Erratum II to our paper "A simplification problem in manifold theory"

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Alexander Kupers reecently pointed out that our proof of Lemma 5.8 in [2] is not correct. As a consequence, the lemma is currently unsettled. Note that this lemma is not used elswhere in [2].

Lemma 5.8. *(Unsettled) Let M be a smooth closed connected manifolds of dimension 4. Then, the map $\mathcal{T}\colon \mathcal{B}(M) \to \mathrm{Wh}(M)$ is surjective.*

Recall that $\mathcal{B}(M)$ is the set of invertible cobordisms starting from M, up to diffoemorphism relative to M (see [2, (3.6)]). The point is that our proof of Lemma 5.8 uses [1, Theorem 11.1A], which only produces a topological semi-s-cobordism. In addition, it requires some "good" fundamental group hypothesis. Thus, our proof establishes the following topological version of Lemma 5.8, where $\mathcal{B}_{Top}(M)$ is the analogue of $\mathcal{B}(M)$ for topological manifolds and cobordisms.

Lemma 5.8. *(Topological version) Let M be a closed connected topological manifolds of dimension 4 with poly-(finite or cyclic) fundamental group. Then, the map $\mathcal{T}\colon \mathcal{B}_{Top}(M) \to \mathrm{Wh}(M)$ is surjective.*

Proof. As said above, our original proof provides, for each $\sigma \in \mathrm{Wh}(M)$, a topological h-cobordism (W, M, M') with $\tau(W, M) = \sigma$. The same can be done for M', providing, for each $\sigma' \in \mathrm{Wh}(M') \approx \mathrm{Wh}(M)$ a topological h-cobordism (W', M', M'') with $\tau(W', M') = \sigma'$. The topological s-cobordism theorem holds for closed 4-manifold with poly-(finite or cyclic) fundamental group [1, Theorem 7.1A]. As in [2, proof of Theorem 3.15], this permits us to prove that W is invertible. □

The above topological Lemma 5.8 would imply the smooth one (with a good fundamental group hypothesis) if the following problem has a positive answer.

Problem. Let (W, M, M') be a topological invertible cobordism, where M is a closed smooth 4-manifold. Does the smooth structure on M extend to W?

References

[1] M. H. Freedman and F. Quinn. *Topology of 4-manifolds*, volume 39 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1990.

[2] JC. Hausmann and B. Jahren. A simplification problem in manifold theory. *Enseign. Math.*, 64(1-2):207–248, 2018.