Geometric proofs of numerical stability for delay equations

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In this paper, asymptotic stability properties of implicit Runge Kutta methods for delay differential equations are considered with respect to the test equation $y'(t) = a\,y(t) + b\,y(t-1)$ with $a, b \in \mathbb{C}$.

In particular, we prove that symmetric methods and all methods of even order cannot be unconditionally stable with respect to the considered test equation, while many of them are stable on problems where $a \in \mathbb{R}$ and $b \in \mathbb{C}$. Furthermore, we prove that Radau IIA methods are stable on the subclass of equations where $a = \alpha + i\gamma$ with $\alpha, \gamma \in \mathbb{R}$, γ sufficiently small, and $b \in \mathbb{C}$.

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1. Introduction

In this work we study the asymptotic stability behaviour of numerical methods when applied to the test equation

$$y'(t) \;=\; a\,y(t) + b\,y(t-1), \qquad t > 0, \tag{1.1}$$

with initial condition $y(t) = g(t)$, $-1 \le t \le 0$, where the constants a, b and the function $g(t)$ are all complex. We are mainly interested in methods that produce unconditionally stable numerical solutions for all a and b, for which the exact solution tends to zero.

Most of the publications on this subject restrict their analysis to the cone $|b| < -\Re a$, where A-stability has been proven to be a necessary and sufficient condition for preserving the asymptotic stability of the solution. Concerning the purely retarded equation, that is (1.1) with $a = 0$, this kind of topic has first been investigated by Cryer (1974) and van der Houwen & Sommeijer (1984), who have considered linear multistep methods. For the case of real a and b, a stability analysis has recently been given by Guglielmi (1997) and Guglielmi & Hairer (1999). There, some numerical experiments on the complex coefficient test equation indicated that

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Gauss methods are not stable, while Radau IIA methods could be stable. Recently, Maset (1999) proved the stability of the Backward Euler method for (1.1) with complex coefficients.

The paper is organized as follows: In Sect. 2 we give the stability definitions and introduce the root locus curves, which are essential for the technique used in this work. In Sect. 3 we give an instability result for the class of symmetric methods. In the following part of this paper (Sect. 4), we give stability results for both symmetric and non-symmetric methods when restricted to equations with $a \in \mathbb{R}$ but still $b \in \mathbb{C}$. Finally, in Sect. 5, we show by a local perturbation analysis that no method with even order can be stable for the general complex coefficient test equation (1.1). For methods with odd order, we discuss the stability for values of a close to the real axis.

We consider Runge-Kutta methods applied with constant stepsize $h = 1/m$, and we use the internal stage value g_i of a previous step as an approximation to $y(t_n + c_i h - 1)$. For the test equation (1.1) this yields the recursion

$$\begin{aligned} g_i^{(n)} &= y_n + h \sum_{j=1}^{s} a_{ij} \Big(a\, g_j^{(n)} + b\, g_j^{(n-m)} \Big) \\ y_{n+1} &= y_n + h \sum_{i=1}^{s} b_i \Big(a\, g_i^{(n)} + b\, g_i^{(n-m)} \Big), \end{aligned} \tag{1.2}$$

where s is the number of stages and b_i, c_i, a_{ij} are the usual coefficients of the Runge-Kutta method (see Zennaro (1986)).

2. Stability regions

It is difficult to represent geometrically the set of all $(a, b) \in \mathbb{C} \times \mathbb{C}$ such that (1.1) is stable. Following Guglielmi (1998) we therefore fix $a \in \mathbb{C}$, and we consider the set of all $b \in \mathbb{C}$ such that stability occurs.

2.1 Analytical stability region

For a fixed $a \in \mathbb{C}$ we denote by $Q_\star[a]$ the set of complex numbers b such that the solution of (1.1) satisfies $\lim_{t\to\infty} y(t) = 0$ for all initial functions $g(t)$. By looking at solutions of the form $y(t) = e^{\lambda t}$ we are led to the characteristic equation

$$\mathcal{K}(\lambda; a, b) := \lambda - a - b\, e^{-\lambda} = 0. \tag{2.1}$$

It is known (e.g., El'sgol'ts & Norkin (1973)) that this set is given by

$$Q_\star[a] = \big\{ b \in \mathbb{C}\ ;\ \text{all roots of (2.1) satisfy } \Re\lambda < 0 \big\}. \tag{2.2}$$

