COAMOEBAS OF BIVARIATE MULTI-AFFINE POLYNOMIALS

JENS FORSGÅRD

ABSTRACT. We describe, explicitly, the coamoeba of a generic bivariate multi-affine polynomial.

The coamoeba of a variety $Z(f)$, denoted $\mathscr{C}(f)$, is by definition its image under the componentwise argument map. Coameobas were introduced by Passare and Tsikh as a "dual" object of the amoeba of $Z(f)$, which by definition is its image under the componentwise logarithmic map. We wish to describe the coamoeba of a generic bivariate multi-affine polynomial f explicitly. Our conclusions are reached through the computation of two resultants and, therefor, the approach is not feasible in more complicated examples.

After an unorthodox normalization, we can assume that our polynomial is of the form

$$f(z) = c + z_1 + z_2 + cz_1z_2,$$

for some constant c. We can assume that $\Re(c) \geq 0$, for otherwise we can consider the polynomial $-f(-z)$, which corresponds in terms of coefficients to the map $c \mapsto -c$ and in terms of the coamoeba to a translation in the universal cover $\mathbb{R}^2$ of $\mathbf{T}^2$. Before describing the coamoeba, we will perform a number of auxiliary computations.

Consider a fix point $\theta \in \mathbf{T}^2$. The containment $\theta \in \mathscr{C}(f)$ is equivalent to that there exists a positive number $r = (r_1, r_2)$ such that

$$\begin{cases} \Re(c) + r_1\cos(\theta_1) + r_2\cos(\theta_2) + r_1r_2\big(\Re(c)\cos(\theta_1+\theta_2) - \Im(c)\sin(\theta_1+\theta_2)\big) & = 0 \\ \Im(c) + r_1\sin(\theta_1) + r_2\sin(\theta_2) + r_1r_2\big(\Re(c)\sin(\theta_1+\theta_2) + \Im(c)\cos(\theta_1+\theta_2)\big) & = 0 \end{cases}$$

Let us denote the quotient r_2/r_1 by λ. Then, the above system can be considered as system of two polynomials in degree two in the variable r_1, with coefficients in c, θ, and λ. Computing the resultant with respect to r_1, we obtain

$$\begin{aligned} R(c,\theta,\lambda) = \lambda\cdot\Big(&\lambda|c|^4\sin^2(\theta_1+\theta_2) - \lambda\Re(c^2) \\ &+ \lambda|c|^2\big(\cos^2(\theta_1) + \cos^2(\theta_2) - 1\big) \\ &+ (\lambda^2-1)\Im(c)\Re(c)\sin(\theta_1-\theta_2) \\ &+ \big(\lambda^2+1)(\Im(c)^2\cos(\theta_1)\cos(\theta_2) - \Re(c^2)\sin(\theta_1)\sin(\theta_2))\Big). \end{aligned}$$

We note that the vanishing of the first factor, λ, corresponds to roots on the boundary of $(\mathbb{C}^\times)^2$.

We wish to deduce an expression for the boundary $\partial\mathscr{C}(f)$. Since the inverse image of a point $\theta \in \partial\mathscr{C}(f)$ is contained in $(\mathbb{C}^\times)^2$, the mapping Arg is locally proper. Hence, the fiber of a small neighbourood of a point $\theta \in \partial\mathscr{C}(f)$ is, locally, a two sheeted folding. In particular, this implies that for $\theta \in \partial\mathscr{C}(f)$, the vanishing of the resultant $R(c,\theta,\lambda)$ at λ is of order (at least) two. Computing

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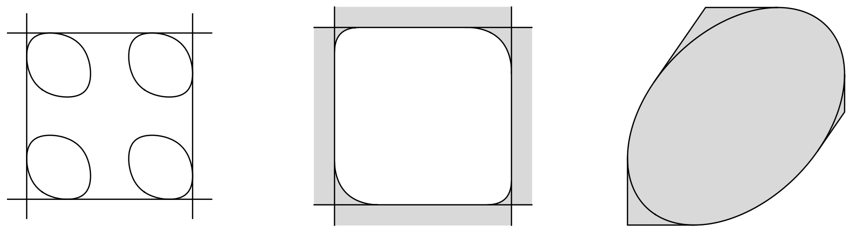

FIGURE 1. For $c = 1 + \frac{2i}{3}$; The shell and analytic locus from (1), the coamoeba $\overline{\mathscr{C}}(f)$, and its image under the map C.

