

Torsion groups of subexponential growth cannot act on finite-dimensional CAT(0)-spaces without a fixed point

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Abstract

We show that finitely generated torsion groups of subexponential growth must have a global fixed point whenever they act by isometry on a finite dimensional complete CAT(0)-space. The same statement for simple groups of subexponential growth is also shown. The method of proof uses ultralimits, equivariant harmonic maps, subharmonic functions, horofunctions and random walks.

1 Introduction

The existence of infinite, finitely generated, torsion groups has a long and rich history which started with a problem posed by Burnside around the year 1900. The discovery that such groups actually could exist came in the 1960s by work of Golod and Shafarevich. Since then other infinite torsion groups with additional important features appeared, notably the papers of Novikov-Adian on infinite Burnside groups and the Grigorchuk groups having intermediate growth, see [CSD21, Os22] for recent discussions.

From the viewpoint of geometric group theory it is a natural question as to whether such groups can act on CAT(0)-spaces, indeed this topic already appeared in the foundational essay by Gromov [Gr87, sect. 4.5.C]. By a theorem of Schur [S11] (or by Selberg's lemma) it is known that any linear finitely generated torsion group must be finite, thus one can say that every action by a finitely generated torsion group on the classical symmetric spaces of nonpositive curvature must fix a point by the Cartan fixed point theorem. Another instance is the elegant argument in Serre's book [Se80, sect. 6.5] that finitely generated torsion groups acting on a tree must have a global fixed-point. By the end of the 1990s it was a well-known open problem formulated as follows (see [Sw99, Be00]):

Can a finitely generated group that acts properly discontinuously (and cocompactly) by isometry on a proper CAT(0)-space contain an infinite torsion group?

At least at that time the expected answer depended on whether the cocompact assumption is included or not. It was again referred to as one of the key open problems in [Ca14].

Grigorchuk constructed a continuum of finitely generated torsion groups of growth strictly between polynomial and exponential (thus amenable) in the early 1980s, see [G80, G84, dlH00,

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CSD21] and references therein. Amenable groups admit proper actions on Hilbert spaces ([BCV95]). Both Grigorchuk groups and Burnside groups, that is, infinite torsion groups of bounded exponent, can act without a global fixed point on infinite-dimensional CAT(0) cubical complexes (see [Sa95, Os18, Sc22]). Also note that [Se80, Theorem 25] shows that every infinitely generated group acts on a tree without a global fixed-point, see also [BH99, II.7.11] for another infinitely generated torsion group example. We refer to the recent paper [HO22] for a historical survey on this problem, its variants and with references to the many previous partial results. In [NOP22, HO22] the following conjecture is stated:

Conjecture 1. ([NOP22]) *Every finitely generated group acting without a global fixed point on a finite-dimensional CAT(0) complex contains an element of infinite order.*

For CAT(0) cubical complexes this is known from a paper by Sageev [Sa95] (see [LV20] for a discussion), and recently extended to CAT(0) cubical complexes without infinite cubes in [GLU24]. Norin, Osajda, and Przytycki recently proved the conjecture for two-dimensional CAT(0) complexes (with certain additional minor assumptions) in their paper [NOP22].

We establish the following, that applies in particular to Grigorchuk’s torsion groups of intermediate growth which seems previously not known:

Theorem 2. *Let Γ be a finitely generated torsion group of subexponential growth. Whenever Γ acts by isometries on a complete CAT(0)-space Y of finite dimension, it must have a global fixed point in Y .*

As pointed out above, the finite generation and finite-dimensionality assumptions are necessary, on the other hand, since we assume subexponential growth, we do not answer the full general question nor settle all of Conjecture 1. However, let us emphasize that our result is more general in the sense that:

- the space Y is not necessarily proper or a CAT(0)-complex, and it is not assumed to have a large isometry group.
- no assumption on the action is made, such as being non-elementary (in the sense of assuming no fixed-point in ∂Y), properly discontinuous, or being part of a cocompact action.

