Filosofie licentiatavhandling

On Hypersurface Coamoebas and Integral Representations of A-Hypergeometric Functions

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Till minne av Mikael Passare.

Abstract

This thesis is concerned with coamoebas of hypersurfaces, and their connection to integral representations of A-hypergeometric functions. In the first part, we introduce the lopsidedness criterion for coamoebas, and define the lopsided coamoeba. We show that the set of connected components of the complement of the closed lopsided coamoeba comes naturally equipped with an order map. Using this order map we obtain new results concerning the geometry of coamoebas. In the second part, we study a class of Euler type hypergeometric integrals arising from connected components of the complement of the coamoeba, known as Euler–Mellin integrals. Through the order map for the lopsided coamoeba, we find a relation to so called Mellin–Barnes integrals. We end with a motivating example showing that Euler–Mellin integrals can be used to study the A-hypergeometric system also at rank jumping parameters.

Sammanfattning

Denna avhandling behandlar koamöbor av hyperytor, samt deras relation till integralrepresentationer av A-hypergeometriska funktioner. I den första delen introducerer vi den sidotunga koamöban, och visar att mängen av sammanhängande komponenter av dess komplement har en naturlig ordningsavbildning. Genom denna avbildning kan vi dra nya slutsatser kring geometrin för koamöbor. I den andra delen behandlar vi en klass av hypergeometriska integraler, så kallade Euler–Mellin-integraler, associerade till de sammanhängande komponenterna av komplementet till en koamöba. Genom ordningsavbildningen visar vi en relation mellan Euler–Mellin-integraler och Mellin–Barnes-integraler. Vi avslutar med ett motiverande exempel som visar att Euler–Mellin-integraler kan användas för att studera lösningsrummet till det A-hypergeometriska systemet även vid icke-generiska parametrar.

Acknowledgements

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Contents

CHAPTER 1

Background and definitions

In 1994 Gelfand, Kapranov and Zelevinsky published the modern classic *Discriminants, Resultants, and Multidimensional Determinants*. Starting with the problem of constructing a general theory for hypergeometric functions, they discovered, and in some cases rediscovered, several important concepts from algebraic geometry, homological algebra and combinatorics. Let us give a brief exposition of parts of the work presented therein, serving both as an introduction and as motivation for the theory developed in this thesis.

The "A-philosophy". The general setting is as follows. Consider q point configurations $A_1, \ldots, A_q \subset \mathbb{Z}^n$ with $N_j = |A_j| < \infty$. To each A_j we associate a family $\mathbb{C}^{A_j}$ of polynomials of the form

$$f_j(z) = \sum_{\alpha \in A_j} c_\alpha z^\alpha,$$

where we employ multi-index notation for the monomials z^α. Identifying each A_j with the $n \times N_j$-matrix

$$A_j = (\alpha_1, \ldots, \alpha_{N_j}),$$

we construct the $(n+q) \times N$-matrix

$$A = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ A_1 & A_2 & \ldots & A_q \end{pmatrix},$$

where $N = N_1 + \cdots + N_q$, and consider its related family $\mathbb{C}^A = \mathbb{C}^{A_1} \times \cdots \times \mathbb{C}^{A_q}$ of polynomials of the form

$$f(z) = f_1(z) \cdots f_q(z).$$

We will assume that each index set A_j is minimal, thus focusing on the family $(\mathbb{C}^*)^A$ rather than $\mathbb{C}^A$. In the case $q = 1$ we drop the index j, and by abuse of notation identify A with A_1. The integers n and $m = N - n - q$ are known as the *dimension* and *codimension* of the point configuration A. The "A-philosophy", which is prevailing in [**7**], can be summed up as the practice of studying polynomials through properties of the families $\mathbb{C}^A$.

An important player within the "A-philosophy" is the Newton polytope Δ_f, defined as the convex hull of the support of f in $\mathbb{R}^n$. With our notation,

$$\Delta_{f_j} = \mathrm{Conv}(A_j),$$

and we can present Δ_f as the Minkowski sum

$$\Delta_f = \Delta_{f_1} + \cdots + \Delta_{f_q}.$$

We always assume that Δ_f is of full dimension, or equivalently that the affine hull of the column span of A is of dimension $n+q-1$, however this need not be the case for each individual Δ_{f_j}.

The normalized volume of A, here denoted by $\mathrm{Vol}(A)$, is the volume of $\mathrm{Conv}(A)$ in its affine hull, normalized such that the standard $n+q-1$-simplex has volume 1. If $q=1$ then this integer agrees with the normalized volume of the Newton polytope Δ_f, that is $\mathrm{Vol}(A) = n!\,\mathrm{Vol}(\Delta_f)$.

To the point configuration A one associates the so called *A-hypergeometric system*. For full generality one consider an integer matrix $A = (\alpha_{ik}) \in \mathbb{Z}^{(n+q)\times N}$ with the vector $(1,\ldots,1)$ in its $\mathbb{Q}$ row span. For a vector $u \in \mathbb{Z}^N$, denote by u_+ and u_- the unique vectors in $\mathbb{N}^N$ with disjoint support such that $u = u_+ - u_-$. Define the differential operators $\Box_u$ and E_i to be

$$\Box_u := \Big(\frac{\partial}{\partial c}\Big)^{u_+} - \Big(\frac{\partial}{\partial c}\Big)^{u_-} \quad \text{and} \quad E_i := \sum_{k=1}^{N} \alpha_{ik} \frac{\partial}{\partial c_k}.$$

The A-hypergeometric system, or *GKZ-hypergeometric system*, $H_A(\beta)$ at $\beta \in \mathbb{C}^{n+q}$ is given by

$$\begin{aligned} \Box_u F(c) &= 0 \quad \text{for } u \in \mathbb{Z}^N \text{ with } Au = 0,\\ \text{and} \quad (E_i - \beta_i)F(c) &= 0 \quad \text{for } 1 \le i \le n+q. \end{aligned}$$

A local multivalued analytic function F that solves this system is called an *A-hypergeometric function* with homogeneity parameter β.

The A-hypergeometric system generalizes many of the classical hypergeometric functions, such as the Gauss hypergeometric ${}_2F_1$, the Appell and the Lauricella functions, and Horn's functions, and they have been studied by numerous authors over the last decades. We feel no need to motivate the study of hypergeometric functions in general, however one should mention that the theory developed by GKZ is only one of the possible approaches available. One benefit of the above setup is that many properties of the system $H_A(\beta)$ is captured by the combinatorics of the point configuration A, it is for example shown in [**8**] that, for generic homogeneity parameters β, and under the assumptions that A span $\mathbb{Z}^{n+q}$ over $\mathbb{Z}$, the rank of the solution space of $H_A(\beta)$ is $\mathrm{Vol}(A)$.

We will refer to an integer $N \times m$-matrix B, of full rank, and such that $AB = 0$, as a *dual matrix* of A. If in addition the greatest common divisor of the maximal minors of B are relatively prime, then B is known as *Gale dual*. Two objects related to a dual matrix B and of great importance to us is the lattice

$$\mathbb{Z}[B] = \big\{ kB \,|\, k \in \mathbb{Z}^N \big\}$$

and the zonotope

$$\mathcal{Z}_B = \left\{ \frac{\pi}{2} \sum_{i=1}^{N} \mu_i b_i \,\middle|\, |\mu_i| \le 1 \right\},$$

where b_i denotes the ith row of B. We should mention that B is a Gale dual if and only if $\mathbb{Z}[B] = \mathbb{Z}^m$, which is the case if and only if the columns of B span the $\mathbb{Z}$-kernel of A. Note that Gale duals occurs implicitly in the definition of the A-hypergeometric system, as we there consider the box operators $\Box_u$ for u in their column span.

Amoebas and coamoebas. The book [**7**] also saw the introduction of the term *amoeba* $\mathcal{A}_f$ of the hypersurface $V(f) \subset (\mathbb{C}^*)^n$, which by definition is its image under the componentwise Log-map. Amoebas arise naturally in several contexts, for example when describing the convergence domains of the Laurent series expansions of the rational function $1/f$; such a domain consists precisely of the inverse image of a connected component of $\mathcal{A}_f^c$. In a similar fashion, amoebas also describe convergence domains of series solutions to the A-hypergeometric system, see [**14, 22**].

One important result concerning amoebas was the discovery in [**6**] of the *order map* ν. This is an injective map from the set of connected components of the complement of the amoeba $\mathcal{A}_f$ to the set $\mathbb{Z}^n \cap \Delta_f$. If E is a component of the complement of $\mathcal{A}_f$, then the jth component of $\nu(E)$ is given by the integral

$$\nu(E)_j = \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(x)} \frac{z_j f_j'(z)}{f(z)} \frac{dz}{z}, \quad x \in E.$$

In the univariate case evaluating $v(E)$ amounts to counting zeros of f by the argument principle, yielding an analogous interpretation of ν for the multivariate case. With this in mind it is not hard to see that the set of vertices $\mathrm{vert}(\Delta_f)$ is always contained in the image of ν, and further more it was shown in [**25**] that any subset of $\mathbb{Z}^n \cap \Delta_f$ containing the set $\mathrm{vert}(\Delta_f)$ appears as the image of ν for some polynomial with the given Newton polytope. Thus, even though the image of ν is non trivial to determine, this map gives a good understanding of the structure of the set of connected components of the complement of $\mathcal{A}_f$. In particular, we have the sharp upper bound on the maximal number of connected components of the complement of $\mathcal{A}_f$ given by $|\mathbb{Z}^n \cap \Delta_f|$.

Let us mention that the order $\nu(E)$ is the gradient of the so called *Ronkin function* $N_f(x)$ [**20, 24**] at a point $x \in E$. The Ronkin function can be approximated by a piecewise linear function, which yields the concept of the *spine* S_f of the amoeba, and in turn unveils a strong connection to tropical geometry. Through these connections and others, the study of amoebas has wandered far beyond the study of hypergeometric series.

The (hypersurface) coamoeba of the polynomial f is the image of its zero-set $V(f) \subset (\mathbb{C}^*)^n$ under the componentwise argument mapping, that is $\mathcal{A}'_f = \mathrm{Arg}(V(f))$. This can be considered either as a subset of the real n-torus $\mathbf{T}^n$, or by considering the multivalued Arg-map as a subset of $\mathbb{R}^n$. The term coamoeba, coined by M. Passare and A. Tsikh in 2005, was chosen to stress that this is to be considered as a dual object to the amoeba. For example, each connected components of the complement of the coamoeba describes the convergence domain of certain integral expansion of the rational function $1/f$ [**16**]. However, being the little brother, the knowledge of coamoebas is still to many aspects immature.

Outline of thesis. This text is divided into two main chapters. In the first we focus on geometrical properties of coamoebas. The progress so far is restricted to that an upper bound on the number of connected components of the complement of the closed coamoeba is given by the normalized volume $n!\,\mathrm{Vol}(\Delta_f)$, see [**17**], however the proof therein offers no further understanding of the structure of this set, in terms of an order map or a Ronkin function for coamoebas. Such a description is sought for, as it is expected that similar results as the relation between amoebas and tropical geometry should exist also when one is concerned with coamoebas.

We will introduce a *lopsidedness criterion* for coamoebas, and study an object which we denote as the *lopsided coamoeba*. We show that there is a map v from the set of connected components of the complement of the closed lopsided coamoeba to a translation of the lattice $2\pi\mathbb{Z}[B]$ inside the zonotope $\mathcal{Z}_B$.

Through the order map v we then investigate lopsided coamoebas of codimension 1 and 2, for the latter unveiling a striking connection with the coamoeba of the so called dehomogenized discriminant $D_B(x)$, see Section 2.4. For special choices of the point configuration A we can conclude that the bound $n!\,\mathrm{Vol}(\Delta_f)$ is sharp. However, focusing on the cases when this is not possible, we arrive at a number of interesting examples. In particular we show in Example 2.5.4 that in general the bound $n!\,\mathrm{Vol}(\Delta_f)$ is *not* sharp, and in Example 2.5.2 that, given a fix Newton polytope Δ_f, there need not exist a maximally sparse polynomial whose complement has the maximal number of connected components.

In the second chapter we focus on integral representations of A-hypergeometric functions, and introduce *Euler–Mellin integrals*. These are A-hypergeometric Euler type integrals which arise from the components of the complement of the coamoeba. After treating their most fundamental properties, we unveil a connection between Euler–Mellin integrals arising from the components of the complement of the lopsided coamoeba to a set of so called Mellin–Barnes integrals, studied in for example [**2, 14**]. Finally, we pursue an important example which shows that Euler–Mellin integrals can be used to study integral representations of A-hypergeometric functions also at non-generic homogeneity parameters.

Sections 2.1–2.3.1 and 2.4 originates from the joint paper [**5**] with P. Johansson, while Sections 3.1, 3.3, and 3.5 as well as Examples 3.2.1 and 3.2.2 originates from the joint paper [**1**] with C. Berkesch and M. Passare.

1.1. Notation

$(\mathbb{C}^*)^n$	denotes the algebraic torus $(\mathbb{C}\setminus\{0\})^n$.
$\mathbf{T}^n$	denotes the real torus $(\mathbb{R}/2\pi\mathbb{Z})^n$.
I_n	denotes the $n\times n$ identity matrix.
M^t	denotes the transpose of the matrix M.
g_M	denotes the greatest common divisor of the maximal minors of M.
$\langle\cdot,\cdot\rangle$	denotes the standard scalar product.
$e_1,\dots,e_s$	denotes the vectors of the standard basis.
$\arg_\pi$	denotes the principal branch of the arg-map.
$\mathrm{CC}(\mathcal{A}_f)$	denotes the set of connected components of the complement of the amoeba.
$\mathrm{CC}(\mathcal{A}'_f)$	denotes the set of connected components of the complement of the coamoeba in $\mathbf{T}^n$.
$\mathrm{CC}(\overline{\mathcal{A}'_f})$	denotes the set of connected components of the complement of the closed coamoeba in $\mathbf{T}^n$.

CHAPTER 2

Hypersurface coamoebas

This chapter is concerned with geometrical properties of coamoebas. The central object of our study is the lopsided coamoeba, introduced in Section 2.2. To warm up we will recall some known properties of coamoebas, also phrasing them in the language provided by [**7**].

2.1. Preliminaries

Consider a binomial

$$f(z) = c\, z^{\alpha} + 1,$$

whose coamoeba $\mathcal{A}'_f$ is the set of $\theta \in \mathbb{R}^n$ such that

$$\langle \theta, \alpha \rangle = \pi - \arg_{\pi}(c) + 2\pi k, \quad k \in \mathbb{Z}.$$

Hence $\mathcal{A}'_f$ consists of a family of parallel hyperplanes, whose normal vector α is parallel to the Newton polytope $[0, \alpha]$. By the fundamental theorem of algebra, any polynomial whose Newton polytope is one dimensional factors into binomials, and hence its coamoeba consists of a family of parallel hyperplanes.

Let Γ be a (not necessarily strict) face of Δ_f. The truncated polynomial with respect to Γ is defined as

$$f_{\Gamma}(z) = \sum_{\alpha \in A \cap \Gamma} c_{\alpha} z^{\alpha}.$$

The coamoeba of a hypersurface is in general not closed, and it was shown in [**11**] and [**18**] that the closure of the coamoeba is the union of all the coamoebas of its truncated polynomials, that is

$$\overline{\mathcal{A}'_f} = \bigcup_{\Gamma \subset \Delta_f} \mathcal{A}'_{f_\Gamma}. \tag{1}$$

We will refer to $\mathcal{A}'_{f_\Gamma}$ as the coamoeba of the face Γ. Motivated by the above equality we define the *shell* $\mathcal{H}_f$ of $\mathcal{A}'_f$ as the union

$$\mathcal{H}_f = \bigcup_{\dim(\Gamma)=1} \mathcal{A}'_{f_\Gamma},$$

see [**10, 17**], which is a hyperplane arrangement on $\mathbf{T}^n$. It is natural to focus on $\overline{\mathcal{A}'_f}$ rather than $\mathcal{A}'_f$, one of the reasons being that it follows from Bochner's tube theorem [**9**, Cor. 2.5.12] that the connected components of the complement of the closed coamoeba $\overline{\mathcal{A}'_f}$, as subsets of $\mathbb{R}^n$, are convex.

We will reserve the term *integer affine transformation* of A to mean a matrix $T \in GL_n(\mathbb{Q})$ such that $TA_j \subset \mathbb{Z}^n$ for each j. Identifying T with the matrix

$$\begin{pmatrix} I_q & 0 \\ 0 & T \end{pmatrix}$$

it induces a function $\mathbb{C}^A \to \mathbb{C}^{TA}$ by the monomial change of variables

$$z_j \mapsto z^{T_j},$$

where T_j denotes the jth column of T. Using the notation

$$e^{x+i\theta} = (e^{x_1+i\theta_1}, \dots, e^{x_n+i\theta_n})$$

we find that

$$T(f_j)\big(e^{(x+i\theta)T^{-1}}\big) = \langle c_j, e^{(x+i\theta)T^{-1}TA_j}\rangle = \langle c_j, e^{(x+i\theta)A_j}\rangle = f_j\big(e^{x+i\theta}\big),$$

where c_j denotes the vector of the coefficients of the polynomial f_j. Thus a point $\theta \in \mathbb{R}^n$ belongs to $\mathcal{A}'_{f_j}$ if and only if $(T^{-1})^t\theta$ belongs to $\mathcal{A}'_{T(f_j)}$. We conclude the following relation previously described in [**19**].

Proposition 2.1.1. *As subsets of $\mathbb{R}^n$, we have that $\mathcal{A}'_{T(f)}$ is the image of $\mathcal{A}'_f$ under the linear transformation $(T^{-1})^t$.* □

Corollary 2.1.2. *As subsets of $\mathbf{T}^n$, the coamoeba $\mathcal{A}'_{T(f)}$ consists of $|\det(T)|$ linearly transformed copies of $\mathcal{A}'_f$. In particular we have the equality*

$$\big|\,\mathrm{CC}\left(\mathcal{A}'_{T(f)}\right)\big| = |\det(T)|\,|\,\mathrm{CC}(\mathcal{A}'_f)|.$$

Proof. The transformation $(T^{-1})^t$ acts with a scaling factor $1/|\det(T)|$ on $\mathbb{R}^n$, now consider a fundamental domain. □

It is noted in [**7**] that any point configuration A can be shrunk, by means of an integer affine transformation, to a point configuration whose maximal minors are relatively prime.