Fig. 1 shows $Q_\star[a]$ (white region) for $a = 0.2$ (left picture) and for $a = -3$ (right picture). For real a, the set $Q_\star[a]$ has the form of a lying drop. It satisfies $Q_\star[a_1] \subset Q_\star[a_2]$ whenever $a_1 \geq a_2$, and the intersections with the horizontal axis are at $b = -a$ and $b = -\phi_a / \sin\phi_a$ where $\phi_a \in (0, \pi)$ is defined by $a = \phi_a \cot\phi_a$. This is a consequence of the shape of the stability region for the real case. It is interesting

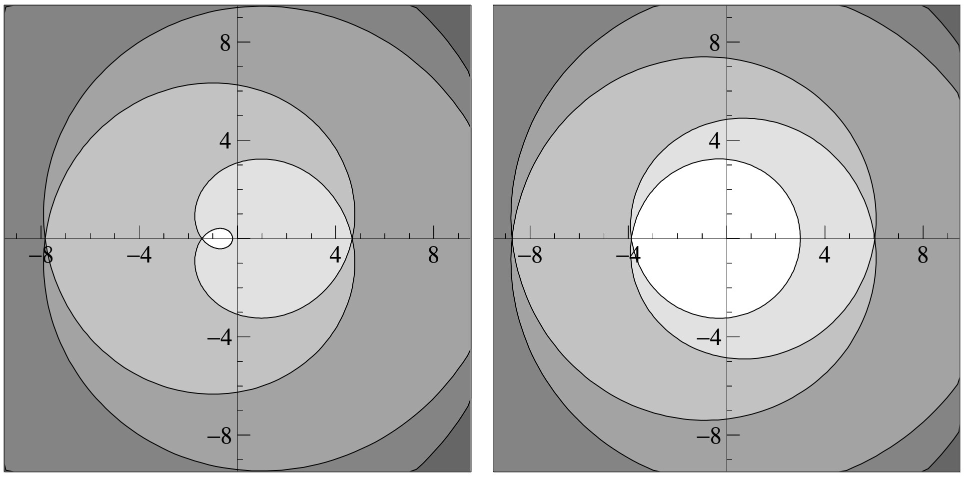

FIG. 1. The stability set $Q_\star[a]$ (white region) for $a = 0.2$ (left picture) and $a = -3$ (right picture). The intensity of grey indicates the increasing number of roots of (2.1) lying in the right half plane.

to note that the size of the stability region $Q_\star[a]$ only depends on the real part of a. Indeed, we have for $\gamma \in \mathbb{R}$ that

$$Q_\star[a + i\gamma] \;=\; e^{i\gamma} Q_\star[a],$$

so that $Q_\star[a + i\gamma]$ is obtained from $Q_\star[a]$ by a simple rotation around the origin. This follows from the fact that $\mathcal{K}(\lambda; a + i\gamma, b) = \mathcal{K}(\lambda - i\gamma; a, be^{-i\gamma})$.

In Fig. 1 we have also drawn the curve, which corresponds to the set of b where at least one root of (2.1) lies on the imaginary axis. It is given by

$$b_\star(\phi) \;=\; e^{i\phi}\,(i\phi - a), \qquad \phi \in \mathbb{R}, \tag{2.3}$$

and it is commonly called *root locus curve* (e.g., Baker & Paul (1994)). Since the roots of (2.1) depend continuously on b, the curve (2.3) separates regions with different numbers of roots lying in the right half plane.

2.2 Numerical stability region

For fixed $a \in \mathbb{C}$ and $m \in \mathbb{Z}^+$, we denote by $Q_m[a]$ the set of complex numbers b such that the numerical solution $\{y_n\}_{n \geq 0}$ of (1.2), with constant stepsize $h = 1/m$, satisfies $\lim_{n\to\infty} y_n = 0$ for all initial functions $g(t)$. If we look at numerical approximations of the form $y_n = \zeta^n y_0$, $g_i^{(n)} = \zeta^n g_i$, we are led to the characteristic equation

$$\zeta = R(z), \qquad mz = a + b\,\zeta^{-m}, \tag{2.4}$$

where $R(z)$ is the stability function of the method (see for example Sect. IV.2 of Hairer & Wanner (1996)). Similar to the analysis for the analytical solution, one can show that the set $Q_m[a]$ is given by

$$Q_m[a] = \big\{ b \in \mathbb{C} \;;\; \text{all roots of (2.4) satisfy } |\zeta| < 1 \big\}. \tag{2.5}$$

