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Order stars and stability for delay differential equations

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Summary We consider Runge-Kutta methods applied to delay differential equations $y'(t) = ay(t) + by(t-1)$ with real a and b. If the numerical solution tends to zero whenever the exact solution does, the method is called $\tau(0)$-stable. Using the theory of order stars we characterize high-order symmetric methods with this property. In particular, we prove that all Gauss methods are $\tau(0)$-stable. Furthermore, we present sufficient conditions and we give evidence that also the Radau methods are $\tau(0)$-stable. We conclude this article with some comments on the case where a and b are complex numbers.

Mathematics Subject Classification (1991): 65L20

1 Introduction

In this work we study asymptotic stability properties of one-step methods when applied to the linear problem

$$y'(t) = a\,y(t) + b\,y(t-1), \tag{1}$$

where $a, b \in \mathbb{R}$ and $y(t) = g(t)$ on $[-1, 0]$. By looking at solutions of the form $y(t) = e^{\lambda t}$ we are led to the characteristic equation

$$\lambda = a + b\,e^{-\lambda}. \tag{2}$$

A Fourier-like analysis [Wri46] (see also [BC63]) then shows that the set of pairs (a, b), for which the solution $y(t)$ of (1) tends to 0 for

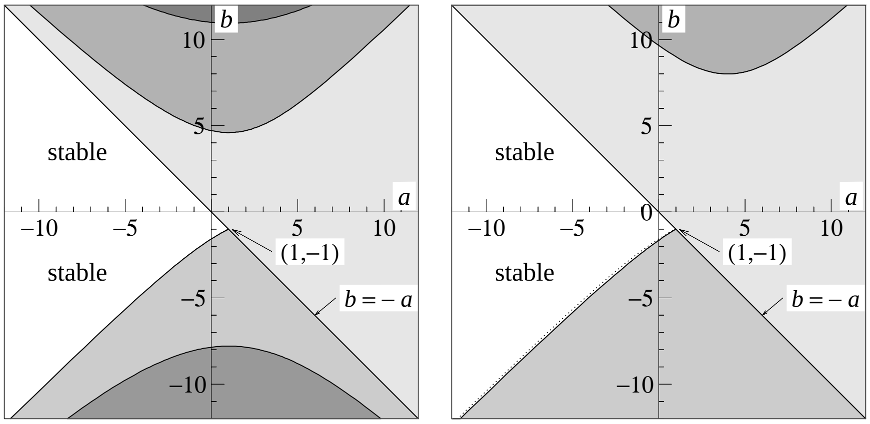

Fig. 1. The set Σ_* (left picture) together with the curves separating sets with different numbers of roots in the right half-plane; the right picture shows the set Σ_m for $m = 2$ corresponding to the trapezoidal rule (Example 2).

$t \to \infty$, is given by

$$\Sigma_* = \{(a, b) \ ; \ \text{all roots of (2) satisfy } \Re\lambda < 0\,\} \tag{3}$$

(see left picture of Fig. 1). This set is bounded to the right by the straight line $a + b = 0$ and by the transcendental curve $a = \varphi \cot \varphi$, $b = -\varphi/\sin\varphi$ for $\varphi \in (0, \pi)$. It can be written as $\Sigma_* = \Sigma_\triangle \cup \Sigma$, where $\Sigma_\triangle$ is the cone given by $a + |b| < 0$ and

$$\Sigma = \{(a, b) \ ; \ |a| < -b \ \text{ and } \ \sqrt{b^2 - a^2} < \arccos(-a/b)\,\} \tag{4}$$

(see [Hay50]). From this representation Σ_* is seen to be star-shaped with respect to the origin. The intensity of grey in Fig. 1 (left picture) indicates the increasing number of roots of (2) in the right half-plane.

