# The resolvent kernel on the discrete circle and twisted cosecant sums

Jay Jorgenson* Anders Karlsson† Lejla Smajlović

April 15, 2024

**Abstract**

In this paper we present a general unifying principle for computing finite trigonometric sums of types that arise in physics and number theory. We obtain formulas that are more general than previous expressions and deduce linear recursions, which are computationally more efficient than the degree two recursions proved by Zagier. As an application, we provide an answer to a question recently posed by Xie-Zhao-Zhao concerning special values of Dirichlet $L$-functions. The proofs use the combinatorial Laplacian on cyclic graphs and their twisted coverings. The techniques therefore connect the trigonometric sums to spectral invariants of graphs and open up for future investigations.

Keywords: Finite trigonometric sum, twisted trigonometric sum, Chebyshev polynomials, graph Laplacian, heat kernel.

## 1 Introduction

Finite trigonometric sums of the type

$$C_m(n) := \sum_{j=1}^{m-1} \frac{1}{\sin^n(j\pi/m)}$$

have a long history and appear in various contexts. Two early points of reference are in Eisenstein's work and in the study of Dedekind sums [BY02]. Modern appearances of these sums include the Hirzebruch signature defects and the Verlinde formulas in topology and mathematical physics [HZ74, Ve88, Do92, Za96], resistance in networks [Wu04, EW09, Ch12, Ch14b] as well as modeling angles in proteins and circular genomes [F-DG-D14]. Many further instances are described in [BY02], such as the chiral Potts model in statistical physics [MO96], [Ch14a]. Finite trigonometric sums are also related to Dedekind and Hardy sums and their generalizations. Several of those sums seem not to have known evaluations, but it is possible to establish reciprocal relations, see for example [BC13], [Ch18] or [MS20].

There also seems to be a growing interest in the evaluation of trigonometric sums in the combinatorics literature, in particular illustrated by the recent contributions [dFGK18], [EL21],

*Supported by PSC-CUNY Awards, which are jointly funded by the Professional Staff Congress and The City University of New York.

†Supported in part by the Swedish Research Council grant 104651320 and the Swiss NSF grants 200020-200400 and 200021-212864.

[GLY22], [AZ23], and [CHJSV23] relevant for our paper. Spectral graph theory and discrete Green functions have been important topics for quite some time, see for example [Ba79] and [CY00]. What our paper shows is how to use the spectral theory of graphs, including various twisting procedures and Bessel functions which may be less common in standard graph theory, to give a unifying procedure to evaluate significant classes of trigonometric sums.

The sums $C_m(n)$ are also discrete analogs of the Riemann zeta function, as observed by Dowker in [Do92] and further developed in [FK17]. This link is already implicitly present in [Ap73] where the asymptotics of the cotangent sums

$$\sum_{k=1}^{m} \cot^n(k\pi/(2m+1))$$

as $m \to \infty$ are used to evaluate the Riemann zeta values $\zeta(n)$, ultimately recovering Euler's formula in case $n$ is even. These computations indicate the delicate nature of these trigonometric sums, including $C_m(n)$, because the values of $\zeta(2n)$ are known while the values of $\zeta(2n+1)$ are far from understood.

In [XZZ24], the authors found a precise formula for $\zeta(2n)$ as a finite linear combination, with universal constants, of the sums $C_m(2k)$ for $0 < k < n$. These formulas do not involve asymptotic expansions. In the same paper, the authors obtain similar formulas for special values of Dirichlet $L$-functions and ask for a direct evaluation of the corresponding twisted trigonometric sums. One of the results in the present article provides an answer to this question posed by [XZZ24]; see section 6.

We use the notation of the cosecant function $\csc(x) = 1/\sin(x)$ and the secant function $\sec(x) = 1/\cos(x)$ throughout this article. Let $m$ and $n$ be positive integers, and $\beta$ be a positive real number, which we call a shift. Define

$$C_m(\beta, n) := \sum_{j=\delta(\beta)}^{m-1} \csc^n\left(\frac{(j+\beta)}{m}\pi\right), \tag{1}$$

where $\delta(\beta) = 1$ if $\beta \in \mathbb{Z}$ and 0, otherwise.

Wang and Zheng proved in [WZ07] that

$$\sum_{n=1}^{\infty} C_m(\beta, 2n) y^{2n} = \frac{my}{\sqrt{1-y^2}} \frac{\sin(2m\arcsin(y))}{\cos(2m\arcsin(y)) - \cos 2m\beta}. \tag{2}$$

and deduced a similar formula for the generating function of the alternating cosecant sums. Other authors have studied twists of powers of cosecants by cosine function, see for example [Do92], Section 3 of [BY02] as well as Section 3 of [He20]. Those authors also study secant sums and derive similar results.

In this paper we will study the cosecant sums with shift $\beta \geq 0$ and twisted by an additive character. In doing so, we also derive results for analogously defined secant sums by suitably adjusting the shift $\beta$.

More precisely, let $m > 1$ be an integer, let $\beta$ be a positive nonintegral real number and take $r \in \{-(m-1), \ldots, 0, \ldots, (m-1)\}$. We define (the average value of) the twisted cosecant sums associated to those parameters and a positive integer $n$ by

$$C_{m,r}(\beta, n) := \frac{1}{m} \sum_{j=0}^{m-1} \csc^{2n}\left(\frac{j+\beta}{m}\pi\right) e^{2\pi i r j/m}. \tag{3}$$

The cosecant sums without the shift $\beta$ are defined as

$$C_{m,r}(n) := \frac{1}{m} \sum_{j=1}^{m-1} \csc^{2n}\left(\frac{j}{m}\pi\right) e^{2\pi i r j/m}. \tag{4}$$

Note that to get (4) from (3), one omits the term where $j = 0$ and then sets $\beta = 0$. The sums (4) appear in the formulas deduced in [Ta92] for the dimensions of a certain complex vector space at level $k$ associated to a labeled Riemann surface of genus $g \geq 2$. Specifically, in statement (12) of [Ta92] the aforementioned dimension is expressed in terms of (4) with $r = 0$, $m = k + 2$ and $n = g - 1$, while in statement (18) of [Ta92], the appropriate dimension of the "twisted" space is expressed in terms of (4) with even $k$, $m = k + 2$, $r = m/2$, and $n = g - 1$. Both expressions are special cases of Verlinde sums; see [Ve01, pp. 11, 14].

In this article we will also study powers (that are not necessarily even) of cosecant and secant functions evaluated at doubled arguments. We will derive an explicit evaluation of their generating functions as well as a finite recursion formula for computation. More precisely, for real number $\alpha$ such that $\alpha \notin \mathbb{Z}$ when $m \equiv 0 \pmod 4$, $\alpha \notin \mathbb{Z} + \frac{1}{2}$ when $m \equiv 2 \pmod 4$ and $2\alpha \notin \mathbb{Z} + \frac{1}{2}$ when $m$ is odd, we will study the sum

$$\tilde{S}_{m,r}(\alpha, n) := \frac{1}{m} \sum_{j=0}^{m-1} \sec^{n}\left(\frac{2(j+\alpha)}{m}\pi\right) e^{2\pi i r j/m}. \tag{5}$$

When $m$ is not divisible by 4 and by taking $\alpha = 0$ in (5), we immediately obtain the secant sums of double argument without the shift. If $m \equiv 0 \pmod 4$, then one needs to exclude the value of $j$ for which $j \equiv \frac{m}{4} \pmod m$ from the range $0, \ldots, m-1$ of summation in (5). Such a sum equals zero when $n + r$ is odd and equals $S_{m/2,r/2}(n/2)$, defined by (14) below, when both $r$ and $n$ are even. We leave the study of the special case $\alpha = 0$ of the sum $\tilde{S}_{m,r}(\alpha, n)$ when $m \equiv 0 \pmod 4$ and both $r$, $n$ are odd to the interested reader.

For any real number $\beta$ such that $2\beta \notin \mathbb{Z}$ when $m$ is odd and such that $\beta \notin \mathbb{Z}$ when $m$ is even, we consider the sums

$$\tilde{C}_{m,r}(\beta, n) := \frac{1}{m} \sum_{j=0}^{m-1} \csc^{n}\left(\frac{2(j+\beta)}{m}\pi\right) e^{2\pi i r j/m}. \tag{6}$$

The (average) cosecant sums of double argument, without the shift $\beta$ are defined by

$$\tilde{C}_{m,r}(n) := \frac{1}{m} \sum_{j \in \{1,\ldots,m-1\} \setminus \{j_m\}^*} \csc^{n}\left(\frac{2j}{m}\pi\right) e^{2\pi i r j/m}, \tag{7}$$

where $\{j_m\}^*$ is the empty set if $m$ is odd and contains the single number $j_m$ such that $j_m \equiv \frac{m}{2} \pmod m$ in the case when $m$ is even.

The cosecant and secant sums, both with and without the shift $\beta$ or twist by an additive character, have been extensively studied using various methods. For example, the authors in [CM99], [BY02], [WZ07], [CS12], [Do15] used contour integration, generating series and partial fraction decomposition to evaluate those sums as well as their generating functions. The approach in [dFGK17], [dFGK18] uses recurrence relations and generating series, while [He20] starts with Taylor series expansions of powers of tangent and cotangent. In [AH18] the starting point is to use various results in the theory of certain special functions. Also, a discrete form of sampling theorem was used in [Ha08], while [AZ23] describes an "automated approach" for proving some trigonometric identities.

In this article, we offer a different point of view and also study a more general situation, which includes series which may include a twist by an additive character. The approach is inspired by Dowker's computation of the heat kernel on a generalized cone [Do89] and the key observation is that the resolvent for the twisted heat kernel on a cycle graph can be viewed as a generating function for certain secant and cosecant sums.

Let us now describe our approach and state our main results.

### 1.1 Overview of methods and illustration of results

Let $X_m$ denote the weighted Cayley graph with vertex set $\mathbb{Z}/m\mathbb{Z}$, generator set $S = \{-1, 1\}$, and weights given by the uniform probability distribution on $S$. Let $\beta \in \mathbb{R}$ be an arbitrary real parameter. Our starting point is the "twisted by an additive character" $\chi_\beta(x) := \exp(2\pi i\beta x)$ heat kernel on $X_m$. We compute the heat kernel using two different means. First, we employ the method of averaging, by which we mean that we view $\mathbb{Z}/m\mathbb{Z}$ as being covered by $\mathbb{Z}$ and then we sum the heat kernel on $\mathbb{Z}$ by the covering group $m\mathbb{Z}$. Second, we use the discrete spectral expansion of the standard Laplacian on $X_m$. Since the heat kernel under consideration is unique, the two different evaluations yield an identity. From this identity, we then compute the resolvent kernel $G_{X_m,\chi_\beta}$ twisted by the character $\chi_\beta$ (or twisted Green's function, see [CY00]) for the Laplace operator on the graph $X_m$. Essentially, the resolvent kernel is equal to the Laplace transform in the time variable of the heat kernel.