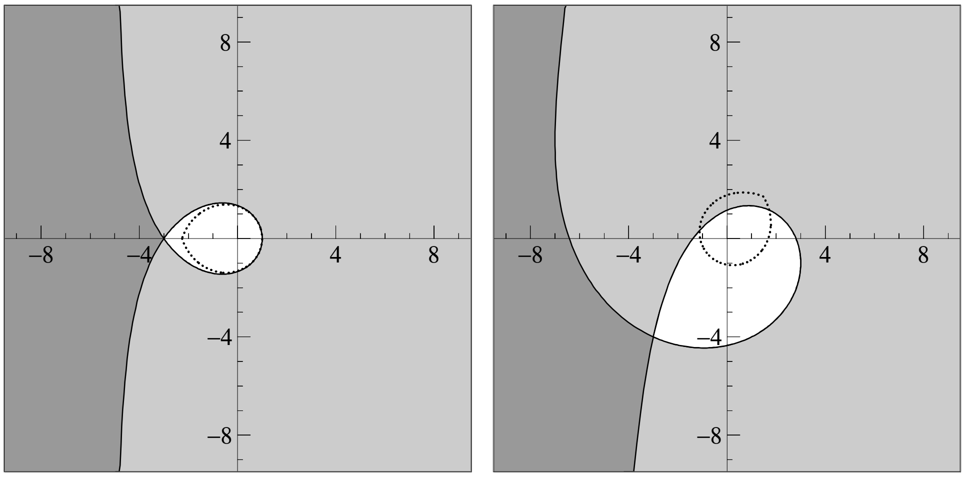

FIG. 2. The numerical stability set $Q_m[a]$ with $m = 1$ (white region) for the trapezoidal rule, for $a = -1$ (left picture) and $a = -1 + 4i$ (right picture). The dotted curve indicates the exact stability domain $Q_\star[a]$.

DEFINITION 2.1 A numerical one-step method with stability function $R(z)$ is called *τ-stable* if $Q_m[a] \supseteq Q_\star[a]$, for all $a \in \mathbb{C}$ and for $m = 1, 2, 3, \ldots$.

In order to plot the region $Q_m[a]$ we have to parametrize the set of z satisfying $|R(z)| = 1$. For this purpose we recall a result from Guglielmi & Hairer (1999), which is fundamental for the stability analysis of later sections.

LEMMA 2.2 Suppose that the stability domain $S = \{z\,;\,|R(z)| \le 1\}$ is simply connected, let $z(t) = x(t) + iy(t)$ for $t \in (-c, c)$ be a smooth parametrization of ∂S such that $z(-t) = \bar{z}(t)$ and $z(0) = (0, 0)$, and define the smooth function $\varphi(t)$ by $\varphi(0) = 0$ and

$$R\big(z(t)\big) = e^{i\varphi(t)}. \tag{2.6}$$

Then, the function $\varphi(t)$ is strictly monotonically increasing and it satisfies $\varphi(-t) = -\varphi(t)$ and $\lim_{t\to c} \varphi(t) = s\pi$, where s is the number of poles of $R(z)$.

The set of values b, for which at least one of the roots ζ of (2.4) satisfies $|\zeta| = 1$, is given by

$$b_m(t) \;=\; e^{i\,m\,\varphi(t)}(i\,m\,y(t) + m\,x(t) - a), \tag{2.7}$$

where $x(t)$, $y(t)$, and $\varphi(t)$ are as in Lemma 2.2. We call this the (numerical) root locus curve.

EXAMPLE 2.3(TRAPEZOIDAL RULE) Fig. 2 shows the curve $b_m(t)$ for $m = 1$ and for the trapezoidal rule

