy to describe, and their gluing is harder. From Proposition 5, the space $\mathcal{C}(H)$ consists of three parts:

→ the space $\mathcal{A}(H)$ of closed abelian subgroups,
→ the space $\mathcal{C}_{\geq Z}(H)$ of closed subgroups of H containing $Z(H)$,
→ the space of lattices $\mathcal{L}(H) = \bigsqcup_{n=1}^{\infty} \mathcal{L}_n(H)$.

Moreover, $\mathcal{L}(H)$ is disjoint from $\mathcal{A}(H) \cup \mathcal{C}_{\geq Z}(H)$.

VI.1. The space $\mathcal{A}(H)$ of abelian subgroups.

From Proposition 5.ii, the space $\mathcal{C}_{\mathbf{R}^2}(H)$ of maximal abelian subgroups of H is homeomorphic to the projective line $\mathcal{C}_{\mathbf{R}}(\mathbf{C}) \approx \mathbf{P}^1$ of lines in the space $H/Z(H) \approx \mathbf{R}^2$. For a closed abelian subgroup A of H, there are two cases to distinguish:

(i) if $A \not\subseteq Z(H)$, then A is contained in an unique maximal abelian subgroup of H,
(ii) if $A \subseteq Z(H)$, then A is contained in all maximal abelian subgroups of H.

If one could identify coherently each maximal abelian subgroup of H to a copy of $\mathbf{R}^2$, the space $\mathcal{A}(H)$ would simply be the quotient of $\mathcal{C}_{\mathbf{R}^2}(H) \times \mathcal{C}(\mathbf{C}) \approx \mathbf{S}^1 \times \mathbf{S}^4$ by the relation identifying each of the circle $(A_{\max}, C)$ to a point (for $A_{\max}$ in the circle $\mathcal{C}_{\mathbf{R}^2}(H)$, and for $C \subset Z(H)$ fixed). This identification is possible locally, but not globally; indeed, even once H is furnished with some Riemannian structure, each $A_{\max}$ can be identified to $\mathbf{C}$ in two ways, corresponding to the two orientations of $A_{\max}$. However, the following can be shown (Proposition 6.1.i of [BrHK]):

7. Proposition. *There exists in $\mathbf{S}^4$ a tame closed interval I, corresponding in $\mathcal{C}(\mathbf{C})$ to the subspace of closed subgroups contained in $\mathbf{R}$, such that*

$$\mathcal{A}(H) \approx \left(S^1 \times S^4\right) / \left((\varphi, x) \sim (\varphi', x), \text{ for all } \varphi, \varphi' \in \mathbf{S}^1 \text{ and } x \in I \right).$$

In particular, $\mathcal{A}(H)$ is a space of dimension 5.

VI.2. The space $\mathcal{C}_{\geq Z}(H)$ of groups containing the centre $Z(H)$.

As a straightforward consequence of the results of Section IV, we have:

8. Proposition. *The space $\mathcal{C}_{\geq Z}(H)$ is homeomorphic to $\mathcal{C}(\mathbf{C})$, namely to a 4–dimensional sphere. Moreover, the intersection $\mathcal{A}(H) \cap \mathcal{C}_{\geq Z}(H)$ is embedded in $\mathcal{C}_{\geq Z}(H)$ as a closed 2–disc.*

More precisely, this intersection $\mathcal{A}(H) \cap \mathcal{C}_{\geq Z}(H)$ is the lower hemisphere of the 2–sphere described in the comments which follow Theorem 4. Observe that the complement of this 2–sphere in $\mathcal{C}_{\geq Z}(H)$ is precisely the space $\mathcal{L}_\infty(H)$ of subgroups of the form $p^{-1}(L)$ for $L \in \mathcal{L}(\mathbf{C})$, as defined just before Example 6.

VI.3. The space $\mathcal{L}(H)$ of lattices in H.

