CORRECTIONS AND UPDATES
(JANUARY 2003 – JULY 2004)

A first set of “Corrections and updates” has appeared in the 2003 printing of my book [Harpe–00], as well as in the 2003 list of Geneva’s preprints [Harpe–03]. Here is a smaller second set.

General references.

Relevant “books” to the subject include [Kapov–01] and the book-like paper [Wall–03].

II.18, and groups acting properly on trees (a correction).

As Guido Mislin observed to me, the statement which appears on Line 16 of Page 22, namely “A group has a free subgroup of finite index if and only if it can act properly on a tree”, is only true for finitely generated groups. For groups in general (namely without any cardinality restriction), it is a result of Dunwoody [Dunwo–79] that

A group $G$ can act (simplicially) with finite stabilisers on a tree
if and only if it has rational cohomological dimension less than or equal to 1.

If $G$ is countable, the tree in question can be chosen to be locally finite so that the action will be proper with respect to either topology on the tree (CW, or metric – they agree in the locally finite case). A nice example to keep in mind is the group $\mathbb{Q}/\mathbb{Z}$ (rationals mod 1), which acts on a locally finite tree with vertices corresponding to the cosets of the finite subgroups of $\mathbb{Q}/\mathbb{Z}$. Note also that, by a simple argument, every infinite, countable locally finite group has rational cohomological dimension equal to 1.

II.41, and dense free subgroups of subgroups of linear groups.

For the results on dense free subgroups in Lie groups already reported in [Harpe–03], see [BreGe–03]. For related results about dense free subgroups in $GL_n$ over a local field, see [BreGe].

II.42, and Tits’ alternative for 3-manifold groups.

More on this in [Butto–04].

II.46, and the Kervaire conjecture

See [OuldH–b] for a recent presentation of the conjecture.

III.5, and groups which are centres.

Here is a strong answer to Remeslennikov’s question. Let $\Gamma$ be a countable group; then $\Gamma$ is embeddable in a finitely generated group $\Delta$ such that the centre of $\Gamma$ coincides with the centre of $\Delta$. If $\Gamma$ is recursively presented, then $\Delta$ can be moreover chosen to be finitely presented. These are some of the results of [OuldH–a].

III.16.ii, is this group finitely-generated?

Let $K_g$ denote the subgroup of the mapping class group $\text{Mod}(g)$ generated by all Dehn twists about boundary curve (namely about simple closed curves which are null-homologous). Then $K_1$ is reduced to one element, but it has been a problem to know whether $K_g$ is finitely
generated for \( g \geq 2 \). From work of KcCullough and Miller, and Mess, it is known that \( K_2 \) is a free group of infinite rank. For \( g \geq 3 \), we know now that \( K_g \) is of infinite index in the corresponding Torelli group, and that \( K_g \) is not finitely generated [BisFa].

Here is one more group for which we do not know whether it is finitely generated: the group \( \Gamma_n \) of homeomorphisms of the \( n \)-torus \( T^n = \mathbb{R}^n / \mathbb{Z}^n \) for which there is a decomposition of \( T^n \) into finitely many pieces on which the homeomorphism agrees with an element of \( GL_n(\mathbb{Z}) \) [FisWh].

III.45, and the existence of uncountably many finitely-generated groups with pairwise non-isomorphic \( II_1 \) factors.

N. Ozawa has solved this problem [Ozawa–03]. More precisely, from work of Gromov and Ol’shanskii, it is known that there exists a countable group \( \Gamma \) with Kazhdan’s Property (T) which has uncountably many pairwise nonisomorphic quotient groups all of which are simple and icc (= with all conjugacy classes distinct from \{1\} infinite). Ozawa’s solution follows from the fact that, among these uncountably many groups, subfamilies of groups with isomorphic factors are at most countable.

IV.1, and differentiating left from right.

The following exercise could save later confusions about left-invariant and right-invariant word metrics on groups.

Let \( F \) be a non-abelian free group freely generated by a finite set \( S \). For each integer \( n \geq 1 \), find elements \( x, y \in F \) such that \( d_S(x, y) = 1 \) and \( d_S(x^{-1}, y^{-1}) \geq n \).

IV.3.i, on commutator lengths.

Commutator lengths on many groups of diffeomorphisms are unbounded: see [GamGh].

IV.13–14, on dead ends in Cayley graphs.