The polynomial f, and the point configuration A, is called *maximally sparse* if $A = \mathrm{vert}(\Delta_f)$. If in addition Δ_f is a simplex, then $V(f)$ is known as a *simple hypersurface*, and we will say that f a *simple polynomial*. Let us describe the coamoeba of a simple hypersurface. Consider first when Δ_f is the standard 2-simplex. After the change of variables $c_i z_i \mapsto z_i$, which corresponds to a translation of the coamoeba, we can assume that $f(z_1, z_2) = 1 + z_1 + z_2$. If the coamoebas of the truncated polynomials of the edges of Δ_f are drawn, with orientations given by the outward normal vectors of Δ_f, then $\mathcal{A}'_f$ consists of the interiors of the oriented regions, together with all intersection points [**18**]. An arbitrary simple trinomial

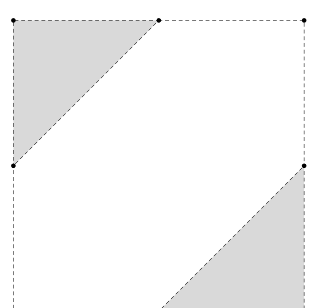

Figure 1. The coamoeba of $f(z_1, z_2) = 1 + z_1 + z_2$ in the domain $[-\pi, \pi]^2$

differs from the standard 2-simplex only by an integer affine transformation, hence its coamoeba consists of a certain number of copies of $\mathcal{A}'_f$, and is given by the same recipe as above.

Consider now when Δ_f is the standard n-simplex, with $n > 2$, that is $f(z) = 1 + z_1 + \cdots + z_n$. Let $\mathrm{Tri}(f)$ denote the set of all trinomials obtained by removing all but three monomials of f, i.e. the set of all truncated polynomials of 2-dimensional faces of Δ_f. It was shown in [**10**] that we have the identity

$$\overline{\mathcal{A}'_f} = \bigcup_{g \in \mathrm{Tri}(f)} \overline{\mathcal{A}'_g}, \tag{2}$$

which holds already for the non-closed coamoebas if $n \neq 3$. Again, an arbitrary simple polynomial is only an integer affine transformation away, and hence the identity (2) holds for all simple hypersurfaces.

It is known, and can be seen from (2) also using Proposition 2.5.1, that if Δ_f is the standard n-simplex, then $(\overline{\mathcal{A}'_f})^c$ has exactly one connected component. Thus the number of connected components of the complement of $\overline{\mathcal{A}'_f}$ equals the normalized volume $n!\,\mathrm{Vol}(\Delta_f) = 1$ in this case. An arbitrary simple polynomial in n variables is given by TA, where A is the point configuration of the standard n-simplex and $T \in GL_n(\mathbb{Z})$. As

$$\mathrm{Vol}\left(\Delta_{T(f)}\right) = |\det(T)|\,\mathrm{Vol}(\Delta_f),$$

it follows that for any simple hypersurface the number of connected components of the complement of its closed coamoeba will be equal to the normalized volume of its Newton polytope.

Let us end this section with a fundamental property of the shell $\mathcal{H}_f$, which we have not seen a proof of elsewhere.

Lemma 2.1.3. *Let $n \geq 2$, and let $l \subset \mathbb{R}^n$ be a line segment with endpoints in $(\overline{\mathcal{A}'_f})^c$ that intersects $\overline{\mathcal{A}'_f}$. Then l intersects $\mathcal{A}'_{f_\Gamma}$ for some edge $\Gamma \subset \Delta_f$. In particular, each cell of the hyperplane arrangement $\mathcal{H}_f$ contains at most one connected component of $(\overline{\mathcal{A}'_f})^c$.*

Proof. When the inclusion $\overline{\mathcal{A}'_f} \subset \bigcup_\Gamma \mathcal{A}'_{f_\Gamma}$ is proven in [**10**], it is shown that if $\{z_j\}_{j\in\mathbb{N}} \subset V(f)$ is such that

$$\lim_{j\to\infty} z_j \notin (\mathbb{C}^*)^n \quad \text{and} \quad \lim_{j\to\infty} \mathrm{Arg}(z_j) = \theta \in \mathbf{T}^n,$$

then $\theta \in \mathcal{A}'_{f_\Gamma}$ for some strict subface $\Gamma \subset \Delta_f$. Hence, by using induction on the dimension n, it is enough to show that the set

$$P = \{z \in V(f) \mid \mathrm{Arg}(z) \in N(l) \cap \mathcal{A}'_f\},$$

where $N(l)$ is an arbitrarily small neighbourhood of l in $\mathbb{R}^n$, is such that $\mathrm{Log}(P)$ is unbounded. That this is the case follows by Hartogs theorem on domains of holomorphy [**13**, Thm. 1.8], but let us fill in the details. Consider the function $g(w) = f(e^w)$, where $w_k = x_k + i\theta_k$. We can assume that l is parallel to the θ_1-axis and, by a translation of the coamoeba, that there are $\rho_1, \ldots, \rho_n > 0$ and $0 < r < \rho_1$ such that the set

$$S = \{-\rho_1 \leq \theta_1 \leq \rho_1\} \times \cdots \times \{-\rho_n \leq \theta_n \leq \rho_n\}$$

fulfils $l \subset S \subset N(l)$. If in addition we denote by

$$\tilde{S} = \{-r \leq \theta_1 \leq r\} \times \{-\rho_2 \leq \theta_2 \leq \rho_2\} \times \cdots \times \{-\rho_n \leq \theta_n \leq \rho_n\}$$

then $S \setminus \tilde{S} \subset (\overline{\mathcal{A}'_f})^c$. Let us now assume that $\mathrm{Log}(P)$ is bounded. Then there exists an R such that if

$$D = \{x \in \mathbb{R}^n \text{ such that } |x| > R\},$$

then $g(w)$ has no zeros in $D + iS \subset \mathbb{C}^n$. Denote by $w' = (w_2, \dots, w_n)$ and let $(D + iS)'$ be the projection of $D + iS$ onto the last $n-1$ components. Then in particular, $g(w)$ has no zeros when $w' \in (D + iS)'$ and w_1 lies in the domain given by $\{r < |\mathrm{Im}\, w_1| < \rho_1\} \cup (\{|\mathrm{Re}\, w_1| > R\} \cap \{|\mathrm{Im}\, w_1| < \rho\})$, see Figure 2. Consider a curve γ as in the figure, and the integral

$$k(w') = \frac{1}{2\pi i} \int_\gamma \frac{g_1'(w_1, w')}{g(w_1, w')} dw_1, \quad w' \in (D + iS)'$$

For a fix w' this counts the number of roots of $g(w)$ inside the box in Figure 2. As it depend continuously on w' in the domain $(D + iS)'$ it is constant, and by considering w' with $|x'| > R$ we conclude that it is zero. However this contradicts the assumption that l intersects $\mathcal{A}'_f$. □

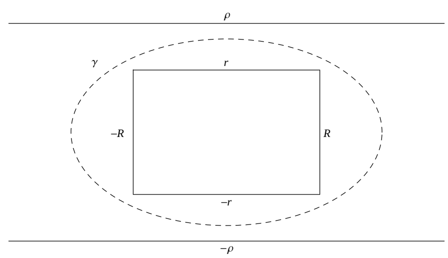

FIGURE 2. The curve $\gamma \subset \mathbb{C}$.

2.2. The lopsidedness criterion

Let us first focus on the case $q = 1$, viewing the polynomial f as a sum of monomials. For shorter notation we will write $c_j = c_{\alpha_j}$. Recall the lopsidedness criterion for amoebas; for a point $x \in \mathbb{R}^n$, consider the list of the moduli of the monomials of f at x,

$$f\{x\} = \left\{e^{\log|c_1| + \langle \alpha_1, x\rangle}, \dots, e^{\log|c_N| + \langle \alpha_N, x\rangle}\right\}.$$

This list is said to be *lopsided* if one component is greater than the sum of the others, the point being that this is a sufficient condition for $x \notin \mathcal{A}_f$. The term ”lopsided” was coined by Purbhoo [**23**], however the criterion was considered earlier by Rullgård [**25**] and in works by Passare and Tsikh. The *lopsided amoeba* $\mathcal{LA}_f$ is defined as the set of points $x \in \mathbb{R}^n$ such that $f\{x\}$ is *not* lopsided. The name can be misleading, lopsided amoebas are not necessarily amoebas, but per se they constitute different class of objects. There is an inclusion $\mathcal{A}_f \subset \mathcal{LA}_f$, and in particular each connected component of $\mathcal{LA}_f^c$ is contained in a unique connected component of $\mathcal{A}_f^c$. While the point of the paper [**23**] is that lopsidedness can be used to find computational methods to approximate the amoeba, our interest lies instead in the relation between the lopsided amoeba and the order map ν. Namely, if the list $f\{x\}$ is dominated in the sense of lopsidedness by the monomial with exponent α, then it follows by Rouché's theorem that $\nu(E) = \alpha$. Hence the order map ν restricted to the set of connected components of the complement of $\mathcal{LA}_f$ is an injective map into the point configuration A. In this sense, the lopsided amoeba lies closer to the "A-philosophy" than the amoeba itself.

We reserve the term half-plane $H \subset \mathbb{C}$ to refer to a rotation of the right half-plane, that is

$$H = H_\phi = \left\{ z \in \mathbb{C} \,\middle|\, \mathrm{Re}\,(e^{i\phi} z) > 0 \right\}$$

where $\phi \in \mathbb{R}$. For each point $\theta \in \mathbf{T}^n$, consider the list

$$f\langle\theta\rangle = \left\{ e^{i(\arg(c_1) + \langle \alpha_1, \theta\rangle)}, \dots, e^{i(\arg(c_N) + \langle \alpha_N, \theta\rangle)} \right\},$$

where we by abuse of notation consider this also as a set $f\langle\theta\rangle \subset S^1 \subset \mathbb{C}$. We say that the list $f\langle\theta\rangle$ is *lopsided* if there exist a half-plane H such that, as a set, $f\langle\theta\rangle \subset \overline{H}$ but $f\langle\theta\rangle \not\subset \partial H$.

DEFINITION 2.2.1 (In the case $q = 1$). The lopsided coamoeba $\mathcal{LA}'_f$ is defined as the set of points $\theta \in \mathbf{T}^n$ such that $f\langle\theta\rangle$ is *not* lopsided.

When necessary we will consider $\mathcal{LA}'_f$ as a subset of $\mathbb{R}^n$. The formulation of Definition 2.2.1 was partly chosen to stress the analogy with the lopsided amoeba, a more natural description is as follows. Denote the components of $f\langle\theta\rangle$ by $t_1, \dots, t_N$, and consider the convex cone

$$\mathbb{R}_+ f\langle\theta\rangle = \left\{ r_1 t_1 + \cdots + r_N t_N \mid r_1, \dots, r_N \in \mathbb{R}_+ \right\}.$$

LEMMA 2.2.2. *We have that $\theta \in \mathcal{LA}'_f$ if and only if $0 \in \mathbb{R}_+ f\langle\theta\rangle$.*

PROOF. If $\theta \in (\mathcal{LA}'_f)^c$, then $\mathbb{R}_+ f\langle\theta\rangle \subset \mathrm{int}(H)$, where H is the half-plane such that $f\langle\theta\rangle \subset \overline{H}$ but $f\langle\theta\rangle \not\subset \partial H$. Conversely, if $\mathbb{R}_+ f\langle\theta\rangle$ does not contain the origin, then it follows from the convexity of $\mathbb{R}_+ f\langle\theta\rangle$ that there exist a half-plane H such that $\mathbb{R}_+ f\langle\theta\rangle \subset \mathrm{int}(H)$. $\square$

COROLLARY 2.2.3. *We have the inclusion $\mathcal{A}'_f \subset \mathcal{LA}'_f$.*

PROOF. If $f(re^{i\theta}) = 0$ then $0 \in \mathbb{R}_+ f\langle\theta\rangle$. $\square$

COROLLARY 2.2.4. *If A is simple, then $\mathcal{A}'_f = \mathcal{LA}'_f$.*

PROOF. By considering integer affine transformations it is enough to prove the statement when Δ_f is the standard n-simplex, that is $f(z) = 1 + z_1 + \cdots + z_n$. We have that $0 \in \mathbb{R}_+ f\langle\theta\rangle$ if and only if we can find $r_0, \dots, r_n \in \mathbb{R}_+$ such that $r_0 + r_1 e^{i\theta_1} + \cdots + r_n e^{i\theta_n} = 0$, and this is equivalent to $\theta \in \mathcal{A}'_f$. $\square$

Simple hypersurfaces are not the only ones for which the identity $\mathcal{A}'_f = \mathcal{LA}'_f$ holds. It will of course be the case as soon as $\mathcal{A}'_f = \mathbf{T}^n$, and such examples are easy to construct by considering products of polynomials. A less trivial example is given by $f(z_1, z_2) = 1 + z_1 + z_2 - r z_1 z_2$ for any $r \in \mathbb{R}_+$, however we leave the details to the reader.

Denote by

$$F(c, z) = \sum_{\alpha \in A} c_\alpha z^\alpha,$$

the polynomial obtained when we consider also the coefficients c to be variables. This polynomial has a coamoeba $\mathcal{A}'_F \subset \mathbf{T}^{N+n}$ which, as F is simple, coincides with its lopsided coamoeba $\mathcal{LA}'_F$. As the convex cone $\mathbb{R}_+ f\langle\theta\rangle$ coincides with the cone $\mathbb{R}_+ F\langle\arg(c), \theta\rangle$, we see that $\mathcal{LA}'_f$ is nothing but the intersection of $\mathcal{A}'_F$ with the sub n-torus of $\mathbf{T}^{N+n}$ given by fixing $\mathrm{Arg}(c)$. In this manner, the lopsided coamoeba inherits some properties of simple coamoebas.

Proposition 2.2.5. *Let* $\mathrm{Tri}(f)$ *denote the set of all trinomials* g *one can construct from the set of monomials of* f. *Then*

$$\overline{\mathcal{LA}'_f} = \bigcup_{g \in \mathrm{Tri}(f)} \overline{\mathcal{A}'_g}.$$

Proof. By the previous discussion we can view $\mathcal{LA}'_f$ is the intersection of $\mathcal{A}'_F$ with the sub n-torus of $\mathbf{T}^{N+n}$ given by fixing $\mathrm{Arg}(c)$. This is of course also the case for each trinomial g in $\mathrm{Tri}(f)$, and hence the identity follows from (2). □

As was the case in (2), also this identity holds in the non-closed case if $N \neq 4$. Lopsided coamoebas first appeared under this disguise in [**10**]. For our sake this proposition implies a naive algorithm for determining lopsided coamoebas, by determining the coamoebas of each trinomial in $\mathrm{Tri}(f)$, however as the number of trinomials grows as N^3 it is not very effective.

Figure 3. Above: the coamoeba and lopsided coamoeba of $f(z_1, z_2) = z_1^3 + z_2 + z_2^2 - z_1 z_2$. Below: the coamoeba and lopsided coamoeba of $f(z_1, z_2) = 1 + z_1 + z_2 + i z_1 z_2$.

Definition 2.2.6. Let $\mathrm{Bin}(f)$ denote the set of all binomials that can be obtained by removing all but two monomials of f. The *shell* $\mathcal{LH}_f$ of the lopsided coamoeba $\mathcal{LA}'_f$ is defined as the union

$$\mathcal{LH}_f = \bigcup_{g \in \mathrm{Bin}(f)} \mathcal{A}'_g$$

In the case $n \geq 2$ Proposition 2.2.5 states that $\overline{\mathcal{LA}'_f}$ is the closure of the coamoeba of the polynomial $\prod_{g \in \mathrm{Tri}(f)} g(z)$, and the name *shell* is motivated by the fact that $\mathcal{LH}_f$ is the ordinary shell of this polynomial.

PROPOSITION 2.2.7. *The boundary of* $\overline{\mathcal{LA}'_f}$ *is contained in* $\mathcal{LH}_f$.

PROOF. The boundary of $\overline{\mathcal{LA}'_f}$ consists of points θ for which $f\langle\theta\rangle$ contains two antipodal points, however this implies that θ is contained in the coamoeba of the corresponding binomial. $\square$

The focus on $\overline{\mathcal{A}'_f}$ rather than $\mathcal{A}'_f$ leads us naturally to consider $\overline{\mathcal{LA}'_f}$ in more detail. Its complement has the following characterization.

PROPOSITION 2.2.8. *We have that* $\theta \in (\overline{\mathcal{LA}'_f})^c$ *if and only if there is a half-plane* H *with* $f\langle\theta\rangle \subset H$.

PROOF. The "if" part is clear. To show "only if", note that if $\theta \in (\mathcal{LA}'_f)^c$ is such that there is no open half-plane H with $f\langle\theta\rangle \subset H$, then $f\langle\theta\rangle$ contains two antipodal points. Then we can find a simple trinomial $g \in T_f$ such that $\theta \in \overline{\mathcal{A}'_g}$, and by the description of simple trinomials in the previous section there is a sequence $\{\theta_n\} \subset \operatorname{int}(\mathcal{A}'_g)$ such that $\lim_{n\to\infty}\theta_n = \theta$. As g is simple we have that $\mathcal{A}'_g = \mathcal{LA}'_g$, hence for each θ_n the list $g\langle\theta_n\rangle$ is not lopsided. Then neither is $f\langle\theta_n\rangle$, showing that $\{\theta_n\} \subset \mathcal{LA}'_f$, and as a consequence that $\theta \in \overline{\mathcal{LA}'_f}$. $\square$

Let us now describe the relation between the sets $\mathrm{CC}(\overline{\mathcal{A}'_f})$ and $\mathrm{CC}(\overline{\mathcal{LA}'_f})$, beginning with yet another characterization of $\mathcal{LA}'_f$.

LEMMA 2.2.9. *Denote by* $f_r(z) = \sum_{\alpha\in A} r_\alpha a_\alpha z^\alpha$ *the polynomial obtained by varying the radii of the coefficients of* f *by* $r = (r_\alpha) \in (\mathbb{R}_+)^N$. *Then*

$$\mathcal{LA}'_f = \bigcup_{r\in(\mathbb{R}_+)^N} \mathcal{A}'_{f_r}.$$

PROOF. The statement follows from Lemma 2.2.2. If $\theta \in \mathcal{A}'_{f_r}$, then $0 \in \mathbb{R}_+ f_r\langle\theta\rangle = \mathbb{R}_+ f\langle\theta\rangle$. Conversely, if $0 \in \mathbb{R}_+ f\langle\theta\rangle$, then there exist an $r \in \mathbb{R}_+^N$ such that $f(e^{i\theta}) = 0$. $\square$

PROPOSITION 2.2.10. *Each connected component of* $(\overline{\mathcal{A}'_f})^c$ *contains at most one connected component of* $(\overline{\mathcal{LA}'_f})^c$.