the discriminant of $R(c,\theta,\lambda)$ with respect to λ, we obtain

$$\begin{aligned} D(c,\theta) =& |c|^4 \cdot \sin^2(\theta_1+\theta_2) \\ &\cdot \big(\Im(c)\cos(\theta_1) - \Re(c)\sin(\theta_1)\big)^2 \cdot \big(\Im(c)\cos(\theta_2) + \Re(c)\sin(\theta_2)\big)^2 \\ &\cdot \Big(2\Re(c^2) - 1 - |c|^4 + \big(|c|^2\cos(\theta_1+\theta_2) - \cos(\theta_1-\theta_2)\big)^2\Big). \end{aligned}$$

The first two factors contain no relevant information. The third and fourth factors describe the shell $\mathscr{H}(f)$. Thus, only the last factor is relevant for our purposes. We conclude that the boundary pieces not contained in the shell are contained in the analytic locus

$$\big(|c|^2\cos(\theta_1+\theta_2) - \cos(\theta_1-\theta_2)\big)^2 = 1 + |c|^4 - 2\Re(c^2). \tag{1}$$

We will now introduce some auxiliary notation and state our main theorem. Let T denote the square $[-1,1]^2$, with coordinates $t = (t_1, t_2)$. Consider the map $C\colon \mathbb{R}^2 \to T$ defined by

$$C(\theta) = \big(\cos(\theta_1+\theta_2), \cos(\theta_1-\theta_2)\big).$$

Furthermore, for $k = (k_1, k_2) \in \mathbb{Z}^2$, let $S_k \subset \mathbb{R}^2$ denote the square defined by

$$k_1\pi \le \theta_1+\theta_2 \le (k_1+1)\pi \quad \text{and} \quad k_2\pi \le \theta_1-\theta_2 \le (k_2+1)\pi.$$

Then, restricted to the interior of a rectangle S_k, the map C induces a diffeomorphism $C_k\colon S_k^\circ \to T^\circ$. The map C_k is orientation preserving if $|k| = k_1 + k_2$ is odd, and orientation reversing if $|k|$ is even. Finally, let $E(f)$ denote the ellipse with equation

$$\frac{(t_1+t_2)^2}{\cos^2(\arg(c))} + \frac{(t_1-t_2)^2}{\sin^2(\arg(c))} = 4. \tag{2}$$

Theorem 1. *The closed coamoeba $\overline{\mathscr{C}}(f)$ is mapped by C onto the convex hull of the ellipse $E(f)$, the point $(-1,-1)$, and the control points*

$$\Big(1, |c|^2 + \sqrt{1+|c|^4 - 2\Re(c^2)}\Big) \quad \textit{and} \quad \Big(|c|^{-2} + \sqrt{1+|c|^{-4} - 2\Re(c^{-2})}, 1\Big).$$

To obtain a description of the coamoeba $\mathscr{C}(f)$, one should *essentially* consider the inverse image under the map C of the convex set from Theorem 1. That is, in nontrivial rectangles S_k one should consider the inverse image under the map C: compare the two pictures in Figure 2.

Let $L(f)$ denote the arrangement of lines in T with defining equations

$$|c|^2 t_1 - t_2 = \pm\sqrt{1+|c|^4 - 2\Re(c^2)}. \tag{3}$$

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Lemma 2. *Each line in the shell $\mathscr{H}(f)$ is mapped by C onto the ellipse $E(f)$, and the analytic locus of* (1) *is mapped by C into the arrangement $L(f)$.*

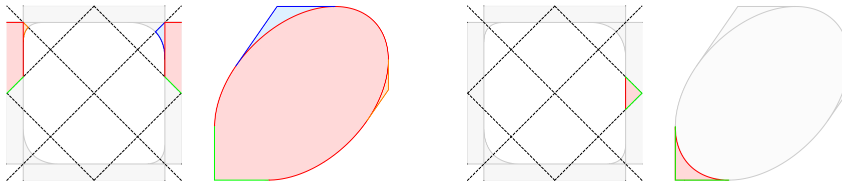

FIGURE 2. Two regions and their images under the map C.

Proof. Solving the binomial equations describing the shell $\mathscr{H}(f)$, it is straightforward to verify the formulas (2) and (3) by use the obtained solutions, (1), and the standard trigonometric addition formulas. □

Lemma 3. *The lines $L(f)$ are tangential to the ellipse $E(f)$.*

Proof. Eliminating t_2 from (2) using (3), the obtained quadratic polynomial has vanishing discriminant. □

Proof of Theorem 1. Recall that the shell $\mathscr{H}(f)$ is contained in the closure $\overline{\mathscr{C}}(f)$, and that the connected components of the complement of $\overline{\mathscr{C}}(f)$ are convex when viewed in $\mathbb{R}^n$. Using these two facts, that Căinduces a diffeomorphisms C_k, and Lemmas 2 and 3, it is straightforward to describe the image of the part of the coamoeba contained in any square S_k. We will settle with the pictures in Figure 2. □

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843.
E-mail address: jensf@math.tamu.edu