If we apply a Runge-Kutta method with constant stepsize $h = 1/m$ to (1), it is natural to use the internal stage value $g_i^{(n-m)}$ as an approximation to $y(t_n + c_i h - 1)$. Hence, we consider the method

$$\begin{aligned} g_i^{(n)} &= y_n + h \sum_{j=1}^{s} a_{ij} \Big(a\, g_j^{(n)} + b\, g_j^{(n-m)} \Big) \\ y_{n+1} &= y_n + h \sum_{i=1}^{s} b_i \Big(a\, g_i^{(n)} + b\, g_i^{(n-m)} \Big). \end{aligned} \tag{5}$$

For a stability analysis of this method, [Zen86] pointed out that it is sufficient to look at numerical approximations of the form $y_n = \zeta^n y_0$, $g_i^{(n)} = \zeta^n g_i$, so that we are led to the characteristic equation

$$\zeta = R(z), \qquad z = \frac{1}{m}\Big(a + b\,\zeta^{-m} \Big), \tag{6}$$

where $R(z)$ is the stability function of the method (see for example [HW96, Sect. IV.2]). The numerical solution of (5) is therefore asymptotically stable, if and only if $|\zeta| < 1$ whenever ζ satisfies (6). We denote

$$\Sigma_m = \{(a,b) \ ; \ \text{all roots } \zeta \text{ of (6) satisfy } |\zeta| < 1\}. \tag{7}$$

It is natural to study the question whether $\Sigma_* \subset \Sigma_m$ for all $m \geq 1$, a property called $\tau(0)$-stability in [Gug97].

Among the publications addressing this question let us mention [Wie76,HS84,AM84,BMR96,Gug97,Gug98]. Many authors consider complex-valued (a,b) and they restrict their analysis to the cone $\Re a < -|b|$ (see [Bar75]). In this case, A-stability is necessary and sufficient for stability [Zen86].

This paper is organized as follows: In Sect. 2 we explain the use of the root locus curve and at two examples we illustrate the techniques used in this paper. In Sect. 3 we discuss necessary stability conditions arising from a local analysis. In the main part of this paper (Sect. 4), we study the $\tau(0)$-stability of symmetric Runge-Kutta methods with the help of order stars, and we prove that the Gauss methods (unique methods of order $p = 2s$) all satisfy this property. The Radau methods are discussed in Sect. 5. Although their stability region seems to be larger than for the Gauss methods, the proof of $\tau(0)$-stability is more complicated. We present a proof for $s = 2$ and we give evidence for larger s. Sect. 6 contains some comments concerning the case of complex-valued a and b in the test equation (1).

2 Use of the root locus curve

In order to prove $\Sigma_* \subset \Sigma_m$ it is useful to apply the so-called root locus technique [BP94]. Since z of (6) depends continuously on a and b (for $R(z) = P(z)/Q(z)$ it is the root of the polynomial equation $P(z)^m(mz - a) - bQ(z)^m = 0$), also $\zeta = R(z)$ depends continuously on a and b. Therefore, it is sufficient to prove that the values of (a,b) satisfying (6) with $|\zeta| = 1$ lie all outside the analytical stability region Σ_*.

Lemma 1 *Suppose that the stability domain $S = \{z \, ; |R(z)| \leq 1\}$ is connected, let $z(t) = x(t) + iy(t)$ for $t \in (-c, c)$ be a smooth parametrization of ∂S such that $z(-t) = \bar{z}(t)$, $z(0) = 0$, and let $z(t)$ be oriented such that S lies to the left. Furthermore, let $\varphi(t)$ be the argument of $R(z(t))$,*

$$R(z(t)) = e^{i\varphi(t)}, \tag{8}$$

in such a way that $\varphi(0) = 0$ *and* $\varphi(t)$ *is continuous. Then, the function* $\varphi(t)$ *is strictly monotonically increasing and it satisfies* $\varphi(-t) = -\varphi(t)$ *and* $\lim_{t\to c}\varphi(t) = s\pi$*, where* s *is the number of poles of* $R(z)$*.*

Proof This result follows from the proof of Lemma IV.4.5 in [HW96]. We write $R(x+iy) = r(x,y)e^{i\varphi(x,y)}$ and we let $\mathbf{a} = (x'(t), y'(t))$ be the tangent vector to $z(t)$ and $\mathbf{n} = (y'(t), -x'(t))$ the exterior normal vector. Since $|R(x+iy)| < 1$ inside S and $|R(x+iy)| > 1$ outside S, it holds $\partial(\log r)/\partial\mathbf{n} \geq 0$. By the Cauchy-Riemann differential equations this implies $\partial\varphi/\partial\mathbf{a} \geq 0$ on ∂S, with strict inequality except at a finite number of points because $R'(z(t))z'(t) = iR(z(t))\frac{\partial\varphi}{\partial\mathbf{a}}(z(t))$. This proves the monotonicity of $\varphi(t)$. The antisymmetry of $\varphi(t)$ follows from the fact that the coefficients of $R(z)$ are real, and the third statement is a consequence of the principle of the argument. $\square$