A vertex \( \gamma \) in a Cayley graph (or more generally in a connected graph given with a base vertex \(*\)) is a dead end if a geodesic segment from \(*\) to \( \gamma \) cannot be extended beyond \( \gamma \). Such a dead end, say at distance \( n \) from \(*\), is of depth \( d \) if \( d + 1 \) is the shortest length of a path from \( \gamma \) to a vertex of the sphere of radius \( n + 1 \) around \(*\). There are dead ends of depth 2 in one of the standard Cayley graphs on the Thompson group \( F \) [CleTa–a], and dead ends of arbitrary large depth in appropriate Cayley graphs of lamplighter groups [CleTa–b].

IV.20–22, on the definition of quasi-isometric groups.

This notion makes sense for arbitrary countable groups, when reformulated as follows (see Theorem 2.1.2 in [Shalo]). Given two countable groups \( \Gamma \) and \( \Delta \), a mapping \( \phi : \Gamma \rightarrow \Delta \) is a quasi-isometry if

(i) for every sequence of pairs \((\alpha_i, \beta_i)_{i \in \mathbb{N}} \) in \( \Gamma \times \Gamma \), we have

\[
\alpha_i^{-1} \beta_i \to \infty \quad \text{in} \quad \Gamma \quad \iff \quad \phi(\alpha_i)^{-1} \phi(\beta_i) \to \infty \quad \text{in} \quad \Delta,
\]

(ii) there exists a finite subset \( C \) in \( \Delta \) such that \( \phi(\Gamma)C = \Delta \) (as sets).

IV.24, and finite generation of cocompact lattices.

Let \( G \) be a \( \sigma \)-compact locally compact group and \( \Gamma \) a discrete subgroup of \( G \) such that the quotient space \( G/\Gamma \) is compact. The last statement of IV.24 could be more precise: \( \Gamma \) is finitely generated if \textit{and only if} \( G \) is compactly generated; this is part of Theorem 2 in [MacSw–60].
IV.29, and groups commensurable to lattices in Lie groups.

Let $G$ be a real Lie group which has a finite number (possibly $> 1$) of connected components, let $\Gamma$ be a lattice in $G$, and let $\Delta$ be a finite extension of $\Gamma$. It is a natural question to ask whether $\Delta$ is again a lattice in some real Lie group with finitely many connected components. The answer, depending on $G$, is

- yes if $G = \text{PSL}(2, \mathbb{R})$ and if $\Gamma$ is cocompact,
- no if $G = \text{PSL}(2, \mathbb{R})$ and if $\Gamma$ is not cocompact,
- yes if the connected component $G^0$ of $G$ has no simple factor isomorphic to $\text{PSL}(2, \mathbb{R})$ and such that the projection of $\Gamma \cap G^0$ in this factor is discrete and not cocompact.

See [GruPl–04].

IV.34, a misprint.

Page 98, Line 10, $\delta K$ should be $K\delta$.

IV.35, complement.

It is important (see [Shalo]) to observe that the locally compact space $X_{\lambda,C}$ is totally disconnected. Also, if the space of all injective $(\lambda, C)$-quasi-isometry from $\Gamma$ to $\Delta$ is not empty, then $\Gamma$ and $\Delta$ operate freely on it.

Moreover, all this discussion should be extended from quasi-isometries to uniform embeddings (see again [Shalo]).

(If $\Gamma$ and $\Delta$ are two finitely generated groups which are biLipschitz equivalent, then the actions of these two groups on the corresponding space $X_{\lambda}$ have a common fundamental domain.)

Question IV–36: See [Ersch–04] and [Harpe–03].

IV.42, and quasi-isometry of S-arithmetic groups.

The discussion of quasi-isometry for lattices in semi-simple real Lie groups should be extended to S-arithmetic lattices. Besides the work of J. Taback quoted in IV.41 of [Harpe–03], there are partial solutions in [Wortm–a] and [Wortm–b].

IV.46, and Lipschitz equivalences of amenable groups

T. Dymarz has shown that, in an amenable group, the inclusion of a proper subgroup of finite index is never at bounded distance from a bijection [Dymarz].

IV.50, and further examples of groups which are not almost convex.

The Thompson’s group $F$ is not almost convex [CleTa–04] (see also http://math.albany.edu/~jtaback/).

V.7 and V.22, finitely presented mapping class groups, and finiteness conditions on outer automorphism groups.
The group of outer automorphisms of a group $\Gamma$ is the quotient $\text{Out}(\Gamma) = \text{Aut}(\Gamma)/\text{Int}(\Gamma)$ of the group of all automorphisms of $\Gamma$ by the normal subgroup of inner ones.