It was previously shown by Caprace-Monod in [CM13, Corollary E] that finitely generated groups of intermediate growth cannot be a *discrete* subgroup of $Isom(Y)$ under the assumption that the space Y is a proper CAT(0)-space having a *cocompact* isometry group. On the other hand, Grigorchuk groups are by construction subgroups of the automorphism groups of rooted binary trees, thus they are subgroups of the isometry groups of certain proper CAT(0)-spaces possessing a global fixed-point, the root. One should also keep in mind that Caprace-Lytchak, [CL10, Theorem 1.6], extended theorems of Burger-Schroeder and Adams-Ballmann to CAT(0) spaces of finite telescoping dimension showing that amenable groups either fix a point at infinity or leave a flat invariant. Our theorem by-passes the trouble with that there might be a fixed point on the visual boundary at infinity. In this connection, we can mention a paper of Papasoglu-Swenson [PS18, Theorem 3.17] which shows the following: Assume that G acts properly and cocompactly on a proper CAT(0)-space. If Γ is an infinite torsion subgroup of G then it cannot fix a point in the limit set of Γ . (Note that in our setting, that is, assuming finite generation and subexponential growth but no properness or cocompactness, the limit set is empty since by Theorem 2 any orbit is bounded.) Groups which have a global fixed point whenever they act on simplicial trees are said to have Serre’s property FA. This is the

1-dimensional version of property FA_n in [F09], that is, having a global fixed point whenever acting by isometry on n -dimensional $CAT(0)$ cell complexes. The literature on these topics is vast, and we cannot survey this here. For property FA and the groups of the type of most immediate relevance for our paper, we can refer to [DG08, GS23] for results and more information.

With a similar proof we also have:

Theorem 3. *Let Γ be a finitely generated simple group of subexponential growth. Whenever Γ acts by isometries on a complete $CAT(0)$ -space Y of finite dimension, it must have a global fixed point in Y .*

Nekrashevych showed the existence of simple groups of intermediate growth in [Ne18].

Our proof uses the technique of ultralimits of spaces and equivariant harmonic maps, heavily inspired by Bourdon's paper [Bo12], combined with substantial arguments and results from the first author's paper [Iz23] (much of what is not repeated here, making our paper shorter than what it would be if all needed details from [Iz23] were reproduced). The use of random walks might be somewhat surprising. We think that the present method, in particular the combination of ultralimits, equivariant harmonic maps, subharmonic functions on the group, horofunctions and random walks, could have further applicability in geometric group theory.

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2 Ultralimits of metric spaces

For a detailed exposition on this topic we refer to [CSD21, Chapter 11] or [Bo12, Iz23, Ly05], and for basics on $CAT(0)$ -spaces we refer to [BH99]. Let ω denote a non-principal ultrafilter on \mathbb{N} . Let Y_j be metric spaces and p_j a point in each Y_j . The associated ultralimit metric space (Y_∞, d) is defined as follows. First, let Y^∞ be all (one-sided) sequences $y_j \in Y_j$ such that $d(y_j, p_j)$ is bounded in j . Via the triangle inequality, one can therefore define the following function on Y^∞ ,

$$d((x_j), (y_j)) := \omega - \lim d(x_j, y_j).$$

This function is clearly symmetric and satisfies the triangle inequality. Now we identify all sequences for which $d((x_j), (y_j)) = 0$. The set of equivalence classes, denoted by Y_∞ , is thus a metric space with the same d well-defined on the quotient. It is rather immediate that if each Y_j is a complete $CAT(0)$ -space, then so is (Y_∞, d) .

The dimension of a $CAT(0)$ -space Y is defined as follows (see [Kl99] for various equivalent definitions of dimension for $CAT(0)$ -spaces). Given a set of points x_0, x_1, \dots, x_n in Y , one has a barycentric simplex $\sigma : \Delta_n \rightarrow Y$ defined by mapping the point $(\alpha_0, \dots, \alpha_n)$ in the standard simplex to the unique minimizer (the *barycenter*), which exists by standard $CAT(0)$ -space theory, see [BH99, Ch. II.2], of the uniformly convex function

$$f(y) = \sum \alpha_i d(x_i, y)^2.$$

The dimension of Y is the greatest number D such that for some points x_0, x_1, \dots, x_D the corresponding barycentric simplex is non-degenerate. A simplex σ is degenerate if $\sigma(\Delta) = \sigma(\partial\Delta)$, see [Kl99, Definition 4.7]. We repeat the argument by Lytchak in [Ly05, Lemma 11.1] (alternatively see [Iz23, Proposition 6.10]) to show:

Proposition 4. *If the dimensions of Y_j is at most $D < \infty$, then the dimension of Y_∞ is at most D .*

Proof. Let x_0, x_1, \dots, x_n be points in Y_∞ such that the associated barycentric simplex σ is not degenerate. Represent these classes by actual sequences of points in Y , denoted $z_{i,j}$, here $j \rightarrow \infty$. By the continuity of barycenters in CAT(0)-spaces and the definition of barycentric simplices implies that σ is the ultralimit of the simplices defined by $z_{i,j}$ in Y_j . If all of these simplices would be degenerate then so would σ . \square

3 Existence of equivariant harmonic maps with positive energy when passing to certain ultralimits

Let Γ be a group generated by a finite set S . Assume that Γ acts by isometry on a complete metric space Y , that is, there is a homomorphism $\rho : \Gamma \rightarrow Isom(Y)$. A (ρ) -equivariant map is a map $f : \Gamma \rightarrow Y$ such that

$$f(g) = \rho(g)f(e)$$

for all $x \in Y$ and $g \in \Gamma$. These maps are determined by $f(e)$ and hence are just the orbit maps, $g \mapsto \rho(g)f(e)$.

Consider the following (S) -energy of maps $f : \Gamma \rightarrow Y$,

$$E(f) = \sum_S d(f(e), f(s))^2,$$

which clearly is finite. We are looking to find a map with minimal energy among the maps which are equivariant. Such maps are called *harmonic*. Intuitively we want to place our orbit in a minimal position, which is a generalization of the consideration of the infimal displacement for single isometries.

When Y is a complete CAT(0)-spaces it is well-known, and already used in the previous section about barycenters, that a convex function such as

$$x \mapsto \sum_S d(x, \rho(s)x)^2,$$

which is the energy of the map defined by $f(e) = x$, either has a minimum in Y or a minimizing sequence p_j that escapes to infinity, that is $d(x, p_j) \rightarrow \infty$ for any $x \in Y$.

In order to deal with the second scenario, we assume that Γ acts on a finite-dimensional CAT(0)-space Y without a global fixed point. We now follow Bourdon's techniques for L^p -spaces in [Bo12]. Let

$$\delta(x) = \max_{s \in S} d(x, s(x)),$$

where we from now on suppress ρ in the notation. Later on, we want $\inf_{x \in Y} \delta(x) > 0$, which clearly would imply that

$$\inf E(f) > 0$$

over all equivariant maps f , see below. Since S is a generating set, the action has a global fixed point in Y if and only if there exists an equivariant map f such that $E(f) = 0$.

Suppose instead that $\inf_{x \in Y} \delta(x) = 0$. We will now do the initial scaling of the action using ultralimits.

First, we rewrite a lemma that Bourdon attributes to Shalom [Sh00, Lemma 6.3].

Lemma 5. *Assume that $\inf_{x \in Y} \delta(x) = 0$ but there is no global fixed point. Then for any integer $n > 0$, there exists $r_n > 0$ and $v_n \in Y$ such that $\delta(v_n) \leq r_n/n$ and for all v with $d(v_n, v) < r_n$, it holds that*

$$\delta(v) \geq r_n/2n.$$

Proof. Let n be a positive integer. Take $w_1 \in Y$ such that $\delta(w_1) \leq 1/2n$. If $\delta(v) \geq 1/4n$ for every v with $d(v, w_1) < 1/2$, we let $v_n = w_1$ and $r_n = 1/2$. Otherwise, there is w_2 with $d(w_1, w_2) < 1/2$ and $\delta(w_2) \leq 1/4n$. If $\delta(v) \geq 1/8n$ for every v with $d(v, w_2) < 1/4$, then we let $v_n = w_2$ and $r_n = 1/4$. This procedure terminates, because otherwise we would get a sequence w_k such that $d(w_k, w_{k+1}) < 2^{-k}$. By completeness of Y this Cauchy sequence converges to a point $w \in Y$. But note that

$$|d(w, s(w)) - d(w_k, s(w_k))| \leq 2d(w, w_k),$$

which implies that $\delta(w) = 0$ since $\delta(w_k) \leq 1/(2^k n)$. This contradicts the assumption that the orbit did not have a global fixed point. \square