Example 1 In this article we mainly consider Padé-approximations to the exponential function (see [HW96, Sect. IV.3]). They are defined by $R_{s-j,s}(z) = P_{s-j,s}(z)/Q_{s-j,s}(z)$, where

$$P_{k\ell}(z) = 1 + \frac{k}{\ell+k}\,z + \ldots + \frac{k(k-1)\ldots 1}{(\ell+k)\ldots(\ell+1)}\,\frac{z^k}{k!} \qquad (9)$$

and $Q_{k\ell}(z) = P_{\ell k}(-z)$. Fig. 2 (left picture) illustrates Lemma 1 for the 3-stage diagonal Padé-approximation $R_{33}(z)$. Its stability domain is the negative half-plane. We see that the argument φ (indicated by arrows) is an increasing function of y and it makes $3/2$ rotations for y between 0 and ∞.

In order to study the values of (a,b) for which $|\zeta| = 1$ in (6), we insert $z = x+iy$ and $\zeta = e^{i\varphi}$ and we separate real and imaginary parts. This yields

$$mx = a + b\cos(m\varphi), \qquad my = -b\sin(m\varphi).$$

For $x = y = \varphi = 0$ we get the straight line $a + b = 0$ (as for the analytic stability region, see Fig. 1). If $\sin(m\varphi) = 0$ and $y = 0$ (this typically happens for $\varphi = s\pi$), we get another straight line $a \pm b = mx$. For $0 < \varphi < s\pi$ we obtain the curve

$$a_m(\varphi) = my\cot(m\varphi) + mx, \qquad b_m(\varphi) = -my/\sin(m\varphi), \qquad (10)$$

which passes through infinity when $m\varphi$ is an integer multiple of π.

Often it is advantageous to compare the values (a_m, b_m) to those of (2) which correspond to $\lambda = im\varphi$, defining the root locus for the true solution. They are given by

$$a_*(m\varphi) = m\varphi\cot(m\varphi), \qquad b_*(m\varphi) = -m\varphi/\sin(m\varphi).$$

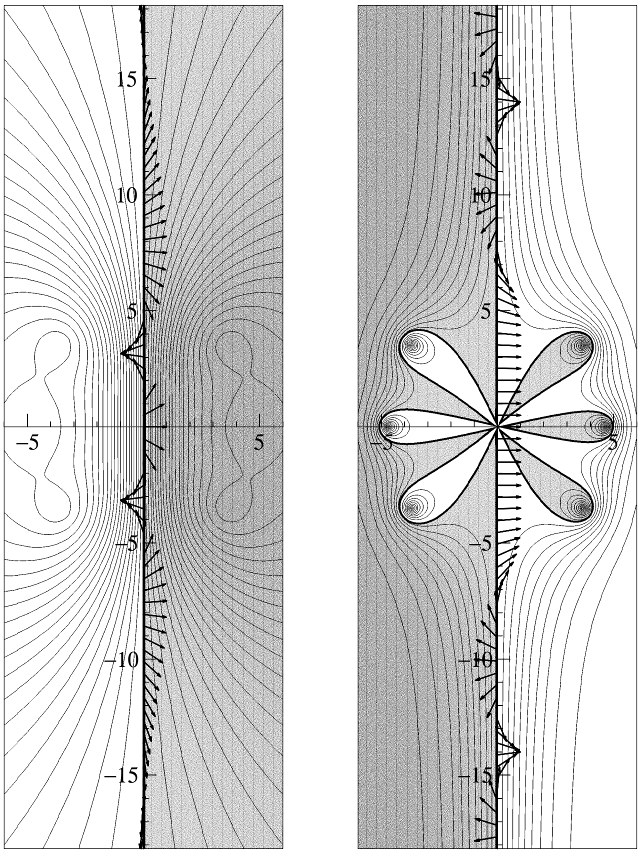

Fig. 2. Stability domain (left) and order star (right) of the 3-stage diagonal Padé-approximation $R_{33}(z)$