Let $M$ be a connected manifold, or a CW-complex. Denote by $\mathcal{H}(M)$ the group of isotopy classes of homeomorphisms of $M$; there is a natural homomorphism from $\mathcal{H}(M)$ to $\text{Out}(\pi_1(M))$. In case $M$ is an orientable manifold, the mapping class group $\mathcal{H}_+(M)$ is the subgroup of orientation preserving classes in $\mathcal{H}(M)$.

Denote by $\mathcal{E}(M)$ the group of homotopy classes of base-point preserving homotopy self-equivalences of $M$ and by $\mathcal{E}_f(M)$ its quotient of homotopy classes of free homotopy self-equivalences [Arkow–90]; in case $M$ is simply connected, $\mathcal{E}_f(M) = \mathcal{E}(M)$. There are natural homomorphisms from $\mathcal{E}(M)$ to $\text{Aut}(\pi_1(M))$ and from $\mathcal{E}_f(M)$ to $\text{Out}(\pi_1(M))$; if $M$ is an Eilenberg-MacLane space $K(\Gamma, n)$, it is easy to see that $\mathcal{E}(M)$ is naturally isomorphic to $\text{Aut}(\Gamma)$; in the particular case $M$ is a $K(\Gamma, 1)$, the group $\mathcal{E}_f(M)$ is naturally isomorphic to $\text{Out}(\Gamma)$.

If $M$ is a closed orientable surface of genus $g \geq 1$, with fundamental group denoted here by $\Gamma_g$, it is a result of Nielsen that the natural homomorphism $\mathcal{H}(M) \to \text{Out}(\Gamma_g)$ is an isomorphism, so that the mapping class group of $M$ is isomorphic to the appropriate subgroup of index 2 in $\text{Out}(\Gamma_g)$; in particular, $\mathcal{H}(M)$ is isomorphic to $\mathcal{E}_f(M)$.

For manifolds of higher dimensions, there are various results on each of $\mathcal{H}(M)$, $\mathcal{E}(M)$, and $\mathcal{E}_f(M)$. For example, if $M$ is a finite CW-complex, $\mathcal{E}(M)$ is finitely presented if $M$ is simply connected [Wilke–76] and infinitely generated in general [FraKa–77]. If $M$ is a closed orientable irreducible 3-manifold which is sufficiently large, then the natural homomorphism $\mathcal{H}(M) \to \text{Out}(\pi_1(M))$ is an isomorphism (see [Waldh–68]), which has also a result for a 3-manifold with boundary). There are larger classes of 3-manifolds of which mapping class groups are known to be finitely presented; see [HatMc–90].

Here are more recent results. (i) Let $M$ be a closed aspherical manifold of dimension $n$ with $n \not\in \{3, 4\}$. Assume that $\pi_1(M)$ is isomorphic to the fundamental group of a complete non-positively curved Riemannian manifold which is $A$-regular (a technical condition, fulfilled by compact manifolds). Then the natural homomorphism $\mathcal{H}(M) \to \text{Out}(\pi_1(M))$ is onto (Corollary 4 of Lecture 5 in [Farre–02]). (ii) Let now $M$ be a closed irreducible 3-manifold and let $N$ be a closed hyperbolic 3-manifold; then two homeomorphisms $f, g : M \to N$ which are homotopic are necessarily isotopic [GaMeT–03].

It is an easy consequence of Mostow rigidity that $\text{Out}(\Gamma)$ is a finite group when $\Gamma$ is the fundamental group of an oriented connected hyperbolic $n$-manifold of finite volume, with $n \geq 3$; see, e.g., C.5.6 and C.5.7 in [BenPo–92]. Another sufficient condition for a hyperbolic group $\Gamma$ to have a finite outer automorphism group is that $\Gamma$ has Kazhdan’s Property (T) [Pauli–91].

Here are three more related references, communicated to me by Ian Hambleton: [HamKr], [McCul–80], and [MarRu–01].

V.30, and compact 3-manifolds with nontrivial free fundamental groups

Let $M$ be a compact 3-manifold with fundamental group a free group of rank $r > 0$. Let $k$ be the number of connected components of the boundary of $M$. Then $M$ is the connected sum of $k$ cubes with handles, of a finite number of 2-sphere bundles over the circle, and possibly of a homotopy sphere. See Chapter 5 in [Hempe–76].