$$R(z) = \frac{1 + z/2}{1 - z/2}\ .$$

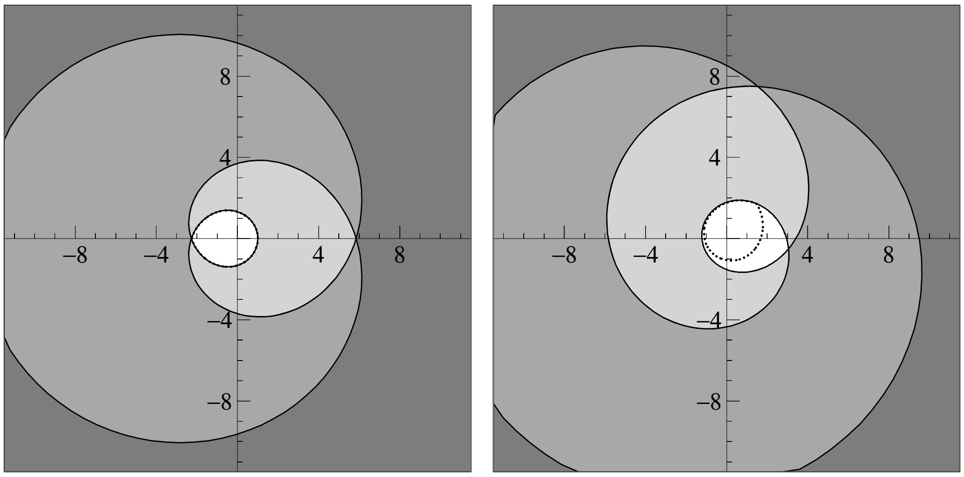

FIG. 3. The numerical stability set $Q_m[a]$ with $m = 1$ (white region) for the subdiagonal Padé approximation $R_{2,3}(z)$, for $a = -1$ (left picture) and $a = -1 + 4i$ (right picture). The dotted curve indicates the exact stability domain $Q_\star[a]$.

This curve divides the complex plane into three domains, containing 0, 1, and 2 roots ζ of (2.4) outside the unit disc. The reason for the *finite* number of subdomains (which is in contrast to the situation of Fig. 1) is the following: if the stability function is given by $R(z) = P(z)/Q(z)$ with polynomials $P(z)$ and $Q(z)$, then (2.4) is equivalent to

$$\zeta = R(z), \qquad P(z)^m(mz - a) = bQ(z)^m, \tag{2.8}$$

which is a polynomial in z and therefore has only finitely many solutions. In the case of the trapezoidal rule and $m = 1$, (2.8) is a polynomial of degree 2. The comparison with the analytical stability region shows that $Q_m[a] \supseteq Q_\star[a]$ for the real value $a = -1$, but $Q_m[a] \not\supseteq Q_\star[a]$ for $a = -1 + 4i$. Hence, the trapezoidal rule is not τ-stable (Guglielmi (1998)).

EXAMPLE 2.4(PADÉ APPROXIMATION) We repeat the experiment of the previous example with the subdiagonal Padé approximation

$$R_{2,3}(z) = \frac{1 + 2z/5 + z^2/20}{1 - 3z/5 + 3z^2/20 - z^3/60}\ . \tag{2.9}$$

Since $R_{2,3}(\infty) = 0$, the root locus curve (2.7) is bounded (Fig. 3). This time we observe that for both values of a the analytical stability region is completly contained in the numerical stability region. Due to the high order (order 5 for (2.9) in contrast to order 2 for the trapezoidal rule) the numerical stability region approximates much better the set $Q_\star[a]$.

3. Instability results

We start our analysis by considering A-stable symmetric Runge-Kutta methods. Although most of them are stable for (1.1) when a and b are real, we shall see in this section that none of them can be τ-stable.

For A-stable symmetric stability functions ($R(-z)R(z) = 1$) the stability region S is exactly the negative half-plane and the border ∂S can conveniently be parametrized by y. Since $x = 0$ for values on ∂S, we get

$$b_m(y) = e^{i\,m\,\varphi(y)}(i\,m\,y - a), \qquad y \in \mathbb{R}, \tag{3.1}$$

where the function $\varphi(y)$ is defined by $R(iy) = e^{i\varphi(y)}$ (see Lemma 2.2). Properties of $\varphi(y)$ and of its inverse $y(\varphi)$ are discussed in Guglielmi & Hairer (1999). Note also that the only difference in formulæ (3.1) and (2.3) (when we set $\phi = m\,y$) is in the function $\varphi(y)$, which in fact is an approximation to y.

THEOREM 3.1 Symmetric Runge-Kutta methods cannot be τ-stable.

Proof. For $b = 0$, the equation (1.1) becomes the well-known Dahlquist test equation for A-stability. Hence, A-stability is a necessary condition for τ-stability, and the numerical root locus curve can be assumed to be given by (3.1). The idea of the proof is to look at the point b_0 of this curve, which has minimal distance to the origin. Since $|b_m(y)|^2 = \Re a^2 + (my - \Im a)^2$, it is obtained for $y = \Im a/m$.