Only for $0 < m\varphi < \pi$ the point (a_*, b_*) is on the boundary of Σ_*. For $m\varphi > \pi$ it lies on a curve that separates regions with different numbers of roots of (2) in the right half-plane (Fig. 1, left picture). Comparing the above formulas for (a_m, b_m) and (a_*, b_*), yields the representation

$$a_m(\varphi) = \frac{y}{\varphi}\, a_*(m\varphi) + mx, \qquad b_m(\varphi) = \frac{y}{\varphi}\, b_*(m\varphi), \tag{11}$$

which will be fundamental in our stability analysis.

Example 2 For the rational function

$$R(z) = \frac{1 + z/2}{1 - z/2},$$

which corresponds to the trapezoidal rule or to the implicit midpoint rule, we consider the parametrization $x(\varphi) = 0$ and $y(\varphi) = 2\tan(\varphi/2)$ in order to get $R(x+iy) = e^{i\varphi}$. In this case we have $y(\varphi) > \varphi$ for all $\varphi \in (0,\pi)$. Since $x(\varphi) = 0$ and the set Σ_* is star-shaped with respect to the origin, this relation together with (11) implies that (a_m, b_m) is always outside the set Σ_*. Consequently, we have $\Sigma_* \subset \Sigma_m$ for all $m \geq 1$ and the method is $\tau(0)$-stable. In Fig. 1 (right picture) we can see Σ_2 together with the curve $(a_2(\varphi), b_2(\varphi))$. It approximates very accurately $(a_*(2\varphi), b_*(2\varphi))$ for $0 < \varphi < \pi/2$ (dotted curve) and, for $\pi/2 < \varphi < \pi$, it separates the regions with 1 and 3 roots of (6) lying outside the unit disk.

Example 3 The $\tau(0)$-stability of the θ-method

$$R(z) = \frac{1 + (1-\theta)z}{1 - \theta z}$$

for $1/2 \leq \theta \leq 1$ has been shown by [Gug98] in a very technical proof. We shall outline here a different proof for values of θ between $1/2$ and $3/4$. We first remark that, multiplying (a_*, b_*) by a factor $y/\varphi < 1$, the point can enter the region Σ_*, but if we then add at least 2 to the a-component, it is again outside of Σ_*. Therefore, whenever $mx \geq 2$ the point (a_m, b_m) is outside Σ_* because of (11). The proof of $\Sigma_* \subset \Sigma_m$ for $m = 1$ is easy. For $m \geq 2$ and for values of φ for which $mx \leq 2$, it is possible to show by elementary computations that $y(\varphi) > \varphi$, but only for $1/2 \leq \theta \leq \theta_0$ with $\theta_0 \approx 3/4$. Hence, also $\Sigma_* \subset \Sigma_m$ for $m \geq 2$.

3 Local stability analysis

We study here the condition $\Sigma_* \subset \Sigma_m$ close to the cusp point $(a, b) = (1, -1)$, which corresponds to $z = 0$ and $\zeta = 1$. Assume that the rational function satisfies

$$R(z) = e^z (1 - Cz^{p+1} - Dz^{p+2} - \ldots), \tag{12}$$

where $C \neq 0$ is the error constant of the approximation. The following result is a special case of [Gug97]. We write it here in terms of the error constants C and D and we give a slightly different proof.

Theorem 1 (necessary condition for $\tau(0)$-stability) *We assume that $p > 1$. If, close to $(a,b) = (1,-1)$, we have $\Sigma_* \subset \Sigma_m$ for all $m \geq 1$, then it holds*

$$(-1)^k C > 0 \qquad \textit{if } p = 2k \textit{ is even,}$$
$$(-1)^k C > 0 \;\; \textit{and} \;\; (-1)^{k+1} D < (-1)^k C/3 \qquad \textit{if } p = 2k-1 \textit{ is odd.}$$

Proof Close to $z = 0$ we can parametrize ∂S as a function of y:

$$x(y) = \begin{cases} (-1)^{k+1} D y^{2k+2} + \dots & \text{if } p = 2k \\ (-1)^k C y^{2k} + \dots & \text{if } p = 2k-1. \end{cases}$$