V.A, and the variety of finite presentations of a finitely presented groups.

A point which is clearly missing from the discussion is the comparison between various presentations of one group.

On the other hand, a given group usually has distinct presentations which are useful for different purposes. A first example is given by the infinite dihedral group. The presentation $\langle r, s \mid r^2 = s^2 = 1 \rangle$ makes it obvious that it is generated by two involutions, whereas the
presentation \langle s, t \mid s^2 = 1, sts = t^{-1} \rangle makes it obvious it has an infinite cyclic subgroup of index two. Another example is given by the braid groups, their Artin presentations [Artin–25], related to the presentations of symmetric groups as Coxeter groups, and their presentations from [BiKoL–98], related to the presentations of symmetric groups with all transpositions as generators.

On the other hand, a basic theorem of Tietze shows that, given two finite presentations of one group, we can pass from one to the other by a finite sequence of (now so-called) Tietze transformations, which are, in loose terms:

(i) add one relation which follows from those already written, and conversely;
(ii) add one generator together with a relation expressing that it is some group element, and conversely.

See [Tietz–08] and, e.g., Proposition II.2.1 in [LynSc–77].

VI.14, and Mellin transform.

There are well-known relationships of some series with each other. For example, let us assume appropriate technical conditions on \((a_k)_{k \geq 0}\) so that the formal computations below converge; assume also \(a_0 = 0\). Set

\[
f(z) = \sum_{k=1}^{\infty} a_k e^{2i\pi kz} \quad z \in \mathbb{C}, \Im(z) > 0.
\]

Then the Dirichlet series defined by \((a_k)_{k \geq 0}\) is essentially the Mellin transform of \(f\):

\[
(Mf)(s) = \int_0^\infty f(iy)y^sy^{-s}dy = \int_0^\infty \sum_{k=1}^{\infty} a_k e^{-2\pi ky}y^sy^{-s}dy = \sum_{k=1}^{\infty} a_k \int_0^\infty e^{-2\pi ky}y^sy^{-s}dy = (2\pi)^{-s}\Gamma(s) \sum_{k \geq 1} \frac{a_k}{k^s}.
\]

As usual, \(\Gamma(s) = \int_0^\infty e^{-y}y^{s-1}dy \) denotes Euler’s Gamma function.

For background on the Mellin transform, see e.g. [ColLa–72].

VI.20, and growth series related to the Richard Thompson group.

“The” growth series of Thompson’s group \(F\) has not yet been computed, but the following comments could make the picture slightly more clear.

On the one hand, the group has a standard presentation with two generators \(s_0, s_1\) and two relations; the positive semi-group defined by these two generators has a growth series which has been computed by José Burillo [Buril–04], and which is rational: \((1-z^2)/(1-2z-z^2+z^3)\). This growth series refers to the group word length, not the semigroup word length; as a semigroup, the positive semi-group is not finitely generated. Thanks to J. Burillo for these informations.

On the other hand, the group has another standard presentation involving an infinite sequence \(s_0, s_1, s_2, \ldots\) of generators; with respect to these, any element in \(F\) has a normal form which is a product \(pq^{-1}\), where both \(p\) and \(q\) are positive words in the generators. Since the number of these generators is infinite, there is no growth series associated to them, at least in the sense discussed in this chapter. This normal form \(pq^{-1}\) has not been of any use (so far?) to compute the growth series of the group \(F\) with respect to \(\{s_0, s_1\}\).

One more recent work on Thompson’s group \(F\): [Guba].
VI.42, and proper weights on arbitrary countable groups.

Given a countable group $\Gamma$ and a function $w : \Gamma \rightarrow \mathbb{R}^*_+$ which is proper (namely such that \(\{\gamma \in \Gamma \mid w(\gamma) \leq c\}\) is a finite subset of $\Gamma$ for every $c > 0$), define a weight $\lambda_w$ on $\Gamma$ by

$$
\lambda_w(\gamma) = \min \left\{ \ell \in \mathbb{R} \mid \begin{array}{l}
\text{there exist } \gamma_1, \ldots, \gamma_n \in \Gamma \text{ with } \\
\gamma = \gamma_1 \cdots \gamma_n \text{ and } \\
\ell = \sum_{1 \leq j \leq n} w(\gamma_j)
\end{array} \right\}.
$$

Then $\lambda_w$ is a proper weight on $\Gamma$ and the formula $d_w(\gamma, \gamma') = \lambda_w(\gamma^{-1} \gamma')$ defines a left-invariant proper metric on $\Gamma$ (where “proper” means now that any closed ball of finite radius is finite).