We now let $a \in \mathbb{C}$ be such that $\Re a < 0$, and $y = \Im a/m$ satisfies $m\varphi(y) \neq my + 2k\pi$. The assumption $\Re a < 0$ implies that $b_0 \in \partial Q_m[a]$, because in this case $0 \in Q_m[a]$, and the segment joining 0 and b_0 does not intersect the root locus curve. The assumption on $\Im a$ implies that b_0 does not lie on the analytical root locus curve. Since the minimal distance of the analytical root locus curve to the origin is also $|\Re a|$, this implies that b_0 is in the interiour of $Q_\star[a]$. Consequently, the necessary condition for τ-stability, $Q_m[a] \supseteq Q_\star[a]$, cannot hold. □

As a special case of this theorem we obtain that none of the Gauss methods can be τ-stable. This extends a result of Guglielmi (1998), which states that the trapezoidal rule (and the implicit mid-point rule) is not τ-stable (see Fig. 2).

The above proof also shows that the condition $Q_m[a] \supseteq Q_\star[a]$ is violated by symmetric Runge-Kutta methods for *nearly all* $a \in \mathbb{C}$. An exceptional case will be treated in the following section.

4. Stability results

In this section we consider the case where $a \in \mathbb{R}$, but $b \in \mathbb{C}$. Here, the analytical and numerical stability regions are symmetric with respect to the real axis (see Figs. 1, 2, and 3). Similar to the case where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ (Guglielmi & Hairer (1999)) we shall derive positive stability results for symmetric as well as for non-symmetric methods.

We mention that the test equation considered in this section is a generalization of the purely retarded test equation (that is when $a = 0$). The stability of numerical methods for the purely retarded test equation has been previously studied by Cryer

(1974), Barwell (1975), Al Mutib (1984) and van der Houwen & Sommeijer (1984). In particular, Barwell called the numerical method Q-stable if the numerical stability region contains the analytical stability region. In our notation this means that $Q_m[0] \supseteq Q_\star[0]$ for all $m \in \mathbb{Z}^+$.

4.1 Symmetric methods

Our analysis for symmetric methods is based on Lemma 2 of Guglielmi & Hairer (1999), which we recall here.

LEMMA 4.1 Let $R(z)$ be symmetric and assume that the order star

$$A = \{z \in \mathbb{C}\ ;\ |R(z)| > |e^z|\}$$

has the whole imaginary axis as boundary, with A lying to the left. Then, the function $\varphi(y)$ defined by $R(iy) = e^{i\varphi(y)}$ and $\varphi(0) = 0$ satisfies

$$\varphi(y) < y \qquad \text{for} \quad y > 0. \tag{4.1}$$

In that paper it is also proved that Gauss methods fulfil the assumptions of Lemma 4.1. In particular, for the mid-point rule (one-stage Gauss method) we have $\varphi(y) = 2\arctan(y/2)$.

THEOREM 4.2 If a is real, and if the symmetric stability function $R(z)$ satisfies the assumptions of Lemma 4.1, then

$$Q_m[a] \supset Q_\star[a] \qquad \text{for all} \quad m \geq 1.$$

Proof. Due to the symmetry of the stability regions with respect to the real axis, it is sufficient to consider the upper half plane $\Im b > 0$. We use

$$b_m(y) = e^{i\,m\,(\varphi(y)-y)}\, b_\star(my), \tag{4.2}$$

which follows immediately from (3.1) and (2.3). We shall show that the whole root locus curve $b_m(y)$ (with the exception of the point $b = -a$) lies outside of $Q_\star[a]$. This is obvious for $my \geq \pi$, because $|b_m(y)|^2 = |b_\star(my)|^2 = (my)^2 + a^2$ is monotonically increasing, and the maximal distance of $Q_\star[a]$ from the origin is $|b_\star(\phi_a)|$ with $\phi_a \in (0, \pi)$ given by $a = \phi_a \cot \phi_a$ (Fig. 1). For $0 < my < \pi$, this follows from $0 < \varphi(y) < y$ (Lemma 4.1) and from (4.2), because the factor $e^{i\,m\,(\varphi(y)-y)}$ represents a clock-wise rotation with an angle of less than my degrees. Since the intersection of $Q_\star[a]$ with the circle $\{b\,;\,|b| = |b_\star(my)|\}$ is a connected arc containing the left intersection point with the real axis, this implies that $b_m(y)$ lies outside $Q_\star[a]$ also in this case. The statement of the theorem now follows from the fact that all $b \in Q_\star[a] \cap \mathbb{R}$ lie in $Q_m[a] \cap \mathbb{R}$ (Theorem 2 of Guglielmi & Hairer (1999)). □

REMARK 4.3 An interesting consequence of the previous result is that Gauss methods are stable when a is real and in particular, according to the definition given by Barwell (1975), they are Q-stable.