Taking the logarithm in (12) we obtain for $\varphi(y)$, defined by (8), that

$$\varphi(y) - y = \begin{cases} (-1)^{k+1} C y^{2k+1} + \dots & \text{if } p = 2k \\ (-1)^{k+1} D y^{2k+1} + \dots & \text{if } p = 2k-1. \end{cases} \tag{13}$$

Inserting $x(y)$ and $\varphi(y)$ into (10) and looking at the first terms in the Taylor expansion, the condition $(a_m, b_m) \notin \Sigma_*$ (see (4)) becomes equivalent to

$$m(\varphi(y) - y) < (1 - a_m(y))x(y)/y \qquad \text{for } \; y \to 0$$

with $a_m(y) = 1 - m^2y^2/3 + \dots$. The above asymptotic relations for $x(y)$ and $\varphi(y)$ then yield the statement. □

For the Padé-approximations to the exponential function $R_{s-j,s}(z)$ of (9) we have $p = 2s - j$ and

$$C = (-1)^s \, \frac{s!\,(s-j)!}{(2s-j)!\,(2s-j+1)!}, \qquad D = C\,\frac{j(2s-j+1)}{(2s-j)(2s-j+2)}.$$

Hence, the necessary condition of Theorem 1 is satisfied, if $j = 0 \bmod 4$ and $j = 1 \bmod 4$, but it is not satisfied for $j = 2 \bmod 4$ and $j = 3 \bmod 4$. The Gauss and Radau methods satisfy this condition, while the Lobatto IIIC methods do not [Gug97].

4 Symmetric stability functions

For A-stable, symmetric stability functions (i.e., $R(-z)R(z) = 1$), such as the diagonal Padé-approximations, the stability region is exactly the negative half-plane and the border ∂S can conveniently be parametrized by y. Since $x = 0$ for values on ∂S, Eq. (11) tells us that the condition $\varphi(y) < y$ for $y > 0$ is sufficient for having $\Sigma_* \subset \Sigma_m$ for all $m \geq 1$. For values of y such that $0 < \varphi(y) < \pi$ it is also necessary to have $\varphi(y) < y$. For the study of this property the use of the order star

$$A = \{ z \in \mathbb{C} \; ; \; |R(z)| > |e^z| \}, \tag{14}$$

as introduced in [WHN78] (see also [HW96, Sect. IV.4]), turns out to be very useful.

Lemma 2 *Let $R(z)$ be symmetric and assume that the order star A has the whole imaginary axis as boundary with A lying to the left. Then, the function $\varphi(y)$ defined by $R(iy) = e^{i\varphi(y)}$ and $\varphi(0) = 0$ satisfies*

$$\varphi(y) < y \qquad \text{for} \quad y > 0. \tag{15}$$

Proof We consider the function $S(x+iy) := R(x+iy)e^{-(x+iy)} = r(x,y)e^{-x}e^{i(\varphi(x,y)-y)}$ with $r(x,y)$ and $\varphi(x,y)$ as given in the proof of Lemma 1. Since $|S(x+iy)| > 1$ to the left and $|S(x+iy)| < 1$ to the right of the imaginary axis, it follows from the Cauchy-Riemann differential equations that $\partial\varphi/\partial y - 1 \le 0$ at $x = 0$ (strict inequality with the exception of a finite number of points). This proves the statement. □

The above proof shows that we not only have $\varphi(y) < y$, but even that $\varphi(y) - y$ is monotonically decreasing for $y > 0$. Fig. 2 (right picture) shows the order star of the 3-stage Padé-approximation. There we included the argument $\varphi(y) - y$ (with arrows) along the imaginary axis, which is nicely seen to be monotonically decreasing.