PROOF. It is clear that each connected component of $(\overline{\mathcal{LA}'_f})^c$ is included in some connected component of $(\overline{\mathcal{A}'_f})^c$, we only have to show that this inclusion is injective. Note first that for a univariate polynomial g, the order of the zero at the origin does not depend on the radii of the coefficients. Thus, with notation as in the previous lemma, the arguments of the zeros of f_r vary continuously with r. Especially, this implies that any line segment in $\mathbb{R}$ between distinct components of the complement of $\mathcal{LA}'_g$ intersects $\mathcal{A}'_g$.

Now consider a multivariate polynomial $f(z)$, and let l be a line segment in $\mathbb{R}^n$ between distinct components of the complement of $\overline{\mathcal{LA}'_f}$. By the previous lemma there exists an $r \in (\mathbb{R}_+)^N$ such that l intersects $\mathcal{A}'_{f_r}$, then Lemma 2.1.3 gives an edge Γ of Δ_f such that l intersects $\mathcal{A}'_{(f_r)_\Gamma}$ and it follows from the univariate case that l intersects $\mathcal{A}'_{f_\Gamma}$. Hence l intersects $\overline{\mathcal{A}'_f}$. $\square$

It is an integrated part of the definition of the lopsided coamoeba that we consider f as a sum of monomials. When turning to the general situation of a product of q factors, it is beneficial to consider the lopsidedness criterion for each factor by itself.

DEFINITION 2.2.11. Let $f = f_1 \cdots f_q$. The lopsided coamoeba $\mathcal{LA}'_{f_1\cdots f_q}$ is defined as the union of the lopsided coamoebas of $f_1, \ldots, f_q$, that is

$$\mathcal{LA}'_{f_1\cdots f_q} = \bigcup_{j=1}^{q} \mathcal{LA}'_{f_j},$$

where each lopsided coamoeba $\mathcal{LA}'_{f_j}$ is defined as above. Similarly, the shell of the lopsided coamoeba $\mathcal{LA}'_{f_1\cdots f_q}$ is defined as the union

$$\mathcal{LH}_{f_1\cdots f_q} = \bigcup_{j=1}^{q} \mathcal{LH}_{f_j},$$

where each shell $\mathcal{LH}_{f_j}$ is defined as above.

We use the subscript $f_1 \cdots f_q$ to mark that we consider f as a product of polynomials. In difference to ordinary coamoebas the inclusion

$$\mathcal{LA}'_{f_1\cdots f_q} \subset \mathcal{LA}'_f$$

is in general not an equality, see Figure 4.

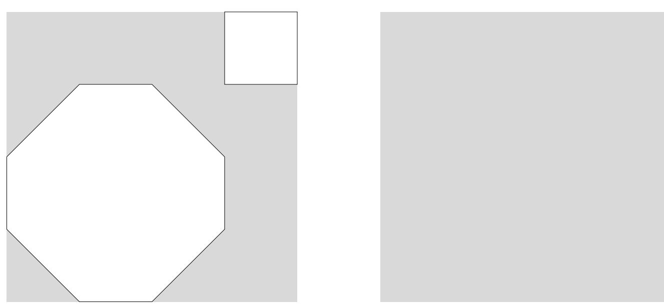

FIGURE 4. The lopsided coamoebas of $f(z_1, z_2) = z_1^3 + z_2 + z_2^2 - z_1 z_2$ and of $f(z_1, z_2)^2$, when the latter is *not* considered as a product.

2.3. An order map for the lopsided coamoeba

The aim of this section is to provide an order map for the lopsided coamoeba. The role played by the point configuration A in the order map of the lopsided amoeba is here given to a dual matrix B. Assuming that B is a Gale dual will make our statements more streamlined, however it is not a necessary assumption in order to develop the theory.

2.3.1. The case $q = 1$. For a fix point $\alpha \in A$, consider the function $p_\alpha^k : \mathbb{R}^n \to 2\pi\mathbb{Z}$ given by

$$p_\alpha^k(\theta) = \arg_\pi\left(\frac{c_{\alpha_k} e^{i\langle \alpha_k, \theta\rangle}}{c_\alpha e^{i\langle \alpha, \theta\rangle}}\right) - \arg_\pi(c_{\alpha_k}) + \arg_\pi(c_\alpha) - \langle \alpha_k - \alpha, \theta\rangle.$$

Note that p_α^k is constant on the complement of the coamoeba of the binomial $c_\alpha z^\alpha + c_{\alpha_k} z^{\alpha_k}$, and especially the vector valued function

$$p_\alpha(\theta) = \big(p_\alpha^1(\theta), \dots, p_\alpha^N(\theta)\big)$$

is constant on the cells of the hyperplane arrangement $\mathcal{LH}_f$, considered as subsets of $\mathbb{R}^n$.

THEOREM 2.3.1. *For each $\alpha \in A$, there is a map*

$$v_\alpha : \mathrm{CC}(\overline{\mathcal{LA}'_f}) \to \mathrm{int}(\mathcal{Z}_B) \cap \big\{ \operatorname{Arg}_\pi(c)B + 2\pi\mathbb{Z}[B]\big\}$$

given by

$$v_\alpha(\Theta) = \big(\operatorname{Arg}_\pi(c) + p_\alpha(\theta)\big)B, \quad \theta \in \Theta. \tag{3}$$

Moreover, the map v_α is independent of the choice of α.

PROOF. As

$$\begin{aligned}\operatorname{Arg}_\pi(c) + p_\alpha(\theta) &= \left(\arg_\pi\left(\frac{c_{\alpha_k} e^{i\langle \alpha_k, \theta\rangle}}{c_\alpha e^{i\langle \alpha,\theta\rangle}}\right) + \arg_\pi(c_\alpha) - \langle \alpha_k - \alpha, \theta\rangle\right)_k \\ &= \left(\arg_\pi\left(\frac{c_{\alpha_k} e^{i\langle \alpha_k, \theta\rangle}}{c_\alpha e^{i\langle \alpha,\theta\rangle}}\right)\right)_k + (\arg_\pi(c_\alpha) + \langle\alpha,\theta\rangle, -\theta)A,\end{aligned}$$

we find that

$$\big(\operatorname{Arg}_\pi(c) + p_\alpha(\theta)\big)B = \left(\arg_\pi\left(\frac{c_{\alpha_1} e^{i\langle \alpha_1, \theta\rangle}}{c_\alpha e^{i\langle \alpha,\theta\rangle}}\right), \dots, \arg_\pi\left(\frac{c_{\alpha_N} e^{i\langle \alpha_N, \theta\rangle}}{c_\alpha e^{i\langle \alpha,\theta\rangle}}\right)\right)B.$$

Hence the right hand side of (3) is well-defined on $\mathbf{T}^n$. As p_α is constant on the cells of $\mathcal{LH}_f$ we have that v_α is well-defined on $\mathrm{CC}(\overline{\mathcal{LA}'_f})$. Given a $\theta \in (\overline{\mathcal{LA}'_f})^c$, the components of $f\langle\theta\rangle$ are contained in one half-plane H. While $p_\alpha^k(\theta)$ is not invariant under multiplication of f with a Laurent monomial, the function $v_\alpha(\Theta)$ in fact is. Hence we can assume that $\alpha = 0$ and that $H = H_0$ is the right half space. Since $\arg_\pi(x_1x_2) = \arg_\pi(x_1) + \arg_\pi(x_2)$ for any two elements $x_1, x_2 \in H_0$ we have that

$$\left\{\begin{array}{rcl} \arg_\pi(c_1) + \langle \alpha_1, \theta\rangle + p_\alpha^1(\theta) &=& \frac{\pi}{2}\mu_1 \\ &\vdots& \\ \arg_\pi(c_N) + \langle \alpha_N, \theta\rangle + p_\alpha^N(\theta) &=& \frac{\pi}{2}\mu_N, \end{array}\right. \tag{4}$$

where $\mu_k = \frac{2}{\pi}\arg_\pi(c_{\alpha_k} e^{i\langle\alpha_k,\theta\rangle}) \in (-1,1)$. Hence

$$\big(\operatorname{Arg}_\pi(c) + p_\alpha(\theta)\big)B = \frac{\pi}{2}\mu B \in \mathrm{int}(\mathcal{Z}_B).$$

Finally, that v_α is independent of the choice of α follows since

$$p_0^k(\theta) - p_\alpha^k(\theta) = \arg_\pi(c_\alpha e^{i\langle\alpha,\theta\rangle}) - \arg_\pi(c_\alpha) - \langle \alpha, \theta\rangle$$

is independent of k, and hence $(p_0(\theta) - p_\alpha(\theta))B = 0$. □

DEFINITION 2.3.2. The map $v = v_\alpha$ is called the *order map* of the lopsided coamoeba $\mathcal{LA}'_f$.

In order to show the statements on surjectivity and injectivity of v, we have to use a more detailed notation. After multiplication with a Laurent monomial, we can assume that A is of the form

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & A_{\mathrm{I}} & A_{\mathrm{II}} \end{pmatrix},$$

where A_{I} is a non singular $n \times n$ matrix. We can also assume that $c_0 = 1$, i.e. that the constant 1 is a monomial of f. Any dual matrix B of A can be presented in the form

$$B = \begin{pmatrix} a_0 \\ -A_{\mathrm{I}}^{-1} A_{\mathrm{II}} \\ I_m \end{pmatrix} T, \tag{5}$$

where $a_0 \in \mathbb{Q}^m$ is defined by the property that each column of B should sum to zero, and $T \in GL_m(\mathbb{Q})$ is chosen such that B becomes an integer matrix.

LEMMA 2.3.3. *Let A be under the assumptions imposed above. Denote by $c_I = (c_1, \ldots, c_n)$ and $c_{II} = (c_{n+1}, \ldots, c_{n+m})$, and similarly for $l \in \mathbb{Z}^N$ and $\mu \in \mathbb{R}^N$. Consider the system*

$$\begin{cases} \operatorname{Arg}_\pi(c_I) + \theta A_I + 2\pi l_I &= \dfrac{\pi}{2}\,\mu_I \\ \operatorname{Arg}_\pi(c_{II}) + \theta A_{II} + 2\pi l_{II} &= \dfrac{\pi}{2}\,\mu_{II} \end{cases} \tag{6}$$

Then $\theta \in (\overline{\mathcal{LA}'_f})^c$ if and only if θ solves (6) for some integers l and some numbers $\mu_0, \ldots, \mu_{n+m}$ such that $\mu_0, \mu_1 + \mu_0, \ldots, \mu_{n+m} + \mu_0 \in (-1, 1)$

PROOF. If $\theta \in (\overline{\mathcal{LA}'_f})^c$, then there is a halfplane H_ϕ such that $f\langle\theta\rangle \subset H_\phi$. As the constant 1 is a term of f, we can choose $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Considering the polynomial $e^{-i\phi} f(z)$, we find that this is lopsided at θ for H_0. Thus there are numbers $\lambda_1, \ldots, \lambda_{n+m} \in (-1, 1)$ and integers $l_1, \ldots, l_{n+m}$ such that

$$\arg_\pi(c_k) + \langle\theta, \alpha_k\rangle + 2\pi l_k = \frac{\pi}{2}\,\lambda_k + \phi, \quad k = 1, \ldots, n+m.$$

This shows that θ fulfils (6) with l as above, $\mu_0 = -\frac{2}{\pi}\,\phi$ and $\mu_k = \lambda_k + \frac{2}{\pi}\,\phi$ for $k = 1, \ldots, n+m$. Conversely, if θ fulfils (6) for such l and μ, then $f\langle\theta\rangle \subset H_{-\frac{\pi}{2}\,\mu_0}$. $\square$

PROPOSITION 2.3.4. *The order map v is a surjection.*

PROOF. Let A be under the assumptions imposed above. Formally solving the first equation of (6) for θ by multiplication with A_{I}^{-1} and eliminating θ in the second equation, also applying the transformation T, one arrives at the equivalent system

$$\begin{cases} \theta &= \dfrac{\pi}{2}\,\mu_{\mathrm{I}} A_{\mathrm{I}}^{-1} - \operatorname{Arg}(c_{\mathrm{I}}) A_{\mathrm{I}}^{-1} - 2\pi l_{\mathrm{I}} A_{\mathrm{I}}^{-1} \\ \operatorname{Arg}_\pi(c) B + 2\pi(0, l_{\mathrm{I}}, l_{\mathrm{II}}) B &= \dfrac{\pi}{2}\,(0, \mu_{\mathrm{I}}, \mu_{\mathrm{II}}) B. \end{cases} \tag{7}$$

To see that v is surjective, consider a point $\operatorname{Arg}_\pi(c)B + 2\pi l B = \frac{\pi}{2}\lambda B \in \operatorname{int}(\mathcal{Z}_B)$, and note that we can assume that $l_0 = 0$. Define μ by $\mu_k = \lambda_k - \lambda_0$ for $k = 0, \ldots, n+m$. It follows that the pair (l, μ) fulfils the second equation of (7). Let $\theta \in \mathbb{R}^n$ be defined by the first equation of (7), it then follows that the triple (θ, l, μ) fulfils (6), and thus by Lemma 2.3.3 we have that $\theta \in (\overline{\mathcal{LA}'_f})^c$. By tracing backwards we find that the order of the component containing θ is $\operatorname{Arg}_\pi(c)B + 2\pi l B$, and hence the map v is surjective. $\square$

PROPOSITION 2.3.5. *If $g_A = 1$ then v is an injection.*

PROOF. For any point $p \in \operatorname{int}(\mathcal{Z}_B)$, the set of all $\mu \in (-1,1)^N$ such that $2\pi\mu B = p$, is a convex set. This implies that for fix integers l, the set of $\theta \in \mathbb{R}^n$ such that (6) is fulfilled with $\mu_0, \mu_1 - \mu_0, \ldots, \mu_N - \mu_0 \in (-1,1)$ is in turn also convex. As the right hand side of (3) is constant on each cell of $\mathcal{LH}_f$, this set is exactly one connected component of $(\overline{\mathcal{LA}'_f})^c$ in $\mathbb{R}^n$. Thus if we consider two points θ and $\tilde{\theta}$ in $\mathbb{R}^n$ which both maps to $\operatorname{Arg}(c)B + 2\pi l B$, then we can assume that θ and $\tilde{\theta}$ fulfils (6) for the same numbers μ, however possibly for different integers l. Under this assumption there are integers $s_1, \ldots, s_N$ such that

$$\langle \alpha_k, \theta \rangle = \langle \alpha_k, \tilde{\theta} \rangle + 2\pi s_k, \quad k = 1, \ldots, N.$$

The assumption that $g_A = 1$ is equivalent to that the columns of A span $\mathbb{Z}^{n+1}$ over $\mathbb{Z}$. Thus for each vector e_i there are integers $r_i = (r_{i1}, \ldots, r_{iN})$ such that $e_i = \sum_k r_{ik}\alpha_k$. Hence

$$\theta_i = \langle e_i, \theta \rangle = \sum_{k=1}^{N} r_{ik} \langle \alpha_k, \theta \rangle = \sum_{k=1}^{N} r_{ik} \langle \alpha_k, \tilde{\theta} \rangle + 2\pi r_{ik} s_k = \tilde{\theta}_i + 2\pi \langle r_i, s \rangle,$$

which shows that θ and $\tilde{\theta}$ correspond to the same point in $\mathbf{T}^n$. □

REMARK 2.3.6. In general, the map v will be g_A to one. Thus if one considers v as a map from $\operatorname{CC}(\overline{\mathcal{LA}'_f})$ into the full translated lattice $\operatorname{int}(\mathcal{Z}_B) \cap (\operatorname{Arg}_\pi(c)B + 2\pi\mathbb{Z}^m)$, then injectivity is measured in terms of g_A, while surjectivity is measured in terms of g_B. In view of Corollary 2.1.2, if one is interested in counting the number of connected components of the complement, we find it natural to assume that v is a bijection.

EXAMPLE 2.3.7. Let us determine the map v explicitly in the first example shown in Figure 3, that is we consider the polynomial $f(z_1, z_2) = z_1^3 + z_2 + z_2^2 - z_1 z_2$. The point configuration is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix},$$

and a Gale dual of A is given by

$$B = (-1, -1, -1, 3)^t.$$

The corresponding zonotope is the interval $\mathcal{Z}_B = [-3\pi, 3\pi]$. As the translation $\operatorname{Arg}_\pi(c)B = 3\arg_\pi(-1) = 3\pi$, the image of the map v will be the doubleton $\{-\pi, \pi\}$. To determine v, it is enough to evaluate v_α for some α and one point in each of the two components of the complement of $\overline{\mathcal{LA}'_f}$, and we see from the picture in Figure 3 that a natural choice of points is $\theta_1 = (-\frac{2\pi}{3}, 0)$ and $\theta_2 = (\frac{2\pi}{3}, 0)$. We find that

$$\begin{array}{rcrcr} v_{\alpha_1}(\Theta_1) & = & (0, -2\pi, -2\pi, -\pi)B & = & \pi \\ v_{\alpha_1}(\Theta_2) & = & (0, 2\pi, 2\pi, \pi)B & = & -\pi. \end{array}$$

EXAMPLE 2.3.8. Let us also consider a univariate case of codimension 1, namely

$$f(z) = 1 + z^3 + iz^5.$$

A Gale dual of A is given by $B = (2, -5, 3)^t$, hence the zonotope is the interval $\mathcal{Z}_B = [-5\pi, 5\pi]$. As the translation term is $(0, 0, \pi/2)B = \frac{3\pi}{2}$, the image of v is $\{-\frac{9\pi}{2}, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}\}$. The lopsided coamoeba $\mathcal{LA}'_f$ can be seen in Figure 5. We

FIGURE 5. $\mathcal{LA}'_f$ in the fundamental domain $[-\pi, \pi]$.

choose one point from each connected component of $(\overline{\mathcal{LA}'_f})^c$, namely

$$\theta_1 = -\frac{7\pi}{8}, \quad \theta_2 = -\frac{\pi}{2}, \quad \theta_3 = 0, \quad \theta_4 = \frac{5\pi}{16}, \quad \theta_5 = \frac{3\pi}{4},$$

and find that

$$\begin{array}{rclcr} v_0(\Theta_1) & = & (0, -\frac{5\pi}{8}, \frac{\pi}{8})B & = & \frac{7\pi}{2} \\ v_0(\Theta_2) & = & (0, \frac{\pi}{2}, 0)B & = & -\frac{5\pi}{2} \\ v_0(\Theta_3) & = & (0, 0, \frac{\pi}{2})B & = & \frac{3\pi}{2} \\ v_0(\Theta_4) & = & (0, \frac{15\pi}{16}, \frac{\pi}{16})B & = & -\frac{9\pi}{2} \\ v_0(\Theta_5) & = & (0, \frac{\pi}{4}, \frac{\pi}{4})B & = & -\frac{\pi}{2}. \end{array}$$

It is notable that the orders does not reflect the positions of the components of the complement of $\overline{\mathcal{LA}'_f}$ on $\mathbf{T}^1$.