Question VI.64: See [Ersch–04] and [Harpe–03].

VII.B, V.31, free subgroups in groups of positive deficiency, and uniformly exponential growth.

Let $\Gamma$ be a group given by a presentation involving a set $S$ of $n$ generators, and $m$ relations; assume that $n > m$. It is a result of J. Wilson that there exists a subset of $S$ of $n - m$ elements which freely generates a free subgroup of $\Gamma$ [Wils–04b]. If $n \geq m - 2$, it follows that $\Gamma$ has uniformly exponential growth, and more precisely that $\omega(\Gamma) \geq 2(n - m) - 1$; this solves positively Conjecture 5.14 in [GroLP–81] (reprinted in [GroLP–99]).

My report on uniformly exponential growth has appeared in [Harpe–02].

VII.14, and large groups

Say that a group is large if it has a finite index subgroup which maps onto a non abelian free group. It is a reformulation of Proposition VII.14.iii that finitely generated groups which are large have uniformly exponential growth. Marc Lackenby [Lacke] has shown the following characterization.

Let $\Gamma$ be a finitely presented group. Then $\Gamma$ is large if and only if there exists a sequence $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ of finite index subgroups of $\Gamma$, with $\Gamma_i$ normal in $\Gamma_1$ for all $i \geq 1$, which has the three following properties:

(i) $\Gamma_i/\Gamma_{i+1}$ is abelian for all $i \geq 1$,

(ii) $\lim_{i \to \infty} \frac{\log[\Gamma_i : \Gamma_{i+1}]}{[\Gamma_i : \Gamma_1]} = \infty$

(iii) $\limsup_{i \to \infty} d(\Gamma_i/\Gamma_{i+1}) > 0$

where $d(\Gamma_i/\Gamma_{i+1})$ denotes the minimal number of generators of the quotient group $\Gamma_i/\Gamma_{i+1}$.

Similarly, $\Gamma$ has a finite index subgroup which has an infinite abelian quotient if and only if there exists a sequence $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ as above which has properties (i) and (ii).

VII.19, on groups of exponential growth which are not uniformly of exponential growth.

John Wilson’s answer to Gromov’s question, already mentionned in [Harpe–03], has now appeared [Wils–04a]. See also [Wils–04c] and [Barth–03].

VII.39 1/2, on growth and unitary representations.

Though the subject has not been investigated properly, growth of groups seems to be also relevant in the theory of unitary representations.

Let us first reproduce a few lines from [Harpe–02]. Corollary 3.2 in [Haage–79] shows the following “phase transition”: for a free group $F_n$, $n \geq 2$, a free set $S_n$ of generators, a real number $\lambda > 0$, and the function of positive type $\phi_\lambda : F_n \rightarrow \mathbb{R}$ defined by $\phi_\lambda(\gamma) = \exp(-\lambda \ell_{S_n}(\gamma))$, the three following conditions are equivalent:

(1) $\phi_\lambda$ is weakly associated with the regular representation of $F_n$,
\[ \gamma \mapsto \phi_\lambda(\gamma) \left(1 + \ell_{S_n}(\gamma)\right)^2 \text{ belongs to } \ell^2(F_n), \]
\[ \lambda \geq \frac{1}{2} \log \left(\omega(F_n)\right) \]
(with \(\omega(F_n) = 2n - 1\)). This carries over to other groups, including Gromov-hyperbolic Coxeter groups, as observed in Section 2 of [JolVa–91].>

Let us also add the following statement (a particular case of results in [Jawor–04]). Let \(\Gamma\) be an amenable finitely generated group. Then \(\Gamma\) has polynomial growth if and only if the only irreducible representation \(\pi\) of \(\Gamma\) which contains weakly the unit representation \(1_{\Gamma}\) is the unit representation itself.

VIII, and the offspring of the first Grigorchuk group.

Part of it is reviewed in [BaGrN–02].

VIII, on commensurability of subgroups of the Grigorchuk group.

It is already cited in [Harpe–03] that every infinite finitely generated subgroup of \(\Gamma\) is commensurable with \(\Gamma\). Also, \(\Gamma\) is subgroup separable. The paper with the proof of these results has now appeared [GriWi–03].

VIII.59, a misprint.

One of the indices \(i_2\) should be read as \(i_3\).

References


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