Consider an integer $n \geq 1$, a lattice $\Lambda_0 \in \mathcal{L}_n(H)$, and denote by $L_0 = p(\Lambda_0) \in \mathcal{L}(\mathbf{C})$ its projection. Choose a positively oriented basis (z_0, z_0') of L_0 and two points $(z_0, t_0) \in \Lambda_0 \cap p^{-1}(z_0)$, $(z_0', t_0') \in \Lambda_0 \cap p^{-1}(z_0')$. The infinite cyclic subgroup $[\Lambda_0, \Lambda_0]$ is generated by the commutator

$$[(z_0, t_0), (z_0', t_0')] \;=\; (0, \mathrm{Im}(\overline{z_0} z_0')) = (0, \mathrm{vol}(\mathbf{C}/L_0)),$$

and Λ_0 itself is generated by (z_0, t_0), (z_0', t_0'), and $(0, \frac{1}{n}\,\mathrm{vol}(\mathbf{C}/L_0))$.

For any nearby lattice $\Lambda \in \mathcal{L}_n(H)$, there exist a unique positively oriented basis (z, z') of $L \doteqdot p(\Lambda)$ near (z_0, z_0') and numbers t, t' near t_0, t_0' such that (z, t), (z', t'), and $(0, \frac{1}{n}\,\mathrm{vol}(\mathbf{C}/L))$ generate Λ. Moreover, any lattice in $\mathcal{L}_n(H)$ with projection L is generated by elements of the form (z, t), (z', t'), and $(0, \frac{1}{n}\,\mathrm{vol}(\mathbf{C}/L))$, for some $t, t' \in [0, \frac{1}{n}\,\mathrm{vol}(\mathbf{C}/L)[$.

It is easy to check that the subgroup $GL_2(\mathbf{R})$ of $\mathrm{Aut}(H)$, see Proposition 5.vi, operates on $\mathcal{L}_n(H)$ in such a way that, for any orbit $\mathcal{O}$, we have

$$\{L \in \mathcal{L}(\mathbf{C}) \mid L = p(\Lambda) \;\text{ for some }\; \Lambda \in \mathcal{O}\} \;=\; \mathcal{L}(\mathbf{C}).$$

Moreover, the group of inner automorphisms operates transitively on the set of possible choices for t, t'. We have essentially proved the first two claims of the following proposition; for details and for the last claim, see [BrHK, Section 7].

9. Proposition. *For each $n \geq 1$, the space $\mathcal{L}_n(H)$ is both*

(i) a torus bundle with base space $\mathcal{L}(\mathbf{C})$,

(ii) a homogeneous space of the 6–dimensional Lie group $\mathrm{Aut}(H)$ by a discrete subgroup. For $n, n' \geq 1$, the spaces $\mathcal{L}_n(H), \mathcal{L}_{n'}(H)$ are homeomorphic to each other; moreover, the torus bundles $\mathcal{L}_n(H) \longrightarrow \mathcal{L}(\mathbf{C})$ and $\mathcal{L}_{n'}(H) \longrightarrow \mathcal{L}(\mathbf{C})$ are isomorphic.

VI.4. Summing up for $\mathcal{A}(H)$, $\mathcal{C}_{\geq Z}(H)$, and $\mathcal{L}(H)$.

Let $(\varphi_s)_{s>0}$ be the one–parameter subgroup of $\mathrm{Aut}(H)$ defined by $\varphi_s(z, t) = (sz, s^2 t)$. We have $\lim_{s\to\infty} \varphi_s(D) = \{e\}$ for any discrete subgroup D of H and $\lim_{s\to 0} \varphi_s(\Lambda) = H$ for any $\Lambda \in \mathcal{L}(H)$. Since $\mathcal{A}(H)$ and $\mathcal{C}_{\geq Z}(H)$ are clearly arc–connected (Propositions 7 and 8), and contain respectively $\{e\}$ and H, it follows that $\mathcal{C}(H)$ is arc–connected. This is the very first part of the following theorem, one of the two theorems which summarise the results in [BrHK]; the space $\mathcal{C}(H)$ is not locally connected because any neighbourhood of a point in $\mathcal{L}_\infty(H)$ is disconnected, containing points from $\mathcal{L}_n(H)$ for n large enough.