The above calculations yield an explicit identity for the resolvent kernel $G_{X_m,\chi_\beta}(x, y; s)$ for real $\beta$ which is obtained by equating the two evaluations. The resulting formula admits a meromorphic continuation to all complex values of $s$. We then determine its analytic properties for different values of real parameter $\beta$ at $s = 0$ and $s = -1$. The properties at $s = 0$ will yield results related to twisted even powers of secants and cosecants. The properties at $s = -1$ will yield results related to twisted, though not necessarily even, powers of shifted secants and cosecants at double arguments. Going further, we will apply the Gauss formula for primitive Dirichlet characters to get an explicit evaluation of the Dirichlet $L$-function associated to the cycle graph at positive integers.

#### 1.1.1 Generating functions for twisted sums of even powers

To illustrate our results let us state the first main theorem. With the notation as above, let $\ell \in \{0, \dots, m-1\}$ be such that $\ell \equiv r \pmod{m}$. For $\beta \notin \mathbb{Z}$ define the generating functions

$$f_{m,r}(s, \beta) = \sum_{n=0}^{\infty} C_{m,r}(\beta, n+1) s^n$$

and

$$f_{m,r}(s) = \sum_{n=0}^{\infty} C_{m,r}(n+1) s^n$$

for the cosecant sums (3) and (4). The first main result is the following theorem.

**Theorem 1.** *The generating function $f_{m,r}(s, \beta)$ can be expressed as*

$$f_{m,r}(s, \beta) = 2e^{-2\pi i\beta\ell/m} \cdot \frac{U_{m-\ell-1}(1-2s) + e^{2\pi i\beta} U_{\ell-1}(1-2s)}{T_m(1-2s) - \cos 2\pi\beta},$$

*where $T_n$ and $U_n$ denote the Chebyshev polynomials of the first and the second kind, with the convention that $U_{-1}(x) \equiv 0$.*

*Similarly, for the generating function $f_{m,r}(s)$ we have that*

$$f_{m,r}(s) = 2\frac{U_{m-\ell-1}(1-2s)+U_{\ell-1}(1-2s)}{T_m(1-2s)-1} + \frac{1}{ms}. \tag{8}$$

For relevant information about Chebyshev polynomials see for example [GR07, Section 8.94]. For the convenience of the reader, we state the most relevant results regarding Chebyshev polynomials in the concluding section 7.3. With the contents of section 7.3 to the side, we can give a simple qualitative description of Theorem 1, which is the following:

*Both of the power series $f_{m,r}(s,\beta)$ and $f_{m,r}(s)$ are rational functions in $s$ with numerators and denominators given in terms of classical Chebyshev polynomials with parameters $m$, $r$ and $\beta$.*

As the notation suggestions, (2) is a special case of Theorem 1 when $r = 0$, after one employs classical formulas for Chebyshev polynomials in terms of trigonometric and inverse trigonometric functions.

From Theorem 1 one can derive a recurrence formula for the coefficients in the series expansion of $f_{m,r}(s,\beta)$. More or less, if $P(s)$ is a convergent Taylor series at $s = 0$, and if we have that $P(s) = Q_1(s)/Q_2(s)$ where $Q_1(s)$ and $Q_2(s)$ are polynomials, then one simply needs to equate the coefficients of $s$ in the expression $Q_2(s)P(s) = Q_1(s)$. As it turns out in this case, there are convenient formulas for the series expansions of the Chebyshev polynomials $T_m(z)$ and $U_m(z)$ at $z = 1$; see 7.3. From these computations, we arrive at the following corollary.

**Corollary 2.** *Define the parameters $m$ and $r$ as above. Set the constants $a_m(j)$ and $b_m(j)$ as in equations* (50) *and* (51)*, respectively. For $\beta \notin \mathbb{Z}$ and any integer $n \geq 0$, define the numbers*

$$c_{m,r}(\beta,n) := e^{2\pi i\beta\ell/m}(-1)^n 2^{-(n+1)} C_{m,r}(\beta,n+1).$$

*Then we have the recurrence relation that*

$$\sum_{j=0}^{n}\binom{n}{j}\tilde{a}_m(n-j)c_{m,r}(\beta,j) = b_{m-\ell-1}(n) + e^{2\pi i\beta}b_{\ell-1}(n),$$

*where $\tilde{a}_m(0) = 1-\cos(2\pi\beta)$, $\tilde{a}_m(k) = a_m(k)$ for $k \geq 1$.*

*Similarly, when $\beta \in \mathbb{Z}$ and $n \geq 0$, define the numbers*

$$c_{m,r}(n) := (-1)^n 2^{-(n+1)} C_{m,r}(n+1). \tag{9}$$

*Then we have the recurrence relation that*

$$\sum_{j=0}^{n}\binom{n}{j}a_m(j+1)c_{m,r}(n-j) = b_{m-\ell-1}(n+1) + b_{\ell-1}(n+1) - \frac{a_m(n+2)}{m}. \tag{10}$$

From Theorem 1 and Corollary 2 one can obtain an abundance of specific formulas, each one of which can be described as mathematically appealing. For example, we will show that for any $k \geq 1$ one has that

$$\sum_{j=1}^{3k-1}\csc^4\left(\frac{j\pi}{3k}\right)\cos\left(\frac{2\pi j}{3}\right) = -\frac{1}{45}\left(39k^4+30k^2+11\right) \tag{11}$$

as well as that

$$\sum_{j=0}^{3k-1}\csc^2\left(\frac{2j+1}{6k}\pi\right)\omega^j = 3k^2 e^{-\frac{i\pi}{3}} \tag{12}$$

where $\omega$ is a primitive third root of unity. The recursive formulas in Corollary 2 allow one to readily evaluate series with higher powers.

**Remark 3.** In [Za96] Zagier proved a different recursion relation between certain cosecant sums. Our formula is simpler in the sense that it is linear and whereas the formula in [Za96] is quadratic. Our formulas are thus analogous to linear recursion relations between zeta values like those found from, for example, [F16], [FK17], [Me17] and references therein.

We shall now consider secant sums. Let $m$ and $r$ be as above, and let $\alpha$ be such that $\alpha - \frac{m}{2} \notin \mathbb{Z}$. The (average value of) the twisted secant sums associated to those parameters and a positive integer $n$ are defined as

$$S_{m,r}(\alpha, n) := \frac{1}{m} \sum_{j=0}^{m-1} \sec^{2n}\left(\frac{j+\alpha}{m}\pi\right) e^{2\pi i r j/m}. \tag{13}$$

The (average) secant sums without the shift $\alpha$ are defined as

$$S_{m,r}(n) := \frac{1}{m} \sum_{j\in\{0,\ldots,m-1\}\setminus\{j_m\}^*} \sec^{2n}\left(\frac{j}{m}\pi\right) e^{2\pi i r j/m}, \tag{14}$$

where the notation for $\{j_m\}^*$ is introduced above.
We define the generating function

$$h_{m,r}(s,\alpha) = \sum_{n=0}^{\infty} S_{m,r}(\alpha, n+1) s^n$$

associated to the sequence of series (13). Additionally, define the generating function

$$h_{m,r}(s) = \sum_{n=0}^{\infty} S_{m,r}(n+1) s^n \tag{15}$$

associated to the sequence of secant sums (14). By taking $\beta = \alpha - m/2$ in Theorem 1, we immediately deduce the following corollary.

**Corollary 4.** *The generating function $h_{m,r}(s,\alpha)$ can be expressed as*

$$h_{m,r}(s,\alpha) = 2(-1)^{\ell} e^{-2\pi i \alpha \ell/m} \cdot \frac{U_{m-\ell-1}(1-2s) + (-1)^m e^{2\pi i \alpha} U_{\ell-1}(1-2s)}{T_m(1-2s) - (-1)^m \cos 2\pi\alpha}.$$

*As for* (15)*, there are two cases to consider. If $m$ is odd, then $h_{m,r}(s) = h_{m,r}(s,0)$. If $m$ is even, then $S_{m,r}(n) = C_{m,r}(n)$; hence, the evaluation for* (15) *in this case is given by* (8)*.*

### 1.1.2 Generating functions for twisted sums at double arguments

As we will show, the resolvent kernel $G_{X_m,\chi_\beta}(x,y;s)$ at $s=-1$ yields the generating function the powers of secants and cosecants at double arguments. In particular, see Section 5, Theorem 11 for our second main result, which is the evaluation of the generating functions associated to the sequences of the sums (5) and of the sums (6). As an application of Theorem 11, we obtain the succinct formulas that

$$\frac{1}{3k} \sum_{j=0}^{3k-1} \sec\left(\frac{4j}{3k}\pi\right) \omega^j = (-1)^{\frac{k-1}{2}} \tag{16}$$

and

$$\frac{1}{3k}\sum_{j=0}^{3k-1}\sec^2\left(\frac{4j}{3k}\pi\right)\omega^j = -k \tag{17}$$

where $\omega$ is a primitive third root of unity and $k \geq 1$. As in the previous section, we state and prove recursive relations for the sequences of these sums.

### 1.1.3 Evaluation of the Dirichlet $L$-function of a cycle graph

Let $m > 1$ be an integer. The Dirichlet $L$-function of a cycle graph $X_m$ is the *spectral* $L$-function corresponding to the spectrum of a combinatorial Laplacian. Specifically, the function is defined for any even Dirichlet character $\chi$ of modulus $m$ and any complex number $s$ by

$$L_{X_m}(s,\chi) = \sum_{j=1}^{m-1}\chi(j)\csc^{2s}\left(\frac{j\pi}{m}\right); \tag{18}$$

see [F16, XZZ24]. For odd Dirichlet characters the similar sum is identically 0. However, the authors in [XZZ24] propose a replacement. Specifically, it is suggested that one should consider the function

$$\tilde{L}_{X_m}(s,\chi) = \sum_{j=1}^{m-1}\chi(j)\csc^{2s}\left(\frac{j\pi}{m}\right)\cot\left(\frac{j\pi}{m}\right). \tag{19}$$

The functions (18) and (19) can be used to evaluate the classical Dirichlet $L$-functions at even and odd integers, respectively; see [XZZ24]. Hence, it is of interest to deduce an explicit evaluation of those functions. In Section 6 we will prove that for any even, primitive Dirichlet character $\chi$ of modulus $m$ one has

$$L_{X_m}(n,\chi) = (-1)^{n+1}2^n\tau(\chi)\sum_{r=0}^{m-1}\overline{\chi(r)}c_{m,r}(n-1), \quad n \in \mathbb{N}, \tag{20}$$

where $\tau(\chi)$ is the Gauss sum associated to the character $\chi$ and the coefficients $c_{m,r}(n-1)$ are explicitly computable for all positive integers $n$ when using the linear recurrence (10).

