ON RULLGÅRD'S WILD GUESS

JENS FORSGÅRD

ABSTRACT. We give a counterexample to Rullgård's wild guess regarding the structure of the complement of a hypersurface amoeba.

Let $f \in \mathbb{C}[z]$ be an n-variate polynomial defining a hypersurface $Z(f) \subset \mathbb{C}_*^n$. The amoeba $\mathscr{A}(f)$ is the image of $Z(f)$ under the componentwise logarithmic absolute value map. Let A be the support of f and let $\mathscr{L}_A$ denote the lattice generated by A. We can assume that $\mathscr{L}_A$ has rank n. Let $\mathscr{N}$ denote the Newton polytope of f, so that $\mathscr{N} = \operatorname{conv}(A)$ in $\mathscr{L}_\mathbb{R} = \mathbb{R} \otimes \mathscr{L}_A$. Notice that $\mathscr{L}_\mathbb{R} \simeq \mathbb{R}^n$ as $\mathbb{R}$-vector spaces. Let $\mathbb{C}^A$ denote the space of all polynomials with support A, where a polynomial f is identified with its coefficient vector.

Amoebas were introduced in [2], where it was noted that the complement $\mathbb{R}^n \setminus \mathscr{A}(f)$ consist of a finite number of open convex sets each corresponding to a Laurent series expansions of the rational function $1/f$. Thus, it is an important problem to understand the structure of $\mathbb{R}^n \setminus \mathscr{A}(f)$. An excellent tool was provided by Forsberg, Passare, and Tsikh in [1] with the introduction of the *order map*; there exists an injective map from the set of connected components of $\mathbb{R}^n \setminus \mathscr{A}(f)$ into $\mathscr{L}_A \cap \mathscr{N}$. If f is a Laurent polynomial with support A, and $\alpha \in \mathscr{L}_A \cap \mathscr{N}$, then the component of $\mathbb{R}^n \setminus \mathscr{A}(f)$ of order α is denoted $E_\alpha(f)$. Let $\alpha \in \mathscr{L}_A \cap \mathscr{N}$, and define $U_\alpha \subset \mathbb{C}^A$ to be the set of all polynomials f such that $\mathbb{R}^n \setminus \mathscr{A}(f)$ has a component of order α. In Rullgård's thesis [3] we find the following theorem and problem.

Theorem 1 (Rullgård)**.** *A sufficient condition for U_α to be nonempty is that there exists a line l such that $a \in \operatorname{conv}(A \cap l) \cap \mathscr{L}_{A\cap l}$.*

Problem 2 (Rullgård)**.** *Find a necessary and sufficient condition for the existence of a Laurent polynomial $f \in \mathbb{C}^A$ with $E_\alpha(f) \neq \emptyset$.*

Note that the condition in Theorem 1 is both necessary and sufficient if $n = 1$. Also, that the condition is sufficient in the multivariate case is deduced from the univariate case. It was perhaps with these facts fresh in mind that Rullgård made the following comment [3, p. 60]: "A rather wild guess would be that [the condition in Theorem 1] is also a necessary condition". Though, stating a wild guess rather than a conjecture indicates that not much energy was spent investigating the matter. We will in an example show that Rullgård's wild guess was indeed wild; the condition of Theorem 1 is not sufficient.

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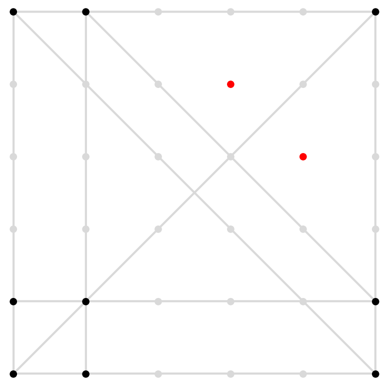

FIGURE 1. In black: the support set $A \times A$ of f. In gray: the lines which intersect $A \times A$ in at least two points and $\mathbb{Z}^2 \cap \mathcal{N}$ in at least three points.

Example 3. Consider the support set $A = \{0, 1, 5\}$. By Theorem 1 there exist univariate polynomials f_1 and f_2 such that $\mathbb{R} \setminus \mathscr{A}(f_1)$ has a component of order three and $\mathbb{R} \setminus \mathscr{A}(f_2)$ has a component of order four. Let $f(z_1, z_2) = f_1(z_1) f_2(z_2)$. It follows that $\mathscr{A}(f) \setminus \mathbb{R}^2$ has a component of order $(3, 4)$. The support of f is contained in the cartesian product $A \times A$. However, no line through $(3, 4)$ in $\mathbb{R}^2$ intersects $A \times A$ in more than one point. If the reader wish for the example to be irreducible, then it suffices to wiggle the coefficients of f; the set $E_{(3,4)} \subset \mathbb{C}^{A \times A}$ is open.

Remark 4. Any reader who wishes to attack Problem 2 should be aware of [3, Theorem 12, p. 36] where conditions on A were given which ensures that $U_\alpha \neq \emptyset$ if and only if $\alpha \in A$.

Problem 5. *The following example would be enlightening to sort out. Let*

$$A = \{(0,0), (2,0), (3,0), (1,2), (0,3), (0,1)\}.$$

Is there a bivariate polynomial $f \in \mathbb{C}^A$ such that $\mathbb{R}^2 \setminus \mathscr{A}(f)$ has a component of order $(1,1)$?

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DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843.
E-mail address: jensf@math.tamu.edu