10. Theorem (Theorem 1.3 in [BrHK]**).** *The compact space $\mathcal{C}(H)$ is arc–connected but not locally connected. It can be expressed as the union of the following three subspaces.*

(i) *$\mathcal{L}(H)$, which is open and dense in $\mathcal{C}(H)$; this has countably many connected components $\mathcal{L}_n(H)$, each of which is homeomorphic to a fixed aspherical 6–manifold that is a 2–torus bundle over $\mathcal{L}(\mathbf{C}) \approx GL_2(\mathbf{R})/GL_2(\mathbf{Z})$.*

(ii) $\mathcal{A}(H)$, *which is homeomorphic to the space obtained from* $\mathbf{S}^4 \times \mathbf{P}^1$ *by fixing a tame arc* $I \subset \mathbf{S}^4$ *and collapsing each of the circles* $\{\{i\} \times \mathbf{P}^1 : i \in I\}$ *to a point.*

(iii) $\mathcal{C}_{\geq Z}(H)$, *from which there is a natural homeomorphism to* $\mathbf{S}^4$*; the complement of* $\mathcal{L}_\infty(H)$ *in* $\mathcal{C}_{\geq Z}(H)$ *is a* 2*–sphere* $\Sigma^2 \subset \mathbf{S}^4$ *(which fails to be locally flat at two points).*

The union $\mathcal{A}(H) \cup \mathcal{C}_{\geq Z}(H)$ *is the complement of* $\mathcal{L}(H)$ *in* $\mathcal{C}(H)$*. The intersection*

$$\mathcal{A}(H) \cap \mathcal{C}_{\geq Z}(H) = \{C \in \mathcal{C}_{\geq Z}(H) \mid p(C) \subset \mathcal{C}_{\{0\},\mathbf{Z},\mathbf{R}}(\mathbf{C})\}$$

is a closed 2*–disc in* Σ^2*. The space* $\mathcal{L}(H) \cup \mathcal{L}_\infty(H)$ *is precisely* $\{C \in \mathcal{C}(H) \mid p(C) \in \mathcal{L}(\mathbf{C})\}$*.*

$\mathcal{C}_{\geq Z}(H)$ *is a weak retract of* $\mathcal{C}(H)$*: there exists a continuous map* $f : \mathcal{C}(H) \longrightarrow \mathbf{S}^4$*, constant on* $\mathcal{A}(H)$*, such that*[4] $f \circ j \simeq \mathrm{id}_{\mathbf{S}^4}$*, where* $j : \mathbf{S}^4 \longrightarrow \mathcal{C}_{\geq Z}(H)$ *is the homeomorphism of (iii). In particular,* $\pi_4(\mathcal{C}(H))$ *surjects onto* $\mathbf{Z}$*.*

The subspace $\mathcal{N}(H)$ *of normal closed subgroups of* H *is the union of* $\mathcal{C}_{\geq Z}(H)$ *(which is homeomorphic to* $\mathcal{C}(\mathbf{C}) \approx \mathbf{S}^4$*) and the closed interval* $\{C \in \mathcal{C}(H) \mid C \subset Z(H)\}$*, attached to the sphere* $\mathcal{C}_{\geq Z}(H)$ *by one of its endpoints.*

We would like to know more generally when $\mathcal{L}(G)$ is dense in $\mathcal{C}(G)$, say for a unimodular connected Lie group G.

VII. The space $\mathcal{C}(H)$

As an illustration of the density of $\mathcal{L}(H)$ in $\mathcal{C}(H)$, let us describe the following simple

11. Example. *Let* $n \geq 1$ *be fixed. For any integer* $k \geq 1$*, let* A_k *denote the subgroup of* $\mathbf{R} \times \mathbf{R}$ *generated by* $(1, 0)$ *and* $(-\frac{1}{k}, 1)$*; let* A *denote the subgroup* $\mathbf{Z}^2$ *of* $\mathbf{R} \times \mathbf{R}$*; and let* Λ_k *denote the subgroup of* H *generated by* A_k *and* $(-ik^2 n, 0)$*.*

Then $\Lambda_k \in \mathcal{L}_n(H)$ *for all* $k \geq 1$ *and* $\lim_{k\to\infty} \Lambda_k = \lim_{k\to\infty} A_k = A$ *in* $\mathcal{C}(H)$*.*

Remark, and consequence of this example. Here, $\mathbf{R} \times \mathbf{R}$ is viewed as a subgroup of $\mathbf{C} \times \mathbf{R} = H$; in particular, it should not be confused with $\mathbf{C}$. (Inside $H = \mathbf{C} \times \mathbf{R}$, note that $\mathbf{C}$ is not a subgroup.) Observe that $p(A_k)$ is a lattice in $\mathbf{C}$ for each $k \geq 1$, but the projection of the limit, $p(A)$, is a cyclic subgroup of $\mathbf{C}$.