In summary, from Theorem 1 and Corollary 2 one has a method by which (20) is explicitly computable in terms of coefficients of Chebyshev polynomials. The main theorem in [XZZ24] proves a relation involving the values of the Dirichlet $L$-functions at positive integers in terms of the values (20); see Theorem A of [XZZ24]. In Section 5 of [XZZ24] the authors posed the question of determining a direct way by which one can evaluate (20), so then one can evaluate Dirichlet $L$-functions. Our results from Section 6 answer this question as stated in [XZZ24].

An explicit expression for values of (19) can be proved by differentiating the shifted $L$-function with respect to $\beta$. This computation is described in Section 7.2.

## 1.2 Organization of the article

In the next section we recall material from the literature regarding the continuous time heat kernel on a Cayley graph. As stated, for this paper the Cayley graph we consider is associated to $\mathbb{Z}/m\mathbb{Z}$, which is the group of integers modulo $m$ with edges given by connecting an edge to its two nearest neighbors. In Section 3 we define and study the corresponding resolvent kernel, which amounts to the Laplace transform in the time variable of the heat kernel. In

Section 4 we prove the main results as stated above, and in Section 5 we develop further general results associated to secant and cosecant sums with doubled arguments. In Section 6 we answer the aforementioned question posed in [XZZ24] which involves certain special values of spectral $L$-functions with a Dirichlet character. Finally, in Section 7, we present a few concluding remarks which suggest further studies which could be undertaken based on the results and methods presented in this article.

# 2 Twisted heat kernel on a discrete circle

## 2.1 Heat kernel on weighted Cayley graphs

Let $G$ be a finite or countably infinite abelian group with composition law which is written additively. Let $S \subseteq G$ be a finite symmetric subset of $G$. The symmetry condition means that if $s \in S$ then $-s \in S$.

Let $\pi_S\colon S \to \mathbb{R}_{>0}$ be a probability distribution on $S$ such that $\pi_S(s) = \pi_S(-s)$. The weighted and undirected Cayley graph $X = \mathcal{C}(G, S, \pi_S)$ of $G$ with respect to $S$ and $\pi_S$ is constructed as follows. The vertices of $X$ are the elements of $G$, and two vertices $x$ and $y$ are connected with an edge if and only if $x - y \in S$. The weight $w(x, y)$ of the edge $(x, y)$ is defined to be $w(x, y) := \pi_S(x - y)$. One can show that $X$ is a regular graph of degree 1.

A function $f : G \to \mathbb{C}$ is an $L^2$-function if $\sum_{x\in G} |f(x)|^2 < \infty$. The set of $L^2$-functions on $G$ is a Hilbert space $L^2(G, \mathbb{C})$ with respect to the classical scalar product of functions

$$\langle f_1, f_2\rangle = \sum_{x\in G} f_1(x)\overline{f_2(x)}.$$

The adjacency operator $\mathcal{A}_X\colon L^2(G, \mathbb{C}) \to L^2(G, \mathbb{C})$ of the graph $X$ is defined as

$$(\mathcal{A}_X f)(x) = \sum_{x-y\in S} \pi_S(x - y)f(y).$$

When $X$ is finite, the adjacency operator when written with respect to the standard basis is called the adjacency matrix $A_X$ of the graph $X$.

Given $x$ in the finite abelian group $G$, let $\chi_x$ denote the character of $G$ corresponding to $x$ in a chosen isomorphisms between $G$ and its dual group; see, for example, [CR62]. As proved in Corollary 3.2 of [Ba79], the character $\chi_x$ is an eigenfunction of the adjacency operator $\mathcal{A}_X$ of $X$ with corresponding eigenvalue

$$\eta_x = \sum_{s\in S} \pi_S(s)\chi_x(s).$$

Let $X$ denote the weighted Cayley graph $\mathcal{C}(G, S, \pi_S)$. The standard, or random walk, Laplacian $\Delta_X$ is defined to be the operator on $L^2(G, \mathbb{C})$ given by

$$\Delta_X f(x) = f(x) - (\mathcal{A}_X f)(x).$$

The heat kernel $K_X : G \times G \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ on $X$ is defined to be a solution to the equation

$$(\partial_t + \Delta_X)K_X(x, y; t) = 0 \quad \text{for } t > 0, \tag{21}$$

when viewed as a function of $x \in G$ for a fixed $y \in G$, and with initial condition

$$\lim_{t\downarrow 0} K_X(x, y; t) = \delta_x(y). \tag{22}$$

Here, $\delta_x$ denotes the standard delta function, meaning $\delta_x(x) = 1$ and $\delta_x(y) = 0$ for $x \neq y$.
It can be shown that (21) and (22) also holds if we interchange the roles of $x$ and $y$.
When the graph X is countable with bounded vertex degree, it is shown in [Do06] and [DM06] that the continuous time heat kernel exists and is unique among all bounded functions.

## 2.2 Twisted heat kernel on $\mathbb{Z}/m\mathbb{Z}$

Let $G = \mathbb{Z}$, and consider the Cayley graph $X = \mathcal{C}(G, S, \pi_S)$ when $S = \{-1, 1\}$ and with $\pi_S(1) = \pi_S(-1) = 1/2$. Then an elementary computation involving properties of the $I$-Bessel function shows that the heat kernel on $X$ is given by

$$K_X(x, y; t) = e^{-t} I_{x-y}(t);$$

see section 3 of [KN06]. In subsequent computations, we will use that $I_\nu(t) = I_{-\nu}(t)$ for any $\nu \in \mathbb{N}$. For an explicit solution of a more general type of diffusion equation on $X$, we refer the interested reader to [SS14] and [SS15].
Let $m > 1$ be a positive integer, and let $G_m = \mathbb{Z}/m\mathbb{Z}$ be the cyclic group of order $m$ with addition modulo $m$. Denote by $X_m$ the Cayley graph $\mathcal{C}(G_m, S, \pi_S)$ where $S = \{-1, 1\}$ and $\pi_S(1) = \pi_S(-1) = 1/2$; in case $m = 2$ then $X_2$ has two edges.
For $\beta \in [0, 1)$, $\chi_\beta(x) := \exp(2\pi i\beta x)$ is an additive character of $\mathbb{Z}$. The $\chi_\beta$-*twisted heat kernel* on the Cayley graph $X_m$ is defined to be a function

$$K_{X_m,\chi_\beta}(x, y; t) : G_m \times G_m \times \mathbb{R}_{\geq 0} \to \mathbb{R}, \tag{23}$$

and it has the following properties. For a fixed $y \in G_m$, and viewed as a function of $x$, (23) satisfies the transformation property

$$K_{X_m,\chi_\beta}(x + km, y; t) = \chi_\beta(k) K_{X_m,\chi_\beta}(x, y; t), \quad \text{for all } k \in \mathbb{Z}. \tag{24}$$

Similarly, one has the analogue of (24) when the heat kernel is viewed as a function of $y$ for a fixed $x \in G_m$ after replacing $\chi_\beta$ by its complex conjugate. Additionally, when viewed as a function of $t$, (23) satisfies the heat equation (21) with the initial condition $\lim_{t\downarrow 0} K_{X_m,\chi_\beta}(x, y; t) = \delta_x(y)$.
Using the method of images, as in [KN06], [Do12] and [CHJSV23], one has the following expression for the twisted heat kernel $K_{X_m,\chi_\beta}(x, y; t)$.

**Lemma 5.** *With the notation as above, the twisted heat kernel $K_{X_m,\chi_\beta}(x, y; t)$ is given by*

$$K_{X_m,\chi_\beta}(x, y; t) = \sum_{k\in\mathbb{Z}} e^{-2\pi i\beta k} e^{-t} I_{x-y+km}(t). \tag{25}$$

*Proof.* First, we observe that the series on the right-hand side of (25) converges uniformly and absolutely for all $t \geq 0$, due to the property that $I_\nu(t) = I_{-\nu}(t)$ for $\nu \in \mathbb{N}$ and the bound

$$\sum_{k=0}^{\infty} |I_{x+km}(t)| \leq e^t \tag{26}$$

which is valid for all (fixed) integers $x$; see [KN06], section 5. The transformation property (24) follows from the definition (25). Namely, for any $\ell \in \mathbb{Z}$ we have, by a substitution

$j = k + \ell$, that

$$\begin{aligned} K_{X_m,\chi_\beta}(x+\ell m, y; t) &= \sum_{k\in\mathbb{Z}} e^{-2\pi i\beta k} e^{-t} I_{x-y+(k+\ell)m}(t) \\ &= \sum_{j\in\mathbb{Z}} e^{-2\pi i\beta(j-\ell)} e^{-t} I_{x-y+jm}(t) \\ &= e^{2\pi i\beta\ell} K_{X_m,\chi_\beta}(x, y; t). \end{aligned}$$

Finally, we have that $e^{-t}I_{x-y+km}(t)$ satisfies the equation

$$\partial_t(e^{-t}I_{x-y+km}(t)) = -\left(e^{-t}I_{x-y+km}(t) - \frac{1}{2}\left(e^{-t}I_{x-y+km+1}(t) + e^{-t}I_{x-y+km-1}(t)\right)\right),$$

for all $k \in \mathbb{Z}$. With all this, we conclude that (25) is indeed the heat kernel on $X_m$ twisted by $\chi_\beta$. □

We can reformulate the lemma to give a slightly different expression for the twisted heat kernel $K_{X_m,\chi_\beta}$ that is more suitable for our purposes.