Let us make a short sidestep and consider the non-closed lopsided coamoeba, $\mathcal{LA}'_f$. The map v extends to a map on $\mathrm{CC}(\mathcal{LA}'_f)$ if one allows for points on the boundary of $\mathcal{Z}_B$, however the vertices of $\mathcal{Z}_B$ will not lie in the image of this map.

THEOREM 2.3.9. *Let f be a Laurent polynomial, and let B be a dual matrix of A. Then each function v_α can be considered as a surjective map*

$$v_\alpha : \mathrm{CC}(\mathcal{LA}'_f) \to (\mathcal{Z}_B \setminus \mathrm{vert}(\mathcal{Z}_B)) \cap \{\mathrm{Arg}(c)B + 2\pi\mathbb{Z}[B]\},$$

where $\mathrm{vert}(\mathcal{Z}_B)$ is the set of vertices of $\mathcal{Z}_B$. The map v_α is independent of choice of α, and further more if $g_A = 1$ then it is an injection.

PROOF. The proof is by following the same steps as in the proofs of Theorem 2.3.1, and Propositions 2.3.4 and 2.3.5, with the only difference that we allow for $|\mu_i| \leq 1$. We only note that p is a vertex of $\mathcal{Z}_B$ if and only if any $\mu \in [-1, 1]^N$ such that $p = \frac{\pi}{2}\mu B$ has $|\mu_k| = 1$ for each k. This implies that $f\langle\theta\rangle$ is contained in one line (but not in an open half-plane), and hence that $\theta \in \mathcal{LA}'_f$. □

Hence we also have a description of the set $\mathrm{CC}(\mathcal{LA}'_f)$, where we note especially that the bound $n!\,\mathrm{Vol}(\Delta_f)$ does not hold for $|\,\mathrm{CC}(\mathcal{LA}'_f)|$. However we should remark that the corresponding result to Proposition 2.2.10 also fails, leaving the question of whether the normalized volume of the Newton polytope is the correct bound also for $|\,\mathrm{CC}(\mathcal{A}'_f)|$ as an open problem.

2.3.2. The general case. Consider now a product $f(z)$ with q factors. Fix a point $\alpha = (\alpha_1, \dots, \alpha_q) \in A_1 \times \cdots \times A_q$. Define each of the functions $p_{\alpha_j} : \mathbb{R}^n \to (2\pi\mathbb{Z})^{N_j}$ as in the previous section and consider the function $p_\alpha : \mathbb{R}^n \to (2\pi\mathbb{Z})^N$ given by

$$p_\alpha(\theta) = \big(p_{\alpha_1}(\theta), \dots, p_{\alpha_q}(\theta)\big)$$

As before, this is constant on the cells of the hyperplane arrangement $\mathcal{LH}_{f_1 \cdots f_q}$ (as considered in $\mathbb{R}^n$).

THEOREM 2.3.10. *For each α, there is a map*

$$v_\alpha : \mathrm{CC}(\overline{\mathcal{LA}'_{f_1\cdots f_q}}) \to \mathrm{int}(\mathcal{Z}_B) \cap \big\{ \mathrm{Arg}_\pi(c)B + 2\pi\mathbb{Z}[B]\big\}$$

given by

$$v_\alpha(\Theta) = \big(\mathrm{Arg}_\pi(a) + p_\alpha(\theta)\big)B, \quad \theta \in \Theta$$

The map v_α is independent of the choice of α. Moreover, the map $v = v_\alpha$ is surjective, and if $g_A = 1$ it is also injective.

PROOF. Compare the two point configurations

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ A_1 & A_2 & \dots & A_q \end{pmatrix} \quad \text{and} \quad \widetilde{A} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ A_1 & A_2 & \cdots & A_q \end{pmatrix},$$

and note that $\ker(A) = \ker(\widetilde{A})$. Hence A and $\widetilde{A}$ have the same set of dual matrices B. The first point configuration we associate to the product $f(z) = f_1(z)\cdots f_q(z)$, while the second we associate to the polynomial

$$\tilde{f}(z, w) = f_1(z) + w_1 f_2(z) + \cdots + w_{q-1} f_q(z).$$

We have that $\theta \in (\overline{\mathcal{LA}'_f})^c$ if and only if there exist $\phi_1, \dots, \phi_q$ such that $f_j\langle\theta\rangle \subset H_{\phi_j}$. This is equivalent to that $\tilde{f}\langle\theta, \phi_1 - \phi_2, \dots, \phi_1 - \phi_q\rangle \subset H_{\phi_1}$. Further more we have that $\tilde{v}(\theta, \phi_1 - \phi_2, \dots, \phi_1 - \phi_q) = v(\theta)$. Thus the statement follows from Theorem 2.3.1 and Propositions 2.3.4 and 2.3.5. $\square$

We will write $A \sim \widetilde{A}$ if A and $\widetilde{A}$ has the same set of dual matrices.

EXAMPLE 2.3.11. Consider polynomials of the form

$$f(z_1, z_2, z_3) = 1 + z_1 + z_2 + z_3 + c_1 z_2 z_3 + c_2 z_1 z_3,$$

where $3!\,\mathrm{Vol}(\Delta_f) = 3$. After reordering the monomials, we have that

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

where the latter is the system for the product of the three univariate polynomials

$$f_1(z) = 1 + z \qquad f_2(z) = 1 + c_2 z \qquad f_3(z) = 1 + c_1 z.$$

We conclude that $(\overline{\mathcal{LA}'_f})^c$ has three connected components unless either $\arg(c_1) = 0$, $\arg(c_2) = 0$ or $\arg(c_1) = \arg(c_2)$. After yet again reordering the rows and the columns, we find that

$$A \sim \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

which is the system for the product of the two trinomials

$$f_4(z_1, z_2) = 1 + z_1 + z_2 \qquad f_5(z_1, z_2) = 1 + c_2 z_1 + c_1 z_2.$$

Hence the complement of the union of the closed coamoebas of two trinomials has at most, and generically, three connected components. This could also be compared to the normalized volume $2!\,\mathrm{Vol}(\Delta_{f_4 f_5}) = 4$.

EXAMPLE 2.3.12. As a generalization of the previous example, we have that the product of two polynomials in n-variables whose Newton polytopes are the standard n-simplexes, is equivalent to the product of $n+1$ univariate binomials of degree one. Hence there are at most, and generically, $n+1$ connected components of the complement of the closed lopsided coamoeba, which in this case coincides with the ordinary closed coamoeba.

EXAMPLE 2.3.13. Let us stay in the world of multiaffine polynomials and consider the case when Δ_f is the unit 3-cube, that is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Thus we can consider instead the product of two polynomials both of whose Newton polytope is the unit square;

$$f_1(z_1, z_2) = 1 + z_1 + z_2 + c_1 z_1 z_2 \qquad f_2(z_1, z_2) = 1 + c_2 z_1 + c_3 z_2 + c_4 z_1 z_2.$$

By Example 2.3.11 the complement of the coamoeba of the product of the trinomials $1 + z_1 + z_2$ and $1 + c_2 z_1 + c_3 z_2$ generically has three connected components. These all lie in distinct cells of the hyperplane arrangement given by the coamoebas of the four binomials

$$1 + z_1, \quad 1 + z_2, \quad 1 + c_2 z_1, \quad \text{and} \quad 1 + c_3 z_2.$$

If any such cell contains two connected components of the complement of the coamoeba of the product $f_1 f_2$, then by the forthcoming Proposition 2.5.1 the hyperplanes given by the coamoebas of the two binomials $1 + c_1 z_2$ and $1 + c_1 z_1$ must both intersect this cell. As this is seen to be possible for at most one of the four cells, we find that the maximal number of connected components of $(\overline{\mathcal{LA}'_f})^c$ is four. Note that $\mathrm{Vol}(A) = 6$, thus this is our first example when it is not possible to construct a lopsided coamoeba whose complement has $\mathrm{Vol}(A)$ many connected components.

The treatment of hypersurface coamoebas of products of polynomials in this manner, rather than as a sum of monomials, makes it natural to pose the following sharpened version of the upper bound theorem:

> *Is the number of connected components of the complement of the union of the closed coamoebas of $f_1, \ldots, f_q$ bounded by the integer* $\mathrm{Vol}(A)$*?*

In the case $q = 1$, we have the inequalities

$$|\,\mathrm{CC}(\overline{\mathcal{LA}'_f})| \leq |\,\mathrm{CC}(\overline{\mathcal{A}'_f})| \leq n!\,\mathrm{Vol}(\Delta_f) = \mathrm{Vol}(A).$$

As the number of connected components of the complement of the lopsided coamoeba is in a direct dependence of the dual matrix B, we find that the bound $\mathrm{Vol}(A)$ holds for lopsided coamoebas and arbitrary q. We will see in the last chapter of this section that the analogous argument is not valid for the original coamoeba $\mathcal{A}'_f$. That is, the maximal number of connected components of the complement of the coamoebas of two point configurations who share the same dual matrix B need not agree with eachother.

2.4. Coamoebas of polynomials of small codimension

When A is simple the coamoeba $\mathcal{A}'_f$ is well known, and as noted earlier $\mathcal{A}'_f = \mathcal{L}\mathcal{A}'_f$. As a first application of the order map v, let us consider coamoebas of polynomials of codimension one and two.

2.4.1. Circuits. Consider the case of codimension one, imposing also the assumption that A is maximally sparse. In particular A is a *circuit*, a case treated exhaustively in [**7**, Chap. 7.1B]. We can write A in the form

$$A = \left(\begin{array}{ccccc} 1 & 1 & \dots & 1 & 1 \\ 0 & \alpha_1 & \dots & \alpha_n & \alpha_{n+1} \end{array} \right),$$

where Δ_f is the union of the two full dimensional simplices $(0, \alpha_1, \dots, \alpha_n)$ and $(\alpha_1, \dots, \alpha_n, \alpha_{n+1})$. A dual matrix B is given by the column vector

$$B = (\mathrm{Vol}(A_{\widehat{0}}), -\mathrm{Vol}(A_{\widehat{1}}), \dots, -\mathrm{Vol}(A_{\widehat{n}}), \mathrm{Vol}(A_{\widehat{n+1}}))^t,$$

where $\mathrm{Vol}(A_{\widehat{j}})$ denotes the normalized volume of the simplex obtained by removing α_j from A. It follows from [**7**] that the zonotope $\mathcal{Z}_B$ is an interval of length

$$\pi(\mathrm{Vol}(A_{\widehat{0}}) + \dots + \mathrm{Vol}(A_{\widehat{n+1}})) = 2\pi n! \,\mathrm{Vol}(\Delta_f).$$

The elements of B are the maximal minors of A, and hence $g_A = g_B$, both which we can assume equals 1. We see that for generic coefficients

$$\left| \operatorname{int}(\mathcal{Z}_B) \cap (\operatorname{Arg}(c)B + 2\pi\mathbb{Z}) \right| = n! \,\mathrm{Vol}(\Delta),$$

and conclude that the complement of the closed lopsided coamoeba has $n! \,\mathrm{Vol}(\Delta_f)$ many connected components. It follows that the maximal number of connected components of the complement of the closed coamoeba is obtained for generic coefficients. This is an affirmative answer to an in general disproved conjecture by Passare [**12**] in the case when the Newton polytope Δ_f has $n+2$ vertices.

When $n \geq 2$, and for generic coefficients, the topological equivalence between $\overline{\mathcal{A}'_f}$ and $\overline{\mathcal{L}\mathcal{A}'_f}$ implied by the above conclusion also yields a method to construct a set of *base points* for the set of connected components of the complement of the coamoeba, by which we mean a set with exactly one element in each connected component. Given a polynomial

$$f(z) = c_0 + c_1 z^{\alpha_1} + \dots + c_n z^{\alpha_n} + c_{n+1} z^{\alpha_{n+1}},$$

under the above assumptions, consider the n polynomials given by

$$f_i(z) = f(z) - n c_i z^{\alpha_i} - 2c_{n+1} z^{\alpha}, \quad i = 1, \dots, n,$$

and the system

$$f_1(z) = \dots = f_n(z) = 0.$$

Avoiding the discriminant locus of this system, the BKK theorem tells us that there are exactly $n! \,\mathrm{Vol}(\Delta_f)$ distinct solutions in $(\mathbb{C}^*)^n$. Let S be the set of arguments of these solutions. The above system is equivalent to

$$\left\{ \begin{array}{rcll} c_i z^{\alpha_i} - c_j z^{\alpha_j} & = & 0 & 1 \leq i, j \leq n \\ c_0 - c_{n+1} z^{\alpha_{n+1}} & = & 0 & \end{array} \right. , \tag{8}$$

which shows that for each $\theta \in S$ the set $f\langle\theta\rangle$ contains at most two points. Thus under the genericity assumption $f\langle\theta\rangle$ is lopsided for each $\theta \in S$. It also follows that $|S| = n!\,\mathrm{Vol}(\Delta_f)$, and that the numbers

$$\phi_\theta = \arg_\pi\left(\frac{c_1 e^{i\langle\alpha_1,\theta\rangle}}{c_0}\right) = \cdots = \arg_\pi\left(\frac{c_n e^{i\langle\alpha_n,\theta\rangle}}{c_0}\right), \quad \theta \in S$$

are distinct. Hence the orders

$$v_0(\Theta) = \phi_\theta\,(0,1,\ldots,1,0)B$$

are also distinct. We conclude that S has exactly one element in each connected component of $(\overline{\mathcal{A}'_f})^c$.

After applying an integer affine transformation, the polynomials $f_i(z)$ differ from the toric derivatives $f'_i(z) = z_i\frac{\partial f}{\partial z_i}$ only by a change of radii of the coefficients. It follows by a continuity of the roots argument that, if this is the case, then the set of arguments of the solutions to the system

$$f'_1(z) = \cdots = f'_n(z) = 0,$$

is in a 1–1 relation with the connected components of $(\overline{\mathcal{A}'_f})^c$.

2.4.2. The case $m = 2$ and a relation to discriminants. Let us move up one step in the complexity chain and consider the case when $m = 2$. We will assume that $g_A = 1$. Recall that related to the point configuration A is the so-called A-discriminant $D_A(c)$, which is a polynomial in the coefficients c vanishing if and only if the hypersurface $V(f) \subset (\mathbb{C}^*)^n$ is singular, see [**7**]. The polynomial $D_A(c)$ enjoys a number of homogeneities, one for each row of the matrix A, and choosing a Gale dual of A yields a dehomogenization of $D_A(c)$ in the following manner; introducing the variables

$$x_i = c_1^{b_{1i}}\cdots c_N^{b_{Ni}}, \quad i = 1,\ldots,m,$$

after multiplication with a Laurent monomial in c, the A-discriminant $D_A(c)$ can be viewed as a polynomial $D_B(x)$. In [**15**], and with a second method in [**21**], it is shown that the zonotope $\mathcal{Z}_B$ together with the coamoeba $\mathcal{A}'_{D_B}$ of the dehomogenized discriminant generically covers $\mathbf{T}^2$ precisely $n!\,\mathrm{Vol}(\Delta_f)$ many times. Hence if $\overline{\mathcal{A}'_{D_B}} \neq \mathbf{T}^2$, then we can construct a coamoeba whose complement has the maximal number of connected components. As the next example shows, this is not always the case.

Example 2.4.1. Consider the point configuration

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 3 & 3 & 2 \end{pmatrix}.$$

where we note that $2!\,\mathrm{Vol}(\Delta_f) = 11$. The dehomogenized discriminant related to the Gale dual

$$B = \begin{pmatrix} 1 & 2 \\ -1 & -3 \\ -2 & -2 \\ 2 & 0 \\ 0 & 3 \end{pmatrix}$$

is

$$\begin{aligned} D_B(x) = {} & 729x_1^2 + 2187x_1^3 + 2187x_1^4 + 729x_1^5 + 1728x_2 + 4752x_1x_2 \\ & + 5400x_1^2x_2 - 1404x_1^3x_2 - 864x_1^4x_2 + 3456x_2^2 - 5616x_1x_2^2 \\ & + 576x_1^2x_2^2 + 256x_1^3x_2^2 + 1728x_2^3. \end{aligned}$$

Its coamoeba covers the torus $\mathbf{T}^2$ completely, and hence the complement of the closed lopsided coamoeba can not have more than 10 connected components.

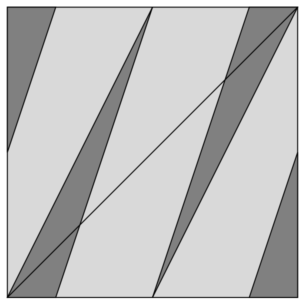

FIGURE 6. The coamoeba of $D_B(x)$ drawn with multiplicity, darker areas are covered twice.

The connection between the zonotope $\mathcal{Z}_B$ and the dehomogenized discriminant $D_B(x)$ is believed to be true also in higher codimensions, however this is still an open problem. For the latest development, we refer the reader to [**21**].

2.5. Examples concerning the geometry of coamoebas

The examples of lopsided coamoebas given in the previous sections, especially the cases when the upper bound $\mathrm{Vol}(A)$ on the number of connected components of the complement cannot be attained, leads us to the quest of finding the sharp upper bound of the number of connected components of the complement of the closed coamoeba. We will consider several examples, not offering a complete solution, however straightening out some of the question marks surrounding this problem. Our method is to use a sharpening of Lemma 2.1.3, for which we need some additional notation.