Using this lemma we can now show, what essentially is part of [Bo12, Prop. 3.1]:

Proposition 6. *Let Γ be a group generated by a finite set S . Assume that it acts on a finite-dimensional complete CAT(0)-space Y , with*

$$\inf_{x \in Y} \max_{s \in S} d(x, s(x)) = 0$$

but there is no global fixed point. Then by passing to a certain rescaled ultralimit action, there is a finite-dimensional complete CAT(0)-space Y_∞ on which Γ acts without a global fixed point and

$$\inf_{x \in Y_\infty} \max_{s \in S} d(x, s(x)) > 0.$$

Proof. Let $\lambda_n = n/r_n$ with r_n and v_n from Lemma 5 with $\delta(x) = \max_{s \in S} d(x, s(x))$ whose infimum is zero. Consider the metric spaces $\lambda_n Y = (Y, \lambda_n d)$, which of course again is a complete CAT(0) on which Γ acts by isometry via the same action. Take the ultralimit space Y_∞ of the sequence of pointed spaces $(\lambda_n Y, v_n)$ as in the previous section. Thanks to Proposition 4 it is still finite-dimensional. Let

$$\delta_n(x) = \max_{s \in S} \lambda_n d(x, s(x)),$$

so $\delta_n(v_n) \leq 1$ and $\delta_n(v) \geq 1/2$ for all v in the $\lambda_n d$ -ball of radius n around v_n . Thanks to the condition $\delta_n(v_n) \leq 1$ we have that Γ still acts (by isometry) on the ultralimit and thanks to the inequality $\delta_n(v) \geq 1/2$ on larger and larger balls around v_n we have that this new action of Γ on Y_∞ satisfy $\inf_{x \in Y_\infty} \delta(x) > 0$. \square

In summary, possibly by replacing Y with an ultralimit we have that Γ acts on a finite dimensional CAT(0)-space Y such that

$$\delta(x) = \max_{s \in S} d(x, s(x))$$

is bounded away from 0, which of course in particular excludes the existence of a global fixed-point. This implies that the energy of any equivariant map f is also bounded away from 0, since

$$E(f) = \sum_S d(f(e), f(s))^2 = \sum_S d(f(e), sf(e))^2 \geq \delta(f(e))^2.$$

Now we pass to the other part of [Bo12, Prop. 3.1]:

Proposition 7. *Let Γ be a group generated by a finite set S . Assume that it acts on a finite-dimensional complete CAT(0)-space Y without a global fixed point. Then there is a finite-dimensional complete CAT(0)-space Z on which Γ acts by isometry, such that there is a harmonic equivariant map $h : \Gamma \rightarrow Z$ with strictly positive energy.*

Proof. In view of Proposition 6, by passing to a certain ultralimit if necessary, we may assume that $\delta(x) = \max d(x, s(x))$ bounded away from 0 and hence the energy $E(f)$ of equivariant maps f is also bounded away from 0 in view of the above displayed inequality.

Let p_j be a sequence such that the equivariant maps with $f_j(e) := p_j$ have energy approaching the minimum. As remarked above in case the p_j stay bounded there is an equivariant harmonic map because of the uniform convexity property of complete CAT(0)-spaces. In any case, we can argue as follows, consider the ω -ultralimit of (Y, p_j) . By the above displayed inequality we have $\delta(p_j)$ is bounded from above, say by twice the square root of the minimal energy. This guarantees that the Γ action extends to this limit space by isometry, and as before this limit is a complete finite dimensional CAT(0)-space (in particular in view of Proposition 4). Likewise we may also define the map $h : \Gamma \rightarrow Y_\infty$ by $h(g) = \omega - \lim f_j(g)$. Since for each j this is equivariant so is g . We now claim it is moreover harmonic. Clearly, $E(h) = \omega - \lim E(f_j) = \inf_f E(f)$. To show that this value of the energy is also minimal in the limit space, take h_1 an equivariant map $\Gamma \rightarrow Y_\infty$ and consider its energy. Take a sequence w_j representing $h_1(e)$. Define the corresponding equivariant maps $f_{1,j}(e) = w_j$. For a given $g \in \Gamma$ let z_j be a sequence representing $g(w_j)$, by equivariance we now have