It follows from Example 11 that the frontier of $\mathcal{L}_n(H)$ in $\mathcal{C}(H)$ contains the closure of the $\mathrm{Aut}(H)$–orbit of A, and it is a fact that this closure coincides with $\mathcal{A}(H)$ (Proposition 6.1.ii in [BrHK]). Moreover, it can be seen that the frontier of $\mathcal{L}_n(H)$ contains $\mathcal{C}_{\geq Z}(H)$.

The previous argument shows part of the following theorem, again copied from [BrHK], by which we will end this account. Recall that we denote here by Σ^2 the topologically embedded sphere in $\mathcal{C}_{\geq Z}(H)$ which corresponds to the $\mathbf{S}^2 \subset \mathbf{S}^4$ in the comments on Theorem 4. The symbols $\mathbf{P}^2$ and $\mathbf{K}$ stand respectively for a real projective plane and a Klein bottle.

12. Theorem (Theorem 1.4 in [BrHK]**).** *The spaces* $\mathcal{L}_n(H)$ *are homeomorphic to a common aspherical homogeneous space, namely the quotient of the* 6*–dimensional automorphism group* $\mathrm{Aut}(H) \cong \mathbf{R}^2 \rtimes GL_2(\mathbf{R})$ *by the discrete subgroup* $\mathbf{Z}^2 \rtimes GL_2(\mathbf{Z})$*.*

The frontier of $\mathcal{L}_n(H)$*, which is independent of* n*, consists of the following subspaces:*

(i) *the trivial group* $\{e\}$*;*

[4] Here, $\simeq$ denotes homotopy equivalence.

(ii) $\mathcal{C}_{\mathbf{R}}(H) \approx \mathbf{P}^2$;
(iii) $\mathcal{C}_{\mathbf{Z}}(H) \approx \mathbf{P}^2 \times]0,\infty[$;
(iv) $\mathcal{C}_{\mathbf{R}^2}(H) \approx \mathbf{P}^1$;
(v) $\mathcal{C}_{\mathbf{R}\oplus\mathbf{Z}}(H) \approx \mathbf{K} \times]0,\infty[$, *which is a* $(\mathbf{P}^1 \times]0,\infty[)$*–bundle over* $\mathbf{P}^1$;
(vi) $\mathcal{C}_{\mathbf{Z}^2}(H)$, *which is a* $(\mathbf{S}^4 \smallsetminus \Sigma^2)$*–bundle over* $\mathbf{P}^1$;
(vii) $p_*^{-1}(\mathcal{C}_{\mathbf{R}\oplus\mathbf{Z}}(\mathbf{C}))$;
(viii) *the full group* H.

In particular, the frontier of $\mathcal{L}_n(H)$ *is the union of* $\mathcal{A}(H)$ *and the complement* Σ^2 *of* $\mathcal{L}_\infty(H)$ *in* $\mathcal{C}_{\geq Z}(H)$*; the part* $\mathcal{A}(H)$ *is itself the union of the subspaces* (i) *to* (vi), *and* $\Sigma^2 \smallsetminus (\Sigma^2 \cap \mathcal{A}(H))$ *is itself the union of the subspaces* (vii) *and* (viii). *The frontier of* $\bigcup_{n=1}^{\infty} \mathcal{L}_n(H)$ *further contains*

(ix) $\mathcal{L}_\infty(H)$.

Each of these spaces, except (vi), *consists of finitely many* $\mathrm{Aut}(H)$*–orbits.*

Observe that, as $\mathcal{L}(H)$ is open dense, the spaces (i) to (ix) of Theorem 1.4 together with the spaces $\mathcal{L}_n(H)$ for $n \geq 1$ constitute a partition of $\mathcal{C}(H)$.

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