Theorem 2 *Let us consider an A-stable, symmetric stability function $R(z) = P(z)/Q(z)$ such that $P(z)$ and $Q(z)$ are polynomials of degree $\le s$. If $R(z) = e^z - Cz^{p+1} + O(z^{p+2})$ with $p \ge 2s-2$, and if the error constant satisfies $(-1)^{p/2}C > 0$, then the corresponding method is $\tau(0)$-stable. In particular, all Gauss methods are $\tau(0)$-stable.*

Proof Due to the symmetry of the function $R(z)$, the whole imaginary axis lies on the boundary of the order star A. The condition $(-1)^{p/2}C > 0$, which is necessary by Theorem 1, means that close to the origin the order star touches the imaginary axis from the left side. There, exactly $p/2+1$ sectors of A lie in the negative half-plane (see Fig. 2). Because of A-stability they have to join infinity. The surrounded sectors of $\mathbb{C} \setminus A$ give thus rise to at least $p/2 \ge s-1$ zeros of $R(z)$. If at some part of the imaginary axis the set $\mathbb{C} \setminus A$ could touch it from the left, this would imply the existence of two additional zeros of $R(z)$ (one in the upper half-plane and one in the lower half-plane). This contradicts the fact that $P(z)$ is a polynomial of degree s.

An application of Lemma 2 shows that $\varphi(y) < y$ for all $y > 0$ and the representation (11) of the root locus curve implies that $(a_m(\varphi), b_m(\varphi))$ lies outside Σ_* for all φ, because $x(\varphi) = 0$ for symmetric methods. □

Let us denote the branches of the root locus curve (10) by

$$\gamma_\ell = \{ (a_m(\varphi), b_m(\varphi)) \ ; \ m\varphi \in ((\ell-1)\pi, \ell\pi) \}$$

with ℓ a positive integer. The curve γ_1 starts at the cusp point $(1,-1)$ and approximates $\partial\Sigma_*$. For ℓ even the curve γ_ℓ lies in the sector $|a|<b$ above the real axis, and for ℓ odd it lies in $|a|<-b$ below the real axis. With the exception of γ_1, the curve γ_ℓ starts at ∞ and ends in ∞.

Theorem 3 *Under the assumptions of Theorem 2, Σ_m is the set which is bounded to the right by the straight line $a+b=0$ ($a\le 1$) and by the curve γ_1.*

Proof We first prove that the curves γ_ℓ are all well separated. For this we look at the ray from the origin parametrized by $a=\mu\cot\alpha$ and $b=-\mu/\sin\alpha$, where $\mu\ge 0$ and α is fixed. For $\alpha\in(0,\pi)$ the ray lies in the sector $|a|<-b$ and it intersects the curve γ_ℓ (for ℓ odd) at $\mu=my(\varphi)$ with $m\varphi=\alpha+(\ell-1)\pi$ (see Eq. (10)). Since $y(\varphi)$ is monotonically increasing (Lemma 1), different values of ℓ cannot give the same μ. Moreover, we see that the curves $\gamma_1,\gamma_3,\gamma_5,\dots$ are ordered in a natural way. The same is true for the curves $\gamma_2,\gamma_4,\gamma_6,\dots$ in the upper sector.

In order to complete the proof of the theorem we shall show that if we cross a curve γ_ℓ outwards, the number of roots ζ of (6) satisfying $|\zeta|>1$ increases by 2. By continuity arguments it is sufficient to prove this on the vertical b-axis. Differentiating the characteristic equation (6) gives for $a\equiv 0$

$$\Delta\zeta=R'(z)\Delta z,\qquad m\Delta z=\zeta^{-m}\Delta b-m\,b\,\zeta^{-m-1}\Delta\zeta.$$

Using $\zeta^{-m}=mz/b$ and $\zeta=R(z)$, this yields

$$\frac{\Delta b}{b}=\Big(\frac{1}{z}+m\frac{R'(z)}{R(z)}\Big)\Delta z.$$

If b crosses the curve γ_ℓ, we have to consider the point $z=iy$ on the imaginary axis for which $|\zeta|=|R(iy)|=1$. Differentiating $R(iy)=e^{i\varphi(y)}$ gives $R'(iy)i=i\varphi'(y)R(iy)$, and we see by Lemma 1 that $R'(iy)/R(iy)=\varphi'(y)$ is a real positive number. This implies that (for $z=iy$) $\Re\Delta z>0$ whenever $\Delta b/b>0$. Consequently, the root z (and also its complex conjugate) crosses the imaginary axis from left to right. Since the method is A-stable, this implies that $|\zeta|$ leaves the unit circle. □

For methods with $|R(\infty)|<1$ the statement of Theorem 3 is not true. There, the stability region Σ_m usually consists of two components, and only one of it approximates Σ_* (see for example [Gug98]).