**Lemma 6.** *With the notation as above, let $\ell \in \{0, \ldots, m-1\}$ be such that $\ell \equiv (x-y) \pmod m$. Then*

$$K_{X_m,\chi_\beta}(x, y; t) = e^{-2\pi i\beta\frac{\ell-(x-y)}{m}} \sum_{j=-\infty}^{\infty} e^{-2\pi i\beta j} e^{-t} I_{\ell+jm}(t). \tag{27}$$

The twisted heat kernel on $X_m$ has a spectral expansion in terms of eigenfunctions and eigenvalues of the Laplacian $\Delta_{X_m}$. Namely, the eigenfunctions $\{\psi_j\}_{j=0}^{m-1}$ are given in terms of the normalized twisted characters, meaning that

$$\psi_j(x) = \frac{1}{\sqrt{m}} \exp\left(2\pi i \frac{j+\beta}{m} x\right) \quad \text{for } x \in G_m \text{ and } j = 0, \ldots, m-1. \tag{28}$$

The normalization is chosen so that the $L^2$-norm of $\psi_j(x)$ on $G_m$ equals one. The eigenvalues are described in section 2.1 for the adjacency operator, which gives that

$$\lambda_j = 1 - \frac{1}{2}\left(\exp\left(2\pi i\frac{j+\beta}{m}\right) + \exp\left(-2\pi i\frac{j+\beta}{m}\right)\right) = 2\sin^2\left(\pi\frac{(j+\beta)}{m}\right) \tag{29}$$

for $j = 0, \ldots, m-1$. With this notation, the spectral expansion of $K_{X_m,\chi_\beta}(x, y; t)$ is given by

$$K_{X_m,\chi_\beta}(x, y; t) = \sum_{j=0}^{m} e^{-\lambda_j t} \psi_j(x)\overline{\psi_j(y)} \quad \text{for } x, y \in G_m \text{ and } t \geq 0. \tag{30}$$

This identity can, of course, also be verified directly.

## 3 Twisted resolvent kernel on $X_m$

In this section we compute the twisted resolvent kernel, meaning the Green's function on $X_m$; see [CY00] for related results on the certain graphs which require that the eigenvalues are non-zero and the additive shift $\beta = 0$. Note that throughout this paper $\sqrt{s}$ denotes the principal branch of the square-root.

Our starting point in computing the twisted resolvent kernel on $X_m$ is the spectral expansion (30). For a complex number $s$ with $\mathrm{Re}(s) > 0$, the resolvent kernel, or Green's function, is defined as

$$G_{X_m,\chi_\beta}(x,y;s) := \int_0^\infty e^{-st} K_{X_m,\chi_\beta}(x,y;t)dt. \tag{31}$$

Since the heat kernel is well defined and bounded for all $t \geq 0$, the integral in (31) converges and defines a holomorphic function of $s$ in the half-plane $\mathrm{Re}(s) > 0$.

With all this, we have the following evaluation of the resolvent kernel (31).

**Proposition 7.** *With the notation as above, write $x - y \equiv \ell \in \{0, \dots, m-1\}$. Then for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ we have that*

$$G_{X_m,\chi_\beta}(x,y;s) = \frac{e^{-2\pi i \beta \frac{\ell-(x-y)}{m}}}{\sqrt{s^2+2s}} \cdot \frac{\sinh\big((m-\ell)\cosh^{-1}(s+1)\big) + e^{2\pi i \beta}\sinh\big(\ell\cosh^{-1}(s+1)\big)}{\cosh\big(m\cosh^{-1}(s+1)\big) - \cos 2\pi\beta}. \tag{32}$$

*Proof.* We begin with (27). From the bound (26), it is evident that for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ that the series

$$\sum_{j=-\infty}^{\infty} e^{-2\pi i \beta j} e^{-(s+1)t} I_{\ell+jm}(t) = e^{-st} K_{X_m,\chi_\beta}(x,y;s)$$

can be integrated as in (31) term by term. When computing these integrals, we get the expression that

$$G_{X_m,\chi_\beta}(x,y;s) = e^{-2\pi i \beta \frac{\ell-(x-y)}{m}} \sum_{j=-\infty}^{\infty} e^{-2\pi i \beta j} \int_0^\infty e^{-(s+1)t} I_{|\ell+jm|}(t)dt, \tag{33}$$

where, as stated above, we have used that $I_\nu(t) = I_{-\nu}(t)$ for any integer $\nu$.

The integral (33) is the Laplace transform of the $I$-Bessel function. Hence, we can apply [GR07], formula 109 on p. 1116 with $\nu = |\ell + jm| \geq 0$ and $a = 1$; note that the variable $s$ in this formula from [GR07] is our $s+1$. The assumption from [GR07] that $\mathrm{Re}(s+1) > a = 1$ is fulfilled for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. So then, we have that

$$\int_0^\infty e^{-(s+1)t} I_{|\ell+jm|}(t)dt = \frac{1}{\sqrt{s^2+2s}} \left(s+1-\sqrt{(s+1)^2-1}\right)^{|\ell+jm|}. \tag{34}$$

Since $\ell \in \{0, \dots, m-1\}$, it is immediate that $|\ell + jm| = \ell + jm$ for all $j \geq 0$. Also, we have that $|\ell + jm| = -\ell - jm$ for $j < 0$. Moreover, for real $s > 0$, one has that $\left|s+1-\sqrt{(s+1)^2-1}\right| < 1$. Let $u = (s+1-\sqrt{(s+1)^2-1})^{-1} > 1$. Therefore,

$$\begin{aligned}\sum_{j=-\infty}^{\infty} e^{-2\pi i \beta j}\left(s+1-\sqrt{(s+1)^2-1}\right)^{|\ell+jm|} &= \sum_{j=-\infty}^{\infty} e^{-2\pi i \beta j} u^{-|\ell+jm|} \\ &= u^{-\ell}\sum_{j=0}^{\infty}\left(e^{-2\pi i\beta}u^{-m}\right)^j + u^{\ell}\sum_{j=1}^{\infty}\left(e^{2\pi i\beta}u^{-m}\right)^j \\ &= \frac{u^{m-\ell} - u^{-(m-\ell)} + e^{2\pi i\beta}(u^\ell - u^{-\ell})}{u^{-m} + u^m - 2\cos(2\pi\beta)}.\end{aligned}$$

Using that

$$\exp(\cosh^{-1}(s+1)) = s+1+\sqrt{(s+1)^2-1} = (s+1-\sqrt{(s+1)^2-1})^{-1} = u,$$

we get that

$$\sum_{j=-\infty}^{\infty} e^{-2\pi i\beta bj}\left(s+1-\sqrt{(s+1)^2-1}\right)^{|\ell+jm|}$$
$$= \frac{\sinh\left((m-\ell)\cosh^{-1}(s+1)\right)+e^{2\pi i\beta}\sinh\left(\ell\cosh^{-1}(s+1)\right)}{\cosh\left(m\cosh^{-1}(s+1)\right)-\cos 2\pi\beta}. \tag{35}$$

When combining (35) with (33) and (34), the proof of equation (32) is completed for real and positive $s$. Since the function on the right-hand side of (32) is holomorphic for $\mathrm{Re}(s) > 0$, the proof for such $s$ follows from the principle of analytic continuation. □

We now will show that for $\beta \notin \mathbb{Z}$ the function on the right-hand side of (32) is holomorphic at $s = 0$.

**Lemma 8.** *For any $\ell \in \{0, \ldots, m-1\}$ and real number $\beta$ with $\beta \notin \mathbb{Z}$, the function*

$$g_{m,\ell}(s,\beta) = \frac{1}{\sqrt{(s+1)^2-1}} \cdot \frac{\sinh\left((m-\ell)\cosh^{-1}(s+1)\right)+e^{2\pi i\beta}\sinh\left(\ell\cosh^{-1}(s+1)\right)}{\cosh\left(m\cosh^{-1}(s+1)\right)-\cos 2\pi\beta}$$

*is holomorphic at $s = 0$.*

*Proof.* Since $g_{m,\ell}(s,\beta)$ is holomorphic in the half-plane $\mathrm{Re}(s) > 0$, it suffices to show that $g_{m,\ell}(s,\beta)$ is bounded as $s \to 0$. Indeed, for any positive integer $j$, it is elementary that

$$\begin{aligned}\sinh\left(j\cosh^{-1}(s+1)\right) &= \frac{1}{2}\left((s+1+\sqrt{s^2+2s})^j-(s+1-\sqrt{s^2+2s})^j\right)\\ &= j\sqrt{s^2+2s}+O(s) \quad \text{as } s\to 0.\end{aligned}$$

For $\beta \notin \mathbb{Z}$, $\cos(2\pi\beta) \neq 1$, so then

$$\lim_{s\to 0} g_{m,\ell}(s,\beta) = \frac{(m-\ell)+\ell e^{2\pi i\beta}}{1-\cos 2\pi\beta}.$$

□

With the spectral expansion (30) of the heat kernel, we get another expression for the resolvent kernel upon integrating as in (31). Specifically, we have that

$$G_{X_m,\chi_\beta}(x,y;s) = \sum_{j=0}^{m-1}\frac{1}{s+\lambda_j}\psi_j(x)\overline{\psi_j(y)} \quad \text{for} \quad \mathrm{Re}(s) > 0.$$

From the formulas (28) and (29) for $\psi_j$ and $\lambda_j$, we arrive at the expression that

$$G_{X_m,\chi_\beta}(x,y;s) = \frac{1}{m}\sum_{j=0}^{m-1}\frac{1}{s+2\sin^2\left(\pi\frac{j+\beta}{m}\right)}\exp\left(2\pi i\frac{j+\beta}{m}(x-y)\right). \tag{36}$$

It is immediate that the right-hand-side of (36) is a meromorphic function with simple poles whenever $s$ is one of the finite points for which $s = -2\sin^2\left(\pi\frac{j+\beta}{m}\right)$. In effect, our main results follow from the identity obtained by equating (32) and (36).