$$0 = \omega - \lim d(gw_j, z_j) = \omega - \lim d(f_{1,j}(g), z_j)$$

which shows that we can represent h_1 by the maps $f_{1,j}$, but then it follows that

$$E(h_1) = \omega - \lim E(f_{1,j}) \geq \inf_f E(f) = E(h)$$

□

4 Proofs of the main theorems

Let Γ be a group generated by a finite set S , which we take to be symmetric, $S^{-1} = S$. Assume that the balls in the word metric defined by S grow subexponential in the radius (we refer to [dlH00] for more information about growth in groups). This is independent of the finite generating set. We denote by μ the uniform probability measure on S . In view of the subexponential growth, a simple count implies that the asymptotic entropy of the associated random walk must be 0. It then follows from the standard theory that the linear drift in the word length of this symmetric random walk is zero. All this is very well-known, we refer to the book [La23] for details, proofs and historical references. For any isometric action $\rho : \Gamma \rightarrow \text{Isom}(X, d)$, the orbit map $g \mapsto \rho(g)x$ from the group with its word metric to the metric space (X, d) is Lipschitz. Taken together this implies that

$$\ell = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{g \in \Gamma} d(x, \rho(g)x) \mu^{*n}(g) = 0,$$

where μ^{*n} are the convolution powers giving the distribution of the random walk at time n . We emphasize that this holds for any isometric action of Γ on a metric space.

Let Γ act by isometry on a finite-dimensional complete CAT(0)-space Y . Assume that Γ has no global fixed point in Y . Then the ultralimits, in two steps as explained in the previous section, Propositions 6 and 7, give another finite-dimensional complete CAT(0) space Z on which Γ acts by isometry and with the existence of an equivariant harmonic map with positive energy. Since the energy is strictly positive there is no fixed point.

As is explained in [Iz23, sect. 4] we may modify the measure μ so that we can assume that its support contains all non-identity elements of Γ keeping the drift 0. The notion of horofunctions b_ξ with respect to a base point o also extends to ultralimits in natural way ([Iz23, Lemma 6.6]). We recall the important Proposition 6.9 in [Iz23] slightly tailored to fit the present notations:

Proposition 8. ([Iz23, Prop. 6.9]) *Let Z be a complete CAT(0)-space. Take $o \in Z$ and set $Z_\infty = \omega - \lim(Z, o)$. Let Γ be a countable group equipped with a symmetric probability measure μ whose support contains $\Gamma \setminus \{e\}$. Suppose that Γ acts by isometry on Z and that there exists an equivariant harmonic map $f : \Gamma \rightarrow Z$. If the drift $\ell = 0$, then for almost all random walk trajectories σ , there exists $\xi(\sigma) \in Z_\infty \cup \partial Z_\infty$ such that a function $\varphi_\sigma : \Gamma \rightarrow \mathbb{R}$ defined by*

$$\varphi_\sigma(\xi) = \begin{cases} d(\xi(\sigma), f(g)) - d(\xi(\sigma), f(e)), & \text{if } \xi(\sigma) \in Z_\infty \\ b_{\xi(\sigma)}(f(g), f(e)), & \text{if } \xi(\sigma) \in \partial Z_\infty \end{cases} \quad (1)$$

is μ -harmonic.

The last notion of μ -harmonic means that the Laplacian defined by μ annihilates φ_σ . The proof requires a few pages and since it can be used in a form directly applicable in the present paper, we refer to [Iz23] for the proof.

Using this proposition one can show the following:

Proposition 9. *Let Z be a complete CAT(0)-space of finite dimension and Γ be a countable group equipped with a symmetric probability measure μ whose support contains $\Gamma \setminus \{e\}$. Suppose that Γ acts on Z by isometry and that there exists an equivariant harmonic map $f : \Gamma \rightarrow Z$. If the drift $\ell = 0$, then there is a Γ -invariant convex subset F isometric to a finite dimensional (possibly zero-dimensional) Euclidean space.*

Proof. Although this assertion is not explicitly stated in [Iz23], it can be proved by following the arguments in [Iz23, §6.4] appropriately in our setting. We sketch the outline here.