5 Radau methods

Having seen that all diagonal Padé-approximations (Gauss methods) are $\tau(0)$-stable, we consider next the first subdiagonal Padé-approximations (Radau methods, see [HW96, Sect. IV.5]). Since we have $y \to 0$ for $\varphi \to s\pi$, the condition (15) of Lemma 2 is no longer satisfied on the whole boundary of the stability region and the stability analysis becomes more complicated. Our proof is based on the following sufficiency condition.

Lemma 3 *Suppose that the stability domain $S = \{z\ ;\ |R(z)| < 1\}$ is connected and that the method is A-stable. Further let $z(t) = x(t) + iy(t)$ be a parametrization of ∂S as in Lemma 1, and let the smooth function $\varphi(t)$ be defined by (8) with $\varphi(0) = 0$. If for all t with $y(t) > 0$ at least one of the following three conditions*

$$\varphi(t) \le y(t), \qquad m\,x(t) \ge 2, \qquad \sin(m\varphi(t)) \le 0 \tag{16}$$

is satisfied, then it holds $\Sigma_* \subset \Sigma_m$.

Proof The proof is based on the representations (10) and (11) of the root locus curve. If $\sin(m\varphi) \le 0$, the point $(a_m(\varphi), b_m(\varphi))$ lies in the upper sector $|a| < b$ and it is therefore outside Σ_*. If $mx \ge 2$, the argumentation of Example 3 shows that $(a_m(\varphi), b_m(\varphi))$ cannot be in Σ_*. Finally, the representation (11) implies that for $y \ge \varphi$ and $x \ge 0$ (A-stability) the pair $(a_m(\varphi), b_m(\varphi))$ is outside Σ_*. $\square$

Instead of verifying the condition $\varphi(t) \le y(t)$ of (16), we find it more convenient to check

$$\varphi'(t) \le y'(t) \qquad \text{on} \quad (0, T). \tag{17}$$

Because of $\varphi(0) = y(0) = 0$, this of course implies $\varphi(t) \le y(t)$ for $t \in (0, T)$. By differentiation of (8) we get $R'(z(t))z'(t) = iR(z(t))\varphi'(t)$ and the condition (17) becomes equivalent to

$$\Re\Big(\frac{R(z)}{R'(z)}\Big) \ge 1 \tag{18}$$

for $z = z(t)$. Since $R(z)/R'(z) = 1 + C(p+1)z^p + O(z^{p+1})$, the set of all z satisfying (18) is an "order star" with p equally spaced sectors at the origin.

This order star (grey-shaded regions) is plotted in Fig. 3 for the subdiagonal Padé-approximations $R_{12}(z)$ and $R_{23}(z)$. It follows from (13) that, close to the origin, the boundary ∂S of the stability region is inside this order star, and therefore the condition (17) is fulfilled. In

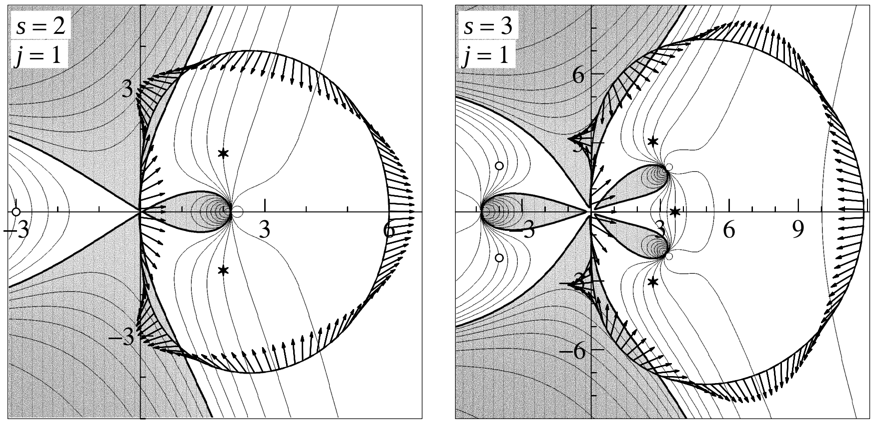

Fig. 3. "Order stars" $\Re(R(z)/R'(z)) > 1$ for subdiagonal Padé-approximations together with the stability domains and the arguments of $R(z)$ (as arrows)