# 4 Proof of Theorem 1

We start by proving the first part of Theorem 1. Assume $\beta \notin \mathbb{Z}$. As stated, for $x, y \in X_m$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, we have two expressions (36) and (32) for the Green's function $G_{X_m,\chi_\beta}(x, y; s)$. Therefore, the right-hand sides of those formulas are equal. Set $r = (x - y)$, and $\ell$ as before. With this, we get, upon cancelling a factor $\exp(2\pi i\beta r/m)$, the identity that

$$\frac{1}{m}\sum_{j=0}^{m-1}\frac{e^{2\pi i\frac{jr}{m}}}{s + 2\sin^2\left(\pi\frac{(j+\beta)}{m}\right)} = \frac{e^{-2\pi i\beta\frac{\ell}{m}}}{\sqrt{s^2+2s}}\frac{\sinh\left((m-\ell)\cosh^{-1}(s+1)\right) + e^{2\pi i\beta}\sinh\left(\ell\cosh^{-1}(s+1)\right)}{\cosh\left(m\cosh^{-1}(s+1)\right) - \cos 2\pi\beta}. \tag{37}$$

From the definition of the Chebyshev polynomials of the first and the second kind, we have for $\mathrm{Re}(s + 1) > 1$ that

$$T_m(s+1) = \cosh\left(m\cosh^{-1}(s+1)\right) \quad \text{and} \quad U_{m-1}(s+1) = \frac{\sinh\left(m\cosh^{-1}(s+1)\right)}{\sqrt{(s+1)^2-1}}.$$

Let

$$F_{m,r}(s,\beta) = e^{-2\pi i\beta\ell/m}\cdot\frac{U_{m-\ell-1}(s+1) + e^{2\pi i\beta}U_{\ell-1}(s+1)}{T_m(s+1) - \cos 2\pi\beta}. \tag{38}$$

Then, for $\mathrm{Re}(s + 1) > 1$ we have that

$$F_{m,r}(s,\beta) = \frac{e^{-2\pi i\beta\frac{\ell}{m}}}{\sqrt{s^2+2s}}\frac{\sinh\left((m-\ell)\cosh^{-1}(s+1)\right) + e^{2\pi i\beta}\sinh\left(\ell\cosh^{-1}(s+1)\right)}{\cosh\left(m\cosh^{-1}(s+1)\right) - \cos 2\pi\beta}$$

and, by using (37),

$$F_{m,r}(s,\beta) = \frac{1}{m}\sum_{j=0}^{m-1}\frac{e^{2\pi i\frac{jr}{m}}}{s + 2\sin^2\left(\pi\frac{(j+\beta)}{m}\right)} \tag{39}$$

The equality (39), which holds for $\mathrm{Re}(s + 1) > 1$, extends to an equality of meromorphic functions which holds for all values of the complex variable $s$. In particular, for any fixed $\beta \in (0, 1)$, Lemma 8 yields that the function $F_{m,r}(s, \beta)$ is holomorphic at $s = 0$. Moreover by differentiating the right-hand side of (39) $n$ times with respect to $s$ evaluating at $s = 0$ we get

$$\partial_s^n\, F_{m,r}(s,\beta)\big|_{s=0} = (-1)^n n! 2^{-(n+1)}\cdot C_{m,r}(\beta, n+1).$$

This yields that

$$\begin{aligned} F_{m,r}(s,\beta) &= \sum_{n=0}^{\infty}\partial_s^n\, F_{m,r}(s,\beta)\big|_{s=0}\frac{s^n}{n!} \\ &= \sum_{n=0}^{\infty}\left((-1)^n 2^{-(n+1)}\cdot C_{m,r}(\beta, n+1)\right)s^n \end{aligned}$$

for $s$ sufficiently close to zero. This proves the first part of Theorem 1, after the cosmetic change of variable for $s$ obtained by replacing $s$ with $-2s$.

To prove the second part, we notice that from (37) that one has the identity

$$F_{m,r}(s,\beta) - \frac{1}{m\left(s+2\sin^2\left(\pi\frac{\beta}{m}\right)\right)} = \frac{1}{m}\sum_{j=1}^{m-1} \frac{e^{2\pi i\frac{jr}{m}}}{s+2\sin^2\left(\pi\frac{j+\beta}{m}\right)}. \tag{40}$$

For all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq 0$ the function on the right-hand side (40) is continuous at $\beta = 0$ from the right. Hence, the function on the left-hand side of (40) must also be right-continuous, so then we have that

$$\lim_{\beta\downarrow 0}\left(F_{m,r}(s,\beta) - \frac{1}{m\left(s+2\sin^2\left(\frac{\beta}{m}\pi\right)\right)}\right) = \frac{1}{m}\sum_{j=1}^{m-1} \frac{e^{2\pi i\frac{jr}{m}}}{s+2\sin^2\left(\pi\frac{j}{m}\right)}. \tag{41}$$

Trivially, from (38) we obtain that

$$\lim_{\beta\downarrow 0}\left(F_{m,r}(s,\beta) - \frac{1}{m\left(s+2\sin^2\left(\frac{\beta}{m}\pi\right)\right)}\right) = \frac{U_{m-\ell-1}(s+1)+U_{\ell-1}(s+1)}{T_m(s+1)-1} - \frac{1}{ms} = F_{m,r}(s). \tag{42}$$

The function on the right-hand side of (41) is holomorphic at $s = 0$. Therefore, $F_{m,r}(s)$ is also holomorphic at $s = 0$, and then

$$\partial_s^n \left. F_{m,r}(s)\right|_{s=0} = (-1)^n n! 2^{-(n+1)} \cdot C_{m,r}(n+1). \tag{43}$$

This proves the second claim of Theorem 1, again after replacing $s$ with $-2s$.

**Example 9.** Consider any positive $m$, $\beta = 1/2$, $r = 0$ and $n = 1$. Then by taking $s = 0$ in Theorem 1, we get that

$$\frac{1}{2}C_{m,0}(1/2,1) = f_{m,0}(0,1/2)$$

or

$$\frac{1}{2m}\sum_{j=0}^{m-1}\csc^2\left(\frac{2j+1}{2m}\pi\right) = \frac{U_{m-1}(1)}{T_m(1)+1} = \frac{m}{2}.$$

This yields the well-known evaluation that

$$\sum_{j=0}^{m-1}\csc^2\left(\frac{2j+1}{2m}\pi\right) = m^2;$$

see [BY02, Corollary 2.6] and references therein regarding the appearance of those sums elsewhere in the literature.

**Example 10.** For any positive integer $k$, let $m = 3k$. Take $\beta = 1/2$ and $r = k$. Let $\omega$ denote the third root of unity. Then, for all positive integers $n$ one has the identity that

$$\frac{1}{3k}\sum_{j=0}^{3k-1}\csc^{2n}\left(\frac{2j+1}{6k}\pi\right)\omega^j = (-1)^{n-1}2^n\partial_s^{n-1} \left. F_{3k,3}(s,1/2)\right|_{s=0},$$

where

$$F_{3k,3}(s,1/2) = e^{-\frac{i\pi}{3}}\frac{U_{2k-1}(s+1)-U_{k-1}(s+1)}{T_{3k}(s+1)+1}.$$

When $n=1$ this yields the formula (12). For $n\geq 1$, one can use the expansions of $U_{2k-1}(z)$, $U_{k-1}(z)$ and $T_{3k}(z)$ at $z=1$, as provided in Section 7.3 below, to get further evaluations. For example, one gets that

$$\sum_{j=0}^{3k-1}\csc^4\left(\frac{2j+1}{6k}\pi\right)\omega^j = k^2(13k^2+2)e^{-\frac{i\pi}{3}}.$$

If one takes $\beta=0$ and the same values of $m$ and $r$, one gets the formula that

$$\frac{1}{3k}\sum_{j=1}^{3k-1}\csc^{2n}\left(\frac{j\pi}{3k}\right)\omega^j = (-1)^{n-1}2^n\partial_s^{n-1}\, F_{3k,3}(s)|_{s=0}$$

where

$$F_{3k,3}(s)=\frac{U_{2k-1}(s+1)+U_{k-1}(s+1)}{T_{3k}(s+1)-1}-\frac{1}{3ks}.$$

By using the recurrence formula (10) with $n=0$, when combined with evaluations (50) and (51), one immediately derives the identity that

$$\sum_{j=1}^{3k-1}\csc^2\left(\frac{j\pi}{3k}\right)\cos\left(\frac{2\pi j}{3}\right) = -k^2-\frac{1}{3}.$$

The recurrence formula (10) with $n=1$, combined with (50) and (51) below, yields (11).

## 5 Secant and cosecant sums of a double argument

In this section we will study the resolvent kernel $G_{X_m,\chi_\alpha}(x,y;s)$, which equals $F_{m,r}(s,\alpha)$ for $r=x-y$, in the neighbourhood of $s=-1$. In doing so, we will prove the following theorem.

**Theorem 11.** *Let $m\geq 1$ and $r$ be integers. Let $\ell\in\{0,\dots,m-1\}$ be such that $r\equiv \ell \,(\mathrm{mod}\, m)$. Let $\alpha$ be a real number such that $\alpha\notin\mathbb{Z}$ when $m\equiv 0\,(\mathrm{mod}\,4)$, $\alpha\notin\mathbb{Z}+\frac{1}{2}$ when $m\equiv 2\,(\mathrm{mod}\,4)$ and $2\alpha\notin\mathbb{Z}+\frac{1}{2}$ when $m$ is odd. Then the generating function*

$$\tilde{f}_{m,r}(z,\alpha) = -\sum_{n=0}^{\infty}\tilde{S}_{m,r}(\alpha,n+1)z^n \tag{44}$$

*for the sum (5) is equal to*

$$\tilde{f}_{m,r}(z,\alpha)=e^{-2\pi i\alpha\frac{\ell}{m}}\cdot\frac{U_{m-\ell-1}(z)+e^{2\pi i\alpha}U_{\ell-1}(z)}{T_m(z)-\cos 2\pi\alpha}=F_{m,r}(z-1,\alpha),$$

*where as above $U_{-1}(x)\equiv 0$. Moreover, the coefficients*

$$\tilde{c}_{m,r}(\alpha,n) := -e^{2\pi i\alpha\frac{\ell}{m}}\tilde{S}_{m,r}(\alpha,n+1)\quad\text{with } n\geq 0$$

*satisfy the recursive relation that*

$$\sum_{j=0}^{n-1}\binom{n}{j}t_m(n-j)\tilde{c}_{m,r}(\alpha,j)+(t_m(0)-\cos 2\pi\alpha)\tilde{c}_{m,r}(\alpha,n) = u_{m-\ell-1}(n)+e^{2\pi i\alpha}u_{\ell-1}(n), \tag{45}$$

*where $t_n(k)$ and $u_n(k)$ are given for $0\leq k\leq n$ by (52), and $t_n(k)=u_n(k)=0$ for $k>n$.*

*Proof.* Our starting point is the equation

$$\frac{1}{m}\sum_{j=0}^{m-1}\frac{e^{2\pi i\frac{jr}{m}}}{s+2\sin^2\left(\pi\frac{(j+\alpha)}{m}\right)} = e^{-2\pi i\alpha\ell/m}\cdot\frac{U_{m-\ell-1}(s+1)+e^{2\pi i\alpha}U_{\ell-1}(s+1)}{T_m(s+1)-\cos 2\pi\alpha}. \tag{46}$$

Equation (46) stems from (37) and (38) with $\beta=\alpha$, which comes from two different ways to write $F_{m,r}(s,\alpha)$. For real values of $\alpha$ such that $2\alpha\notin\frac{m}{2}\mathbb{Z}$ when $m$ is even and for $2\alpha\notin\mathbb{Z}+\frac{1}{2}$ when $m$ is odd, it is obvious that the left-hand side of (46) is analytic at $s=-1$. Therefore, $F_{m,r}(s,\alpha)$ is analytic at $s=-1$, for the given values of $m$ and $\alpha$.