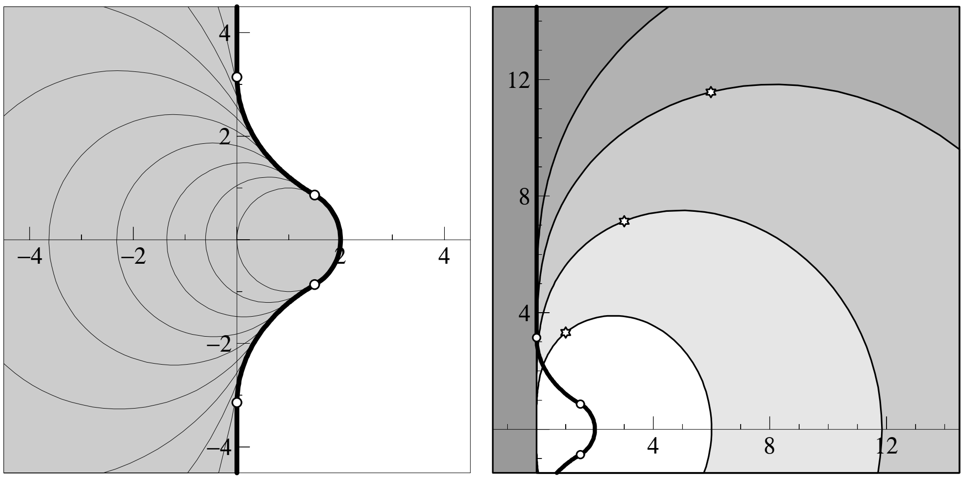

FIG. 4. The left picture shows the discs $D[a]$ of (4.3) and their union D. The picture to the right shows the stablity domains of the Radau IIA methods with $s = 2, 3, 4, 5$ stages together with the border of D. By Table 1 of Guglielmi & Hairer (1999), condition (i) of Theorem 4.4 is satisfied for all values between the origin and the point marked by an asterix.

4.2 Non-symmetric methods

We next consider arbitrary non-symmetric stability functions, and we give sufficient conditions for numerical stability. Similar to the proof of the previous theorem, we study conditions, under which the numerical root locus curve does not enter the analytical stability region. For a fixed $a \le 1$, the point $b_m(t)$ of (2.7) is certainly outside of $Q_\star[a]$, if its distance to the origin is larger than $b_\star(\phi_a)$ with ϕ_a as in the proof of Theorem 4.2. Because of $|b_m(t)| = |mz(t) - a|$, this is the case if $mz(t) \notin D[a]$, where

$$D[a] = \big\{ z \in \mathbb{C}\ ;\ |z-a| < |b_\star(\phi_a)| \big\}. \tag{4.3}$$

The set $D[a]$ for several values of a, and the union

$$D = \bigcup_{a \le 1} D[a] \tag{4.4}$$

are shown in the left picture of Fig. 4. Computing the envelope of this family of circles, one finds that the set D lies to the left of the curve given by the imaginary axis for $|y| \ge \pi$, by

$$x(\phi) = \frac{\phi \sin^2 \phi}{\phi - \sin\phi \cos\phi}, \qquad y(\phi) = \pm \frac{\phi \sqrt{\phi^2 - \sin^2 \phi}}{\phi - \sin\phi\cos\phi} \qquad \text{for} \quad \phi \in (0,\pi)$$

and by the circle with radius 1 and center 1 between the points $(3/2, \pm\sqrt{3}/2)$.

THEOREM 4.4 Let a be real. In the situation of Lemma 2.2, assume that $Q_m[a] \cap Q_\star[a] \ne \emptyset$ for all m, and that for every t at least one of the following two conditions is satisfied:

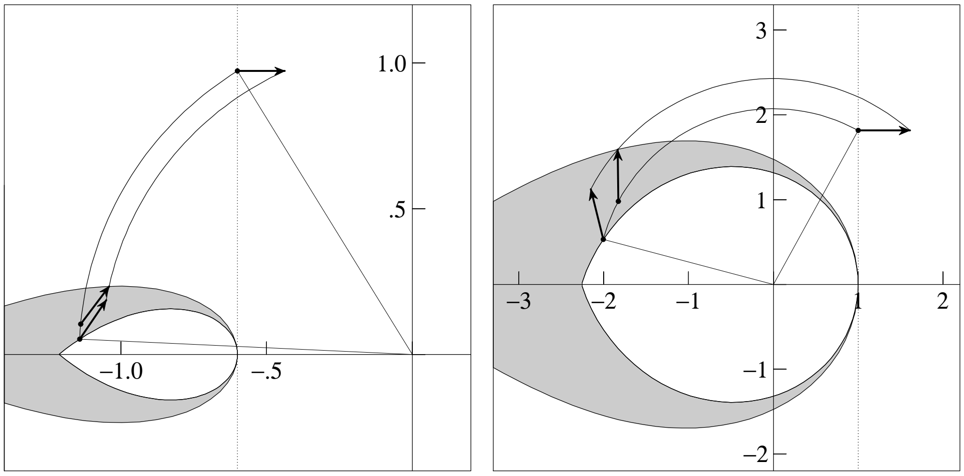

FIG. 5. Illustration of the proof of Theorem 4.4 with the θ-method $R(z) = (1+(1-\theta)z)/(1-\theta z)$ and $\theta = 2/3$ (left picture with $a = 0.6$ and right picture with $a = -1$). The grey region is the set $Q_m[a]$ minus $Q_\star[a]$.

(i) $x(t) \geq 0$ and $y(t) \geq \varphi(t)$,

(ii) $z(t) = x(t) + iy(t) \notin D$.

Then, it holds $Q_m[a] \supset Q_\star[a]$ for all $m \geq 1$.

Proof. Because of $Q_m[a] \cap Q_\star[a] \neq \emptyset$, it is sufficient to prove that the numerical root locus curve $b_m(t)$ never enters the analytical stability region $Q_\star[a]$. If $mz(t) \notin D$, which is the case for $|my(t)| \geq \pi$ or when condition (ii) is satisfied, this is true by definition of D. We therefore have to consider only the case where $x(t) \geq 0$ and $0 \leq m\varphi(t) \leq my(t) < \pi$.

The idea is to write the root locus curve (2.7) as

$$b_m(t) = e^{i\,m\,\varphi(t)} \Big(e^{-i\,m\,y(t)}\, b_\star(m\,y(t)) + m\,x(t) \Big), \tag{4.5}$$

and to interpret this equation geometrically as follows (Fig. 5): we start by considering the point $b_\star(my(t))$ on the boundary of the analytical stability domain $Q_\star[a]$, we multiply it by $e^{-imy(t)}$ (this is a rotation around the origin and yields the point $imy(t) - a$ on the vertical line $\Re b = -a$), then we add the positive number $mx(t)$, and finally we multiply by $e^{im\varphi(t)}$ (a rotation back). In order to prove that $b_m(t)$ lies outside $Q_\star[a]$, we consider the horizontal arrow from the point $imy(t) - a$ to $imy(t) - a + mx(t)$ (Fig. 5). A multiplication with $e^{imy(t)}$ would turn it back with one end of the arrow at $b_\star(my(t))$ and the other end outside of the analytic stability region. This is, because the tangent vector of the analytical root locus curve at $b_\star(\phi)$ with $\phi = my(t)$ is given by $b'_\star(\phi) = i\,e^{i\phi}\,(1 - a + i\phi)$, the exteriour normal vector by $u = e^{i\phi}\,(1 - a + i\phi)$, and the inner product of u with the vector $v = e^{i\phi}\,mx(t)$, which is given by $\Re(\overline{u}\,v)$, is positive because of $a < 1$. A multiplication with $e^{im\varphi(t)}$, which brings the peak of the arrow to $b_m(t)$ on the border of $Q_m[a]$, corresponds to a rotation with a smaller angle ($\varphi(t) \le y(t)$), so that $b_m(t)$ stays surely outside $Q_\star[a]$, also in this case. □

As an interesting application of the previous theorem we consider the s-stage Radau IIA methods for $s \ge 2$. Fig. 4 (right picture) shows the stability domains for $s = 2, 3, 4$, and 5, as well as the boundary of the set D of (4.4). Most of the border of the stability domains is seen to lie outside of D (condition (ii) of Theorem 4.4), so that the condition (i) has to be checked only for a small part of it. In fact, condition (i) of Theorem 4.4 follows immediately from Table 1 of Guglielmi & Hairer (1999) for $2 \le s \le 7$. The case $s = 1$ (Backward Euler method) has recently been investigated by Maset (1999), who proved its stability for $a \in \mathbb{C}$ and $b \in \mathbb{C}$.