the case of $R_{12}(z)$ one can find explicit formulas for the boundary of the order star as well as for the boundary of ∂S. By standard algebraic manipulations one can see that the only intersection points of these curves are the origin and $1 \pm i\sqrt{11}$. It turns out that $R(1 \pm i\sqrt{11}) = -1$, so that the argument of R is $\varphi = \pm\pi$ (horizontal arrow in Fig. 3, left picture). Hence, for $0 < \varphi < \pi$ the condition $\varphi \le y$ is satisfied. For $\pi < \varphi < 2\pi$ it holds $\sin(m\varphi) \le 0$ for $m = 1$ and $mx \ge 2$ for $m \ge 2$. This completes the proof of the $\tau(0)$-stability of the 2-stage Radau method.

The right picture of Fig. 3 shows a similar behaviour for the 3-stage method. In this case the unique intersection points other than the origin are $3 \pm i\sqrt{51}$. They are attained with $\varphi = \pm 2\pi$. Again the method is $\tau(0)$-stable. For larger values of s we computed numerically the first intersection point of ∂S with the boundary of the order star (18). They are given in Table 1. From there we see that the Radau methods are $\tau(0)$-stable as far as we have computed these values.

s	x	y	φ
2	1	$\sqrt{11}$	π
3	3	$\sqrt{51}$	2π
4	5.9828	11.5756	9.4217
5	9.7899	16.6729	12.5409
6	14.2720	22.4309	15.6436
7	19.3730	28.8384	18.7392

Table 1. Points on ∂S satisfying $\Re\big(R(z)/R'(z)\big) = 1$

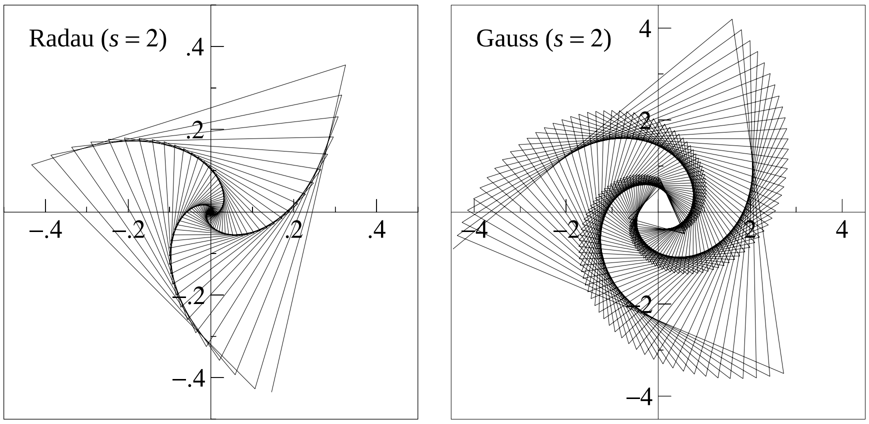

Fig. 4. Numerical solutions of the 2-stage Radau IIA and the 2-stage Gauss methods for $m = 2$ applied to (1) with $a = -1 + 2\pi i$ and $b = -2.25$

6 Comments on the complex case

We remark that the results of this paper do not carry over to the situation where the coefficients a and b in the test equation (1) are complex-valued. This has already been observed in [Gug98], where it is shown that the implicit midpoint rule is not τ-stable (τ-stability is an obvious extension of $\tau(0)$-stability to the complex case).

In the numerical experiments of Fig. 4 we apply the 2-stage Radau method and the 2-stage Gauss method to the equation (1) with $a = -1 + 2\pi i$ and $b = -2.25$. With these values the true solution is asymptotically stable. We use $g(t) \equiv 1$ and the stepsize $h = 1/2$ (i.e., $m = 2$). In the pictures of Fig. 4 we omitted the first five steps in order to emphasize the asymptotic phase. The Gauss method clearly shows an instability which implies that it cannot be τ-stable. On the contrary, the 2-stage Radau method as well as the implicit Euler method seem to preserve the correct asymptotic behaviour.

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