We denote by $e(\Delta_f)$ the set of edges of Δ_f. For each $\Gamma_i \in e(\Delta_f)$, let $(p_i, \gamma_i) \in \mathbb{Z}^n \times \mathbb{Z}^n$ be such that $\Gamma_i = \{p_i + t\gamma_i \,|\, t \in (0,1)\}$. To each line segment

$$l = \{l_0 + \mu t \,|\, t \in [0,1]\}$$

with end points l_0 and $l_1 = l_0 + \mu$ in $\mathcal{H}_f^c$ we associate a vector in $h(l) \in \mathbb{Z}^{e(\Delta_f)}$ given by

$$h(l) = \sum_{\Gamma_i \in e(\Delta_f)} \operatorname{sgn}(\langle \mu, \gamma_i \rangle)\, |\, l \cap \mathcal{A}'_{f_{\Gamma_i}} |\, e_i.$$

The following proposition is merely a reformulation of a result in [**10**], and we state it without proof.

Proposition 2.5.1. [**10**, Thm. 4.7 and Thm. 4.11] *Let $n = 2$ and let $\gamma_1, \dots, \gamma_{|e(\Delta_f)|}$ be chosen such that the boundary of Δ_f is oriented. Let l be a line segment with endpoints in $(\overline{\mathcal{A}'_f})^c$. Then*

$$\langle h(l), (1, \dots, 1) \rangle = 0.$$

Given two connected components Θ_1 and Θ_2 of $(\overline{\mathcal{A}'_f})^c$, choosing any line segment with endpoints $l_0 \in \Theta_1$ and $l_1 \in \Theta_2$ yields the same vector $h(l)$, which we denote by $h(\Theta_1, \Theta_2)$.

There is a (far from sharp) bound on the number of connected components of the complement of the coamoeba $\overline{\mathcal{A}'_f}$ given by Lemma 2.1.3 as the number of cells of the hyperplane arrangement $\mathcal{H}_f$. This bound can be sharpened by taking into account also Proposition 2.5.1, which is the method used in [**3**], though under different notation. There it is shown that if f is a polynomial with support as in Example 2.4.1, then $(\overline{\mathcal{A}'_f})^c$ has at most 10 connected components. This was the first example of a Newton polytope for which there was no maximally sparse polynomial whose complement has $n! \operatorname{Vol}(\Delta_f)$ many connected components.

Example 2.5.2. Consider the polynomial

$$f(z_1, z_2) = 1 + iz_1 + e^{\frac{2i\pi}{3}} z_1^2 + e^{\frac{3i\pi}{7}} z_1 z_2^3 + z_2^3 + z_1^2 z_2^2 - e^{\frac{13i\pi}{30}} z_1^2 z_2 - e^{\frac{7i\pi}{30}} z_1^2 z_2,$$

and note that its Newton polytope coincides with that of the polynomials in Example 2.4.1. Using a computer, it is easily checked that the points

$$\begin{array}{cccc} (0, -\frac{\pi}{2}), & (\frac{\pi}{3}, 0), & (\frac{\pi}{2}, \frac{23\pi}{64}), & (0, \frac{3\pi}{11}), \\ (0, \frac{\pi}{2}), & (\frac{3\pi}{4}, \frac{3\pi}{4}), & (\frac{3\pi}{4}, -\frac{\pi}{2}), & (-\frac{3\pi}{4}, 0), \\ (-\frac{3\pi}{4}, \frac{7\pi}{18}), & (\frac{3\pi}{4}, \frac{5\pi}{18}) & \text{and} & (\frac{\pi}{2}, -\frac{27\pi}{28}) \end{array} \tag{9}$$

belongs to the complement of $\overline{\mathcal{A}'_f}$, and that they also belong to distinct cells of the hyperplane arrangement $\mathcal{H}_f$, see Figure 7. We conclude that $(\overline{\mathcal{A}'_f})^c$ has 11 connected components.

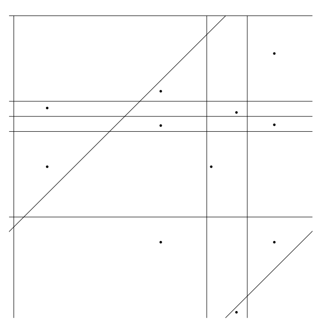

Figure 7. The shell of the coamoeba from Example 2.5.2, with the points (9) marked out.

Example 2.5.3. Consider the product of two multiaffine polynomials in two variables, that is

$$f_1(z_1, z_2) = 1 + z_1 + z_2 + c_0 z_1 z_2 \quad \text{and} \quad f_2(z_1, z_2) = 1 + c_1 z_1 + c_2 z_2 + c_3 z_1 z_2.$$

The shell $\mathcal{H}_{f_1 f_2}$ consists of two families of four hyperplanes parallel to the axis on $\mathbf{T}^2$, and for generic coefficients they divide $\mathbf{T}^2$ into 16 cells. We note that while $2!\,\mathrm{Vol}(\Delta_f) = 8$, we have that $\mathrm{Vol}(A) = 6$, and according to Example 2.3.13 the maximal number of connected components of the complement of the closed lopsided coamoeba is four. Using Proposition 2.5.1 it is not hard to show that the maximal number of connected components of the complement of the closed coamoeba is six, and this number is achieved only if the two coamoebas combine as in Figure 8.

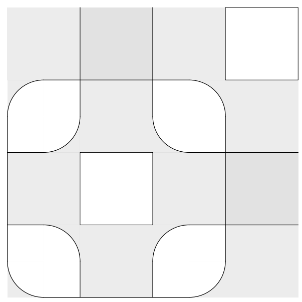

FIGURE 8. The coamoeba of $f(z_1, z_2) = (1 + z_1 + z_2 + iz_1z_2)(1 - z_1 - z_2 + iz_1z_2)$.

EXAMPLE 2.5.4. Let us consider the case when Δ_f is the three dimensional unit cube, that is $f(z)$ of the form

$$f(z_1, z_2, z_3) = 1 + z_1 + z_2 + z_3 + c_3z_1z_2 + c_2z_1c_3 + c_1z_2z_3 + c_0z_1z_2z_3.$$

As in the previous example we know that the complement of the closed lopsided coamoeba can have at most four connected components, however now the normalized volume is $\mathrm{Vol}(A) = 3!\,\mathrm{Vol}(\Delta_f) = 6$. We claim that the complement of the coamoeba $\overline{\mathcal{A}'_f}$ has at most four connected components. Actually, we will show that the complement of the *phase limit set*

$$\mathcal{P}^\infty(f) = \bigcup_{\dim(\Gamma)<3} \mathcal{A}'_{f_\Gamma} = \bigcup_{\dim(\Gamma)=2} \overline{\mathcal{A}'_{f_\Gamma}}$$

has at most four connected components. By the relation (1), Lemma 2.1.3, and the fact that $\mathcal{H}_f \subset \mathcal{P}^\infty(f)$, this implies our claim.

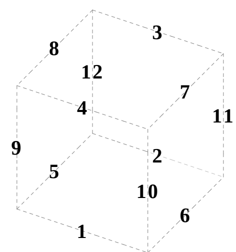

FIGURE 9. The Newton polytope Δ_f with indexed edges.

Let us choose vectors

$$\gamma_1 = \cdots = \gamma_4 = e_1, \quad \gamma_5 = \cdots = \gamma_8 = e_2 \quad \text{and} \quad \gamma_9 = \cdots = \gamma_{12} = e_3,$$

where the indexing of the edges is according to Figure 9. That each connected component of $(\overline{\mathcal{A}'_f})^c$ is contained in a connected component of $\mathcal{P}^\infty(f)^c$ implies that for each two dimensional face of Δ_f the corresponding result of Proposition 2.5.1 holds, that is each vector $h(\Theta_i, \Theta_j)$ lies in the kernel of the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0. \end{pmatrix}$$

Fix a component Θ_1. It is clear that for each component Θ_j we can draw a line segment $l_j = l(\Theta_1, \Theta_j)$ such that $h(l_j) \in \{0,1\}^{12}$. The set $\{0,1\}^{12} \cap \ker(M)$ has 38 elements:

(0,0,0,0,0,0,0,0,0,0,0,0), (0,0,0,0,0,0,0,0,1,1,1,1), (0,0,0,0,0,0,1,1,0,0,1,1), (0,0,0,0,1,1,0,0,1,1,0,0),
(0,0,0,0,1,1,1,1,0,0,0,0), (0,0,0,0,1,1,1,1,1,1,1,1), (0,0,0,1,0,0,0,1,0,1,1,1), (0,0,0,1,1,1,0,1,0,1,0,0),
(0,0,1,0,0,0,1,0,0,0,1,0), (0,0,1,0,1,1,1,0,1,1,1,0), (0,0,1,1,0,0,0,0,0,1,1,0), (0,0,1,1,1,1,1,1,0,1,1,0),
(0,1,0,0,0,1,0,0,1,1,0,1), (0,1,0,0,0,1,1,1,0,0,0,1), (0,1,0,1,0,1,0,1,0,1,0,1), (0,1,1,0,0,1,1,0,0,0,0,0),
(0,1,1,0,0,1,1,0,1,1,1,1), (0,1,1,1,0,1,0,0,0,1,0,0), (0,1,1,1,0,1,1,1,0,1,1,1), (1,0,0,0,1,0,0,0,1,0,0,0),
(1,0,0,0,1,0,1,1,1,0,1,1), (1,0,0,1,1,0,0,1,0,0,0,0), (1,0,0,1,1,0,0,1,1,1,1,1), (1,0,1,0,1,0,1,0,1,0,1,0),
(1,0,1,1,1,0,0,0,1,1,1,0), (1,0,1,1,1,0,1,1,0,0,1,0), (1,1,0,0,0,0,0,1,0,0,1), (1,1,0,0,1,1,1,1,1,0,0,1),
(1,1,0,1,0,0,0,1,0,0,0,1), (1,1,0,1,1,1,0,1,1,1,0,1), (1,1,1,0,0,0,1,0,1,0,1,1), (1,1,1,0,1,1,1,0,1,0,0,0),
(1,1,1,1,0,0,0,0,0,0,0,0), (1,1,1,1,0,0,0,0,1,1,1,1), (1,1,1,1,0,0,1,1,0,0,1,1), (1,1,1,1,1,1,0,0,1,1,0,0),
(1,1,1,1,1,1,1,1,0,0,0,0) and (1,1,1,1,1,1,1,1,1,1,1,1).

Notice that the coamoebas of the parallel edges $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 comes in some order on the torus. This implies that if a subset of the above 38 vectors can be realized as the vectors $h(l_j)$ for a set of connected components of the complement of a coamoeba $\overline{\mathcal{A}'_f}$, then there is a permutation $\sigma_1 \in S_4$ such that

$$0 = h(l_j)_{\sigma_1(1)} \leq h(l_j)_{\sigma_1(2)} \leq h(l_j)_{\sigma_1(3)} \leq h(l_j)_{\sigma_1(4)}. \tag{10}$$

Similar conditions hold for the remaining two families of parallel edges. Thus we can get an upper bound on the maximal number of connected components of the complement of $\overline{\mathcal{A}'_f}$ by determining for each permutation $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in S_4^3$, how many of the 38 vectors given above that fulfils the conditions (10). We check this using the computer program Wolfram Mathematica 8, and find that for each permutation there is at most four such vectors, see Appendix A. This shows that the complement of the coamoeba can have at most four connected components. In this case it is easy to construct a (lopsided) coamoeba whose complement has four connected components, and hence this bound is sharp.

The question of what is the sharp upper bound on the maximal number of connected components of the complement of a coamoeba, given a fix Newton polytope Δ_f, and how this bound is attained, has seen several semi-conjectures during the last decade. Let us sum up the situation in its present state, as far as we are aware of it. In one dimension it is clear that the bound $n! \operatorname{Vol}(\Delta_f)$ is sharp. In two dimensions, we are yet to find an example when this is not the case. In three dimensions, and therefore also in all higher dimensions, we know that the bound need not be sharp. It is natural to ask whether the bound is sharp in two dimensions, or to be more precise, what are the possible obstructions which makes it impossible to construct a coamoeba whose complement has $n!\,\mathrm{Vol}(\Delta_f)$ many connected components? It has also been believed that the maximal number of connected components of the complement of a coamoeba should be attained for maximally sparse polynomials, perhaps even for maximally sparse polynomials with generic coefficients, however we know now that this is not the case. This statement, now disproved, has been considered to be "dual" to the statement that the amoeba of a maximally sparse polynomial is *solid*, that is, its complement does not have any bounded components.

CHAPTER 3

Integral representations of hypergeometric functions

An Euler type integral is an integral of the form

$$\int_C \frac{z^s}{f(z)^t}\,\frac{dz}{z},$$

where C is some domain, usually chosen as a compact cycle to ensure convergence. It is known that for generic homogeneity parameters $\beta = -(t, s)$, one can construct a basis for the solution space to $H_A(\beta)$ using Euler type integrals, choosing domains of integration appropriately. In this chapter we will introduce the *Euler–Mellin integral* which is a variant of the Euler type integral, characterized by the explicit but non-compact domain of integration constructed by considering the coamoeba $\mathcal{A}'_f$. The focus of our treatment is to establish the basic properties of Euler–Mellin integrals, arriving at a relation to Mellin–Barnes integrals through use of the order map v, see Section 3.4. In the last section we give an example that shows that Euler–Mellin integrals, by their explicit nature, can be used as a tool to study integral solutions to the A-hypergeometric system also at non-generic homogeneity parameters. Though this line has not yet been developed further, it is one of the main motivations for studying Euler–Mellin integrals.

3.1. Euler–Mellin integrals

The Euler–Mellin integral can be seen as a natural generalization of the Mellin transform of a rational function $1/f$ of several variables, and is given by

$$M_f(s,t) := \int_{\mathbb{R}^n_+} \frac{z^s}{f(z)^t}\,\frac{dz_1\wedge\ldots\wedge dz_n}{z_1\cdots z_n} = \int_{\mathbb{R}^n} \frac{e^{\langle s,x\rangle}}{f(e^x)^t}\,dx_1\wedge\ldots\wedge dx_n, \tag{11}$$

where $\mathbb{R}^n_+ = (0,\infty)^n$ denotes the positive orthant in $\mathbb{R}^n$. Here we employ multi-index notation also for the polynomials $f_1,\ldots,f_q$; that is for $t\in\mathbb{C}^q$, we write $f(z)^t = f_1(z)^{t_1}\cdots f_q(z)^{t_q}$. As before, whenever there is no risk of confusion, we use the notation $f(z) := f(z)^{(1,\ldots,1)} = \prod_{j=1}^q f_j(z)$.

In order for such an integral to converge, one must place restrictions on both the exponent vector (s,t) and the polynomial f; it is not enough to demand only that each f_j is nonvanishing on $\mathbb{R}^n_+$. We next provide such a domain of convergence for the Euler–Mellin integral (11), generalizing [**16**, Thm. 1].

Definition 3.1.1. The polynomial f is said to be *completely nonvanishing* on a set X if for each face Γ of Δ_f, the truncated polynomial f_Γ has no zeros on X. In particular, the polynomial f itself does not vanish on X.

For a vector $\tau \in \mathbb{R}_+^q$, we denote by $\tau\Delta_f$ the weighted Minkowski sum $\sum_{j=1}^q \tau_j \Delta_{f_j}$ of the Newton polytopes of the f_j with respect to τ. Note that with this notation, the Newton polytope of f satisfies $\Delta_f = (1,\ldots,1)\Delta_f$.

Theorem 3.1.2. *If each of the polynomials $f_1,\ldots,f_q$ are completely nonvanishing on the positive orthant $\mathbb{R}_+^n$, then the integral* (11) *converges and defines an analytic function in the tube domain*

$$\big\{(s,t) \in \mathbb{C}^{n+q} \mid \tau := \mathrm{Re}\, t \in \mathbb{R}_+^q,\ \sigma := \mathrm{Re}\, s \in \mathrm{int}(\tau\Delta_f)\big\}.$$

Proof. It suffices to prove that for any (s,t) with all $\tau_j > 0$ and $\sigma \in \mathrm{int}(\tau\Delta_f)$, there exist positive constants c and k such that

$$\big|f(e^x)^t e^{-\langle s,x\rangle}\big| = \big|f(e^x)^t\big| e^{-\langle \sigma,x\rangle} \geq c e^{k|x|} \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

In fact, it is enough to show that this inequality holds outside of some ball $B(0)$ in $\mathbb{R}^n$.

Since $\sigma \in \mathrm{int}(\tau\Delta_f)$, we can expand it as a sum $\sigma = \sigma_1 + \cdots + \sigma_q$ of q vectors such that $\sigma_j/\tau_j \in \mathrm{int}(\Delta_{f_j})$. It is shown in the proof of [**16**, Thm. 1] that for each $\sigma_j \in \mathrm{int}(\Delta_{f_j})$ there are positive constants c_j and k_j such that

$$\big|f_j(e^x)\big| e^{-\langle \sigma_j,x\rangle} \geq c_j e^{k_j|x|}$$

for x outside of some ball $B_j(0)$. Note that it is essential in that proof that f_j is completely nonvanishing on the positive orthant. Thus for x outside of $B(0) = \bigcup_{j=0}^q B_j(0)$, we have

$$\big|f(e^x)^t\big| e^{-\langle\sigma,x\rangle} \;=\; \prod_{j=1}^q \Big(\big|f_j(e^x)\big| e^{-\left\langle \frac{\sigma_j}{\tau_j},x\right\rangle}\Big)^{\tau_j} \;\geq\; \prod_{j=1}^q c_j^{\tau_j} e^{\tau_j k_j|x|} \;=\; c e^{k|x|}, \tag{12}$$

where $c = c_1^{\tau_1}\cdots c_q^{\tau_q}$ and $k = \tau_1 k_1 + \cdots + \tau_q k_q$ are the desired positive constants. $\square$

Example 3.1.3. By a classical integral representation of the Gauss hypergeometric function ${}_2F_1$,

$$\int_0^\infty \frac{z^s}{(1+z)^{t_1}(c+z)^{t_2}}\frac{dz}{z} \;=\; \frac{\Gamma(t_1+t_2-s)\Gamma(s)}{\Gamma(t_1+t_2)}\, {}_2F_1(t_2, t_1{+}t_2{-}s; t_1{+}t_2; 1{-}c) \tag{13}$$

for $\mathrm{Re}\,(t_1+t_2) > \mathrm{Re}\,(t_1+t_2-s) > 0$ and $|\,\mathrm{Arg}(c)| < \pi$. Note that $|\,\mathrm{Arg}(c)| < \pi$ is equivalent to $f(z) = (1+z)(c+z)$ being completely nonvanishing on $\mathbb{R}_+$. Since $\Delta_{f_1} = \Delta_{f_2} = [0,1]$, the condition that $\sigma \in \mathrm{int}(\tau\Delta_f)$ is the same as $0 < \mathrm{Re}\,(s) < \mathrm{Re}\,(t_1+t_2)$. We also note that the right hand side of (13) is analytic in this domain. Further, since $\mathrm{Re}\,(t_1) > 0$ and $\mathrm{Re}\,(t_2) > 0$, the convergence domain given in Theorem 3.1.2 is not optimal; however, being full-dimensional, it is large enough for our goal of meromorphic continuation.

