Appendix to: The space of clouds in an Euclidean space

by Jean-Claude HAUSMANN and Eugenio RODRIGUEZ

10 Cuts of $\underline{m}$

This appendix contains the details of the note (4.8) of our main paper [HR], of which we keep the notations. Recall that a partial order "$\hookrightarrow$" is defined on $\mathcal{P}(\underline{m})$ by saying that $A \hookrightarrow B$ if and only if there exits a non-decreasing map $\varphi : A \to B$ such that $\varphi(x) \geq x$. If $A \in \mathcal{P}(\underline{m})$, denote by $\bar{A}$ the complementary set $\underline{m} - A$.

Lemma 10.1 *Let $A, B \in \mathcal{P}(\underline{m})$. If $A \hookrightarrow B$, then $\bar{B} \hookrightarrow \bar{A}$.*

PROOF: Suppose first that $|A| = |B|$. The non-decreasing map $\varphi : A \to B$ satisfies $\varphi(x) \geq x$ if and only if

$$|A \cap \underline{n}| \geq |B \cap \underline{n}| \quad \text{for all } n \leq m. \tag{10.1}$$

If φ satisfies (10.1), so does the unique non-decreasing map $\psi : \bar{B} \to \bar{A}$. This proves Lemma 10.1 when $|A| = |B|$.

For the general case, let $\varphi : A \to B$ a map as above and set $B_1 := \varphi(B)$. One has $A \hookrightarrow B_1 \hookrightarrow B$. Since $\bar{B} \subset \overline{B_1}$, one has $\bar{B} \hookrightarrow \overline{B_1}$ and, since $|A| = |B_1|$, one has $\overline{B_1} \hookrightarrow \bar{A}$ by the first case. □

A subset S of $\mathcal{P}(\underline{m})$ is a *cut* (of $\underline{m}$) if, for all $I, J \subset \underline{m}$ the two following conditions are fullfiled:

(A) $I \in S \Leftrightarrow \bar{I} \notin S$.

(B) if $I \in S$ and $J \hookrightarrow I$, then $J \in S$.

For example, in [HR, § 4], if α is a chamber of $\mathbb{R}^m_{\nearrow}$, then $S(\alpha)$ is a cut of $\underline{m}$. The term "cut" is chosen in analogy with Dedeckind's cuts on the rational numbers.

Condition (A) implies that a cut contains 2^{m-1} elements. For example, the subsets of $\underline{m}$ which do not contain the element m form a cut of $\underline{m}$. Denote by Cut_m the set of all cuts of $\underline{m}$.

We shall associate to a cut a virtual genetic code which determines it. If $S \in \mathcal{P}(\underline{m})$, define $S_m := S \cap \mathcal{P}_m(\underline{m})$, where $\mathcal{P}_m(\underline{m}) := \{X \in \mathcal{P}(\underline{m}) \mid m \in X\}$. As in Lemma (4.2) of [HR], one proves, using Lemma 10.1, that

Lemma 10.2 *If $S \in \mathrm{Cut}_m$, then S is determined by S_m.*

By Condition (B) and Lemma 10.2, a cut S is determined by the elements $A_1, \dots, A_k$ of S_m which are maximal with respect to the partial order "$\hookrightarrow$". We denote $S = \langle A_1, \ldots, A_k \rangle$ and, as in [HR, § 4], we write $S = \langle A_1, \ldots, A_k \rangle$ and call $\{A_1, \ldots, A\}$ the *genetic code* of the cut S.

Recall that $\langle A_1, \ldots, A_k \rangle$ is a virtual genetic code if the two following conditions are satisfied:

(i) $A_i \not\hookrightarrow A_j$ for all $i \neq j$ and

(ii) $\bar{A}_i \not\hookrightarrow A_j$ for all i, j.

Lemma 10.3 *The genetic code of a cut S is a virtual genetic code.*

PROOF: Condition (i) is satisfied since the A_i's are maximal elements. For Condition (ii), suppose that $\bar{A}_i \hookrightarrow A_j$ for some i, j. Since A_j belongs to S so does $\bar{A}_i$ by Lemma 10.1. But this is not possible since $A_i \in S$. □

Proposition 10.4 *The correspondance which associates to a cut its genetic code gives a bijection between* Cut_m *and the set* $\mathcal{G}_m$ *of virtual genetic codes of type* m.

PROOF: We already know that a cut is determined by its genetic code. Let let $\{A_1, \ldots, A_k\} \in \mathcal{G}_m$. Define the set $S_m := \{I \in \underline{m} \mid \exists\, j \text{ such that } I \hookrightarrow A_j\}$. Define $S \in \mathcal{P}(\underline{m})$ by

$$I \in S \Leftrightarrow \begin{cases} m \in I & \text{and} \quad I \in S_m \\ & \text{or} \\ m \notin I & \text{and} \quad \bar{I} \notin S_m \end{cases} . \tag{10.4}$$

Let us first show that $I \in S \Leftrightarrow \bar{I} \notin S$. Suppose $I \in S$. If $m \in I$ then $m \notin \bar{I}$ and $\bar{\bar{I}} = I \in S_m$ which implies that $\bar{I} \notin S$. If $m \notin I$, then $m \in \bar{I} \notin S_m$ which also implies that $\bar{I} \notin S$. The reverse implication is obtained by exchanging the role of I and $\bar{I}$.

Let us now show that if $I \in S$ and $J \hookrightarrow I$ then $J \in S$. Consider the following cases:

1. $m \in I, J$. This is obvious by the definition of S_m.

2. $m \in I$, $m \notin J$. Suppose that $J \notin S$. Then $\bar{J} \in S_m$ and $\bar{J} \hookrightarrow A_{k_0}$ for some k_0. Therefore $\bar{A}_{k_0} \hookrightarrow J$ by Lemma 10.1. Since $I \in S_m$ there exists k_1 such that $I \hookrightarrow A_{k_1}$. But then $\bar{A}_{k_0} \hookrightarrow J \hookrightarrow I \hookrightarrow A_{k_1}$ which is impossible.

3. $m \notin I$, $m \in J$. Not possible, since $J \hookrightarrow I$.

4. $m \notin I, J$. If $J \notin S$ then $\bar{J} \in S_m$. As $I \in S$ and $m \notin I$, then $\bar{I} \notin S_m$. But, as $\bar{I} \hookrightarrow \bar{J}$ one would have $\bar{I} \in S_m$ which is a contradiction. □

References

[HR] Hausmann J-CL. and Rodriguez E. The space of clouds in an Euclidean space. *http//:www.unige.ch/math/folks/hausmann/haurod.ps*