By differentiating the left-hand side of (46) $n$ times with respect to $s$, we get, after applying the trigonometric identity $-1+2\sin^2 x=-\cos 2x$, that

$$\begin{aligned}\partial_s^n\, F_{m,r}(s,\alpha)\big|_{s=-1} &= (-1)^n n!\cdot\frac{1}{m}\sum_{j=0}^{m-1}\frac{e^{2\pi i\frac{jr}{m}}}{\left(-1+2\sin^2\left(\pi\frac{(j+\alpha)}{m}\right)\right)^{n+1}}\\ &= -n!\tilde{S}_{m,r}(\alpha,n+1).\end{aligned}$$

Therefore, equation (44) holds for $\tilde{f}_{m,r}(z,\alpha)=F_{m,r}(z-1,\alpha)$. As in previous discussion, the recursion formula (45) follows from the uniqueness of the Taylor series expansion. $\square$

By letting $\alpha=\beta-\frac{m}{4}$ in the above theorem, and using that $\cos x=\sin(\pi/2+x)$, we arrive at the following corollary

**Corollary 12.** *Let $m\geq 1$ and $r$ be integers. Let $\ell\in\{0,\ldots,m-1\}$ be such that $r\equiv\ell\ (\mathrm{mod}\, m)$. Let $\beta$ be a real number such that $2\beta\notin\mathbb{Z}$ when $m$ is odd and $\beta\notin\mathbb{Z}$ when $m$ is even. Then the generating function*

$$\tilde{h}_{m,r}(z,\beta)=-\sum_{n=0}^{\infty}\tilde{C}_{m,r}(\beta,n+1)z^n$$

*for the sum (6) is given (with the convention that $U_{-1}(x)\equiv 0$) by*

$$\tilde{h}_{m,r}(z,\beta)=e^{-2\pi i\beta\frac{\ell}{m}}\cdot\frac{U_{m-\ell-1}(z)+e^{2\pi i\beta}U_{\ell-1}(z)}{T_m(z)-\cos\pi\left(2\beta-\frac{m}{2}\right)}.$$

The generating function for the sum (7) is obtained in a similar manner. Namely, from (46) and by taking $\beta=\alpha$ with $s=z-1$, we get that

$$\frac{1}{m}\sum_{j=0}^{m-1}\frac{e^{\frac{2\pi ir}{m}j}}{z-\cos\left(\frac{2\pi(j+\beta)}{m}\right)}=e^{-2\pi i\beta\ell/m}\cdot\frac{U_{m-\ell-1}(z)+e^{2\pi i\beta}U_{\ell-1}(z)}{T_m(z)-\cos(2\pi\beta)}$$

in a certain vertical strip in the complex $z$-plane depending on parameters $\beta$ and $m$. This yields for $\beta=\frac{m}{4}$ the identity that

$$\frac{1}{m}\sum_{j=0}^{m-1}\frac{e^{\frac{2\pi ir}{m}j}}{z+\sin\left(\frac{2\pi j}{m}\right)}=e^{-i\pi\ell/2}\cdot\frac{U_{m-\ell-1}(z)+e^{i\pi m/2}U_{\ell-1}(z)}{T_m(z)-\cos(m\pi/2)}. \tag{47}$$

where both sides of (47) are holomorphic for all complex $z$ with $0<\mathrm{Re}(z)<\delta$ when

$$0<\delta<\min\{|\sin(2\pi j/m)|\text{ for }j\in\{1,\ldots,m-1\}\setminus\{j_m\}^*\}.$$

When $m$ is odd, this gives

$$\frac{1}{m}\sum_{j=1}^{m-1}\frac{e^{\frac{2\pi ir}{m}j}}{z+\sin\left(\frac{2\pi j}{m}\right)}=e^{-i\pi\ell/2}\cdot\frac{U_{m-\ell-1}(z)+e^{i\pi m/2}U_{\ell-1}(z)}{T_m(z)}-\frac{1}{mz},$$

while for even $m$, we get

$$\frac{1}{m}\sum_{j\in\{1,\dots,m-1\}\setminus\{j_m\}^*}\frac{e^{\frac{2\pi ir}{m}j}}{z+\sin\left(\frac{2\pi j}{m}\right)}=e^{-i\pi\ell/2}\cdot\frac{U_{m-\ell-1}(z)+e^{i\pi m/2}U_{\ell-1}(z)}{T_m(z)-(-1)^{m/2}}-\frac{2}{mz}.$$

The left-hand sides of the above two displayed equations are holomorphic functions at $z=0$, hence so are the right-hand sides. Moreover, for even $m$ we have

$$\partial_z^n\left(\frac{1}{m}\sum_{j=1}^{m-1}\frac{e^{\frac{2\pi ir}{m}j}}{z+\sin\left(\frac{2\pi j}{m}\right)}\right)\Bigg|_{z=0}=(-1)^n n!\tilde{C}_{m,r}(n+1)$$

for any non-negative integer $n$. An analogous conclusion holds true for odd $m$.

With all this, we have proved the following corollary.

**Corollary 13.** *Let $m\geq 1$ and $r$ be integers. Let $\ell\in\{0,\dots,m-1\}$ be such that $r\equiv\ell\,(\mathrm{mod}\,m)$. Then the generating function*

$$\tilde{h}_{m,r}(z)=\sum_{n=0}^{\infty}(-1)^n\tilde{C}_{m,r}(n+1)z^n$$

*for the sum* (7) *is given by*

$$\tilde{h}_{m,r}(z)=e^{-i\pi\ell/2}\cdot\frac{U_{m-\ell-1}(z)+e^{i\pi m/2}U_{\ell-1}(z)}{T_m(z)-\cos(m\pi/2)}-\frac{\delta(m)}{mz},$$

*where $\delta(m)=1$ if $m$ is odd and $\delta(m)=2$ if $m$ is even.*

**Example 14.** For any positive and odd integer $k$, let $m=3k$, and set $r=k$. Let $\omega$ denote the third root of unity. By taking $\alpha=1/2$, from (45) with $n=0$ one gets that

$$\frac{1}{3k}\sum_{j=0}^{3k-1}\sec\left(\frac{2j+1}{3k}\pi\right)\omega^j=(-1)^{\frac{k-1}{2}}e^{-\frac{i\pi}{3}}.$$

Similarly, by setting $\alpha=0$ in (45), one deduces identities (16) and (17) by considering $n=0$ and $n=1$.

**Remark 15.** Let us note here that the secant and cosecant sums (5) and (6) with double argument taken to an even power are closely related to sums (13) and (3). For even values $m=2k$, one has

$$\tilde{C}_{2k,r}(\beta,2n)=C_{k,r}(\beta,n)\left(\frac{1+(-1)^r}{2}\right).$$

Given that we have different generating functions, those relations yield further identities satisfied by functions $f$ and $\tilde{h}$ and their derivatives.

We studied both types of sums because there are instances when one sum cannot be reduced to another one, such as when taking odd powers in (5) and (6) or odd $m$ in (13) and (3).

## 6 Sums twisted by a multiplicative character

In this section we will relate the results in this article to that from [F16, FK17] and [XZZ24]. In particular, we will prove formula (20) for evaluation of the special values of the spectral $L$-function associated to the cycle graphs $X_m$ at positive integers, thus providing an answer to the question raised at the end of [XZZ24].

More precisely, we will consider the generating function for the $L$-function defined on $X_m$ for any even Dirichlet character $\chi$ of modulus $m$ and any complex number $s$. This $L$-function is given in (18). When the character is trivial, $L_{X_m}(s,\chi)$ becomes the spectral zeta function $\zeta_{X_m}(s)$ on $X_m$ which was studied in [FK17]. Note that the special values of $\zeta_{X_m}(s)$ at positive integers $n$ is the non-twisted cosecant sum $C_m(0,2n)$ as defined in (1).

The following corollary evaluates the generating function for the special values of $L_{X_m}(n+1,\chi)$ associated to a primitive Dirichlet character modulo $m$ and $n \geq 0$.

**Corollary 16.** *Let $m > 1$ be an integer and assume $\chi$ is a primitive Dirichlet character modulo $m$. The generating function*

$$F_{m,\chi}(s) = \sum_{n=0}^{\infty}(-1)^n 2^{-(n+1)}\overline{L_{X_m}(n+1,\chi)}s^n$$

*can be evaluated as the following rational function of $s$:*

$$F_{m,\chi}(s) := \frac{m}{\tau(\chi)}\sum_{r=0}^{m-1}\chi(r)\left(\frac{U_{m-r-1}(s+1)+U_{r-1}(s+1)}{T_m(s+1)-1} - \frac{1}{ms}\right), \tag{48}$$

*where $\tau(\chi)$ denotes the Gauss sum associated to the character $\chi$, and the value of $F_{m,\chi}(s)$ at $s=0$ is obtained by taking the limit as $s \to 0$.*

*Proof.* It suffices to relate $L_{X_m}(n+1,\chi)$ to the sum of twists of $C_{m,r}(n+1)$ and apply the second part of Theorem 1. Recall the identity

$$\sum_{r=0}^{m-1}\chi(r)e^{\frac{2\pi i r}{m}j} = \overline{\chi(j)}\tau(\chi),$$

which holds for primitive Dirichlet characters. From this, we immediately deduce that

$$\begin{aligned}\sum_{r=0}^{m-1}\chi(r)C_{m,r}(n) &= \frac{\tau(\chi)}{m}\sum_{j=0}^{m-1}\overline{\chi(j)}\csc^{2n}\left(\frac{j\pi}{m}\right)\\ &= \frac{\tau(\chi)}{m}\overline{L_{X_m}(n,\chi)}.\end{aligned}$$

Therefore, for $s$ in a neighborhood of $s=0$, we deduce from (43) that

$$\sum_{n=0}^{\infty}(-1)^n 2^{-(n+1)}\overline{L_{X_m}(n+1,\chi)}s^n = \frac{m}{\tau(\chi)}\sum_{r=0}^{m-1}\chi(r)F_{m,r}(s). \tag{49}$$

By observing that $F_{m,r}(s)$ is defined by (42), the proof is complete. □

The proof of (20) now readily follows by conjugating (49) and recalling that $|\tau(\chi)|^2 = m$, and that, according to (42) and (43), one has that

$$F_{m,r}(s) = \sum_{n=0}^{\infty} c_{m,r}(n)s^n$$

in a neighbourhood of $s = 0$, where $c_{m,r}(n)$ is defined by (9). Note that the terms $c_{m,r}(n)$ satisfy the recurrence relation (10).