As the right hand side of (13) is a meromorphic function in s and t, it is a meromorphic extension of the Euler–Mellin integral. On this right side, we have the regularized ${}_2F_1$ as one factor, thus the polar locus of the meromorphic extension is contained in two families of hyperplanes given by the polar loci of the Γ-functions. Our next result shows that this kind of meromorphic continuation is possible for all Euler–Mellin integrals.

To obtain the strongest form of this result, we choose a specific presentation for $\tau\Delta_f$. To begin, each Newton polytope Δ_{f_j} can be written uniquely as the intersection of a finite number of halfspaces

$$\Delta_{f_j} = \bigcap_{k=1}^{K_j} \{\sigma \in \mathbb{R}^n \mid \langle \mu_k^j, \sigma\rangle \geq \nu_k^j\}, \tag{14}$$

where the μ_k^j are primitive vectors. Fixing an order, let $\{\mu_1, \dots, \mu_K\}$ be equal to the set $\{\mu_k^j \mid 1 \leq j \leq q, 1 \leq k \leq K_j\}$, where we assume that $\mu_{k_1} \neq \mu_{k_2}$ for all $k_1 \neq k_2$. We now extend the definitions of ν_k^j from (14) to each μ_k; namely, for each k, let $\nu_k := (\nu_k^1, \dots, \nu_k^q)$ with

$$\nu_k^j := \min\{\langle \mu_k, \alpha\rangle \mid \alpha \in \Delta_{f_j}\},$$

and set $|\nu_k| := \nu_k^1 + \cdots + \nu_k^q$. It now follows from the definition of the ν_k that

$$\tau\Delta_f = \bigcap_{k=1}^{K} \{\sigma \in \mathbb{R}^n \mid \langle \mu_k, \sigma\rangle \geq \langle \nu_k, \tau\rangle\} \tag{15}$$

and $\mathrm{int}(\tau\Delta_f) = \sum_{j=1}^q \tau_j \ \mathrm{int}(\Delta_{f_j})$.

THEOREM 3.1.4. *If the polynomials $f_1, \dots, f_q$ are completely nonvanishing on the positive orthant $\mathbb{R}_+^n$, then the Euler–Mellin integral $M_f(s,t)$ admits a meromorphic continuation of the form*

$$M_f(s,t) = \Phi(s,t) \prod_{k=1}^{N} \Gamma(\langle \mu_k, s\rangle - \langle \nu_k, t\rangle), \tag{16}$$

where Φ is an entire function and μ_k, ν_k are given by (15).

PROOF. By Theorem 3.1.2, the original integral (11) converges on the tube domain

$$\left\{(s,t) \in \mathbb{C}^{n+q} \mid \tau \in \mathbb{R}_+^q,\ \sigma \text{ such that } \langle \mu_k, \sigma\rangle > \langle \nu_k, \tau\rangle \text{ for all } 1 \leq k \leq N\right\},$$

where $\tau = \mathrm{Re}\,(t)$ and $\sigma = \mathrm{Re}\,(s)$, which is a domain since Δ_f is of full dimension. Our goal is to expand the convergence domain of the integral (11), at the cost of multiplication by terms corresponding to the poles of the gamma functions appearing in (16). We do this iteratively, integrating by parts in the direction of a vector μ_k at each step. This expands the domain of convergence in the opposite direction of μ_k by a distance d_k, which we determine explicitly.

To begin, we set notation for the first iteration in one direction. Fix k between 1 and N, and let Γ be the face of Δ_{f_i} corresponding to μ_k and ν_k. For $\alpha \in \mathrm{supp}(f)$, set

$$d_k^\alpha := \langle \mu_k, \alpha\rangle - |\nu_k|.$$

Since $\alpha \in \Delta_f$, it follows that $d_k^\alpha \geq 0$. In particular, since there is a decomposition $\alpha = \sum_j \alpha_j$ with $\alpha_j \in \Delta_{f_j}$, we see that $d_k^\alpha = 0$ if and only if $\langle \mu_k, \alpha_j\rangle = \nu_k^j$ for all j.

Denote by $\lambda^{\mu_k} z = (\lambda^{\mu_k^1} z_1, \lambda^{\mu_k^2} z_2, \dots, \lambda^{\mu_k^n} z_n)$ where λ is any nonzero complex number, then for any j, the truncated polynomial $(f_j)_\Gamma$ has the homogeneity $(f_j)_\Gamma(\lambda^{\mu_k} z) = \lambda^{\nu_k^j}(f_j)_\Gamma(z)$. Hence the coefficients of the scaled polynomial

$\lambda^{-\nu_k^j}(f_j)_\Gamma(\lambda^{\mu_k}z)$ are independent of k and λ. In particular, we have that the Newton polytope of

$$f_j'(z) := \frac{d}{d\lambda}\Big(\lambda^{-\nu_k^j} f_j(\lambda^{\mu_k}z)\Big)\Big|_{\lambda=1}$$

is disjoint from Γ. This fact allows us to extend the domain of convergence of (11) over the hyperplane defined by $\langle \mu_k, \sigma\rangle = \langle \nu_k, \tau\rangle$ as follows. Since M_f is independent of λ, we have

$$0 = \frac{d}{d\lambda}\int_{\mathbb{R}_+^n} \frac{(\lambda^{\mu_k}z)^s}{f(\lambda^{\mu_k}z)^t}\frac{dz}{z} = \frac{d}{d\lambda}\left[\lambda^{\langle \mu_k, s\rangle - \langle \nu_k, t\rangle}\int_{\mathbb{R}_+^n} \frac{z^s}{\lambda^{-\langle \nu_k, t\rangle} f(\lambda^{\mu_k}z)^t}\frac{dz}{z}\right].$$

Thus differentiating (11) with respect to λ and setting $\lambda = 1$ yields the identity

$$M_f(s,t) = \frac{1}{\langle \mu_k, s\rangle - \langle \nu_k, t\rangle}\int_{\mathbb{R}_+^n} \frac{z^s g_k(z)}{f(z)^{t+1}}\frac{dz}{z}, \tag{17}$$

where g_k is the polynomial

$$g_k = -\sum_{j=1}^{q} t_j \cdot f_1 \cdots f_j' \cdots f_q.$$

Note that $\mathrm{supp}(g_k)$ is contained in $\mathrm{supp}(f)$; moreover, since Γ is the face of Δ_f corresponding to μ_k and $\mathrm{supp}(f_j')$ is disjoint from $\Delta_{f_j} \cap \Gamma$, we see that $\mathrm{supp}(g_k)$ is disjoint from Γ. In other words, for each $\alpha \in \mathrm{supp}(g_k)$, we have $d_k^\alpha > 0$.

We now rewrite (17) as the sum

$$M_f(s,t) = \sum_{\alpha \in \mathrm{supp}(g_k)} \frac{h_\alpha(t)}{\langle \mu_k, s\rangle - \langle \nu_k, t\rangle}\int_{\mathbb{R}_+^n} \frac{z^{s+\alpha}}{f(z)^{t+1}}\frac{dz}{z}, \tag{18}$$

for some linear polynomials $h_\alpha(t)$, noting that each term of (18) is a translation of the original Euler–Mellin integral. By Theorem 3.1.2, the term corresponding to α converges on the domain given by $\tau + 1 > 0$ and

$$\langle \mu_l, \sigma + \alpha\rangle > \langle \nu_l, \tau + 1\rangle, \quad l = 1, \ldots, N,$$

where the latter is equivalent to

$$\langle \mu_l, \sigma\rangle > \langle \nu_l, \tau + 1\rangle - \langle \mu_l, \alpha\rangle = \langle \nu_l, \tau\rangle - d_l^\alpha, \quad l = 1, \ldots, N.$$

The sum (18) converges on the intersection of these domains, which is given by $\tau + 1 > 0$ and

$$\begin{aligned} \langle \mu_l, \sigma\rangle &> \langle \nu_l, \tau\rangle \qquad \text{if } l \neq k, \\ \langle \mu_k, \sigma\rangle &> \langle \nu_k, \tau\rangle - d_k, \end{aligned}$$

where $d_k := \min\{d_k^\alpha \mid \alpha \in \mathrm{supp}(g_k)\}$. Since d_k is by definition strictly greater than 0, (18) has a strictly larger domain of convergence than (11); we say that it has been extended by the "distance" d_k in the direction determined by μ_k.

Before iterating this procedure, we set some notation. Let G_k be the semigroup generated by the integers $\{d_k^\alpha\} \subseteq \mathbb{N}$. Let $\beta = (\alpha_1, \ldots, \alpha_q)$ be an ordered q-tuple with $\alpha_i \in \mathrm{supp}(f)$ for each i. We sometimes write β as an exponent of z, where we mean the sum $\beta = \alpha_1 + \cdots + \alpha_q$. Similarly, we denote by $d_k^\beta := d_k^{\alpha_1} + \cdots + d_k^{\alpha_q} \in G_k$.

Now after q iterations, let $\mu_{l(i)}$ denote the direction of the extension in the ith iteration. Let $d_{l(i)}^{\beta_i} := d_{l(i)}^{\alpha_1} + \cdots + d_{l(i)}^{\alpha_{i-1}} \in G_{l(i)}$ be the sum of the distances of the first $i-1$ components of β in the direction $\mu_{l(i)}$. Then there is a rational function of the type

$$L_\beta(s,t) = \prod_{i=1}^{r} \frac{h_{\beta_i}(t)}{\langle \mu_{l(i)}, s\rangle - \langle \nu_{l(i)}, t\rangle + d_{l(i)}^{\beta_i}}, \tag{19}$$

where $h_\beta(t) := (h_{\beta_1}(t), \ldots, h_{\beta_q}(t))$ is an ordered q-tuple of linear polynomials such that M_f can be expressed as a finite sum of translations of the original Euler–Mellin integral:

$$M_f(s,t) = \sum_\beta L_\beta(s,t) \int_{\mathbb{R}_+^n} \frac{z^{s+\beta}}{f(z)^{t+q}} \frac{dz}{z}. \tag{20}$$

Fixing k, we next expand the domain of convergence of (20) in the direction determined by μ_k. To achieve this, simultaneously expand the domain of convergence of each term, arguing as above. This yields the expression

$$\begin{aligned} M_f(s,t) &= \sum_\beta L_\beta(s,t) \sum_{\alpha\in\operatorname{supp}(g_k)} \frac{h_{(\beta,\alpha)_{q+1}}(t)}{\langle \mu_k, s\rangle - \langle \nu_k, t\rangle + d_k^\beta} \int_{\mathbb{R}_+^n} \frac{z^{s+\beta+\alpha}}{f(z)^{t+q+1}} \frac{dz}{z} \\ &= \sum_{\beta'} L_{\beta'}(s,t) \int_{\mathbb{R}_+^n} \frac{z^{s+\beta'}}{f(z)^{t+q'}} \frac{dz}{z}, \end{aligned} \tag{21}$$

where $\beta' = (\beta, \alpha)$, $r' = r+1$, and the resulting rational function $L_{\beta'}(s,t)$ is given by

$$L_{\beta'} = L_\beta \frac{h_{\beta'_{r'}}(t)}{\langle \mu_k, s\rangle - \langle \nu_k, t\rangle + d_k^\beta}.$$

Since the convergence domain of each term in (20) is extended by the distance d_k in the direction determined by μ_k, the convergence domain of the sum is similarly extended. In addition, since $d_k^\alpha > 0$, we have that $d_k^{\beta+\alpha} > d_k^\beta$; therefore, the products $L_\beta(s,t)$ will never repeat factors in their denominators. As (21) is in the same form as (20), we may iterate this procedure for the convergence domain extension.

Finally, note that after r iterations, which extended the domain of convergence of $M_f(s,t)$ in the direction determined by μ_k for r_k of the steps, we obtain a meromorphic function on the tube domain given by $\tau + \sum_{k=1}^K r_k = \tau + r > 0$ and

$$\langle \mu_k, \sigma\rangle > \langle \nu_k, \tau\rangle - r_k d_k, \quad k = 1, \ldots, N.$$

Hence $M_f(s,t)$ can be extended to a meromorphic function on $\mathbb{C}^{n+q}$ as in (16). Finally, we note that because the denominator of the products $L_\beta(s,t)$ never has repeated terms, all poles of the extended Euler–Mellin integral (16) are simple. Therefore by the removable singularities theorem, (16) is an entire function, as desired. □

The entire function Φ is of great interest to the study of A-hypergeometric functions. As defined in Theorem 3.1.4, the Gamma functions used there might have introduced some unnecessary zeros, which we wish to remove in this further study.

REMARK 3.1.5. Not all poles of the gamma functions in (16) are necessarily poles of the Euler–Mellin integral. In the proof of the theorem, we see that the linear form $\langle \mu_k, \sigma\rangle - \langle \nu_k, \tau\rangle - d$ appears in the denominator of some rational function L_β

if and only if $d \in G_k$. Hence if $G_k \neq \mathbb{N}$, then we have introduced extra zeros of the entire function Φ.

REMARK 3.1.6. If $q = 1$, then $h_{\beta_i}(t) = k_{\beta_i}(t + i)$ for some constant k_{β_i}, where h_{β_i} is as in (19). Therefore each L_β is divisible by $(t)_{i+1} = t(t + 1) \cdots (t + i)$, which can thus be factored outside of the sum (20). In particular, there is an entire function $\widetilde{\Phi}(s, t)$ such that $\widetilde{\Phi}(s, t) = \Gamma(t)\Phi(s, t)$.

3.2. An example: products of linear forms

Let us determine explicitly the function Φ in the case when f is a product of linear form, in order to illustrate Theorem 3.1.4 and our recent remarks.

EXAMPLE 3.2.1. Consider the case of one linear function of n variables,

$$M_f(s, t) = \int_{\mathbb{R}^n_+} \frac{z_1^{s_1} \cdots z_n^{s_n}}{(1 + z_1 + \cdots + z_n)^t} \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n}.$$

We claim that

$$M_f(s, t) = \frac{\Gamma(s_1) \cdots \Gamma(s_n)\Gamma(t - s_1 - \cdots - s_n)}{\Gamma(t)},$$

and hence $\Phi(s, t) = 1/\Gamma(t)$. When $n = 1$ this is a known identity for the Beta function. For $n > 1$ one can argue by induction, making the change of variables given by $w_n = z_n$ and $w_i = z_i/(1 + z_n)$ for $i \neq n$.

Let us also generalize this example to arbitrary simplex, which is given by an integer affine transformation T of the standard simplex. That is, we consider the Euler–Mellin integral

$$M_f(s, t) = \int_{\mathbb{R}^n_+} \frac{z_1^{s_1} \cdots z_n^{s_n}}{(1 + z^{T_1} + \cdots + z^{T_n})^t} \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n},$$

where T_i denotes the ith column of T. By the change of variables $z \mapsto z^{T^{-1}}$ we find that

$$M_f(s, t) = \frac{\Gamma((T^{-1}s)_1) \cdots \Gamma((T^{-1}s)_n)\Gamma(t - |T^{-1}s|)}{|\det(T)|\,\Gamma(t)}.$$

EXAMPLE 3.2.2. Consider the case of $q + 1$ linear functions of one variable,

$$M_f(s, t) = \int_0^\infty \frac{z^s}{(1 + z)^{t_0}(c_1 + z)^{t_1} \cdots (c_q + z)^{t_q}} \frac{dz}{z}. \tag{22}$$

Note that we have reindexed t for this example. If $q = 0$, then (22) is the Beta function. Here $\Phi(s, t) = 1/\Gamma(t)$, or with the notation of Remark 3.1.6, $\widetilde{\Phi}(s, t) = 1$. For $q = 1$, we have already seen in Example 3.1.3 that

$$\Phi(s, t) = \frac{1}{\Gamma(t_0 + t_1)}\, {}_2F_1(t_1, t_0 + t_1 - s; t_0 + t_1; 1 - c_1).$$

This equality is obtained by the change of variables $w = z/(1 + z)$ and application of the generalized binomial theorem. By similar calculations for $q = 2$,

$$\Phi(s, t) = \frac{1}{\Gamma(t_0 + t_1 + t_2)}\, F_1(t_0 + t_1 + t_2 - s, t_1, t_2; t_0 + t_1 + t_2; 1 - c_1, 1 - c_2),$$

where F_1 denotes the first Appell series. For arbitrary q and $|c_i| < 1$ we have the series expansion,

$$\Phi(s,t) = \frac{1}{\Gamma(t_0+|t|)} \sum_{k\in\mathbb{N}^q} \frac{(t_0+|t|-s)_{|k|}}{(t_0+|t|)_{|k|}} \frac{(t)_k}{k!} (1-c)^k,$$

where $t = (t_1,\dots,t_m)$, $|t| = t_1+\dots+t_q$, $(t)_k = (t_1)_{k_1}\cdots(t_q)_{k_q}$, and $k! = k_1!\cdots k_q!$.

EXAMPLE 3.2.3. Following the line of the examples in [**16**], let us give an expression for the entire function in the previous example as an integral over the standard q-simplex. We will only consider the special case given by $t_0 = 1$, thus considering integrals of the form

$$\widetilde{M}_f(s,t) = \int_{\mathbb{R}^n_+} \frac{z^s}{(1+z)(1+\langle c,z\rangle)^t} \frac{dz}{z},$$

where we use the notation

$$\begin{array}{rcll} f_i(z) & = & 1 + c_{i1}z_1 + \dots + c_{in}z_n & 1 \le i \le q, \\ \alpha_j(\tau) & = & 1-(1-c_{1j})\tau_1 - \dots - (1-c_{qj})\tau_q & 1 \le j \le n \end{array}$$

and $d_{ij} = 1 - c_{ij}$. We will restrict ourselves to the case $|c_i| < 1$, the general result follows by extension.