5. A local perturbation analysis

We conclude our investigation with a local analysis close to $b = -a$. This point lies on the analytical as well as on the numerical root locus curve, and both curves are tangent there. Throughout this section we assume that $|a| < |b_\star(\phi_a)|$ with $\phi_a \in (0, \pi)$ given by $\Re a = \phi_a \cot \phi_a$, so that $b = -a$ lies on the boundary of the analytical stability region $Q_\star[a]$. Let the stability function $R(z)$ satisfy

$$R(z) = e^z \big(1 - Cz^{p+1} + \mathcal{O}(z^{p+2})\big), \tag{5.1}$$

where $C \ne 0$ is the error constant. We parametrize the boundary of the stability domain S in terms of the angle φ, so that $R\big(z(\varphi)\big) = e^{i\varphi}$. Because of (5.1), we get for $\varphi \to 0$ that

$$z(\varphi) = i\varphi + C(i\varphi)^{p+1} + \mathcal{O}(\varphi^{p+2}).$$

Inserted into the root locus curve (2.7) this yields

$$b_m(\varphi) - b_\star(m\varphi) = e^{im\varphi}\big(mz(\varphi) - im\varphi\big) = mC(i\varphi)^{p+1} + \mathcal{O}(\varphi^{p+2}). \tag{5.2}$$

We now distinguish the cases where p is even or odd.

THEOREM 5.1(EVEN ORDER) One-step methods of even order cannot be τ-stable.

Proof. The tangent vector of the analytical root locus curve at $b_\star(0) = -a$ is in the direction $b_\star'(0) = (1 - a)i$, which is non-vertical as soon as $\Im a \ne 0$. For even order $p = 2k$, it follows from (5.2) that

$$b_m(\varphi) - b_\star(m\varphi) = i(-1)^k mC\varphi^{2k+1} + \mathcal{O}(\varphi^{2k+2}),$$

so that close to $b = -a$ (i.e., for positive and negative small φ) the numerical root locus curve lies on both sides of $b_\star(m\varphi)$. This contradicts τ-stability. □

Since the order of symmetric methods is always even, this result gives a second proof of the statement of Theorem 3.1. In Section 3 we have shown more for symmetric methods, namely that for nearly all a we have $Q_m[a] \not\supset Q_\star[a]$ and not only for a satisfying $|a| < |b_\star(\phi_a)|$.

THEOREM 5.2(ODD ORDER) Consider a method of odd order $p = 2k-1$. For every $a \in \mathbb{C}$ there exists a neighbourhood U of $b = -a$ such that $U \cap Q_m[a] \supset U \cap Q_\star[a]$ if and only if

$$(-1)^k C \; > \; 0. \tag{5.3}$$

Moreover, assuming that $Q_m[a_0] \supset Q_\star[a_0]$ for a real $a_0 < 1$, and that the border of $Q_m[a]$ touches $Q_\star[a]$ only at the point $b = -a$, then the condition (5.3) implies that $Q_m[a] \supset Q_\star[a]$ for all a in a complex neighbourhood of a_0.

Proof. The first statement follows from

$$b_m(\varphi) - b_\star(m\varphi) = (-1)^k mC\varphi^{2k} + \mathcal{O}(\varphi^{2k+1}),$$

and from the fact that close to $b_\star(0) = -a$ the region $Q_\star[a]$ lies to the left of the curve $b_\star(\phi)$. The second statement follows from a continuity argument. □

We remark that (5.3) is also a necessary condition for $\tau(0)$-stability (see Guglielmi & Hairer (1999)). The stability function of the Radau IIA methods satisfies (5.3), so that the stability result of Theorem 5.2 applies.

6. Conclusions

In this work we have concluded the characterization for A-stable symmetric methods when applied to the test equation (1.1). In particular, this class of methods which has been proved to be stable for real coefficient delay differential equations (under some further technical assumptions, as shown by Guglielmi & Hairer (1999)) performs stably on equation (1.1) when a is real and b complex, but turns out to be unstable on general complex coefficient problems. Moreover, we have shown by a local perturbation analysis that no method of even order can be stable for the general complex coefficient test equation (1.1).

On the other hand we have shown that Radau IIA methods (with a number of stages $s \leq 7$) perform stably on problems with real a, and also in the case where the imaginary part of a is sufficiently small. A complete proof of τ-stability for the Radau IIA methods is still missing.

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