**Example 17.** When $n = 0$, then a simple computation shows that

$$c_{m,r}(0) = (m^2 - 6mr + 6r^2 - 1)/(6m).$$

This yields an interesting evaluation of $L_{X_m}(1, \chi)$, namely that

$$L_{X_m}(1, \chi) = \frac{2\tau(\chi)}{m} \sum_{r=0}^{m-1} \overline{\chi(r)}(r - m)r.$$

When $n = 1$, further calculations easily produce that $L_{X_m}(2, \chi)$ is given by

$$L_{X_m}(2, \chi) = -\frac{2\tau(\chi)}{3m} \sum_{r=0}^{m-1} \overline{\chi(r)}(r - 2m)(r - m)r(r + m).$$

These two formulas suggest a general pattern. From the recurrence relation (10), it is immediate that $mc_{m,r}(n - 1)$ is a polynomial of degree $2n$ in two variables $m$ and $r$. Hence, $L_{X_m}(n, \chi)$ can be expressed as

$$L_{X_m}(n, \chi) = \sum_{r=0}^{m-1} \overline{\chi(r)} P_{2n}(r, m)$$

for a certain explicitly computable polynomial $P_{2n}(r, m)$ of degree $2n$.

Using results of Section 5, it is possible to deduce further evaluations of secant and cosecant sums of double arguments twisted by multiplicative characters, thus complementing results of [BZ04] and [BBCZ05]. For example, consider a positive integer $m$ which is not divisible by 4 and a primitive Dirichlet character $\chi$ modulo $m$. By reasoning as in the proof of Corollary 16, with the starting point being Theorem 11 with $\alpha = 0$, one will deduce the evaluation of the $L$-function given by

$$\hat{L}_{X_m}(w, \chi) = \sum_{j=1}^{m-1} \chi(j) \sec^w \left(\frac{2j\pi}{m}\right) \quad \text{whenever } w = n \text{ and } n \text{ is a positive integer.}$$

From this, we have the following corollary.

**Corollary 18.** *Let $m > 1$ be an integer not divisible by 4, and assume that $\chi$ is a primitive Dirichlet character modulo $m$. The generating function*

$$\hat{F}_{m,\chi}(s) = \sum_{n=0}^{\infty} \hat{L}_{X_m}(n, \chi)s^n$$

*equals the following rational function:*

$$\hat{F}_{m,\chi}(s) := -\frac{m}{\tau(\overline{\chi})} \sum_{r=0}^{m-1} \overline{\chi(r)} \left(\frac{U_{m-r-1}(s) + U_{r-1}(s)}{T_m(s) - 1}\right).$$

# 7 Concluding remarks

## 7.1 Cotangent and tangent sums

In view of the standard identity $\csc^2 x = 1 + \cot^2 x$, we can also deduce results complementary to [He20, Theorem 2.2], where evaluations of cotangent sums twisted by multiplicative character were obtained.

Specifically, it is clear that computing even powers of cotangent sums reduces to computing even powers of cosecant sums of the same argument and with the same twist by an additive character. In other words, an application of the recurrence relation in Corollary 2 then allows one to compute that

$$\frac{1}{m}\sum_{j=0}^{m-1}\cot^{2n}\left(\frac{j+\beta}{m}\pi\right)e^{\frac{2\pi ir}{m}j} = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}C_{m,r}(\beta,k),$$

where we define $C_{m,r}(\beta,0)$ to be equal to 1 for all values of $m$, $r$, $\beta$. We assume that $m$ and $r$ are chosen as above and that $\beta$ is such that $\beta \notin \mathbb{Z}$. Similar reasoning applies to the computation of cotangent sums without the shift $\beta$ and to the computation of even powers of tangents, which reduces to an application of the binomial theorem to and secant sums.

These results can be compared to those of [EL21] where the authors compute, by using a different method, sums of any powers of cotangent and tangent functions at arguments of the form $\frac{j+\beta}{m}\pi$. Their result is more general in the sense that they treat both even and odd powers. On the other hand, we look only at even powers, but employ a character twist. As shown above, the use of the character twist is necessary in other situations, such as when one wants to apply the Gauss formula and pass to multiplicative character twists; see Section 6 above.

## 7.2 Differentiating or integrating with respect to $\beta$

A further possibility that presents itself is to differentiate or integrate the formulas above with respect to $\beta$. Let us illustrate an approach.

Let $\chi$ be a primitive, odd Dirichlet character, from which we seek to study the function $\tilde{L}_{X_m}(s,\chi)$ defined by (19). To do so, let us start with the shifted $L$-function which we define for $\beta \notin \mathbb{Z}$ by

$$L_{X_m}(s,\chi,\beta) = \sum_{j=0}^{m-1}\chi(j)\csc^{2s}\left(\frac{(j+\beta)\pi}{m}\right) = \sum_{j=1}^{m-1}\chi(j)\csc^{2s}\left(\frac{(j+\beta)\pi}{m}\right).$$

By proceeding analogously as in the proof of Corollary 16, it is immediate that the generating function

$$F_{m,\chi}(t,\beta) = \sum_{n=0}^{\infty}(-1)^n 2^{-(n+1)}\overline{L_{X_m}(n+1,\chi,\beta)}t^n$$

for the special values of $L_{X_m}(s,\chi,\beta)$ at positive integers $s=n$ is given by

$$F_{m,\chi}(t,\beta) := \frac{m}{\tau(\chi)}\sum_{r=0}^{m-1}\chi(r)F_{m,r}(t,\beta)$$

where $F_{m,r}(t,\beta)$ is defined by (38) and $t$ is any complex value where $|t|$ is sufficiently small and $\mathrm{Re}(t) \geq 0$. On the other hand, for any non-zero $s$ one has that

$$\frac{\partial}{\partial \beta} L_{X_m}(s,\chi,\beta)|_{\beta=0} = \frac{-2s\pi}{m} \sum_{j=1}^{m-1} \chi(j) \csc^{2s}\left(\frac{j\pi}{m}\right) \cot\left(\frac{j\pi}{m}\right) = \frac{-2s\pi}{m} \tilde{L}_{X_m}(s,\chi).$$

It is evident that the generating function for the values of $\tilde{L}_{X_m}(s,\chi)$ at positive integers can be expressed in terms of derivatives of $F_{m,\chi}(t,\beta)$ with respect to $\beta$ evaluated as $\beta \to 0$.

For the sake of limiting the length of our paper we have not pursued the computations here. In fact, apart from deriving certain new formulas of special interest, the main goal of our paper is to provide a general method and framework by which one can evaluate a wealth of finite trigonometric sum rather than to catalogue all such formulas that can be established in this way.

### 7.3 Chebyshev polynomials

For the convenience of the reader, we recall some notation and properties of Chebyshev polynomials which are needed above.

Chebyshev polynomials of the first kind are defined for $x \in [-1,1]$ and positive integers $n$ by the relation $T_n(x) = \cos(n \cos^{-1} x)$. For $|x| \geq 1$ the Chebyshev polynomials of the first kind are defined for positive integers $n$ by the relation

$$T_n(x) = \frac{1}{2}\left((x - \sqrt{x^2-1})^n + (x + \sqrt{x^2-1})^n\right).$$

By the principle of analytic continuation, we may assume that $T_n$ is defined for positive integers $n$ and complex numbers $z$ with $\mathrm{Re}(z) \geq 1$ by

$$T_n(z) = \frac{1}{2}\left((z - \sqrt{z^2-1})^n + (z + \sqrt{z^2-1})^n\right) = \cosh(n \cosh^{-1}(z)),$$

where we use the principal branch of the square root.

Chebyshev polynomials of the second kind are defined for $x \in [-1,1]$ and positive integers $n$ by the relation $U_n(x)\sqrt{1-x^2} = \sin((n+1)\cos^{-1} x)$. By extending the definition to $x$ with $|x| \geq 1$ and then using the principle of analytic continuation, it is easy to see that $U_n$ is defined for positive integers $n$ and complex numbers $z$ with $\mathrm{Re}(z) \geq 1$ by

$$U_n(z) = \frac{1}{2\sqrt{z^2-1}}\left((+ - \sqrt{z^2-1})^{n+1} - (z - \sqrt{z^2-1})^{n+1}\right) = \frac{\sinh((n+1)\cosh^{-1}(z))}{\sqrt{z^2-1}},$$

where we use the principal branch of the square root.

For all positive integers $n$, the functions $T_n(z)$ and $U_n(z)$ are holomorphic at $z = 1$. Moreover, let us set

$$T_n(z) = \sum_{k=0}^{\infty} a_n(k)(z-1)^k \quad \text{and} \quad U_n(z) = \sum_{k=0}^{\infty} b_n(k)(z-1)^k.$$

Formula 8.949.2 from [GR07] expresses the derivatives of $T_n(z)$ in terms of the Gegenbauer polynomials, and formula 8.937.4 from [GR07] evaluates the Gegenbauer polynomials at $z = 1$. When combining these results, we arrive at the equality $a_n(0) = 1$ and, moreover, that

$$a_n(k) = \frac{1}{k!} \cdot \prod_{j=0}^{k-1} \frac{n^2 - j^2}{(2j+1)} \quad \text{for } 0 \leq k \leq n. \tag{50}$$

One can proceed in a similar way, this time by applying 8.949.5 from [GR07] which expresses the derivatives of $U_n(z)$ in terms of the Gegenbauer polynomials. In doing so, we arrive at the equality that

$$b_n(k) = \frac{1}{(n+1)k!} \cdot \prod_{j=0}^{k} \frac{(n+1)^2 - j^2}{(2j+1)} \quad \text{for } 0 \le k \le n. \tag{51}$$

Let

$$T_n(z) = \sum_{j=0}^{n} t_n(j) z^j \quad \text{and} \quad U_n(z) = \sum_{j=0}^{n} u_n(j) z^j.$$

Formula 8.994 from [GR07] implies that $t_n(0) = u_n(0) = 0$ for odd $n$ and $t_{2k}(0) = u_{2k}(0) = (-1)^k$ for even integers $n = 2k$. The formulas 9.392.2 and 8.392.3 from [GR07] express the Gegenbauer polynomials in terms of the hypergeometric function. When combining with formulas 8.949.2 and 8.949.2 from [GR07], we get expressions for the coefficients $t_n(j)$ and $u_n(j)$ as follows. First, assume that $n = 2k+1 > 0$ is odd. Then $t_n(j) = u_n(j) = 0$ for all even $0 \le j < n$, and for odd values of $1 \le j \le n$ one has

$$t_n(j) = (-1)^{\frac{n-j}{2}} \frac{n}{n+j} \binom{\frac{n+j}{2}}{\frac{n-j}{2}} \cdot 2^j \quad \text{and} \quad u_n(j) = (-1)^{\frac{n-j}{2}} \binom{\frac{n+j}{2}}{\frac{n-j}{2}} \cdot 2^j. \tag{52}$$

When $n = 2k$ is even, then $t_n(j) = u_n(j) = 0$ for all odd values of $1 \le j < n$, while for even values of $0 \le j \le n$ the coefficients $t_n(j)$ and $u_n(j)$ are given by (52).