PROPOSITION 3.2.4. *We have the equality*

$$\widetilde{M}_f(s,t) = \frac{\Gamma(1+|t|-|s|)\Gamma(s_1)\cdots\Gamma(s_n)}{\Gamma(t_1)\cdots\Gamma(t_q)} \int_{\sigma_q} \frac{\tau^t}{\alpha(\tau)^s}\frac{d\tau}{\tau}, \tag{23}$$

where σ_q stands for the standard q-simplex.

PROOF. Let us begin with the case $n = q = 1$. By Example 3.2.2 we have that

$$\begin{aligned} \widetilde{M}_f(s,t) &= \frac{\Gamma(s)\Gamma(1+t-s)}{\Gamma(1+t)} c^{-t}{}_2F_1\Big(t, 1+t-s; 1+t; 1-\frac{1}{c}\Big) \\ &= \frac{\Gamma(s)\Gamma(1+t-s)}{\Gamma(1+t)} {}_2F_1(t,s;1+t;1-c). \end{aligned}$$

where the last equality is a functional identity for ${}_2F_1$. The classical identity,

$$\int_0^1 \frac{\tau^t}{(1-\tau)^r(1-(1-c)\tau)^s}\frac{d\tau}{\tau} = \frac{\Gamma(t)\Gamma(1-r)}{\Gamma(1+t-r)} {}_2F_1(t,s;1+t-r;1-c)$$

is valid under the restriction to $r = 0$ since $\mathrm{Re}\,(t+1) > \mathrm{Re}\,(t)$, which yields

$$\int_0^1 \frac{\tau^t}{(1-(1-c)\tau)^s}\frac{d\tau}{\tau} = \frac{\Gamma(t)}{\Gamma(1+t)} {}_2F_1(t,s;1+t;1-c),$$

thus proving the claim.

Let us continue by proving the case $n = 1$, using induction over q. Note that we have the equality

$$c^{-t} \sum_{k\in\mathbb{N}^q} \frac{(1+|t|-s)_{|k|}\,(t)_k}{(1+|t|)_{|k|}\,k!}\Big(1-\frac{1}{c}\Big)^k = \sum_{k\in\mathbb{N}^q} \frac{(s)_{|k|}\,(t)_k}{(1+|t|)_{|k|}\,k!}(1-c)^k,$$

which can be shown by induction over q, using the above functional identity for ${}_2F_1$. Here $k! = k_1!\cdots k_q!$ and $(t)_k = (t_1)_{k_1}\cdots(t_q)_{k_q}$ as above. Hence by Example

3.2.2 we have a description of our entire function as a series;

$$\widetilde{\Phi}(s,t) = \frac{1}{\Gamma(1+|t|)} \sum_{k\in\mathbb{N}^q} \frac{(s)_{|k|}\,(t)_k}{(1+|t|)_{|k|}\,k!}\,(1-c)^k$$

Let us now consider the integral in the right hand side of (23) for $q \geq 2$. We use the notation $\tau' = (\tau_1,\dots,\tau_{q-1})$, and begin with making the substitution $\sigma_i = \tau_i(1-\tau_q)$, for $i \neq q$.

$$\begin{aligned}
&\int_{\sigma_q} \frac{\tau^t}{(1-d_1\tau_1-\cdots-d_q\tau_q)^s}\,\frac{d\tau}{\tau} \\
&\quad= \int_0^1 \int_{\sigma_{q-1}} \frac{\tau^t\,(1-\tau_q)^{|t'|}}{(1-d_q\tau_q)^s(1-\frac{d_1(1-\tau_q)\tau_1}{(1-d_q\tau_q)}-\cdots-\frac{d_{q-1}(1-\tau_q)\tau_{q-1}}{(1-d_q\tau_q)})^s}\,\frac{d\tau}{\tau} \\
&\quad= \frac{\Gamma(t_1)\cdots\Gamma(t_{q-1})}{\Gamma(1+|t'|)} \sum_{k'\in\mathbb{N}^{q-1}} \frac{(s)_{|k'|}\,(t')_{k'}\,{d'}^{k'}}{(1+|t'|)_{|k'|}\,k'!} \int_0^1 \frac{\tau_q^{t_q}(1-\tau_q)^{|t'|+|k'|}}{(1-d_q\tau_q)^{s+|k'|}}\,\frac{d\tau_q}{\tau_q} \\
&\quad= \frac{\Gamma(t_1)\cdots\Gamma(t_q)}{\Gamma(1+|t|)} \sum_{k\in\mathbb{N}^q} \frac{(s)_{|k|}\,(t)_k\,d^k}{(1+|t|)_{|k|}\,k!},
\end{aligned}$$

were in the second last step we have used the induction hypothesis, and in the last step we have used the case $q = 1$. This concludes the case $n = 1$.

Finally we will show the general case by an induction over n. Note that the generalized binomial theorem gives

$$\alpha_n(\tau) = \sum_{k\in\mathbb{N}^q} \frac{(s_n)_{|k|}}{k!}\,d_n^k\,\tau^k,$$

and thus we evaluate the right hand side of (23) to

$$\begin{aligned}
&\frac{\Gamma(1+|t|-|s|)\Gamma(s_1)\cdots\Gamma(s_n)}{\Gamma(t_1)\cdots\Gamma(t_q)} \int_{\sigma_q} \frac{\tau^t}{\alpha(\tau)^s}\,\frac{d\tau}{\tau} \\
&\quad= \frac{\Gamma(1+|t|-|s|)\Gamma(s_1)\cdots\Gamma(s_n)}{\Gamma(t_1)\cdots\Gamma(t_q)} \sum_{k\in\mathbb{N}^q} \frac{(s_n)_{|k|}}{k!}\,d_n^k \int_{\sigma_q} \frac{\tau^{t+k}}{\alpha'(\tau)^{s'}}\,\frac{d\tau}{\tau} \\
&\quad= \sum_{k\in\mathbb{N}^q} \frac{\Gamma(1+|t|-|s|)\Gamma(s_n+|k|)(t)_k}{\Gamma(1+|t|+|k|-|s'|)k!}\,d_n^k \int_{\mathbb{R}_+^{n-1}} \frac{{z'}^{s'}}{(1+|z'|)(1+\langle c',z'\rangle)^{t+k}}\,\frac{dz'}{z'},
\end{aligned}$$

where the last equality follows by the induction hypothesis. On the other hand, by first making the change of variables given by $z_n = (1-w)/w$, then dilating each other variable by w^{-1}, and finally using the generalized binomial theorem we have that the left hand side of (23) evaluates to

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} \frac{z^s}{(1+z_1+\cdots+z_n)(1+\langle c,z\rangle)^t}\,\frac{dz}{z} \\
&\quad= \int_{\mathbb{R}_+^{n-1}} \int_0^1 \frac{{z'}^{s'}(1-w)^{s_n-1}w^{|t|-|s|}}{(1+|z'|)(1+\langle c',z'\rangle-(1-c_n)(1-w))^t}\,dw\,\frac{dz'}{z'} \\
&\quad= \sum_{k\in\mathbb{N}^m} \frac{(t)_k}{k!}\,d_n^k \int_{\mathbb{R}_+^{n-1}} \int_0^1 \frac{{z'}^{s'}(1-w)^{s_n-1+|k|}w^{|t|-|s|}}{(1+|z'|)(1+\langle c',z'\rangle)^{t+k}}\,dw\,\frac{dz'}{z'}
\end{aligned}$$

$$= \sum_{k \in \mathbb{N}^q} \frac{\Gamma(1+|t|-|s|)\Gamma(s_n+|k|)(t)_k}{\Gamma(1+|t|+|k|-|s'|)k!} d_n^k \int_{\mathbb{R}_+^{n-1}} \frac{{z'}^{s'}}{(1+|z'|)(1+\langle c', z'\rangle)^{t+k}} \frac{dz'}{z'},$$

which concludes the proof. □

3.3. The relation to coamoebas

For Theorems 3.1.2 and 3.1.4 to hold, each $f_j(z)$ must be completely nonvanishing on the positive orthant. This is a strong restriction that many polynomials will not fulfill, however we can relax this condition by considering the coamoeba $\mathcal{A}'_f$.

Proposition 3.3.1. *If $\theta \in \mathbf{T}^n$, then a polynomial $f(z)$ is completely nonvanishing on the set $\operatorname{Arg}^{-1}(\theta)$ if and only if $\theta \notin \overline{\mathcal{A}'_f}$.*

Proof. The claim is equivalent to the statement

$$\overline{\mathcal{A}'_f} = \bigcup_{\Gamma} \mathcal{A}'_{f_\Gamma},$$

see Section 2.1. □

Thus for polynomials $f_1, \dots, f_q$ such that the closed coamoeba of $f(z) = \prod_{j=1}^q f_j(z)$ is a proper subset of $\mathbf{T}^n$, there is a $\theta \notin \overline{\mathcal{A}'_f}$ for which the Euler–Mellin integral with respect to θ is well-defined:

$$M_f^\theta(s,t) := \int_{\operatorname{Arg}^{-1}(\theta)} \frac{z^s}{f(z)^t} \frac{dz}{z}. \tag{24}$$

As (24) differs from our earlier definition of the Euler–Mellin integral in (11) only by a change of variables, it is immediate that the analogs of Theorems 3.1.2 and 3.1.4 hold. In addition, a slight perturbation of θ does not impact the value of (24).

Theorem 3.3.2. *The Euler–Mellin integral M_f^θ of (24) is a locally constant function in θ. Thus it depends only on the choice of connected component Θ of the complement of $\overline{\mathcal{A}'_f}$, and we thus write $M_f^\Theta := M_f^\theta$, and accordingly, $\Phi^\Theta := \Phi^\theta$, for $\theta \in \Theta$.*

Proof. First consider the case $n = 1$, and suppose that θ_1 and θ_2 lie in the same connected component of the complement of $\overline{\mathcal{A}'_f}$; in fact, assume that the interval $[\theta_1, \theta_2] \subseteq (\overline{\mathcal{A}'_f})^c$. In other words, $f(z)$ has no zeros with arguments in this interval, and hence $z^{s-1}/f(z)^t$ is analytic in the corresponding domain. Connecting the two rays $\operatorname{Arg}^{-1}(\theta_1)$ and $\operatorname{Arg}^{-1}(\theta_2)$ with the circle section of radius r yields a closed curve, and the integral of $z^{s-1}/f(z)^t$ over this (oriented) curve is zero by residue calculus. By the proof of Theorem 3.1.2, the integral over the circle section tends to 0 as $r \to \infty$, so the two Euler–Mellin integrals $M_f^{\theta_1}$ and $M_f^{\theta_2}$ are equal.

In arbitrary dimension, we obtain the desired equality by considering one variable at a time while the remaining variables are fixed. □

Example 3.3.3. For the polynomial $f(z_1, z_2) = c_1 + c_2 z_1 + c_3 z_2 + c_4 z_1 z_2$, we see that if we choose $\theta = (\arg(c_1/c_2), \arg(c_1/c_3))$, then

$$\Phi^\Theta(s_1, s_2, t) = \frac{c_1^{s_1+s_2-t} c_2^{-s_1} c_3^{-s_2}}{\Gamma(t)^2} \, {}_2F_1\Big(s_1, s_2; t; 1 - \frac{c_1 c_4}{c_2 c_3}\Big),$$

where Θ is the connected component of $(\overline{\mathcal{A}'_f})^c$ containing θ. In accordance with Remark 3.1.6, we may ignore one of the factors $\Gamma(t)$ in the denominator, while ${}_2F_1/\Gamma(t)$ is the regularized Gauss hypergeometric function.

We now fix a connected component Θ of $\overline{\mathcal{A}'_f}$ and study the entire function $\Phi = \Phi^\Theta$ from (24). In particular, we consider its dependence on the coefficients c of the polynomials f_j. In order to emphasize this dependence, we write $\Phi(s,t,c)$ rather than $\Phi(s,t)$. Generalizing [**16**, Section 6], we show that Φ is an A-hypergeometric function. More precisely, $c \mapsto \Phi(s,t,c)$ satisfies the A-hypergeometric system of partial differential equations with the desired homogeneity parameter $\beta = -(t,s)$.

Let $\Sigma_A \subseteq \mathbb{C}^N$ denote the singular locus of all A-hypergeometric functions, which is the zero set of the principal A-determinant (also known as the full A-discriminant) [**7**].

Theorem 3.3.4. *Let $c \in \mathbb{C}^N \setminus \Sigma_A$ and let Θ be a connected component of $\mathbb{R}^n \setminus \overline{\mathcal{A}'_f}$, where f is the polynomial $f(z) = \prod_{j=1}^q f_j$. If $s \in \mathbb{C}^n$ with $\tau := \operatorname{Re} t$ and $\sigma := \operatorname{Re} s \in \operatorname{int}(\tau\Delta_f)$, then the analytic germ*

$$\Phi^\Theta(s,t,c) = \frac{1}{\prod_k \Gamma(\langle \mu_k, s\rangle - \langle \nu_k, t\rangle)} \int_{\operatorname{Arg}^{-1}(\theta)} \frac{z^s}{f(z)^t}\frac{dz}{z} \tag{25}$$

for any $\theta \in \Theta$ has a (multivalued) analytic continuation to $(\mathbb{C}^N \setminus \Sigma_A) \times \mathbb{C}^{n+q}$ that is everywhere A-hypergeometric (in the variables c) with homogeneity parameter $\beta = -(t,s)$.

Proof. Fix a representative $\theta \in \Theta$. As θ is disjoint from $\overline{\mathcal{A}'_f}$ for polynomials f with coefficients c near the original ones, say in a small ball $B(c)$, the integral in (25) does indeed define an analytic germ $\Phi = \Phi^\theta(s,t,c)$. By Theorem 3.1.4, Φ is extendable to an entire function with respect to the variables s and t. In other words, we now have an analytic extension of Φ to the infinite cylinder $B(c) \times \mathbb{C}^{n+q}$.

To see that Φ is an A-hypergeometric function with homogeneity parameter β as given, we first fix $\tau > 0$. We then fix s at an arbitrary value with $\operatorname{Re} s \in \operatorname{int}(\tau\Delta_f)$, so that it is away from the polar hyperplanes of the gamma functions. For such a parameter, the product of gamma functions in Φ is simply a nonzero constant. Thus it is enough to show that the integral itself is A-hypergeometric at β. This is accomplished through the argument of [**26**, Theorem. 5.4.2], which applies as we may interchange differentiation and integration since Euler–Mellin integrals are uniformly convergent by the bound in (12). See also [**8**, Remark 2.8(b)].

Having established that Φ is an A-hypergeometric function in the product domain given by $B(c) \times (\mathbb{R}_+ \operatorname{int}(\tau\Delta_f) + i\mathbb{R}^n) \times (\mathbb{R}^q_+ \times i\mathbb{R}^q)$, it follows from the uniqueness of analytic continuation that its extension to the cylinder $B(c) \times \mathbb{C}^{n+q}$ will remain A-hypergeometric. Now for each fixed (s,t), there is a (typically multivalued) analytic continuation of $c \mapsto \Phi^\theta(s,t,c)$ from $B(c)$ to all of $\mathbb{C}^N \setminus \Sigma_A$. As these continuations still depend analytically on s and t, we have now achieved the desired analytic continuation to the full product domain $(\mathbb{C}^N \setminus \Sigma_A) \times \mathbb{C}^{n+q}$. The uniqueness of analytic continuation again guarantees that Φ will everywhere satisfy the A-hypergeometric system with the homogeneity parameter β, as desired. $\square$

3.4. The relation to Mellin–Barnes integrals

Let γ be such that $A\gamma = \beta$, and fix a Gale dual B of A. As before we denote the ith row of B by b_i. Integrals of the form

$$L(c_1, \dots, c_N) = \int_{(i\mathbb{R})^m} \prod_{i=1}^{N} \Gamma(-\gamma_i - \langle b_i, r\rangle) c_i^{\gamma_i + \langle b_i, r\rangle}\, dr,$$

where $dr = dr_1 \wedge \cdots \wedge dr_m$, are known as *Mellin–Barnes integrals* [**2, 14**]. Let us state three important properties of these integrals. Firstly, if $\operatorname{Arg}(c)B \in \operatorname{int}(\mathcal{Z}_B)$, then the integral L converges absolutely [**2**, Corr. 4.2]. Secondly, if this is the case, and in addition if $\gamma_i < 0$ for each i, then $L(c)$ satisfies the A-hypergeometric system $H_A(\beta)$ [**2**, Thm. 3.1] as a germ of a hypergeometric function in c. The third property we formulate as a proposition, however without proof, using the notation

$$L^\theta(c) = L(c_\theta) = L(c_1 e^{\langle \alpha_1, \theta\rangle}, \dots, c_N e^{\langle \alpha_N, \theta\rangle}).$$

Proposition 3.4.1. [**2**, Prop. 4.3] *Let β be* totally non-resonant, *i.e. such that the shifted lattice $\beta + \mathbb{Z}^{n+q}$ has empty intersection with any hyperplane spanned by $n+q-1$ linearly independent elements of A. Let $\theta_1, \dots, \theta_k$ be n-tuples such that the m-tuples* $\operatorname{Arg}(c_{\theta_1})B, \dots, \operatorname{Arg}(c_{\theta_k})B$ *are distinct and contained in* $\operatorname{int}(\mathcal{Z}_B)$. *Then the germs of A-hypergeometric functions defined by the Mellin–Barnes integrals $L^{\theta_1}(c), \dots, L^{\theta_k}(c)$ are linearly independent.*

Thus for each set of points $\operatorname{int}(\mathcal{Z}_B) \cap (\operatorname{Arg}(c)B + 2\pi\mathbb{Z}[B])$ we can associate a set of linearly independent solutions to $H_A(\beta)$. The goal of this section is to show that the order map v can be lifted to a bijection between the Euler–Mellin integrals arising from components of the complement of the lopsided coamoeba and the Mellin–Barnes integrals corresponding to their orders. To achieve this, we need to consider a dual matrix of special kind. As in the proofs of Propositions 2.3.5 and 2.3.4, let us write A of the form

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & A_{\mathrm{I}} & A_{\mathrm{II}} \end{pmatrix},$$

where A_{I} is a nonsingular $n \times n$-matrix. We choose the dual matrix

$$B = \begin{pmatrix} -a_0 \\ A_{\mathrm{I}}^{-1} A_{\mathrm{II}} \\ -I_m \end{pmatrix} D$$

where a_0 is defined by the property that the columns sum to zero, and D is an integer diagonal matrix chosen such that B is an integer matrix.