Suppose $\xi(\sigma) \in Z_\infty$ in Proposition 8. Then, as in the first part of [Iz23, §6.4], there is a point in Z_∞ fixed by Γ , and hence every orbit of Γ in Z_∞ is bounded. In particular, any orbit of Γ in Z is bounded, and this implies the existence of a global fixed point, an invariant zero-dimensional Euclidean space, in Z via the Bruhat-Tits fixed point theorem, [BH99, II.2.8].

If $\xi(\sigma) \in \partial Z_\infty$ in Proposition 8, then the horofunction $\varphi_\sigma : g \mapsto b_{\xi(\sigma)}(f(g), f(e))$ is μ -harmonic. Since horofunctions on CAT(0) spaces are convex, the pull-back of them by a harmonic map are μ -subharmonic functions on Γ in general (see for example [Iz23, Prop. 2.17]). Here a function u on Γ is μ -subharmonic if the Laplacian of u defined by μ is nonpositive. If u is the pull-back of a convex function on a CAT(0) space by a harmonic map, then the absolute value of its Laplacian measures the strength of the convexity of the function around the image of the harmonic map. Thus the fact $\varphi_\sigma : g \mapsto b_{\xi(\sigma)}(f(g), f(e))$ is μ -harmonic means that the horofunction has the weakest convexity around the image $f(\Gamma)$ of f . This suggests that the convex hull of $f(\Gamma)$ is flat, and it can be proved as follows.

Since we assume that Z is finite-dimensional, a compactness principle [Iz23, Theorem 6.12] tells us that $\xi(\sigma)$ actually lies in ∂Z . The compactness principle and [Iz23, Lemma 5.1] imply

that there exist $\xi \in \partial Z$ and a bi-infinite geodesic c_g passing through $f(g)$ and terminating at ξ for each $g \in \Gamma$, and all the c_g geodesics are parallel to each other (in order to prove this, we use the weakest convexity of the horofunction). Then an argument involving the splitting theorem, see [BH99, Ch. II.2], leads us to see that there is a splitting of the convex hull of $f(\Gamma)$ one of whose factor is a Hilbert space, and the action of Γ also splits along this splitting. Taking a maximal flat factor in this splitting, and applying the argument above to the other factor, we see that the action of Γ on the other factor must fix a point; the action leaves the product F of the maximal flat factor and the fixed point in the other factor invariant. Since Z is finite-dimensional, so is F , and it becomes the desired flat subset in Z . \square

Recall now that we have excluded the existence of a global fixed point in Z in the preceding section. Therefore, by Proposition 9, we obtain a representation of Γ into the group of affine isometries of a Euclidean space of finite and strictly positive dimension. Thus we obtain a linear representation of Γ over the real numbers.

In the case of a finitely generated torsion group, by Schur's theorem this representation must have a finite image. Hence by the Cartan-Bruhat-Tits fixed point theorem it must fix a point in this flat. This leads to a contradiction to the construction of the space Z from Proposition 7. The conclusion is then that there must be a global fixed point for the original action, $\rho : \Gamma \rightarrow \text{Isom}(Y)$ which concludes the proof of Theorem 2.

In the case Γ is a simple group of subexponential growth, then by the Tits alternative it must be virtually nilpotent. Virtually nilpotent groups have a finite index normal subgroup, which must be trivial since the group is simple. Hence it follows that the image of any linear representation is trivial or a cyclic group of prime order (which must be Γ itself) and again we have a fixed point in Z , leading to the desired contradiction, that is, there must be a global fixed point for the original action, $\rho : \Gamma \rightarrow \text{Isom}(Y)$.

Remark 10. Suppose $\Gamma = \mathbb{Z}$ then the drift is surely 0 for the simple symmetric random walk. This group may act without a fixed point, precisely when being a parabolic or hyperbolic isometry of Y . Note that the proof here applies all the way until the point that it preserves a flat in Z so it has an axis (and the hypothesis of only possessing finite linear representations is obviously not satisfied). This clearly does not contradict the no fixed-point assumption, and is also consistent with the case when Γ acts parabolically (i.e. without an axis) on the original space Y .

Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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