## References


[AZ23] Allouche, J.-P., Zeilberger, D.: *Human and automated approaches for finite trigonometric sums*, Ramanujan J 62 (2023), no.1, 189–214.

[AH18] Annaby, M. A., Hassan, H. A.: *Trigonometric sums by Hermite interpolations*, Appl. Math. Comput. **330** (2018), 213—-224.

[Ap73] Apostol, Tom M. *Another elementary proof of Euler's formula for $\zeta(2n)$*. Amer. Math. Monthly 80 (1973), 425–431.

[Ba79] Babai, L.: *Spectra of Cayley graphs*, J. Combin. Theory Ser. B **27** (1979), no. 2, 180–189.

[BBCZ05] Beck, M., Berndt, B. C., Chan, O-Y., Zaharescu, A.: *Determinations of analogues of Gauss sums and other trigonometric sums*, Int. J. Number Theory **1** (2005), no. 3, 333–356.

[BY02] Berndt, B.C., Yeap, B. P.: *Explicit evaluations and reciprocity theorems for finite trigonometric sums*, Adv. in Appl. Math. **29** (2002), no. 3, 358–385.

[BZ04] Berndt, B.C., Zaharescu, A.: *Finite trigonometric sums and class numbers*, Math. Ann. **330** (2004), no. 3, 551–575.

[BC13] Bettin, S., Conrey, J. B.: *A reciprocity formula for a cotangent sum*, Int. Math. Res. Not. IMRN **24** (2013) 5709–5726.

[CHJSV23] Cadavid, C., Hoyos, P., Jorgenson, J., Smajlović, L., and Velez, J.: *On an approach for evaluating certain trigonometric character sums using the discrete time heat kernel*, European J. Combin. **108** (2023), Paper No. 103635.

[Ch12] Chair, N.: *Exact two-point resistance, and the simple random walk on the complete graph minus N edges*, Ann. Physics **327** (2012), no. 12, 3116–3129.

[Ch14a] Chair, N.: *Trigonometrical sums connected with the chiral Potts model, Verlinde dimension formula, two-dimensional resistor network, and number theory*, Ann. Physics **341** (2014), 56–76.

[Ch14b] Chair, N.: *The effective resistance of the N-cycle graph with four nearest neighbors*, J. Stat. Phys. **154** (2014), no. 4, 1177–1190.

[Ch18] Chu, W.: *Reciprocal relations for trigonometric sums*, Rocky Mountain J. Math. **48** (2018), no. 1, 121–140.

[CM99] Chu, W. Marini, A.: *Partial fractions and trigonometric identities*, Adv. Appl. Math. **23** (1999), 115—175.

[CY00] Chung, F., Yau, S.-T.: *Discrete Green's functions*, J. Combin. Theory Ser. A **91** (2000), no. 1-2, 191–214.

[CR62] Curtis, C.W., Reiner, I.: *Representation theory of finite groups and associative algebras*, Reprint of the 1962 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1988.

[CS12] Cvijović, D., Srivastava, H. M.: *Closed-form summations of Dowker's and related trigonometric sums*, J. Phys. A **45** (2012), no. 37, 374015, 10 pp.

[Do06] Dodziuk, J.: *Elliptic operators on infinite graphs*, Analysis geometry and topology of elliptic operators. World Sci. Publ, NJ (2006), pp. 353–368.

[DM06] Dodziuk, J:, Mathai, V.: *Kato's inequality and asymptotic spectral properties for discrete magnetic Laplacians*, in *The ubiquitous heat kernel*, J. Jorgenson and L. Walling, eds. volume 398 of Contemp. Math. Amer. Math. Soc. Providence, RI (2006), pp. 69–81.

[Do89] Dowker, J. S.: *Heat kernel expansion on a generalized cone*, J. Math. Phys. **30** (1989), no. 4, 770–773.

[Do92] Dowker, J. S.: *On Verlinde's formula for the dimensions of vector bundles on moduli spaces*, J. Phys. A **25** (1992), no. 9, 2641–2648.

[Do12] Dowker, J. S.: *Heat–kernels on the discrete circle and interval*, https://arxiv.org/pdf/1207.2096.pdf.

[Do15] Dowker, J. S.: *On sums of powers of cosecs*, https://arxiv:1507.01848, 2015.

[EL21] Ejsmont, W., Lehner, F.: *The trace method for cotangent sums*, J. Combin. Theory Ser. A **177** (2021), Paper No. 105324, 32 pp.

[EW09] Essam, J. W., Wu, F. Y.: *The exact evaluation of the corner-to-corner resistance of an $M \times N$ resistor network: asymptotic expansion*, J. Phys. A **42** (2009), no. 2, 025205, 10 pp.

[F-DG-D14] Fernández-Durán, J. J., Gregorio-Domínguez, M. M.: *Modeling angles in proteins and circular genomes using multivariate angular distributions based on multiple nonnegative trigonometric sums*, Stat. Appl. Genet. Mol. Biol. **13** (2014), no. 1, 1–18.

[dFGK17] da Fonseca, C. M., Glasser, M. L., Kowalenko, V.: *Basic trigonometric power sums with applications*, Ramanujan J. **42** (2017), no. 2, 401–428.

[dFGK18] da Fonseca, C. M., Glasser, M. L., Kowalenko, V.: *Generalized cosecant numbers and trigonometric inverse power sums*, Appl. Anal. Discrete Math. **12** (2018), no. 1, 70–109.

[F16] Friedli, Fabien *A functional relation for L-functions of graphs equivalent to the Riemann hypothesis for Dirichlet L-functions.* J. Number Theory **169** (2016), 342–352

[FK17] Friedli, F., Karlsson, A.: *Spectral zeta functions of graphs and the Riemann zeta function in the critical strip*, Tohoku Math. J. (2) **69** (2017), no. 4, 585–610.

[GR07] Gradshteyn, I. S., Ryzhik, I. M.: *Table of Integrals, Series and Products.* Elsevier Academic Press, Amsterdam, 2007.

[GLY22] Grigor'yan, Alexander; Lin, Yong; Yau, Shing-Tung: *Discrete tori and trigonometric sums.* J. Geom. Anal. **32** (2022), no. 12, Paper No. 298, 17 pp.

[Ha08] Hassan, H. A.: *New trigonometric sums by sampling theorem* J. Math. Anal. Appl. **339** (2008), no. 2, 811–827.

[He20] He, Y.: *Explicit expressions for finite trigonometric sums*, J. Math. Anal. Appl. **484** (2020), no. 1, 123702, 24 pp.

[HZ74] Hirzebruch, F.; Zagier, D. *The Atiyah-Singer theorem and elementary number theory.* Mathematics Lecture Series, No. 3. Publish or Perish, Inc., Boston, Mass., 1974. xii+262 pp.

[KN06] Karlsson, A., Neuhauser, M.: *Heat kernels, theta identities, and zeta functions on cyclic groups*, in: Grigorchuk R., Mihalik, M., Sapir M., Šunić, Z. eds. Topological and asymptotic aspects of group theory, 177—189, Contemp. Math., **394**, Amer. Math. Soc., Providence, RI, 2006.

[MO96] McCoy, B. M., Orrick, W. P.: *Analyticity and integrability in the chiral Potts model*, J. Stat. Phys. **83** (1996), 839–865.

[Me17] Merca, M.: *On families of linear recurrence relations for the special values of the Riemann zeta function*, J. Number Theory **170** (2017), 55–65.

[MS20] Milovanović, G.V., Simsek, Y.: *Dedekind and Hardy type sums and trigonometric sums induced by quadrature formulas*, Trigonometric sums and their applications, 183–228, Springer, Cham, 2020.

[SS14] Slavík, A., Stehlík P.: *Explicit solutions to dynamic diffusion-type equations and their time integrals*, Applied Mathematics and Computation **234** (2014), 486–505.

[SS15] Slavík, A., Stehlík P.: *Dynamic diffusion-type equations on discrete-space domains*, J. Math. Anal. Appl. **427** (2015), no. 1, 525–545.

[Ta92] Thaddeus, M.: *Conformal field theory and the cohomology of the moduli space of stable bundles*, J. Differential Geom. **35** (1992), no. 1, 131–149.

[Ve01] Vergne, M.: *Residue formulae for Verlinde sums, and for number of integral points in convex rational polytopes*, in: European women in mathematics (Malta, 2001), World Scientific, 2003, 225–285.

[Ve88] Verlinde, E.: *Fusion rules and modular transformations in 2D conformal field theory*, Nuclear Phys. B **300** (1988), no. 3, 360–376.

[WZ07] Wang, X., Zheng, D. - Y.: *Summation formulae on trigonometric functions*, J. Math. Anal. Appl. **335** (2007), no. 2, 1020–1037.

[Wu04] Wu, F. Y.: *Theory of resistor networks: the two-point resistance*, J. Phys. A **37** (2004), no. 26, 6653–6673.

[XZZ24] Xie B., Zhao Y., Zhao Y.: *Special values of spectral zeta functions of graphs and Dirichlet L-functions*, J. Number Theory **256** (2024), 136–159.

[Za96] Zagier, Don, *Elementary aspects of the Verlinde formula and of the Harder-Narasimhan-Atiyah-Bott formula.* Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), 445–462, Israel Math. Conf. Proc., 9, Bar-Ilan Univ., Ramat Gan, 1996.

Jay Jorgenson
Department of Mathematics
The City College of New York
Convent Avenue at 138th Street
New York, NY 10031 U.S.A.
e-mail: jjorgenson@mindspring.com

Anders Karlsson
Section de mathématiques
Université de Genève
Case postale 64, 1211 Genève, Switzerland;
Mathematics department
Uppsala University
Box 256, 751 05 Uppsala, Sweden
e-mail: anders.karlsson@unige.ch

Lejla Smajlović
Department of Mathematics and Computer Science
and School of Economics and Business
University of Sarajevo
Zmaja od Bosne 35, 71 000 Sarajevo
Bosnia and Herzegovina
e-mail: lejlas@pmf.unsa.ba