Lemma 3.4.2. *With the above notation we have that*

$$\frac{g_B}{g_A} = \frac{|\det(D)|}{|\det(A_I)|}.$$

Proof. We can assume that $g_A = 1$. Following [**14**, Prop. 4.2] this is equivalent to that A can be extended to a $N \times N$ unimodular matrix

$$\widetilde{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & A_{\mathrm{I}} & A_{\mathrm{II}} \\ * & * & * \end{pmatrix},$$

with inverse

$$\widetilde{A}^{-1} = \begin{pmatrix} * & \widetilde{B} \end{pmatrix} = \begin{pmatrix} * & \tilde{b}_0 \\ * & \widetilde{B}_1 \\ * & \widetilde{B}_2 \end{pmatrix}.$$

It follows that $\widetilde{B}$ is a Gale dual of A, and by the Schur complement formula we have that $|\det(A_{\mathrm{I}})| = |\det(\widetilde{B}_2)|$. As $B = \widetilde{B}T$ for some affine transformation T, the relation follows. ☐

THEOREM 3.4.3. *For each $\theta \in (\overline{\mathcal{LA}'_f})^c$ we have the equality of germs of A-hypergeometric functions*

$$g_B\, L^\theta(c) = 2\pi i\, e^{-\langle s,\theta\rangle}\, \Gamma(t)\, g_A\, M_f^\theta(c). \tag{26}$$

PROOF. We will give the proof for the homogeneity parameter $(-t, -A_{\mathrm{I}}s)$. We have that the Euler–Mellin integral evaluates to

$$\begin{aligned} M_f^\theta(c) &= \int_{\mathrm{Arg}^{-1}(\theta)} \frac{z^{A_{\mathrm{I}}s}}{(c_0 + c_1 z^{\alpha_1} + \cdots + c_n z^{\alpha_n} + c_{n+1} z^{\alpha_{n+1}} + \cdots + c_{n+m} z^{\alpha_{n+m}})^t} \frac{dz}{z} \\ &= \frac{c_0^{|s|-t}}{c_{\mathrm{I}}{}^s} \int_{\mathrm{Arg}^{-1}(\tilde{\theta})} \frac{z^{A_{\mathrm{I}}s}}{(1 + z^{\alpha_1} + \cdots + z^{\alpha_n} + x_1^{\frac{1}{d_1}} z^{\alpha_{n+1}} + \cdots + x_m^{\frac{1}{d_m}} z^{\alpha_{n+m}})^t} \frac{dz}{z} \end{aligned}$$

where $x_i = c^{B_i}$, and B_i denotes the ith column of B. Let $\phi = \mathrm{Arg}(x)$. The lopsidedness condition is equivalent to that the integral

$$\int_{\mathrm{Arg}^{-1}(\tilde{\theta},\phi)} \frac{z^{A_{\mathrm{I}}s} x^r}{(1 + z^{\alpha_1} + \cdots + z^{\alpha_n} + x_1^{\frac{1}{d_1}} z^{\alpha_{n+1}} + \cdots + x_m^{\frac{1}{d_m}} z^{\alpha_{n+m}})^t} \frac{dz \wedge dx}{z\, x}$$

converges, however this is precisely the Mellin transform of the function $M_f^\theta(c)$ with respect to x. We find that

$$\begin{aligned} \{\mathcal{M}\, M_f^\theta(x)(r)\} &= \frac{|\det(D)|}{|\det(A_{\mathrm{I}})|} \int_{\mathrm{Arg}^{-1}(\tilde{\theta},\tilde{\phi})} \frac{z^{s - A_{\mathrm{I}}^{-1} A_2 D r} v^{Dr}}{(1 + z_1 + \cdots + z_n + x_1 + \cdots + x_m)^t} \frac{dz \wedge dx}{z\, x} \\ &= \frac{|\det(D)|}{|\det(A_{\mathrm{I}})|} \frac{\Gamma(s - A_{\mathrm{I}}^{-1} A_{\mathrm{II}} Dr)\Gamma(Dr)\Gamma(t - |Dr| - |s| + |A_{\mathrm{I}}^{-1} A_{\mathrm{II}} Dr|)}{\Gamma(t)}, \end{aligned}$$

see Example 3.2.1. Turning to the Mellin–Barnes integral, let us choose parameters $\gamma_{\mathrm{I}} = -s - A_{\mathrm{I}}^{-1} A_{\mathrm{II}} \gamma_{\mathrm{II}}$ and $\gamma_0 = |s| - t + \langle b_0, \gamma_{\mathrm{II}} \rangle$. Under condition $s_j > 0$ for all j, $t > |s|$ and γ_{II} negative and small, we find that $\gamma_k < 0$ for all k. Further more

$$\sum_i \gamma_i \begin{pmatrix} 1 \\ \alpha_i \end{pmatrix} = A\gamma = \begin{pmatrix} -t \\ -A_{\mathrm{I}} s \end{pmatrix}.$$

We have that

$$\begin{aligned} L^\theta(c) &= \int_{i\mathbb{R}^m} \Gamma(t - |Dr| - |s| + |A_{\mathrm{I}}^{-1} A_{\mathrm{II}} Dr|)\Gamma(s - A_{\mathrm{I}}^{-1} A_{\mathrm{II}} Dr)\Gamma(Dr) \\ &\qquad \cdot e^{-\langle A_{\mathrm{I}} s,\theta\rangle} c_0^{-t+|Dr|+|s|-|A_{\mathrm{I}}^{-1} A_{\mathrm{II}} Dr|} c_{\mathrm{I}}^{-s+A_{\mathrm{I}}^{-1} A_{\mathrm{II}} Dr} c_{\mathrm{II}}^{-Dr}\, dr \\ &= \frac{c_0^{|s|-t}}{e^{\langle A_{\mathrm{I}} s,\theta\rangle} c_{\mathrm{I}}^s} \int_{i\mathbb{R}^m} \Gamma(t - |Dr| - |s| + |A_{\mathrm{I}}^{-1} A_{\mathrm{II}} Dr|)\Gamma(s - A_{\mathrm{I}}^{-1} A_{\mathrm{II}} Dr)\Gamma(Dr) \frac{dr}{x^r} \end{aligned}$$

The bounds of Theorem 3.1.2 implies that we can apply the Mellin inversion formula, obtaining the equality

$$|\det(D)|\, L^\theta(c) = 2\pi i\, e^{-\langle A_{\mathrm{I}} s, \theta\rangle}\Gamma(t)\, |\det(A_{\mathrm{I}})|\, M_f^\theta(c),$$

which together with the previous lemma proves (26). □

COROLLARY 3.4.4. *For totally non-resonant homogeneity parameters, the germs of hypergeometric functions arizing from different components of the complement of the lopsided coamoebas are linearly independent.*

PROOF. Let $\theta_1, \ldots, \theta_k$ be a set of base points for the set of components of the complement of the lopsided coamoeba. If there is a vanishing linear combination of the hypergeometric germs $M_f^{\theta_1}(c), \ldots, M_f^{\theta_k}(c)$, say with coefficients $r_1, \ldots, r_k$, then

$$g_B \sum_{j=1}^{k} r_j e^{\langle s, \theta_j\rangle}\, L^{\theta_j}(c) = 2\pi i\, \Gamma(t)\, g_A \sum_{j=1}^{k} r_j\, M_f^{\theta_j}(c) = 0.$$

Thus it follows by Proposition 3.4.1 that $r_1 = \cdots = r_k = 0$. □

As it is not always possible to construct a lopsided coamoeba whose complement has $\mathrm{Vol}(A)$ many connected components, we cannot always construct a Mellin–Barnes basis of solutions. In general, the number of connected components of the complement of the coamoeba is greater than the number of connected components of the complement of the lopsided coamoeba, and it is natural to ask whether the set of hypergeometric function arising from the original Euler–Mellin integrals are linearly independent. The goal is to construct a Euler–Mellin basis of solutions in the cases when there is no Mellin–Barnes basis of solutions, but when we can still find a coamoeba whose complement has $\mathrm{Vol}(A)$ many connected components. We will give a partial answer to this question.

THEOREM 3.4.5. *Let $n = 1$, and consider a polynomial of degree d with full support. Let $\beta = -(t, s)$ be a totally non-resonant homogeneity parameter, and in addition assume that t is non integer. Then the germs of hypergeometric functions arising from different components of the complement of the coamoeba are linearly independent.*

PROOF. Consider a polynomial

$$f(z) = c_0 + c_1 z + \cdots + c_d z^d = c_d(z - r_1)\cdots(z - r_d).$$

The proof is by comparing the two point configurations

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & d \end{pmatrix} \quad \text{and} \quad \widetilde{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 1 & 0 & 1 & \ldots & 0 & 1 \end{pmatrix},$$

where $\widetilde{A}$ is a $(d+1) \times 2d$-matrix. To the first system we associate the integrals

$$M_f^\theta(s, t, c) = \int_{\mathrm{Arg}^{-1}(\theta)} \frac{z^s}{(c_0 + c_1 z + \cdots + c_d z^{k_d})^t}\, \frac{dz}{z},$$

while related to the second system we consider

$$\widetilde{M}_f^\theta(s, t_1, \ldots, t_d, r) = \int_{\mathrm{Arg}^{-1}(\theta)} \frac{z^s}{(z - r_1)^{t_1} \cdots (z - r_{k_N})^{t_{k_N}}} \frac{dz}{z}.$$

We have that $-(t, s)$ is a totally non-resonant parameter of the first system if and only if $s - kt$ is non integer for each $k = 0, 1, \ldots, d$. For the second system, let us note that each set of $d+1$ linearly independent columns must contain at least one column from each factor of f. It follows that the normal vector of any hyperplane spanned by d linearly independent columns of $\widetilde{A}$ is either of the form $(-\delta_1, \ldots, -\delta_d, 1)$, where $\delta_i \in \{0, 1\}$, or of the form $(e_i, 0)$. This implies that $-(t, \ldots, t, s)$ is a totally non-resonant parameter of the second system. As

$$M_f^\theta(s, t, c) = c_N^{-t}\, \widetilde{M}_f^\theta(s, t, \ldots, t, r),$$

any linear relation between the integrals M_f^θ arising from different components of $(\overline{\mathcal{A}'_f})^c$, yields a linear relation between the integrals $\widetilde{M}_f^\theta$. However, as for the second system, the ordinary coamoeba coincides with the lopsided coamoeba, we find that these integrals are linearly independent by Corollary 3.4.4, which finishes the proof. □

3.5. Recovering solutions at non-generic parameters

We conclude with an example studied in [**27**], where it was first seen that some parameters β admit a larger solution space for $H_A(\beta)$ than expected. We illustrate how Euler–Mellin integrals capture these extra solutions at non-generic parameters β, offering a new tool to understand how these special functions arise.

Example 3.5.1. Consider the system $H_A(\beta)$ given by $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$ and the special parameter $\beta = (1, 2)$. Here the Euler–Mellin integral is

$$M_f^\Theta(s, t, c) = \int_{\mathrm{Arg}^{-1}(\theta)} \frac{z^s}{(c_1 + c_2 z + c_3 z^3 + c_4 z^4)^t} \frac{dz}{z} \tag{27}$$

for any connected component Θ of the coamoeba complement in $\mathbb{R}^2$ of the polynomial $f(z) = c_1 + c_2 z + c_3 z^3 + c_4 z^4$ and $\theta \in \Theta$. In order to calculate Φ_f^Θ, we first expand (27) five times in different directions, so that it converges for $(s, t) = (-2, -1)$. Upon expansion, we see that $M_f^\Theta(s, t, c)$ is equal to

(28)
$$\frac{(t)_2}{s} \int \frac{z^s h_1(z)}{f(z)^{t+2}} \frac{dz}{z} + \frac{(t)_3}{s} \int \frac{z^s h_2(z)}{f(z)^{t+3}} \frac{dz}{z} + \frac{(t)_4}{s} \int \frac{z^s h_3(z)}{f(z)^{t+4}} \frac{dz}{z} + \frac{(t)_5}{s} \int \frac{z^s h_4(z)}{f(z)^{t+5}} \frac{dz}{z},$$

where all integrals are taken over $\mathrm{Arg}^{-1}(\theta)$ and $(t)_n = \Gamma(t+n)/\Gamma(t)$ is the Pochhammer symbol. This shows that when $(s, t) = (-2, -1)$, the entire function Φ_f^Θ falls into the situation noted in Remark 3.1.6, and we thus ignore the factor $(t+1)$ in (28). To be explicit,

$$h_1(z) = \frac{3c_2c_3z^4}{s+1} + \frac{3c_2c_3z^4}{s+3} + \frac{4c_2c_4z^5}{s+1} + \frac{4c_2c_4z^5}{s+4},$$

$$
\begin{aligned}
h_2(z) = {}& \frac{36c_1c_3^2z^6}{(s+3)(4t-s+2)} + \frac{48c_1c_3c_4z^7}{(s+3)(4t-s+1)} + \frac{48c_1c_3c_4z^7}{(s+4)(4t-s+1)} \\
&+ \frac{64c_1c_4^2z^8}{(s+4)(4t-s)} + \frac{c_2^3z^3}{(s+1)(s+2)} + \frac{3c_2^2c_3z^5}{(s+1)(s+2)} + \frac{4c_2^2c_4z^6}{(s+1)(s+2)} \\
&+ \frac{27c_2c_3^2z^7}{(s+3)(4t-s+2)} + \frac{36c_2c_3c_4z^8}{(s+3)(4t-s+1)} + \frac{36c_2c_3c_4z^8}{(s+4)(4t-s+1)} \\
&+ \frac{48c_2c_4^2z^9}{(s+4)(4t-s)} + \frac{9c_3^3z^9}{(s+3)(4t-s+2)}, \\
h_3(z) = {}& \frac{48c_1c_3^2c_4z^{10}}{(s+3)(4t-s+1)(4t-s+2)} + \frac{48c_1c_3^2c_4z^{10}}{(s+4)(4t-s+1)(4t-s+12)} \\
&+ \frac{64c_1c_3c_4^2z^{11}}{(s+4)(4t-s+1)^2} + \frac{36c_2c_3^2c_4z^{11}}{(s+3)(4t-s+1)(-s+4t+2)} \\
&+ \frac{36c_2c_3^2c_4z^{11}}{(s+4)(4t-s+1)(4t-s+2)} + \frac{48c_2c_3c_4^2z^{12}}{(s+4)(4t-s)(4t-s+1)} \\
&+ \frac{12c_3^3c_4z^{13}}{(s+3)(4t-s+1)(4t-s+2)} + \frac{12c_3^3c_4z^{13}}{(s+4)(4t-s+1)(4t-s+2)}, \\
\text{and} \quad h_4(z) = {}& \frac{64c_1c_3^2c_4^2z^{14}}{(s+4)(4t-s)(4t-s+1)(4t-s+2)} \\
&+ \frac{48c_2c_3^2c_4^2z^{15}}{(s+4)(4t-s)(4t-s+1)(4t-s+2)} \\
&+ \frac{16c_3^3c_4^2z^{17}}{s(s+4)(4t-s)(4t-s+1)(4t-s+2)}.
\end{aligned}
$$

Each term in (28) corresponds to a translation of our original integral (27) and converges at $(s,t)=(-2,-1)$. In addition, the absence of a degree 2 term in f is manifested in fact that no term of any $h_i(t)$ has both $(s+2)$ and $(4t-s+2)$ as factors in its denominator. Thus there are entire functions Φ_1, Φ_2, and Φ_3 in s and t such that

$$\Phi_f^\Theta = (4t-s+2)\Phi_1^\Theta + (s+2)\Phi_2^\Theta + (s+2)(4t-s+2)\Phi_3^\Theta.$$

From this expression we see that while $\Phi_f^\Theta(c,-2,-1)=0$ independently of c and Θ, we also obtain two functions Φ_1 and Φ_2 that are also solutions of $H_A(\beta)$. Explicit calculation implies that

$$\Phi_1^\Theta(-2,-1,c) = 2\,\frac{c_2^2}{c_1} \qquad \text{and} \qquad \Phi_2^\Theta(-2,-1,c) = 2\,\frac{c_3^2}{c_4},$$

for any choice of Θ. These span the Laurent series solutions of the system $H_A(1,2)$, which has dimension two only at this parameter [**4**].

APPENDIX A

Mathematica code

We present here the Mathematica code for the program used in Example 2.5.4.

```
M = Transpose[{{1,  0,  0, -1,  0,  0,  0,  0, -1,  1,  0,  0},
               {0,  0,  0,  0,  0,  1, -1,  0,  0, -1,  1,  0},
               {0,  0, -1,  1,  0,  0,  1, -1,  0,  0,  0,  0},
               {0,  1, -1,  0,  0,  0,  0,  0,  0,  0,  1, -1},
               {0,  0,  0,  0,  1,  0,  0, -1, -1,  0,  0,  1},
               {1, -1,  0,  0, -1,  1 , 0,  0,  0,  0,  0,  0}}];

L = Select[Tuples[{0, 1}, 12], #.M == {0, 0, 0, 0, 0, 0} &];
P = Permutations[{1, 2, 3, 4}];

CheckPart[v_, p_] := With[{w = v[[p]]},
   Min[Differences[w]] >= 0 && First[w] == 0];

Tally@Flatten@
  Table[Length[
    Select[L,
     CheckPart[#[[1 ;; 4]], p1] && CheckPart[#[[5 ;; 8]], p2] &&
       CheckPart[#[[9 ;; 12]], p3] &]], {p1, P}, {p2, P}, {p3, P}]
```

Running the above program yields the output

```
{{4, 288}, {3, 1056}, {2, 2912}, {1, 9568}}
```

which should be interpreted in the following manner; there is 9568 permutations $\sigma \in (\sigma_1, \sigma_2, \sigma_3) \in S_4^3$ such that the complement of the corresponding coamoeba has at most 1 connected component, there is 2912 permutations such that the complement of the corresponding coamoeba has at most 2 connected components, and so forth.

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