ON RULLGÅRD’S WILD GUESS

JENS FORSGÅRD

ABSTRACT. We give a counterexample to Rullgård’s wild guess regarding the structure of the complement of a hypersurface amoeba.

Let \( f \in \mathbb{C}[z] \) be an \( n \)-variate polynomial defining a hypersurface \( Z(f) \subset \mathbb{C}^n_* \). The amoeba \( \mathcal{A}(f) \) is the image of \( Z(f) \) under the componentwise logarithmic absolute value map. Let \( A \) be the support of \( f \) and let \( \mathcal{L}_A \) denote the lattice generated by \( A \). We can assume that \( \mathcal{L}_A \) has rank \( n \). Let \( \mathcal{N} \) denote the Newton polytope of \( f \), so that \( \mathcal{N} = \text{conv}(A) \) in \( \mathcal{L}_\mathbb{R} = \mathbb{R} \otimes \mathcal{L}_A \). Notice that \( \mathcal{L}_\mathbb{R} \cong \mathbb{R}^n \) as \( \mathbb{R} \)-vector spaces. Let \( \mathcal{C}^A \) denote the space of all polynomials with support \( A \), where a polynomial \( f \) is identified with its coefficient vector.

Amoebas were introduced in [2], where it was noted that the complement \( \mathbb{R}^n \setminus \mathcal{A}(f) \) consists of a finite number of open convex sets each corresponding to a Laurent series expansions of the rational function \( 1/f \). Thus, it is an important problem to understand the structure of \( \mathbb{R}^n \setminus \mathcal{A}(f) \). An excellent tool was provided by Forsberg, Passare, and Tsikh in [1] with the introduction of the order map; there exists an injective map from the set of connected components of \( \mathbb{R}^n \setminus \mathcal{A}(f) \) into \( \mathcal{L}_A \cap \mathcal{N} \). If \( f \) is a Laurent polynomial with support \( A \), and \( \alpha \in \mathcal{L}_A \cap \mathcal{N} \), then the component of \( \mathbb{R}^n \setminus \mathcal{A}(f) \) of order \( \alpha \) is denoted \( \mathcal{E}_\alpha(f) \). Let \( \alpha \in \mathcal{L}_A \cap \mathcal{N} \), and define \( \mathcal{U}_\alpha \subset \mathcal{C}^A \) to be the set of all polynomials \( f \) such that \( \mathbb{R}^n \setminus \mathcal{A}(f) \) has a component of order \( \alpha \). In Rullgård’s thesis [3] we find the following theorem and problem.

**Theorem 1** (Rullgård). A sufficient condition for \( \mathcal{U}_\alpha \) to be nonempty is that there exists a line \( l \) such that \( \alpha \in \text{conv}(A \cap l) \cap \mathcal{L}_A \cap l \).

**Problem 2** (Rullgård). Find a necessary and sufficient condition for the existence of a Laurent polynomial \( f \in \mathcal{C}^A \) with \( \mathcal{E}_\alpha(f) \neq \emptyset \).

Note that the condition in Theorem 1 is both necessary and sufficient if \( n = 1 \). Also, that the condition is sufficient in the multivariate case is deduced from the univariate case. It was perhaps with these facts fresh in mind that Rullgård made the following comment [3, p. 60]: “A rather wild guess would be that [the condition in Theorem 1] is also a necessary condition”. Though, stating a wild guess rather than a conjecture indicates that not much energy was spent investigating the matter. We will in an example show that Rullgård’s wild guess was indeed wild; the condition of Theorem 1 is not sufficient.

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FIGURE 1. In black: the support set $A \times A$ of $f$. In gray: the lines which intersect $A \times A$ in at least two points and $\mathbb{Z}^2 \cap \mathcal{N}$ in at least three points.

**Example 3.** Consider the support set $A = \{0, 1, 5\}$. By Theorem 1 there exist univariate polynomials $f_1$ and $f_2$ such that $\mathbb{R} \setminus \mathcal{A}(f_1)$ has a component of order three and $\mathbb{R} \setminus \mathcal{A}(f_2)$ has a component of order four. Let $f(z_1, z_2) = f_1(z_1) f_2(z_2)$. It follows that $\mathcal{A}(f) \setminus \mathbb{R}^2$ has a component of order $(3, 4)$. The support of $f$ is contained in the cartesian product $A \times A$. However, no line through $(3, 4)$ in $\mathbb{R}^2$ intersects $A \times A$ in more than one point. If the reader wish for the example to be irreducible, then it suffices to wiggle the coefficients of $f$; the set $E_{[3,4]} \subset \mathbb{C}^{A \times A}$ is open.

**Remark 4.** Any reader who wishes to attack Problem 2 should be aware of [3, Theorem 12, p. 36] where conditions on $A$ were given which ensures that $U_\alpha \neq \emptyset$ if and only if $\alpha \in A$.

**Problem 5.** The following example would be enlightening to sort out. Let $A = \{(0, 0), (2, 0), (3, 0), (1, 2), (0, 3), (0, 1)\}$. Is there a bivariate polynomial $f \in \mathbb{C}^A$ such that $\mathbb{R}^2 \setminus \mathcal{A}(f)$ has a component of order $(1, 1)$?

**References**


Department of Mathematics, Texas A&M University, College Station, TX 77843.

E-mail address: jensf@math.tamu.edu